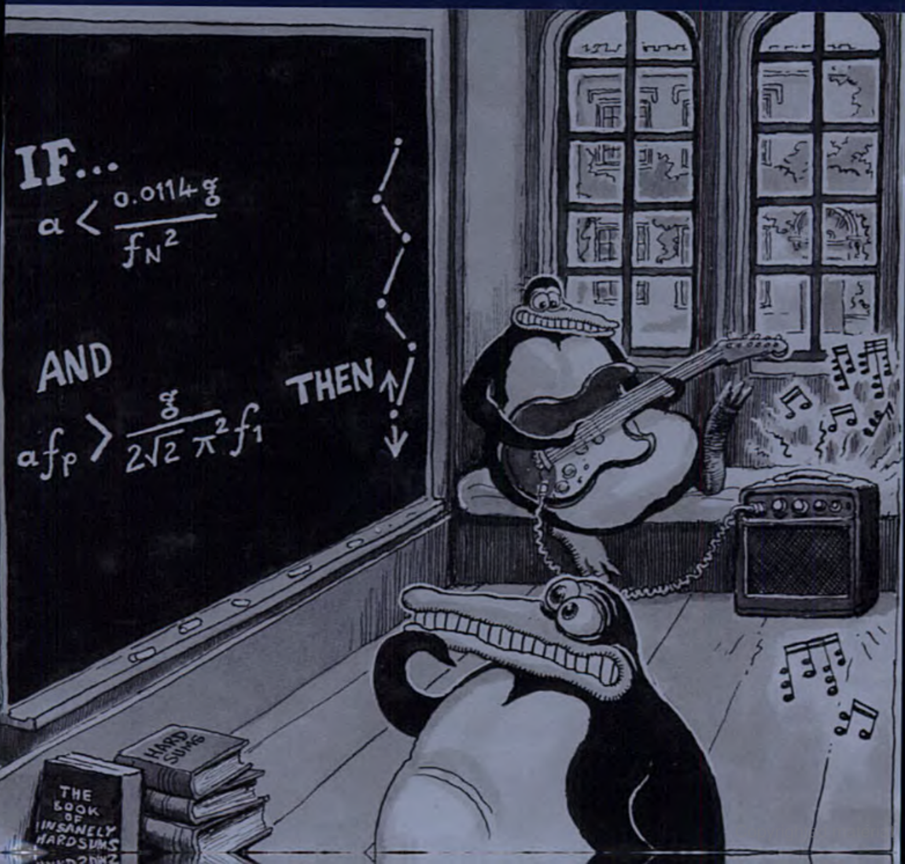
OXFORD

1089 and all that

A JOURNEY INTO MATHEMATICS

David Acheson



1089 and All That

A Journey into Mathematics

David Acheson
Jesus College, Oxford

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2002
Reprinted 2003, 2004

A catalogue record for this book is available from the British Library

Cataloging in Publication Data
(Data available)

ISBN 0 19 851623 1

10 9 8 7 6 5 4 3

Printed in Great Britain
on acid-free paper by Biddles Ltd, King's Lynn, Norfolk

CHAPTER ONE

1089 and All That

Think of a three-figure number.

Any three-figure number will do, so long as the first and last figures differ by 2 or more.

Now reverse it, and subtract the smaller number from the larger. So, for example,

$$782 - 287 = 495.$$

Finally, reverse the new three-figure number, and add:

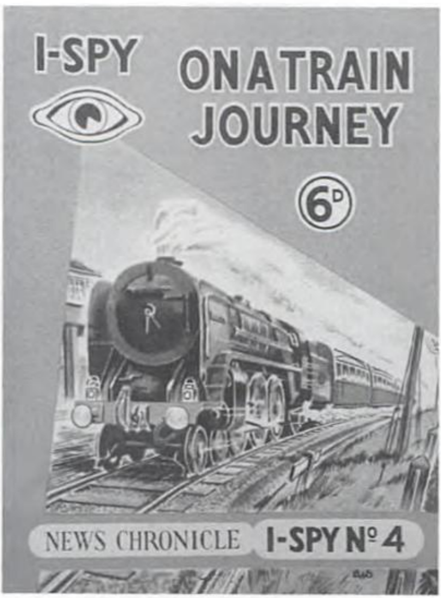
$$495 + 594 = 1089.$$

At the end of this procedure, then, we have a final answer of 1089, though we have to expect, surely, that this final answer will depend on which three-figure number we start with.

But it *doesn't*.

The final answer always turns out to be 1089.

As I remember, the '1089 trick' was the first piece of mathematics that really impressed me, and I came across it at the age of ten, in the *I-SPY Annual* for 1956.



This was a book for children, published by a well-known British newspaper of the time, and it contained a mixture of adventure stories and more educational articles with titles like 'Pond Life'.

But my favourite bit, by a long way, was

ABRACADABRA!

and Uncle Jack turns you into a conjuror

A TRICK WITH NUMBERS

The conjuror writes a number on the blank side of the slate he is holding. A friend is asked to write a number of three different figures on a piece of paper. He must then turn this number round and take the smaller number from the larger and finally turn this number round and add it to the result of the subtraction.

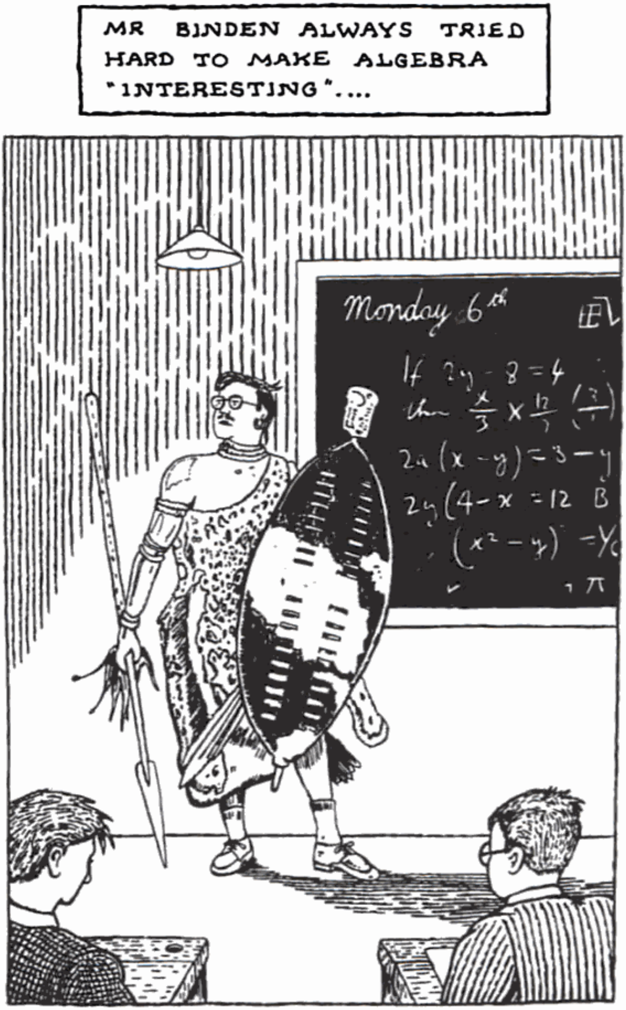
When this has been done, the conjuror turns the slate round and shows that he has written the final number 1089.

SECRET

The number arrived at in this trick is always 1089.

There were other conjuring tricks as well, including 'The Vanishing Glass of Water' and 'Reading the Mind', but somehow it was '1089' that really caught my attention.

It was the element of mystery and surprise, I think, that put this result into a different league from some of the work we were doing in school.


MR BINDEN ALWAYS TRIED HARD TO MAKE ALGEBRA "INTERESTING"....
Monday 6th

Now, I'm not saying that I didn't enjoy 'sums', and other bits of elementary mathematics, for I most certainly did. But if I tell you, for instance, that a typical homework problem at the time went something like this:

> A and B can fill a cistern in 4 hours. A and C can fill the same cistern in 5 hours. B can fill twice as fast as C. Find how long C would take to fill the cistern, working alone.*

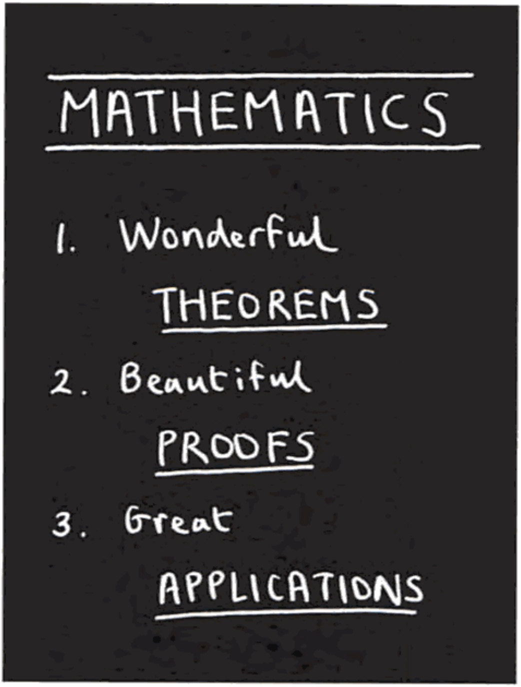
I think you will understand why the '1089' trick made such an impression.

And now, over 40 years later, it seems to me that these same elements of mystery and surprise run through a great deal of mathematics at its best. Some of the first-rate theorems and results really do generate a sense of *wonder*.

I hope to show something of this as we go through the book, and I hope to show, too, how there is much pleasure to be had, from time to time, in the actual deductive arguments by which those theorems and results are *proved*.

In addition to all this, we shall take in several remarkable applications of mathematics to science and nature.

* C would in fact take 20 hours to fill the cistern, poor devil.

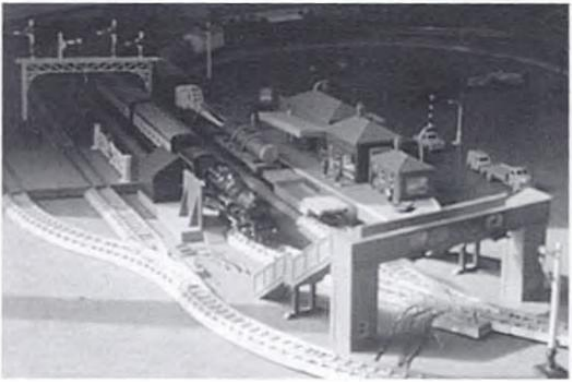


So, whether you are very young or very old, or somewhere in between; whether you are at school or at university, or neither; whether you have a pen in your hand, or a gin and tonic . . . we are about to go on a journey.

Along the way we shall be taking in some of the most important ideas of mathematics, and something of their history.

We will be going, in short, from first steps to the frontiers, and in order to keep track of 'the big picture' while we're doing all this, we will be going along quite fast.

If we imagine that we are on a train, for instance, then it will be the *Mathematics Express* . . .



CHAPTER TWO

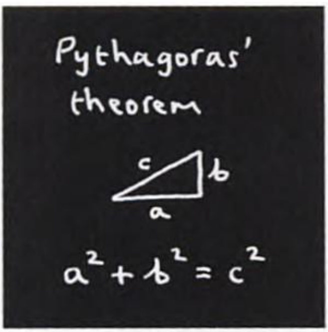
'In Love with Geometrie'

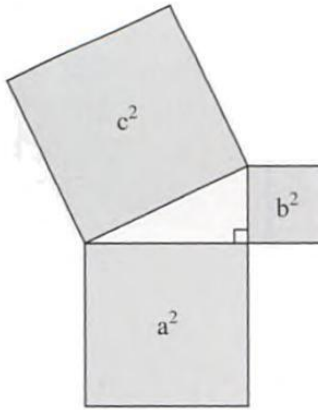
One of the best-documented examples of someone being really surprised by mathematics is to be found in the following anecdote about the philosopher Thomas Hobbes (1588–1679):

> He was 40 yeares old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the *47 El. libri 1*. He read the Proposition. *By G –*, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible!*

Here, then, is an example of mathematics at its best, for Hobbes found the result so stunning that he couldn't quite believe it.

The result in question was, in fact, none other than *Pythagoras' theorem*: if a, b and c are the sides of a right-angled triangle, and c is the longest side, then $a^2 + b^2 = c^2$.





And Hobbes didn't just take somebody's word for this; he read a *proof*. It was this proof, as much as anything else, that made him

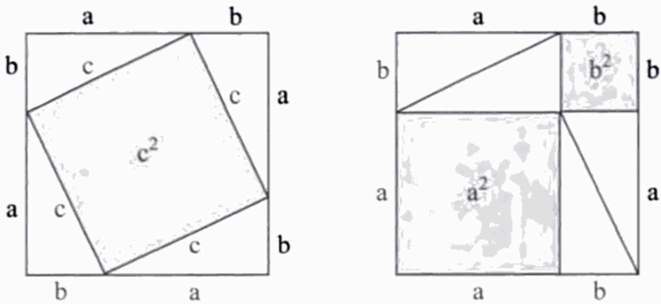
> . . . in love with Geometrie.

So we too will now prove Pythagoras' theorem.

I can see, of course, that this might prompt the question: *why bother*? After all, the theorem has been around for over 2000 years. Everybody knows Pythagoras' theorem. Surely, if it were not true, if there were anything wrong with it, *somebody would have noticed by now*.

In mathematics, however, this kind of argument is virtually worthless.

And in any case, the following delightfully simple proof of Pythagoras' theorem is almost fun.

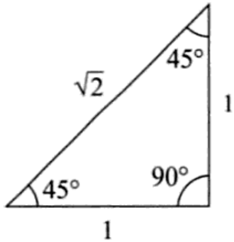


Take a square of side $a + b$ and place 4 copies of the original right-angled triangle within it, as shown. This leaves a square area c^2. Now think of the triangles as white tiles on a dark floor, and move three of them so that they occupy the new positions indicated. The floor area *not* occupied by triangles is now $a^2 + b^2$, yet must be the same as before.

So $a^2 + b^2 = c^2$.

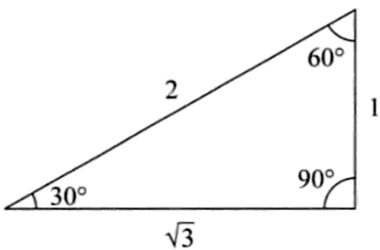
Two special cases of Pythagoras' theorem are of particular interest.

One is when the smaller angles of the right-angled triangle are both 45°:



If the two shorter sides are of length 1, say, then the length of the longest side is $\sqrt{2}$, and this is one common way in which the square root of 2 arises in mathematics.

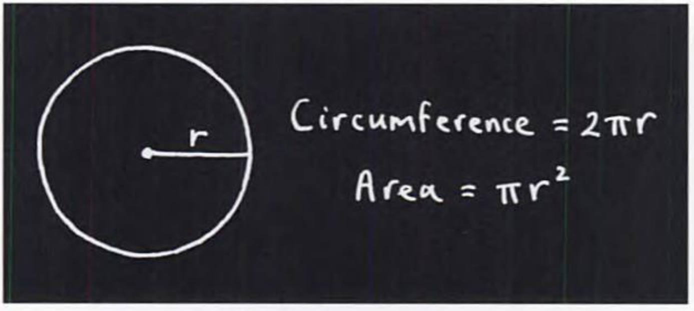
Another commonly-occurring special case is when the two smaller angles are 30° and 60°:



But these *are* just special cases. The real power and importance of Pythagoras' theorem lies in its *generality*; it is equally true whether the right-angled triangle in question is short and fat or long and thin.

And we know this not because Professor X – who is supposedly a world expert – assures us that it is, but because we have seen it for ourselves.

If Pythagoras' theorem is the most well-known result in the whole of geometry, then the next best known must surely be the formulae for the circumference and area of a circle of radius r:



And this is the way in which the special number

$$\pi = 3.14159\ldots$$

first enters mathematics. In 'elementary' mathematics, π is all about circles.

Imagine the surprise, then, in the mid-seventeenth century, when mathematicians found π cropping up in all sorts of places that had, apparently, nothing to do with circles at all.

One of the most famous results of this kind is an extraordinary connection between π and the *odd numbers*:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Here, as the dots indicate, we are meant to keep on adding and subtracting the fractions on the right-hand side *for ever*. First, then, it is not at all obvious that the 'sum' in question settles down to any definite value at all.

But, even given that it does, why should that value be $\frac{\pi}{4}$? What on earth have circles got to do with the odd numbers 1, 3, 5, 7, . . .?

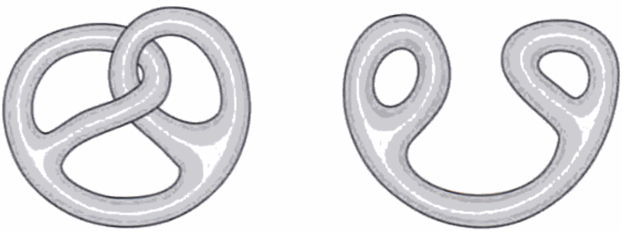
Surprising *connections* of this kind are just the sort of thing that get mathematicians really excited.

Today, the whole subject of geometry extends way beyond the world of right-angled triangles, circles and so on. There are even branches of the subject in which the ideas of length, angle and area don't really feature at all.

One of these is *topology* – a sort of rubber-sheet geometry – where a recurring question is whether some geometric object can be deformed 'smoothly' into another one.

Questions of this kind can be very demanding, and even counter-intuitive.

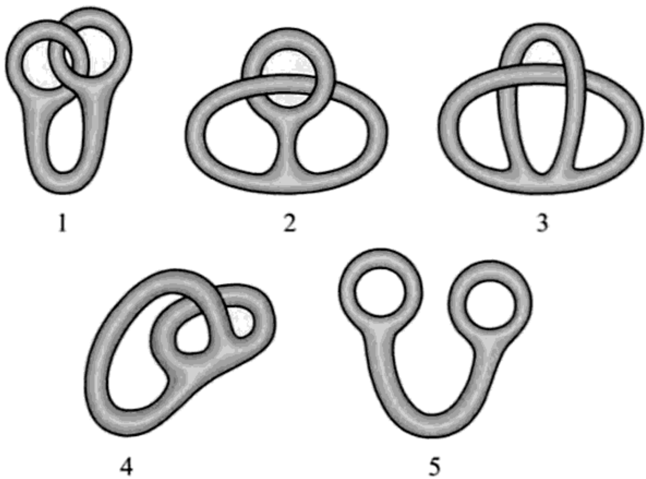
Look, for example, at the two geometric objects below, and ask yourself if the one on the left can be deformed smoothly into the one on the right.



Imagine, if you will, that the object is made of some very elastic material, so that you can stretch or squash it as much as you like.

Is it possible, then, to deform the object – without cutting or tearing it – into its 'unlinked' version?

Well, actually . . . it *is*:



We end this chapter, though, by returning to one of the most important legacies of ancient Greek geometry: the concept of *proof*.

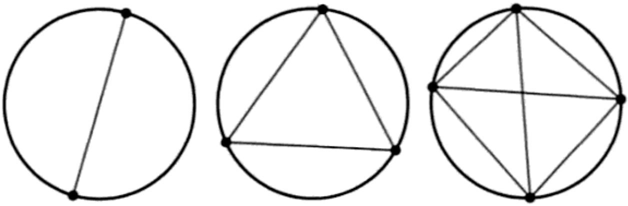
One reason for emphasizing this idea so early in the book is that it is all too easy in mathematics to jump to the wrong conclusion.

And it is particularly dangerous to jump to some general conclusion on the basis of a few special cases.

Here's an example. Take a circle, mark 2 points on the circumference and join them by a straight line. This divides the circle into 2 regions.

Now mark 3 points on the circumference instead, and join each point to *all* the others by straight lines. We get 4 regions.

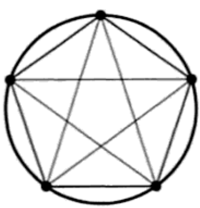
If we do the same thing with 4 points, we get 8 regions.



$n = 2$ $n = 3$ $n = 4$

The pattern seems clear, doesn't it? The number of regions appears to be doubling every time we add an extra point. So we suspect that with $n = 5$ there will be 16 regions.

And so there are:



$n = 5$

At which point, surely, we conclude with rather more confidence that with $n = 6$ the number of different regions will be 32.

But it isn't.

It's 31:

$n = 6$

And the general formula for the number of regions isn't the simple one we had in mind at all.

It's $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$.

And that's why mathematicians need *proof*.

CHAPTER THREE

But . . . that's Absurd . . .

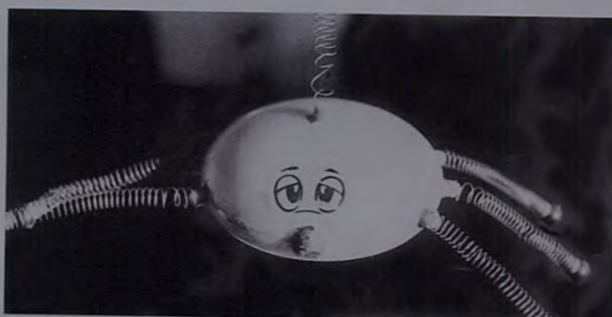
At the end of *The Adventure of the Beryl Coronet*, Sherlock Holmes explains his methods of deduction, as usual, and remarks:

> It is an old maxim of mine that when you have excluded the impossible, whatever remains, however improbable, must be the truth.

This is, in a way, like *proof by contradiction*, which is one of the most elegant and powerful techniques in the whole of mathematics.

'This is a clever book...My enthusiasm for it knows no bounds.'
Ian Stewart, *New Scientist*

'A lovely little book...'
Simon Singh, Author of *Fermat's Last Theorem*



OXFORD
UNIVERSITY PRESS

www.oup.com

ISBN 0-19-851623-1

9 780198 516231