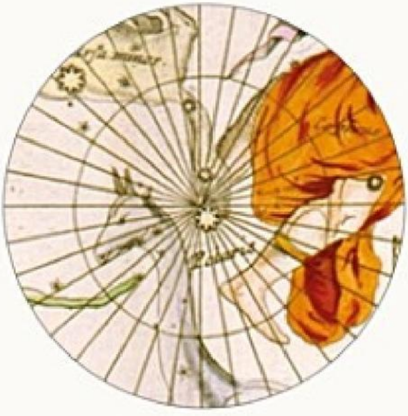
A BEAUTIFUL QUESTION



FINDING NATURE'S DEEP DESIGN

FRANK WILCZEK

WINNER OF THE NOBEL PRIZE IN PHYSICS

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ALSO BY FRANK WILCZEK

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This work was prepared especially for A Beautiful Question *by He Shuifa, a modern master of traditional Chinese art and calligraphy. He is renowned for the vigor and subtlety of his brushwork and for the spiritual depth of his depictions of flowers, birds, and nature. A simple translation of the inscription is this: "Taiji double fish is the essence of Chinese culture. This image was painted by He Shuifa on a lake in early winter." The playful "double fish" aspect of Taiji comes to life in He Shuifa's image. The yin and yang resemble two carp playing together, and there are hints of their eyes and fins. In Henan, on the Yellow River,* *there is a waterfall called Dragon's Gate. Yulong carp attempt to jump the cataract, although it is very difficult for them. Those that succeed transform into lucky dragons. With a sense of humor, we may associate this event with the transformation of virtual into real particles, an essential quantum process that is now thought to underlie the origin of structure in the Universe (see plates XX and AAA). Alternatively we may identify ourselves with the carp, and their strivings with our quest for understanding.*

TO MY FAMILY AND FRIENDS:
BEAUTIFUL ANSWERS OF THE SECOND KIND

traditions. Many motivations have been ascribed to the Creator, but artistic ambition is rarely prominent among them.

In Abrahamic religions, conventional doctrine holds that the Creator set out to embody some combination of goodness and righteousness, and to create a monument to His glory. Animistic and polytheistic religions have envisaged beings and gods who create and govern different parts of the world with many kinds of motives, running the gamut from benevolence to lust to carefree exuberance.

On a higher theological plane, the Creator's motivations are sometimes said to be so awesome that finite human intellects can't hope to comprehend them. Instead we are given partial revelations, which are to be believed, not analyzed. Or, alternatively, God is Love. None of those contradictory orthodoxies offers compelling reasons to expect that the world embodies beautiful ideas; nor do they suggest that we should strive to find such ideas. Beauty can form part of their cosmic story, but it is generally regarded as a side issue, not the heart of the matter.

Yet many creative spirits have found inspiration in the idea that the Creator might be, among other things, an artist whose esthetic motivations we can appreciate and share—or even, in daring speculation, that the Creator is *primarily* a creative artist. Such spirits have engaged our Question, in varied and evolving forms, across many centuries. Thus inspired, they have produced deep philosophy, great science, compelling literature, and striking imagery. Some have produced works that combine several, or all, of those features. These works are a vein of gold running back through our civilization.

Galileo Galilei made the beauty of the physical world central to his own deep faith, and recommended it to all:

> The greatness and the glory of God shine forth marvelously in all His works, and is to be read above all in the open book of the heavens.

… as did Johannes Kepler, Isaac Newton, and James Clerk Maxwell. For all these searchers, finding beauty embodied in the physical world, reflecting God's glory, was the goal of their search. It inspired their work, and sanctified their curiosity. And with their discoveries, their faith was rewarded.

While our Question finds support in spiritual cosmology, it can also stand on its own. And though its positive answer may inspire a spiritual interpretation, it does not require one.

We will return to these thoughts toward the end of our meditation, by which point we will be much better prepared to appraise them. Between now and then, the world can speak for itself.

HEROIC VENTURES

Just as art has a history, with developing standards, so does the concept of the world as a work of art. In art history, we are accustomed to the idea that old styles are not simply obsolete, but can continue to be enjoyed on their own terms, and also offer important context for later developments. Though that idea is much less familiar in science, and in science it is subject to important limitations, the historical approach to our Question offers many advantages. It allows us—indeed, forces us—to proceed from simpler to more complex ideas. At the same time, by exploring how great thinkers struggled and often went astray, we gain perspective on the initial strangeness of ideas that have become, through familiarity, too "obvious" and comfortable. Last but by no means least, we humans are especially adapted to think in story and narrative, to associate ideas with names and faces, and to find tales of conflicts and their resolution compelling, even when they are conflicts of ideas, and no blood gets spilled. (Actually, a little does …)

For these reasons we will sing, to begin, songs of heroes: Pythagoras, Plato, Filippo Brunelleschi, Newton, Maxwell. (Later a major heroine, Emmy Noether, will enter too.) Real people went by those names—very interesting ones! But for us they are not merely people, but also legends and symbols. I've portrayed them, as I think of them, in that style, emphasizing clarity and simplicity over scholarly nuance. Here biography is a means, not an end. Each hero advances our meditation several steps:

- *Pythagoras* discovered, in his famous theorem about right-angled triangles, a most fundamental relationship between numbers, on the one hand, and sizes and shapes, on the other. Because Number is the purest product of Mind, while Size is a primary characteristic of Matter, that discovery revealed a hidden unity between Mind and Matter.

  Pythagoras also discovered, in the laws of stringed instruments, simple and surprising relationships between numbers and musical harmony. That discovery completes a trinity, Mind-Matter-Beauty, with Number as the linking thread. Heady stuff! It led Pythagoras to surmise that All Things Are Number. With these discoveries and speculations, our Question comes to life.

- *Plato* thought big. He proposed a geometric theory of atoms and the Universe, based on five symmetrical shapes, which we now call the Platonic solids. In this audacious model of physical reality, Plato valued beauty over accuracy. The details of his theory are hopelessly wrong. Yet it provided such a dazzling vision of what a positive answer to our Question might look like that it inspired Euclid, Kepler, and many others to brilliant work centuries later. Indeed, our modern, astoundingly successful theories of elementary particles, codified in our Core Theory, are rooted in heightened ideas of symmetry that would surely make Plato smile. And when trying to guess what will come next, I often follow Plato's strategy, proposing objects of mathematical beauty as models for Nature.

  Plato was also a great literary artist. His metaphor of the Cave captures important emotional and philosophical aspects of our relationship, as human inquirers, with reality. At its core is the belief that everyday life offers us a mere shadow of reality, but that through adventures of mind, and sensory expansion, we can get to its essence—and that the essence is clearer and more beautiful than its shadow. He imagined a mediating *demiurge*, which can be translated as *Artisan*, who rendered the realm of perfect, eternal Ideas into its imperfect copy, the world we experience. Here the concept of the world as a work of art is explicit.
- *Brunelleschi* brought new ideas to geometry from the needs of art and engineering. His *projective geometry*, in dealing with the actual appearance of things, brought in ideas—relativity, invariance, symmetry—not only beautiful in themselves, but pregnant with potential.
- *Newton* brought the mathematical understanding of Nature to entirely new levels of ambition and precision.

A common theme pervades Newton's titanic work on light, the mathematics of calculus, motion, and mechanics. It is the method he called Analysis and Synthesis. The method of Analysis and Synthesis suggests a two-stage strategy to achieve understanding. In the analysis stage, we consider the smallest parts of what we are studying their "atoms," using the word figuratively. In a successful analysis, we identify small parts that have simple properties that we can summarize in precise laws. For example:

- In the study of light, the atoms are beams of pure spectral colors.
- In the study of calculus, the atoms are infinitesimals and their ratios.
- In the study of motion, the atoms are velocity and acceleration.
- In the study of mechanics, the atoms are forces.

(We'll discuss these in more depth later.) In the synthesis stage we build up, by logical and mathematical reasoning, from the behavior of individual atoms to the description of systems that contain many atoms.

When thus stated broadly, Analysis and Synthesis may not seem terribly impressive. It is, after all, closely related to common rules of thumb, e.g., "to solve a complex problem, divide and conquer"—hardly an electrifying revelation. But Newton demanded precision and completeness of understanding, saying,

> 'Tis much better to do a little with certainty & leave the rest for others that come after than to explain all things by conjecture without making sure of any thing.

And in these impressive examples, he achieved his ambitions. Newton showed, convincingly, that Nature herself proceeds by Analysis and Synthesis. There really is simplicity in the "atoms," and Nature really does operate by letting them do their thing.

Newton also, in his work on motion and mechanics, enriched our concept of what physical laws are. His laws of motion and of gravity are *dynamical* laws. In other words, they are laws of change. Laws of this kind embody a different concept of beauty than the static perfection beloved of Pythagoras and (especially) Plato.

Dynamical beauty transcends specific objects and phenomena, and invites us to imagine the expanse of possibilities. For example, the sizes and shapes of actual planetary orbits are not simple. They are neither the (compounded) circles of Aristotle, Ptolemy, and Nicolaus Copernicus, nor even the more nearly accurate ellipses of Kepler, but rather curves that must be calculated numerically, as functions of time, evolving in complicated ways that depend on the positions and masses of the Sun and the other planets. There is great beauty and simplicity here, but it is only fully evident when we understand the deep design. The appearance of particular objects does not exhaust the beauty of the laws.

- *Maxwell* was the first truly modern physicist. His work on electromagnetism ushered in both a new concept of reality and a new method in physics. The new concept, which Maxwell developed from the intuitions of Michael Faraday, is that the primary ingredients of physical reality are not point-like *particles*, but rather space-filling *fields*. The new method is *inspired guesswork*. In 1864 Maxwell codified the known laws of electricity and magnetism into a system of equations, but discovered the resulting system was inconsistent. Like Plato, who shoehorned five perfect solids into four elements plus the Universe, Maxwell did not give up. He saw that by adding a new term he could both make the equations appear more symmetric and make them mathematically consistent. The resulting system, known as the Maxwell equations, not only unified electricity and magnetism, but derived light as a consequence, and survives to this day as the secure foundation of those subjects.

  By what is the physicist's "inspired guesswork" inspired? Logical consistency is necessary, but hardly sufficient. Rather it was beauty and symmetry that guided Maxwell and his

Different artists have different styles. We don't expect to find Renoir's shimmering color in Rembrandt's mystic shadows, or the elegance of Raphael in either. Mozart's music comes from a different world entirely, the Beatles' from another, and Louis Armstrong's from yet another. Likewise, the beauty embodied in the physical world is a particular kind of beauty. Nature, as an artist, has a distinctive style.

To appreciate Nature's art, we must enter her style with sympathy. Galileo, ever eloquent, expressed it this way:

> Philosophy [Nature] is written in that great book which ever is before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circles, and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

Today we've penetrated much further into the great book, and discovered that its later chapters use a more imaginative, less familiar language than the Euclidean geometry Galileo knew. To become a fluent speaker in it is the work of a lifetime (or at least of several years in graduate school). But just as a graduate degree in art history is not a prerequisite for engaging with the world's best art and finding that a deeply rewarding experience, so I hope, in this book, to help you engage with Nature's art, by making her style accessible. Your effort will be rewarded, for as Einstein might have said,

> Subtle is the Lord, but malicious She is not.

Two obsessions are the hallmarks of Nature's artistic style:

- Symmetry—a love of harmony, balance, and proportion
- Economy—satisfaction in producing an abundance of effects from very limited means

Watch for these themes as they recur, grow, and develop throughout our narrative and give it unity. Our appreciation of them has evolved from intuition and wishful thinking into precise, powerful, and fruitful methods.

Now, a disclaimer. Many varieties of beauty are underrepresented in Nature's style, as expressed in her fundamental operating system. Our delight in the human body and our interest in expressive portraits, our love of animals and of natural landscapes, and many other sources of artistic beauty are not brought into play. Science isn't everything, thank goodness.

CONCEPTS AND REALITIES; MIND AND MATTER

Our Question can be read in two directions. Most obviously, it is a question about the world. That is the direction we've emphasized so far. But the other direction is likewise fascinating. When we find that *our* sense of beauty is realized in the physical world, we are discovering something about the world, but also something about ourselves.

Human appreciation of the fundamental laws of Nature is a recent development on evolutionary or even historical time scales. Moreover, those laws reveal themselves only after elaborate operations—looking through sophisticated microscopes and telescopes, tearing atoms and nuclei apart, and processing long chains of mathematical reasoning—that do not come naturally. Our sense of beauty is not in any very direct way adapted to Nature's fundamental workings. Yet just as surely, our sense of beauty is excited by what we find there.

What explains that miraculous harmony of Mind and Matter? Without an explanation of that miracle, our Question remains mysterious. It is an issue our meditation will touch upon repeatedly. For now, two brief anticipations:

1. We human beings are, above all, visual creatures. Our sense of vision, of course, and in a host of less obvious ways our deepest modes of thought, are conditioned by our interaction with light. Each of us, for example, is born to become an accomplished, if unconscious, practitioner of projective geometry. That ability is hardwired into our brain. It is what allows us to interpret the two-dimensional image that arrives on our retinas as representing a world of objects in three-dimensional space.

   Our brains contain specialized modules that allow us to construct, very quickly and without conscious effort, a dynamic worldview based on three-dimensional objects located in three-dimensional space. We do this beginning from two two-dimensional images on the retinas of our eyes (which, in turn, are the product of light rays emitted or reflected from the surfaces of external objects, which propagate to us in straight lines). To work back from the images we receive to the objects that cause them is a tricky problem in inverse projective geometry. In fact, as stated, it is an impossible problem, because there's not nearly enough information in the projections to do an unambiguous reconstruction. A basic problem is that even to get started we need to separate objects from their background (or foreground). We exploit all kinds of tricks based on typical properties of objects we encounter, such as their color or texture contrast and distinctive boundaries, to do that job. But even after that step is accomplished, we are left with a difficult geometrical problem, for which Nature has helpfully provided us, in our visual cortex, an excellent specialized processor.

Another important feature of vision is that light arrives to us from very far away, and gives us a window into astronomy. The regular apparent motion of stars and the slightly less regular apparent motion of planets gave early hints of a lawful Universe, and provided an early inspiration and testing ground for the mathematical description of Nature. Like a good textbook, it contains problems with varying degrees of difficulty.

In the more advanced, modern parts of physics we learn that light itself is a form of matter, and indeed that matter in general, when understood deeply, is remarkably light-like. So again, our interest in and experience with light, which is deeply rooted in our essential nature, proves fortunate.

Creatures that, like most mammals, perceive the world primarily through the sense of smell would have a much harder time getting to physics as we know it, even if they were highly intelligent in other ways. One can imagine dogs, say, evolving into extremely intelligent social creatures, developing language, and experiencing rich lives full of interest and joy, but devoid of the specific kinds of curiosity and outlook, based on visual experience, that lead to our kind of deep understanding of the physical world. Their world would be rich in reactions and decays—they'd have great chemistry sets, elaborate cuisines, aphrodisiacs, and, à la Proust, echoing memories. Projective geometry and astronomy, maybe not so much. We understand that smell is a chemical sense, and we are beginning to understand its foundation in molecular events. But the "inverse" problem of working from smell back to molecules and their laws, and eventually to physics as we know it, seems to me hopelessly difficult.

Birds, on the other hand, are visual creatures, like us. Beyond that, their way of life would give them an extra advantage over humans, in getting started on physics. For birds, with their freedom of flight, experience the essential symmetry of three-dimensional space in an intimate way that we do not. They also experience the basic regularities of motion, and especially the role of inertia, in their everyday lives, as they operate in a nearly frictionless environment. Birds are born, one might say, with intuitive knowledge of classical mechanics and Galilean relativity, as well as of geometry. If some species of bird evolved high abstract intelligence—that is, if they ceased being birdbrains—their physics would develop rapidly. Humans, on the other hand, have to unlearn the friction-laden Aristotelean mechanics they use in everyday life, in order to achieve deeper understanding. Historically that involved quite a struggle!

Dolphins, in their watery environment, and bats, with their echolocation, give us other interesting variations on these themes. But I will not develop those here.

A general philosophical point, which these considerations illustrate, is that the world does not provide its own unique interpretation. The world offers many possibilities for different sensory universes, which support very different interpretations of the world's significance. In this way our so-called Universe is already very much a multiverse.

2. Successful perception involves sophisticated inference, because the information we sample about the world is both very partial and very noisy. For all our innate powers, we must also learn how to see by interacting with the world, forming expectations, and comparing our predictions with reality. When we form expectations that turn out to be correct, we experience pleasure

> and satisfaction. Those reward mechanisms encourage successful learning. They also stimulate—indeed, at base they *are*—our sense of beauty.
>
> Putting those observations together, we discover an explanation of why we find interesting phenomena (phenomena we can learn from!) in physics beautiful. An important consequence is that we especially value experience that is surprising, but not too surprising. Routine, superficial recognition will not challenge us, and may not be rewarded as active learning. On the other hand, patterns whose meaning we cannot make sense of at all will not offer rewarding experience either; they are noise.
>
> And here we are lucky too, in that Nature employs, in her basic workings, symmetry and economy of means. For these principles, like our intuitive understanding of light, promote successful prediction and learning. From the appearance of part of a symmetric object we can predict (successfully!) the appearance of the rest; from the behavior of parts of natural objects we can predict (sometimes successfully!) the behavior of wholes. Symmetry and economy of means, therefore, are exactly the sorts of things we are apt to experience as beautiful.

NEW IDEAS AND INTERPRETATIONS

Together with new appreciations of some very old and some less old ideas, you will find in this book several essentially new ones. Here I'd like to mention some of the most important.

My presentation of the Core Theory as geometry, and my speculations about the next steps beyond it, are adaptations of my technical work in fundamental physics. That work builds, of course, on the work of many others. My use of color fields as an example of extra dimensions, and my exploitation of the possibilities they open up for illustrating local symmetry, are (as far as I know) new.

My theory that promotion of learning underlies, and is the evolutionary cause of, our sense of beauty in important cases, and the application of that theory to musical harmony, which offers a rational explanation for Pythagoras's discoveries in music, form a constellation

> and Aristotle, it was not mathematics or science upon which his fame rested. Pythagoras was famous
>
> 1. As an expert on the fate of the soul after death, who thought that the soul was immortal and went through a series of reincarnations
> 2. As an expert on religious ritual
> 3. As a wonder-worker who had a thigh of gold and who could be two places at the same time
> 4. As the founder of a strict way of life that emphasized dietary restrictions, religious ritual and rigorous self discipline

A few things do seem clear. The historical Pythagoras was born on the Greek island of Samos, traveled widely, and became the inspiration for and founder of an unusual religious movement. His cult flourished briefly in Crotone, in southern Italy, and developed chapters in several other places before being everywhere suppressed. The Pythagoreans formed secretive societies, on which the initiates' lives centered. These communities, which included both men and women, promoted a kind of intellectual mysticism that seemed marvelous, yet strange and threatening, to most of their contemporaries. Their worldview centered on worshipful admiration of numbers and musical harmony, which they saw as reflecting the deep structure of reality. (As we'll see, they were on to something.)

THE REAL PYTHAGORAS

Here again is the *Stanford Encyclopedia:*

> The picture of Pythagoras that emerges from the evidence is thus not of a mathematician, who offered rigorous proofs, or of a scientist, who carried out experiments to discover the nature of the natural world, but rather of someone who sees special significance in and assigns special prominence to mathematical relationships that were in general circulation.

Bertrand Russell was pithier:

> A combination of Einstein and Mary Baker Eddy.

To scholars of factual biography, it is a major problem that later followers of Pythagoras ascribed their own ideas and discoveries to Pythagoras himself. In that way they hoped both to give their ideas authority and, by enhancing Pythagoras's reputation, to promote their community—the community he founded. Thus magnificent discoveries in different fields of mathematics, physics, and music, as well as an inspiring mysticism, a seminal philosophy, and a pure morality were all portrayed as the legacy of a single godlike figure. That awesome figure is, for us, the *real* Pythagoras.

It is not altogether inappropriate to assign the (historical) shadow Pythagoras credit for the real Pythagoras, because the latter's great achievements in mathematics and science emerged from the way of life the former inspired, and the community he founded.

(Those so inclined might draw parallels to the differing careers in life, and afterward, of other major religious figures …)

Thanks to Raphael, we know what the real Pythagoras looked like. In plate B* he is captured deep in concentration as he writes in a great book, surrounded by admirers.

ALL THINGS ARE NUMBER

It is difficult to make out what Pythagoras is writing, but I like to pretend it is some version of his most fundamental credo:

> All Things Are Number

It is also difficult to know, at this separation in time and space, exactly what Pythagoras meant by that. So we get to use our imagination.

PYTHAGORAS'S THEOREM

For one thing, Pythagoras was mightily impressed by Pythagoras's theorem. So much so that when he discovered it, in a notable lapse from vegetarianism, he offered a hecatomb—the ritual sacrifice of one hundred oxen, followed by feasting—to the Muses, in thanks.

Why the fuss?

Pythagoras's theorem is a statement about right triangles; that is, triangles that contain a 90-degree angle, or, in other words, a square corner. The theorem tells you that if you erect squares on the different sides of such a triangle, then the sum of the areas of the two smaller squares adds up to the area of the largest square. A classic example is the 3-4-5 right triangle, shown in figure 1:

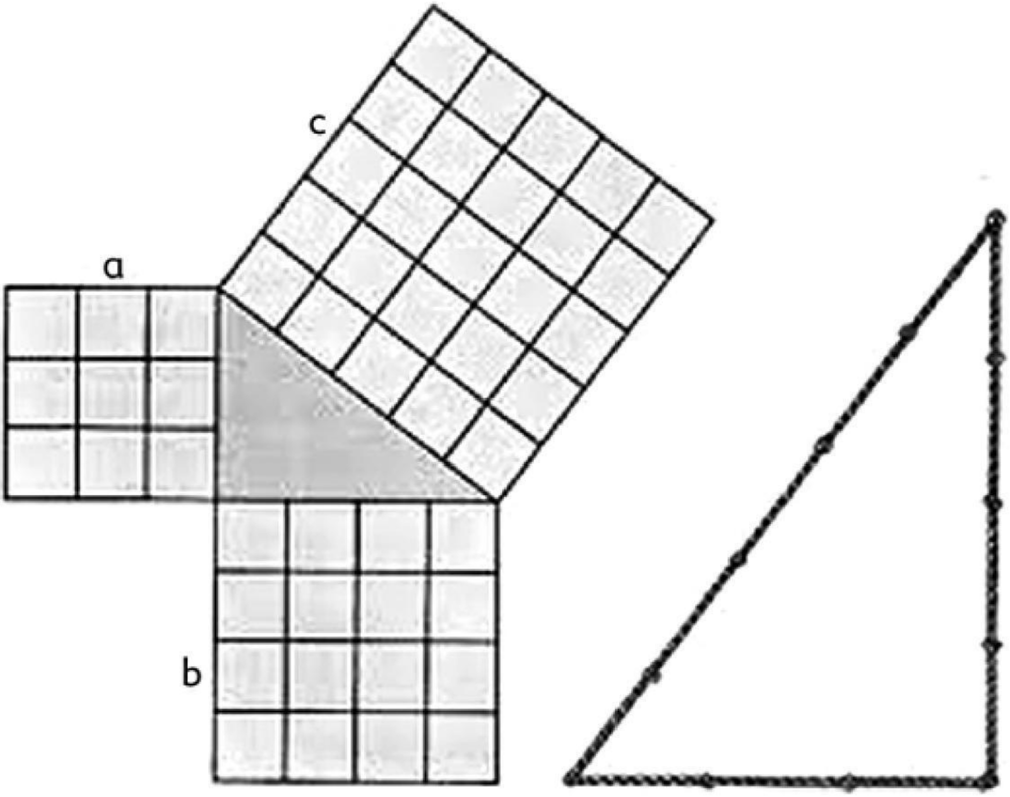

FIGURE 1. THE 3-4-5 RIGHT TRIANGLE, A SIMPLE CASE OF PYTHAGORAS'S THEOREM.

The areas of the two smaller squares are $3^2 = 9$ and $4^2 = 16$, as we can see, in the spirit of Pythagoras, by *counting* their subunits. The area of the largest square is $5^2 = 25$. And we verify $9 + 16 = 25$.

By now Pythagoras's theorem is familiar to most of us, if only as a dim memory from school geometry. But if you listen to its message afresh, with Pythagoras's ears, so to speak, you realize that it is saying something quite startling. It is telling you that the *geometry* of objects embodies hidden *numerical* relationships. It says, in other words, that Number describes, if not yet everything, at least something very important about physical reality, namely the sizes and shapes of the objects that inhabit it.

Later in this meditation we will be dealing with much more advanced and sophisticated concepts, and I'll have to resort to metaphors and analogies to convey their meaning. The special joy one finds in precise mathematical thinking, when sharply defined concepts fit together perfectly, is lost in translation. Here we have an opportunity to experience that special joy. Part of the magic of Pythagoras's theorem is that one can prove it with minimal preparation. The best proofs are unforgettable, and their memory lasts a lifetime. They've inspired Aldous Huxley and Albert Einstein—not to mention Pythagoras!—and I hope they'll inspire you.

Guido's Proof

"So simple!"

That is what Guido, the young hero of Aldous Huxley's short story "Young Archimedes," says, as he describes his demonstration of Pythagoras's theorem. Guido's proof is based on the shapes displayed in plate C.

Guido's Plaything

Let's spell out what was obvious to Guido at a glance.

Each of the two large tiled squares contains four colored triangles that are matched in the other large square. All the colored triangles are right triangles, and all are the same size. Let's say the length of the smallest side is *a,* the next smallest *b,* and the longest (the hypotenuse) *c*. Then it's easy to see that the sides of both large (total) squares have length $a + b$, and in particular that those two squares have equal areas. So the non-triangular parts of the large squares must also have equal areas.

But what are those equal areas? In the first large square, on the left, we have a blue square with side *a,* and a red square with side *b.* They have areas a^2 and b^2, and their combined area is $a^2 + b^2$. In the second large square, on the right, we have a gray square with side *c.* Its area is c^2. Recalling the preceding paragraph, we conclude that

$$a^2 + b^2 = c^2$$

… which is Pythagoras's theorem!

Einstein's Proof(?)

In Einstein's *Autobiographical Notes* he recalls,

> I remember that an uncle told me about the Pythagorean theorem before the holy geometry booklet had come into my hands. After much effort I succeeded in "proving" this theorem on the basis of the similarity of

triangles; in doing so it seemed to me "evident" that the relations of the sides of the right-angled triangles would have to be determined by one of the acute angles.

There is not really enough detail in that account to reconstruct Einstein's demonstration with certainty, but here, in figure 2, is my best guess. That guess deserves to be right, because this is the simplest and most beautiful proof of Pythagoras's theorem. In particular, this proof makes it brilliantly clear why the *squares* of the lengths are what's involved in the theorem.

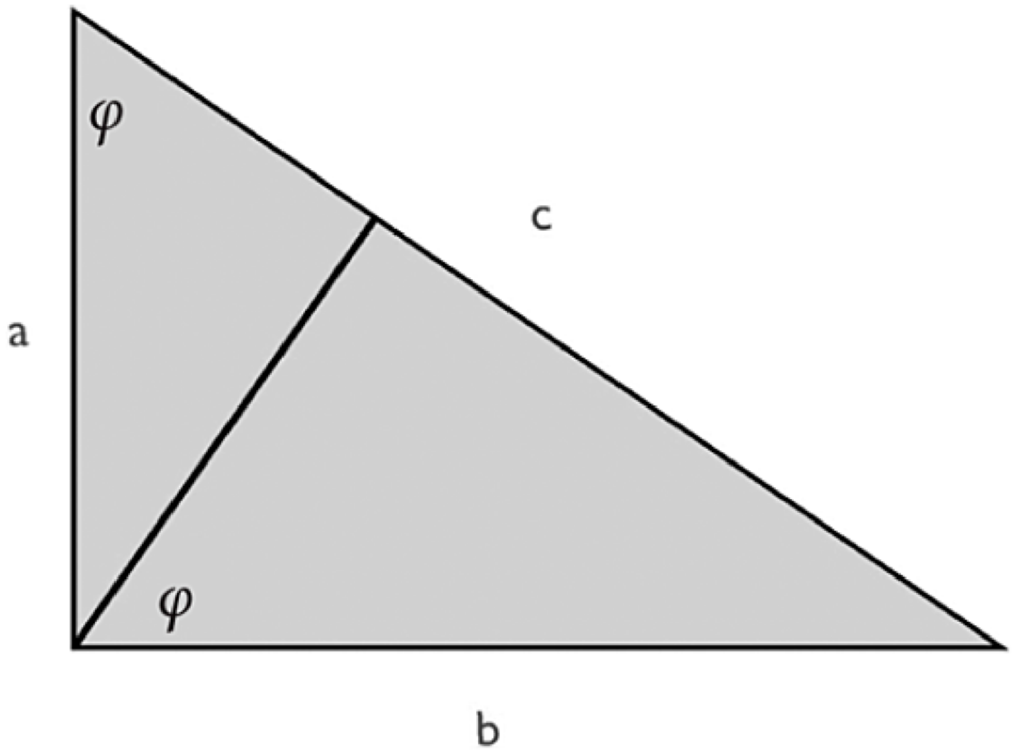

FIGURE 2. A PLAUSIBLE RECONSTRUCTION OF EINSTEIN'S PROOF, FROM AUTOBIOGRAPHICAL NOTES.

A Polished Jewel

We start from the observation that right triangles that include a common angle φ are all similar to one another, in the precise sense that you can get from any one to any other by an overall rescaling (magnification or shrinking). Also: if we rescale the length of the triangle by some factor, then we will rescale the area by the square of that factor.

epitomized in Euclid's *Elements*, was devoted to showing precisely this: that geometry is a system of *logic.*

As we continue our meditation, we'll find that Nature is inventive in her language. She stretches our imagination with new kinds of numbers, new kinds of geometry—and even, in the quantum world, new kinds of logic.

PYTHAGORAS II: NUMBER AND HARMONY

The essence of all stringed instruments, whether ancient lyre or modern guitar, cello, or piano, is the same: they produce sound from the motion of strings. The exact quality of sound, or timbre, depends on many complex factors, including the nature of the material that makes the string, the shapes of the surfaces—“sounding boards”—that vibrate in sympathy, and the way in which the string is plucked, bowed, or hammered. But in all instruments there is a principal tone, or pitch, that we recognize as the note being played. Pythagoras—the real one—discovered that the pitch obeys two remarkable rules. Those rules make direct connections among numbers, properties of the physical world, and our sense of harmony (which is one face of beauty).

The drawing that follows, not by Raphael, shows Pythagoras in action, performing experiments on harmony:

FIGURE 3. AN ETCHING FROM MEDIEVAL EUROPE DEPICTING PYTHAGORAS AT WORK ON MUSICAL HARMONY. WE CAN INFER FROM THE FIGURE THAT PYTHAGORAS LISTENED TO HOW THE SOUNDS PRODUCED BY HIS INSTRUMENT CHANGED AS HE VARIED TWO DIFFERENT THINGS. BY HOLDING A STRING DOWN FIRMLY AT DIFFERENT POINTS, HE COULD VARY THE EFFECTIVE LENGTH OF THE VIBRATING PART. AND BY CHANGING THE WEIGHT THAT STRETCHES A STRING, HE COULD VARY ITS TENSION.

HARMONY, NUMBER, AND LENGTH: AN ASTONISHING CONNECTION

Pythagoras's first rule is a relationship between the length of the vibrating string and our perception of its tone. The rule says that two copies of the same type of string, both subject to the same tension, make tones that sound good together precisely when the lengths of the strings are in ratios of small whole numbers. Thus, for example, when the ratio of lengths is 1:2, the tones form an octave. When the ratio is 2:3, we hear the dominant fifth; when the ratio is 3:4, the major fourth. In musical notation (in the key of C) these correspond to playing two Cs, one above the other, together, a C-G, or C-F, respectively. People find those tone combinations appealing. They are the main building blocks of classical music, and of most folk, pop, and rock music.

In applying Pythagoras's rule, the length that we must consider is of course the effective length, that is, the length of the portion of the string that actually vibrates. By clamping down on the string, creating a dead zone, we can change the tone. Guitarists and cellists exploit that possibility when they "finger" with their left hands. As they do so they are, whether or not they know it, reincarnating Pythagoras. In the drawing, we see Pythagoras adjusting the effective length using a pointed clamp, which is a technique conducive to accurate measurement.

When tones sound good together, we say they are in harmony, or that they are concordant. What Pythagoras discovered, then, is that the perceived harmonies of tones reflect relationships in what might seem to be an entirely different world—the world of numbers.

HARMONY, NUMBER, AND WEIGHT: AN ASTOUNDING CONNECTION

Pythagoras's second rule involves the tension of the string. The tension can be adjusted, in a controlled and readily measurable way, by burdening the string with different amounts of weight, as shown in figure 3. Here the result is even more remarkable. The tones are in harmony if the tensions are ratios of *squares* of small whole numbers. Higher tensions correspond to higher pitches. Thus a 1:4 ratio of tensions produces the octave, and so forth. When string musicians tune their instruments prior to a performance, stretching or relaxing the strings by winding their pegs, Pythagoras returns.

This second relationship is even more impressive than the first as evidence that Things are hidden Numbers. The relationship is better hidden because the numbers must be processed—squared, to be exact—

before the relationship becomes evident. The shock of discovery is accordingly greater. Also, the relationship brings in weight. And weight, more unmistakably than length, links us to Things in the material world.

DISCOVERY AND WORLDVIEW

Now we've discussed three major Pythagorean discoveries: the Pythagorean theorem on right triangles, and two rules of musical consonance. Together, they link shape, size, weight, and harmony, with the common thread being Number.

For the Pythagoreans, that trinity of discoveries was more than enough to anchor a mystic worldview. Vibration of strings is the source of musical sound. These vibrations are nothing but periodic motions; that is, motions which repeat themselves at regular intervals. We also see the Sun and planets move in periodic motions across the sky, and infer their periodic motion in space. So they too must emit sound. Their sounds form the Music of the Spheres, a music that fills the cosmos.

Pythagoras was fond of singing. He also claimed actually to hear the Music of the Spheres. Some modern scholars speculate that the historical Pythagoras suffered from tinnitus, or ringing in the ears. The real Pythagoras, of course, did not.

In any case, the larger point is that All Is Number, and Number supports Harmony. The Pythagoreans, drunk on mathematics, inhabited a harmony-filled world.

THE FREQUENCY IS THE MESSAGE

Pythagoras's musical rules deserve, I think, to be considered the first quantitative laws of Nature ever discovered. (Astronomical regularities, beginning with the regular alternation of night and day, were of course noticed much earlier. Calendar-keeping and casting of horoscopes, using mathematics to predict or reconstruct the positions of the Sun, Moon, and planets, were significant technologies before Pythagoras was born. But empirical observations about specific objects are quite different from general laws of Nature.)

It is ironic, therefore, that we still don't fully understand why they are true. Today we have a much better understanding of the physical processes involved in the production, transmission, and reception of thinner parts prefer to vibrate at higher frequencies. (This effect is responsible for the difference in the overall pitch between typical male and female voices. At puberty the male vocal cords thicken markedly, leading to lower frequencies of vibration and a deepened voice.) Thus when a sound, after its many tribulations, sets the surrounding fluid into motion, the response of the basilar membrane will be different at different places along its length. A low-frequency tone will put the thicker parts into vigorous motion, while a high-frequency tone will put the thinner parts into vigorous motion. In this way, information about frequency gets encoded into information about position!

If the cochlea is the eye of audition, the organ of Corti is its retina. The organ of Corti runs parallel to the basilar membrane, and close by. Its structure is complex in detail, but roughly speaking it consists of hair cells and neurons, one hair cell per neuron. The motion of the basilar membrane, coupled through intermediate fluid, exerts forces on the hair cells. The hair cells move in response, and their motion triggers electrical firing of the corresponding neurons. The frequency of the firing is the same as the frequency of stimulation, which in turn is the same as the frequency of the original tone. (For experts: The firing patterns are noisy, but they contain a strong component at the signal frequency.)

Because the organ of Corti abuts the basilar membrane, its neurons inherit the position-dependent frequency response of that membrane. This is very important for our perception of chords, because it means that when several tones sound simultaneously, their signals do not get completely scrambled. Different neurons respond preferentially to different tones! This is the physiological mechanism that allows us to do such a good job of discriminating different tones.

In other words, our inner ears follow the advice of Newton—and anticipate his analysis of light—by performing an excellent Analysis of the incoming sound into pure tones. (As we'll discuss later, our sensory ability to analyze the frequencies of signals in light, or in other words the color content of light, is based on different principles, and is much poorer.)

This sets the scene for the third stage of our story. In it, signals from the primary sensory neurons in the organ of Corti are combined and passed on to subsequent neural layers in the brain. Here our knowledge is considerably less precise. But it is only here that we can finally come to grips with our main question:

> *Why* do tones whose frequencies are in ratios of small whole numbers sound good together?

Let us consider what the brain is offered when two different sound frequencies play simultaneously. Then we have two sets of primary neurons responding strongly, each firing with the same frequency as the vibrations of the string that excites them. Those primary neurons fire their signals brainward, to "higher" levels of neurons, where their signals are combined and integrated.

Some of the neurons at the next level will receive inputs from both sets of firing primaries. If the frequencies of the primaries are in a ratio of small whole numbers, then their signals will be synchronized. (For this discussion, we will simplify the actual response, ignoring the noise and treating it as accurately periodic.) For example, if the tones form an octave, one set will be firing twice as fast as the other, and every firing of the slower one will have the same predictable relationship to the firing of the former. Thus the neurons sensitive to both will then get a repetitive pattern that is predictable and easy to interpret. From previous experience, or perhaps by inborn instinct, those secondary neurons—or the later neurons that interpret their behavior—will "understand" the signal. For it will be possible to anticipate future input (i.e., more repetitions) in a simple way, and simple predictions for future behavior will be borne out, over many vibrations, until the sound changes its character.

Note that the sound vibrations we can hear have frequencies ranging from a few tens to several thousand per second, so even brief sounds will produce many repetitions, except at the very low-frequency end. And at the low-frequency end our sense of harmony peters out, consistent with the line of thought we are pursuing.

Higher levels of neurons, which combine the combiners, need coherent input to get on with their job. So if our combiners are producing sensible messages, and in particular if their predictions satisfy the test of time, it is in the interest of the higher levels to reward them with some kind of positive feedback, or at least to leave them in peace. On the other hand, if the combiners are producing wrong predictions, the mistakes will propagate up to higher levels, ultimately producing discomfort and a desire to make it stop.

When will the combiners produce wrong predictions? That will happen when the primary signals are almost, but not quite, in synch. For then the vibrations will reinforce each other for a few cycles, and the combiners will extrapolate that pattern. They expect it to continue—but it doesn't! And indeed it is tones that are just slightly off—like C and C#, for example—that sound most painful when played together.

If this idea is right, then the basis of harmony is successful prediction in the early stages of perception. (This process of prediction need not, and usually does not, involve conscious attention.) Such success is experienced as pleasure, or beauty. Conversely, unsuccessful prediction is a source of pain, or ugliness. A corollary is that by expanding our experience, and learning, we can come to hear harmonies that were previously hidden to us, and to remove sources of pain.

Historically, in Western music, the palette of acceptable tone combinations has expanded over time. Individuals can also learn, by exposure, to enjoy tone combinations that at first seem unpleasant. Indeed, if we are built to enjoy *learning* to make successful predictions, then predictions that come too easily will not yield the greatest possible pleasure, which should also bring in novelty.

PLATO I: STRUCTURE FROM SYMMETRY—PLATONIC SOLIDS

The Platonic solids carry an air of magic about them. They have been, and are, literally, objects to conjure with. They reach back deep into human prehistory, and live on as the generators of good or bad luck in some of the most elaborate of games, notably Dungeons & Dragons. Their mystique has inspired, besides, some of the most fruitful episodes in the development of mathematics and science. A worthy meditation on embodied beauty must dwell upon them.

Albrecht Dürer, in his *Melancholia I* (figure 4), alludes to the allure of regular solids, although the solid that appears is not quite a Platonic solid. (Technically, it is a truncated triangular trapezohedron. It can be constructed by stretching out the sides of an octahedron in a peculiar way.) Perhaps the philosopher is melancholy because she can't fathom why a baleful bat dropped that particular, not quite Platonic, solid into her study, rather than a straightforward example.

FIGURE 4. DÜRER'S MELANCHOLIA I. IT FEATURES A TRUNCATED PLATONIC SOLID, A VERY MAGIC SQUARE, AND MANY OTHER ESOTERIC SYMBOLS. TO ME, IT WELL DEPICTS THE FRUSTRATIONS I OFTEN ENCOUNTER WHEN USING PURE THOUGHT TO COMPREHEND REALITY. FORTUNATELY, IT'S NOT ALWAYS THIS WAY.

Regular Polygons

connection to specific numbers. Plato interpreted this profound emergence in an astonishingly creative way, as we shall see.

Prehistory

Famous people often get credit for the discoveries of others. This is the "Matthew Effect" identified by the sociologist Robert Merton, based on this observation from the Gospel of Matthew:

> For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken even that which he hath.

So it is for the Platonic solids.

At the Ashmolean Museum of Oxford University you can see a display of five carved stones dating from 2000 BCE Scotland that appear to be realizations of the Platonic solids (though some scholars dispute this). They were most likely used in some sort of dice game. Let us imagine cave people huddled around the communal fire, rapt in paleolithic Dungeons & Dragons. But it was probably Plato's contemporary Theaetetus (417–369 BCE) who first *proved* mathematically that those five bodies are the only possible regular solids. It's not clear to what extent Theaetetus was inspired by Plato, or vice versa, or whether it was something in the Athenian air they both breathed. In any case, the Platonic solids got their name because Plato used them creatively, in work of imaginative genius, to construct a visionary theory of the physical world.

FIGURE 7. PRE-PLATONIC ANTICIPATIONS OF THE PLATONIC SOLIDS, PROBABLY USED IN DICE GAMES CIRCA 2000 BCE.

Going back much further, we now realize that some of the biosphere's simplest creatures, including viruses and diatoms (not pairs of atoms, but marine algae that often grow elaborate Platonic exoskeletons), not only "discovered" but have literally embodied the Platonic solids since long before humans walked the Earth. The herpesvirus, the virus that causes hepatitis B, the HIV virus, and many other nasties are shaped like icosahedra or dodecahedra. They encase their genetic material—either DNA or RNA—in protein exoskeletons, which determine their external form, as seen in plate D. The exoskeleton is color-coded in such a way that identical colors indicate identical building blocks. The dodecahedron's signature triply meeting pentagons leap to the eye. If we join the centers of the blue regions with straight lines, an icosahedron emerges.

More complex microscopic creatures, including the radiolaria lovingly portrayed by Ernst Haeckel in his marvelous book *Art Forms in Nature*, also embody the Platonic solids. In figure 8 it is the intricate silica exoskeletons of these single-cell organisms that we see. The radiolarians are an ancient life-form, represented in the earliest fossils. They continue to thrive in the oceans today. Each of the five Platonic solids is realized in a number of species. Several species names enshrine those shapes, including *Circoporus octahedrus*, *Circogonia icosahedra*, and *Circorrhegma dodecahedra*.

Euclid's Inspiration

Euclid's *Elements* is, by a wide margin, the greatest textbook of all time. It brought system and rigor to geometry. From a larger perspective it established, by example, the method of Analysis and Synthesis in the domain of ideas.

Analysis and Synthesis is Isaac Newton's, and our, preferred formulation of "reductionism." Here is Newton:

> By this way of Analysis we may proceed from Compounds to Ingredients, and from Motions to the Forces producing them; and in general, from Effects to their Causes, and from particular Causes to more general ones, till the Argument end in the most general. This is the Method of Analysis: And the Synthesis consists in assuming the Causes discover'd, and establish'd as Principles, and by them explaining the Phænomena proceeding from them, and proving the Explanations.

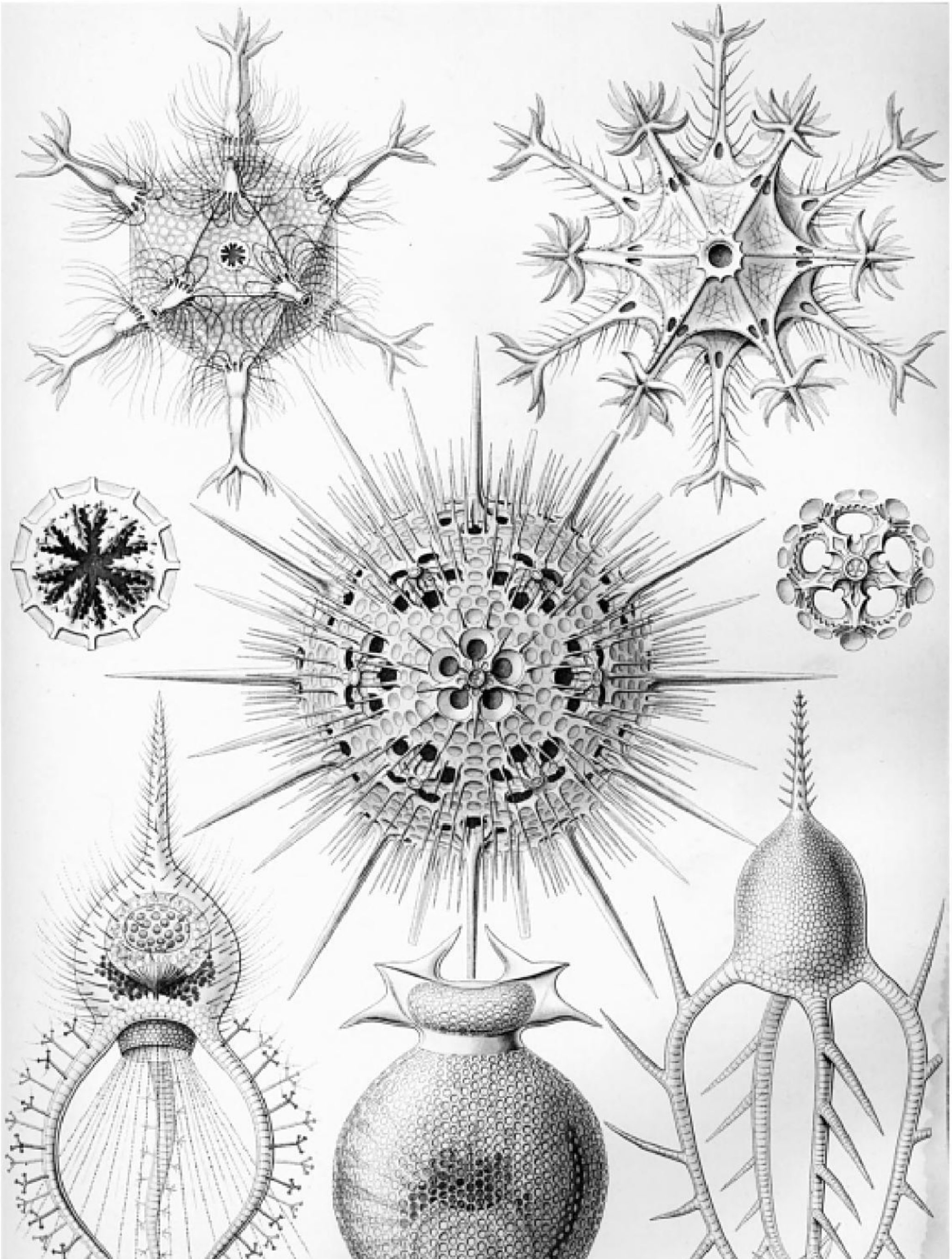

FIGURE 8. RADIOLARIA BECOME VISIBLE UNDER A MODEST MICROSCOPE. THEIR EXOSKELETONS OFTEN EXHIBIT THE SYMMETRY OF PLATONIC SOLIDS.

This strategy parallels Euclid's approach to geometry, where he proceeds from simple, intuitive *axioms* to deduce rich and surprising consequences. Newton's great *Principia,* the founding document of modern mathematical physics, also follows Euclid's expository style, building from axioms to major results step-by-step through logical construction.

It is important to emphasize that axioms (or laws of physics) don't tell you what to do with them. By stringing them together without purpose, it's easy to generate hosts of forgettable, worthless truths—like a play or a piece of music that wanders aimlessly, arriving nowhere. As those who have attempted to deploy artificial intelligence to do creative mathematics have discovered, identifying *goals* is often the hardest challenge. With a worthy goal in mind, it becomes easier to find the means to achieve it. My all-time favorite fortune cookie summed this up brilliantly:

> The work will teach you how to do it.

Also, of course, as a matter of presentation, it's attractive to students and potential readers to have an inspiring goal in sight—and impressive for them to realize, at the start, that they can look forward to experiencing an amazing feat of construction that builds, by inexorable steps, from "obvious" axioms to far-from-obvious conclusions.

So: What was Euclid's goal in the *Elements*? The thirteenth and final volume of that masterpiece concludes with constructions of the five Platonic solids, and a proof that there are only five. I find it pleasant—and convincing—to think that Euclid had this conclusion in mind when he began drafting the whole, and worked toward it. In any case, it is a fitting, fulfilling conclusion.

Platonic Solids as Atoms

The ancient Greeks recognized four building blocks, or elements, for the material world: fire, water, earth, and air. You might notice that four, the number of elements, is close to five, the number of regular solids. Plato certainly did! One finds, in his influential, visionary, inscrutable *Timaeus,* a theory of the elements based on the solids. Here it comes:

Each of the elements is built from a different variety of atom. The atoms take the form of Platonic solids. The atoms of fire are tetrahedra, the atoms of water are icosahedra, the atoms of earth are cubes, and the atoms of air are octahedra.

There is a certain plausibility to these assignments. They have explanatory power. The atoms of fire have sharp points, which explains why contact with fire is painful. The atoms of water are most smooth and well-rounded, so they can flow around one another smoothly. The atoms of earth can pack closely, and fill space without gaps. Air, being both hot and wet, features atoms intermediate between those of fire and water.

Now while five is close to four, it is not quite equal to it, so there cannot be a perfect match between regular solids, regarded as atoms, and elements. A merely brilliant thinker might have been discouraged by that difficulty, but Plato, a genius, was undaunted. He took it as a challenge and an opportunity. The remaining regular solid, the dodecahedron, he proposed, does figure in the Creator's construction, but not as an atom. No, the dodecahedron is no mere atom—rather, it is the shape of the Universe as a whole.

Aristotle, who was forever determined to one-up Plato, put forward a different, more conservative and intellectually consistent variation of that theory. Two of that influential philosopher's big ideas were: that the Moon, planets, and stars inhabit a celestial realm made from stuff different from what we find in the mundane world; and that "Nature abhors a vacuum," so that the celestial spaces could not be empty. Thus consistency required there to be a fifth element, or quintessence, different from earth, air, fire, and water, to fill the celestial realm. Dodecahedra, then, find their place as the atoms of quintessence, or ether.

It is difficult to agree, today, with the details of these theories, in either version. We haven't found it useful, in science, to analyze the world in terms of those four (or five) elements. Nor are modern atoms hard, solid bodies, much less realizations of the Platonic solids. Plato's theory of the elements, seen from today's perspective, is both crude and, in detail, hopelessly misguided.

Structure from Symmetry

And yet, though it fails as a scientific theory, Plato's vision succeeds as prophecy and, I would claim, as a work of intellectual art. To appreciate those larger virtues, we have to step away from the details, and look at the bigger picture. The deepest, core intuition of Plato's vision of the physical world is that the physical world must, fundamentally, embody beautiful concepts. And this beauty must be of a very special kind: the beauty of mathematical regularity, of perfect symmetry. For Plato, as for Pythagoras, that intuition was at the same time a faith, a yearning, and a guiding principle. They sought to harmonize Mind with Matter by showing that Matter is built from the purest products of Mind.

It is important to emphasize that Plato pushed his ideas past the level of philosophical generalities to make specific claims about what matter is. His specific ideas, though wrong, do not fall into the ignominious category of "not even wrong." Plato even made some gestures in the direction of comparing his theory with reality, as we've seen. Fire stings because tetrahedra are sharply

Creation. The planets were supposed to be carried about on celestial spheres. Copernicus and Ptolemy had different views about where those spheres should be centered (the Sun, or Earth), but both took their existence for granted, as did young Kepler. Thus Kepler considered that there were six great spheres centered upon the Sun. He asked: Why six? And why do they have the sizes they do?

The answers struck Kepler one day while he was lecturing to an introductory astronomy class. One can circumscribe a different Platonic solid around each of the first five spheres, and inscribe it within the next. Thus the five Platonic solids can mediate among six spheres! The system will only work, however, if the spheres have appropriate sizes. In this way, Kepler could predict the relative distances between the various planets and the Sun. Convinced he had discovered God's plan, Kepler announced his discovery in a rapturous book, *Mysterium Cosmographicum,* full of quotations like this:

> I feel carried away and possessed by an unutterable rapture over the divine spectacle of the heavenly harmony.

And this:

> God himself was too kind to remain idle, and began to play the game of signatures, signing his likeness into the world; therefore I chance to think that all nature and the graceful sky are symbolized in the art of geometry.

It is indeed a gorgeous system, as you can see from the splendidly realized model in figure 9.

Evidently Kepler had asked himself, and believed he had answered, our Question: The world *does* embody Beauty, very much along the lines Plato anticipated. He went on to discuss, in concrete detail, the precise nature of the music emitted by those rotating spheres—and wrote out the score!

FIGURE 9. THE PLATONIC SOLIDS INSPIRED KEPLER TO PROPOSE A MODEL OF THE SIZE AND SHAPE OF THE SOLAR SYSTEM, EXHIBITED HERE. THE PLANETS ARE CARRIED ABOUT BY THE REVOLUTIONS OF CELESTIAL SPHERES WHOSE SPACING IS CONTROLLED BY PLATONIC SURFACES INTERPOSED BETWEEN THEM AS THEIR SCAFFOLDING.

Kepler's enthusiasm carried him through a life full of woes, both personal and professional. He lived close to the center of the turbulent vortex of war, religion, and politics that swept over middle Europe following the Reformation. His mother was tried as a witch. And his honest, toilsome work to describe the motion of the planets accurately resulted, through his own discoveries, in the overthrow of his youthful dream. For the planets describe not circles, but ellipses (Kepler's first law), and the Sun is not at the center of those ellipses (for experts: rather, it is at one focus). Eventually deeper beauties emerged from Kepler's mature and more accurate portrait of Nature, but they were quite different from the dreams of his youth, and he did not live to see them.

DEEP TRUTHS

The great Danish physicist and philosopher Niels Bohr (1885–1962), a founding figure in quantum theory and author of the complementarity principle highlighted later in this book, was fond of a concept he called "deep truth." It exemplifies Ludwig Wittgenstein's proposal that all of philosophy can, and probably should, be conveyed in the form of jokes.

According to Bohr, ordinary propositions are exhausted by their literal meaning, and ordinarily the opposite of a truth is a falsehood. Deep propositions, however, have meaning that goes beneath their surface. You can recognize a deep truth by the feature that its opposite is also a deep truth. In this sense, the sober conclusion

> The world, alas, is not made according to mathematical principles in the way that Plato guessed.

… expresses a deep truth. As, of course, does its opposite:

> The world *is* made according to mathematical principles, as Plato guessed.

Dalí's Last Supper

It seems fitting to conclude this sub-meditation with a modern work of art that plays on its themes.

Plate E, Salvador Dalí's masterpiece, *The Sacrament of the Last Supper*, contains many hidden geometrical themes. But of these themes, the strangest and most striking is the appearance of several large, but only partly realized, pentagons looming over the scene as a whole. It seems clear that they are meant to come together in a dodecahedron that embraces not only the participants in the supper, but also the viewer. And we are meant to recall Plato's conception that this shape frames the Universe.

PLATO II: ESCAPING THE CAVE

Our Question, in asking after Beauty, hinges in part upon the relationship between physical reality and our perception of it. We have discussed this for hearing, and later we will discuss it for vision.

But there's another dimension to our Question, which is the relationship between physical reality and ultimate reality. Or, if you are (understandably) uncomfortable with the concept of ultimate reality, let's just say the big picture—how we connect the deep nature of physical reality to our hopes and dreams. What, if anything, does it all mean? Those issues are major elements in appreciating (or not) the world's beauty, as we pass beyond the level of raw perception.

Plato suggested some answers to those questions long ago. His answers were based on mystic intuition and dubious logic, rather than science. Nevertheless, they have inspired scientific work, and continue to do so. We'll have many occasions to look back to them. And their influence extends beyond science, to philosophy, art, and religion. Alfred North Whitehead famously wrote:

> The safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato.

So let us now visit Plato's Cave, where we find the mystic core of his worldview, captured in visionary imagery.

ALLEGORY OF THE CAVE

Plato's allegory of the Cave occurs in his weightiest work, the *Republic.* He puts it, as he does many of his thoughts, into the mouth of

Socrates, his revered teacher. Socrates describes the Cave to Glaucon, Plato's elder brother, who was likewise a student of Socrates. This setting, and this cast of characters, emphasize the central importance of the Cave in Plato's thinking.

Here is how he introduces it:

> **Socrates.** And now, I said, let me show in a figure how far our nature is enlightened or unenlightened: Behold! Human beings living in an underground den, which has a mouth open towards the light and reaching all along the den; here they have been from their childhood, and have their legs and necks chained so that they cannot move, and can only see before them, being prevented by the chains from turning round their heads. Above and behind them a fire is blazing at a distance, and between the fire and the prisoners there is a raised way; and you will see, if you look, a low wall built along the way, like the screen which marionette players have in front of them, over which they show the puppets.
>
> **Glaucon.** I see.
>
> **Socrates.** And do you see, I said, men passing along the wall carrying all sorts of vessels, and statues and figures of animals made of wood and stone and various materials, which appear over the wall? Some of them are talking, others silent.
>
> **Glaucon.** You have shown me a strange image, and they are strange prisoners.
>
> **Socrates.** Like ourselves.

The point is clear and simple: The prisoners see a projection of reality, not reality itself. Because that projection is all they know, they take it for granted. It is their world. But we should not feel superior to those benighted prisoners because our own situation is no different, according to Socrates (i.e., Plato). The words "Like ourselves" arrive with the force of a blow.

The story of the Cave does not *prove* that point, of course—it's only a story, after all. But it does persuade us to consider, as a logical possibility, that there's more to reality than our senses detect. And this deeply subversive story issues challenges: Do not accept limitations. Struggle to attempt different ways of viewing things. Doubt your perceptions. Be suspicious of authority.

The Platonic vision of a reality beyond the world of appearances is captured beautifully in plate F, a cosmic version of the Cave.

In Plato's theory of Ideals, these two streams—Pythagorean intuitions of harmony and perfection, and Parmenides's changeless reality—flow together. (Plato's theory is generally called the theory of Ideas, but I think "Ideal" corresponds better to what Plato had in mind, so I'll use that word.)

The Ideals are the perfect objects, of which real objects are imperfect copies. Thus, for example, there is an Ideal Cat. Actual animals are cats to the extent that they share in the properties of that Cat. The Ideal Cat, of course, never dies, nor can it change in any way. This theory embodies Parmenides's metaphysics: There is a realm of Ideals, the deepest reality, which is eternal and unchanging, and provides the source of all we can name or speak of. And it builds upon Pythagoras: We come into close contact with the world of eternal, perfect Ideals when we deal in mathematical concepts like numbers, or Platonic solids.

There is a third, subterranean stream that also surely fed into the theory of Ideals. This is the stream of Orphic religion. That was the serious side, we might say, of Greek mythology. The details of Orphism, which featured secretive rituals, are obscured in the mists of history (that's the fate of secrets!), and they need not concern us here. But its centerpiece was the doctrine of immortality of souls, which had (and, of course, still has) sublime emotional appeal. Wikipedia describes it as

> Characterizing human souls as divine and immortal but doomed to live (for a period) in a "grievous circle" of successive bodily lives through metempsychosis, or the transmigration of souls.

These ideas fit into the theory of Ideals elegantly. Each of us, by our nature, participates in the world of Ideals. The part of us that participates is our soul, and is imperishable. While we live on Earth our attention is diverted by appearances, and if we do not transcend them, we are only dimly aware of the Ideals, and our soul slumbers. But through philosophy, mathematics, and a dose of mysticism (the mysterious ceremonies of Orphism) we can awaken it. There is a Cave—and there is a way out.

Liberation

Plato describes the process of liberation:

Socrates. And now look again, and see what will naturally follow if the prisoners are released.… The glare will distress him, and he will be unable to see the realities of which in his former state he had seen the shadows.… Will he not fancy that the shadows which he formerly saw are truer than the objects which are now shown to him?

Glaucon. Far truer.

. . . .

Socrates. He will require to grow accustomed to the sight of the upper world. And first he will see the shadows best, next the reflections of men and other objects in the water, and then the objects themselves; then he will gaze upon the light of the moon and the stars and the spangled heaven; and he will see the sky and the stars by night better than the sun or the light of the sun by day?

Glaucon. Certainly.

It is noteworthy that Plato (through Socrates) describes liberation as an active process, a process of learning and engagement. This is quite different from ideas that are more popular, though to me less inspiring, where salvation comes about through external grace, or through renunciation.

If liberation comes through engagement with a hidden reality, how are we to achieve it? Here there are two paths, inward and outward.

Along the inward path, we examine our concepts critically, and try to strip them of the dross of mere appearance, to reach their ideal (i.e., Ideal) meaning. This is the path of philosophy and metaphysics.

Along the outward path, we engage appearances critically, and try to strip them of complications, to find their hidden essence. This is the way of science and physics. As we've anticipated, and will discuss in depth, the outward path does in fact lead to liberation.

Undoing Projection: Looking Ahead

In his central intuition, Plato was quite correct—indeed, more profoundly correct than he possibly could have known. Our naturally given view of the world is but a shadowy projection of the world as it truly is.

Our unaided senses take only paltry samples from the cornucopia of information the world puts on offer. With the help of microscopes, we discover a microcosmic universe full of tiny alien creatures, some our friends and some our enemies, and the yet more alien constituents of our material being, things that play by the weird rules of quantum mechanics. With the help of optical telescopes, we discover the vast size of the cosmos dwarfing our Earth, featuring vast, dark, (apparently) empty spaces sprinkled with billions of billions of varied suns and planets. With the help of radio receivers, we come to "see" invisible radiations that fill space, and to put them to use. And so on …

As for our senses, so for our minds. Without training and help, they cannot begin to do justice to the richness of reality we know, let alone what we don't yet know—the unknown unknowns. We go to school, read books, tap into the Internet, and use scratch pads, computer programs, and other tools to help us keep complicated ideas in order, solve the equations that govern the Universe, and visualize their consequences.

Those aids to sensation and imagination open the doors of perception, allowing us to escape from our Cave.

THE TURN TO UNWORLDLINESS

But Plato, knowing nothing of that future, emphasized the inward path. Here he explains why:

> **Socrates.** Accordingly, we must use the embroidered heaven as an example to illustrate our theories, just as one might use exquisite diagrams drawn by some fine artist such as Daedalus. An expert in geometry, faced with such designs, would admire their finish and craftsmanship; but he would not dream of studying them in all earnest, expecting to find all angles and lengths conforming exactly to the theoretical values.
>
> **Glaucon.** That would of course be absurd.
>
> **Socrates.** The genuine astronomer, then, will adopt the same outlook when studying the motions of the planets. He will admit that the sky and all it contains have been framed by their maker as perfectly as things can be.… He will not imagine that these visible, material changes go on for ever without the slightest alteration or irregularity, and waste his efforts trying to find perfect exactitude in them.
>
> **Glaucon.** Now that you put it like that, I agree.
>
> **Socrates.** So if we mean to study astronomy in a way which makes proper use of the soul's inborn intellect, we shall proceed as we do in geometry,

> working at mathematical problems, and not waste time observing the heavens.

We can summarize that one-sided dialogue in an inequality. It states, quite simply, that the Real does not live up to the Ideal. The Real is strictly less:

Real < Ideal

The Artisan, who creates the physical world from the world of Ideals, is an artist, and a good one. Yet ultimately the Artisan is a copyist whose creations reflect the messiness of the available materials. The Artisan paints with a broad brush and blurs details. The physical world is a flawed representation of the ultimate reality we should seek.

To put it another way: Plato recommends unworldliness. If your theories are beautiful, but do not exactly agree with observations—well, then, so much the worse for the observations.

Two Kinds of Astronomy

Why did Plato, in seeking ultimate truth, turn inward, away from the physical world? Part of the reason, no doubt, was that he loved his theories too much, and could not bring himself to contemplate their possible failure. That all-too-human attitude is still with us—it is standard in politics, common in social sciences, and not unknown even in physics.

But part of the reason emerged from the study of Nature, in astronomy, the subject his dialogue alludes to.

Keeping an accurate calendar was important for the societies of the ancient world, whose economic base was agricultural, and especially so for those that relied on irrigation systems. It was also, not coincidentally, important for religious purposes, because rituals were timed to get godly assistance at planting and at harvest. All this required astronomy. So too did the art of divining the future through astrology. The ancient Babylonians became extremely adept at predicting the timing of astronomical events, including the variation of the Sun's position at dawn and on setting, equinoxes, solstices, and eclipses of the Moon and Sun. Their method was simple, in principle, and almost theory-independent. They accumulated centuries of accurate observations, noted regularities (periodicities) in the behaviors, and extrapolated those regularities into the future. In other words, they assumed that future cycles of behavior in the celestial realm would reproduce past behavior, as it had been observed to do repeatedly in the past. “Big data” is all the rage today, but the basic concept goes back a long way, for it is none other than the method of ancient Babylonian astronomy.

The Babylonian work was maturing as Plato wrote, and most likely he had no more than vague knowledge of it. In any case, their “bottom-up,” data-heavy, theory-light approach was completely at odds with his goals and methods.

To Plato, as we’ve seen, what seemed overwhelmingly important is the human soul—its ascent to wisdom, purity, and a transcendent Ideal. Thus in building an account of planetary motion, what is most important is that the theory should be beautiful, not that it should be completely accurate. The primary goal is to identify the Ideals that inspired the Artisan. The compromises that coarse building materials forced upon Her are a secondary concern.

The dominant, as well as the simplest, periodicities in astronomy are the regular cycles of day and night and of the seasons, which associated the apparent motion of stars across the sky and with the Sun’s apparent trajectory. Today, we understand that those cycles are associated with the daily rotation of the Earth around its axis, and its yearly revolution around the Sun. Because both these motions are fairly close to being circular motions at constant speed, the observed phenomena could be described (to an excellent approximation) by an extremely beautiful theory, as follows:

The most perfect geometric figure is a circle. For uniquely among closed figures, a circle has the same appearance everywhere along its extent. Any other figure exhibits differences among its different parts, and so not all of those parts can be the best possible, and therefore neither can the whole. Similarly, the most perfect motion in a circle is motion at constant speed. Also, motion in a circle at constant velocity is as unchanging as motion can possibly be, because it takes the same form at every moment. From these “top-down” considerations, we deduce that the Ideal of motion is motion in a circle at constant velocity. And when we look to the sky, we discover that by combining two such perfect motions, we can match the observed motion of the Sun and stars, pretty nearly.

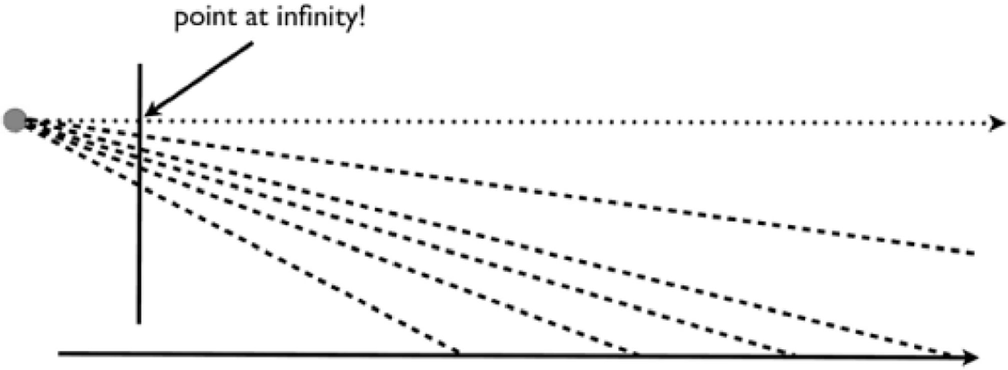

FIGURE 10. THE POINTS ALONG THE HORIZONTAL LINE (A FLOOR) PROJECT ONTO A LINE SEGMENT ALONG THE VERTICAL LINE (A CANVAS). THE INFINITE LIMIT OF THE HORIZONTAL LINE, WHICH IS NEVER ATTAINED IN REALITY, NEVERTHELESS PROJECTS TO A REAL, FINITE "POINT AT INFINITY" ON THE CANVAS.

The points on our maximally simplified landscape—a flat horizontal plane, reducing to a line in cross-section—project light in straight lines to the viewer. They are the dashed lines in the figure. By following those lines to where they intersect the canvas (whose cross-section is the vertical solid line) we determine where the different landscape points should appear in the painting.

As you can see, points that are farther away get projected higher up, vertically, on the canvas. But as we consider more and more distant landscape points, the rate at which their images climb the canvas decreases. The connecting light rays approach a horizontal limit, depicted by the dotted line in the figure. That limiting line does not correspond to an actual point on the landscape—yet it intersects the canvas at a specific point.

Right before our eyes, a conceptual miracle has occurred: We have captured infinity! As we view the landscape, there is a horizon. The horizon is not a physical thing, but an idealization. It represents the boundary of our vision, and lies at infinite distance. Yet the image the horizon casts on our canvas is unquestionably real. It is a unique, specific point—the point at infinity.

Further wonders await, as we restore both the canvas and the base of our landscape (both a plain, and a plane) to their full two dimensions.

Let's suppose, to keep things simple, that the canvas and the plain are perpendicular to one another.

Now we must imagine many straight lines on the landscape. Each will extend to the horizon, and each will project its associated point at infinity to the canvas. One discovers, however, that parallel lines on the plain all approach the same point on the horizon. In figure 11, this leaps to your eye.

FIGURE 11. PARALLELS MEET AT THE HORIZON, IN A COMMON "VANISHING POINT." ONCE YOU ARE ALERT TO THIS PHENOMENON, YOU WILL SEE IT ALL AROUND YOU.

We call that point the *vanishing point* of the family of parallels. In the language appropriate to the canvas, we can say that *parallel lines meet in the point at infinity.*

Here mystical poetry emerges as a straightforward description of artistic reality.

Different families of parallel lines define different vanishing points, which together define the horizon. Projected back to the canvas, the horizon generates a horizontal line, capturing the horizon as a collection of points at infinity. The conceptual horizon, in other words, projects onto canvas as the tangible line at infinity.

Discoveries such as these both excited and empowered the pioneering Renaissance artist/scientist/engineer Brunelleschi. He developed these insights into a powerful technique for producing realistic drawings. In a famous experiment, he used projective geometry to make an accurate representation of how the Baptistery of St. John, in Florence, should appear, as seen from an entrance to the nearby cathedral, then under construction. As shown in figure 12, he arranged so that a viewer could compare the drawing, seen reflected in a mirror, to the actual Baptistery, revealed when the mirror is removed. (A small hole in the drawing permitted viewing.)

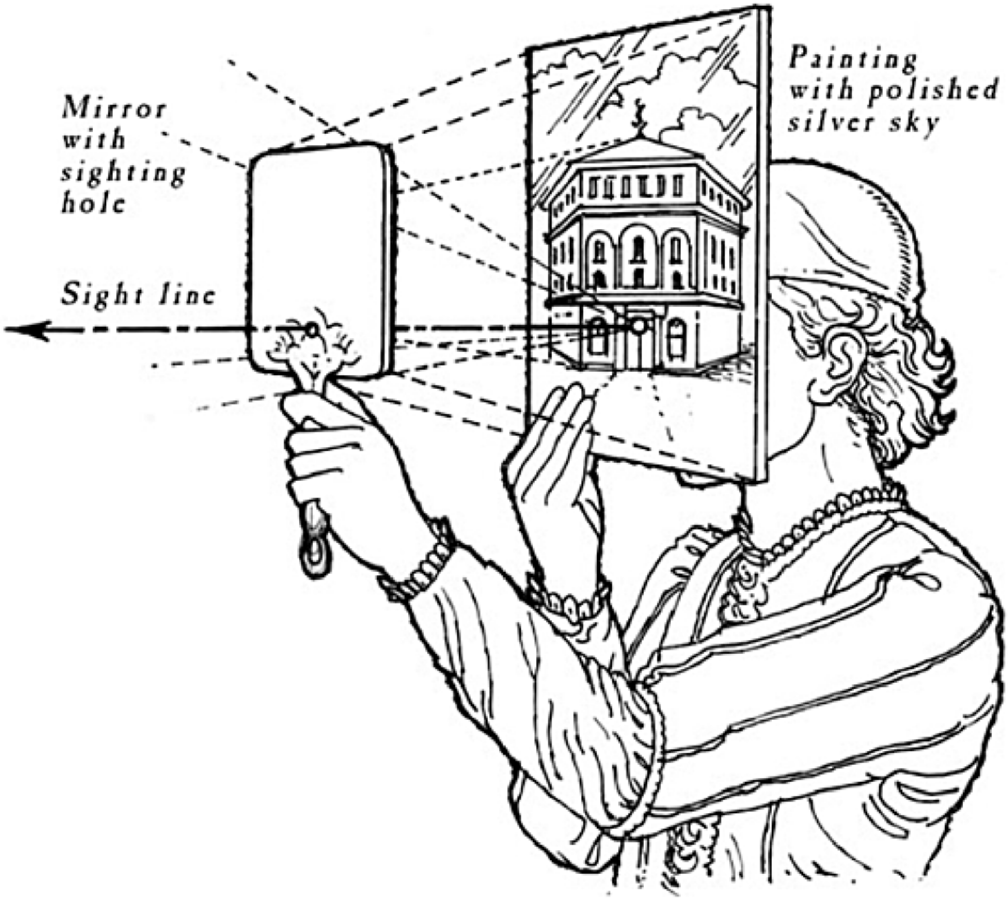

FIGURE 12. BRUNELLESCHI'S DEVICE FOR COMPARING HIS DRAWINGS, BASED ON THE NEW SCIENCE OF PERSPECTIVE, TO REALITY.

This ingenious demonstration made a huge impression on contemporary artists, who took up Brunelleschi's techniques with enthusiasm and developed them energetically. Before long, exuberant joy in perspective infused masterpieces such as *Giving of the Keys to Saint Peter* (plate G), by Pietro Perugino. Here perspective is an active player, lending a special sense of order, harmony, and authority to this founding event for the Catholic Church. This fresco is in the Sistine Chapel.

There's no better way to understand the joy of the artists who discovered and experimented with perspective than to share in one of their simpler creations. In plate H, I've indicated the process by which you can make an accurate perspective drawing of a floor tiled by squares, viewed from in front and above, extending off to an infinite horizon. All you need is a pencil, a straightedge, and an eraser. ("Straightedge" is the term of art for a ruler without distance markings. Of course, a ruler with distance markings will also serve—just ignore the markings!)

The process of construction is indicated in the top portion of the figure. We draw a line, indicated in black, which will be the horizon. We start with one square tile, indicated in blue at the bottom. It isn't drawn square, of course, because we're viewing the floor obliquely. The opposite sides of the "square," when continued, meet at the horizon, at their vanishing points. These continuations are also in blue. So that's what we start with: one tile, and the horizon. The challenge is then to draw all the other equal squares in the tiling as they would appear (in perspective) to an actual viewer.

The key observation is that the diagonals of the squares also form a family of parallel lines. That family of parallels will also meet at the horizon, at their vanishing point. We can draw the red continuation of our original square's diagonal to locate that vanishing point. And then we continue back from the vanishing point, with the orange lines, to get diagonals for the neighboring squares! Having located those diagonals, we know that the intersections of the orange and blue lines are vertices of the neighboring squares. The yellow lines, through those vertices and the appropriate vanishing points, therefore contain the sides of those squares. And now we can keep going—the intersections of the yellow "side" lines with the orange "diagonal" lines are the vertices of new squares.… You can keep going as long as you like, until you lose patience, or your pencil wears down—or your squares shrink down to atomic dimensions.

To complete the construction, you can just erase the diagonals, and (optionally) make all the lines the same color, to arrive at the bottom