A Bouquet of Numbers and Other Scientific Offerings

Jeremy Bernstein

World Scientific

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JEREMY BERNSTEIN

Stevens Institute of Technology, USA

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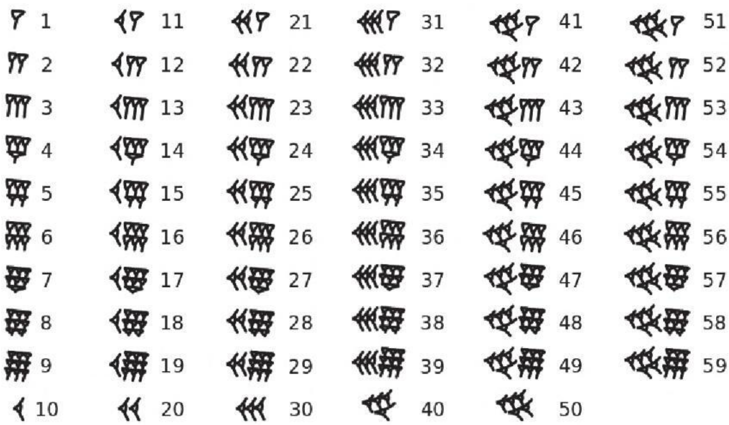
Contents

Some time ago a very intelligent young lady about to enter high school asked me about the work of Archimedes as it is related to his calculation of the areas of polynomials. I had to admit that I did not know but said that I would look into it. This led me to write a whole bouquet of numbers which is the first essay in this collection. It is typical in the sense of how I work. Something strikes my fancy and I am off. I am only satisfied when I understand enough so that I can explain it usually by writing an essay. The subject matter of these essays is very diverse but I will explain how I got to write them as we go along.

1 A Bouquet of Numbers for Olivia

Here are some things that I have learned about numbers. I don't know what kids learn about these things now. Maybe you already know everything I am going to describe but I didn't learn about them until I went to college.

I am going to stick to the decimal system 1,2,3…. by and large. We have this system because we have ten fingers. But it is not the only system that is possible. The Babylonians used a wonderful system based on 60.



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How would you like to do division in that system?

Computers use "bits" — ones and zeros. Here is a short list

1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	You do it.

Adding binary numbers we have to carry the ones. Suppose we add 11 + 11 which should give us six which is 110. So 11 + 11 gives us adding from the right 0 and if we carry the 1 the left hand 1 become 10 to which we add 1 to give 11 and hence the 110. Why don't you try to add 7 to 6 once you get the binary for 13.

It is fun to think of the trinomial system with 0,1,2

0	0
1	1
2	2
3	10
4	11
5	12
6	20
7	21
8	22
9	100
10	101

Now let us stick to ordinary positive decimal integers. How many are there? Put another way is there a biggest one? Here we use a proof by contradiction of which we shall see several examples.

Let us assume there is a biggest one. Call it m. But m + 1 is bigger so the claim that there is a biggest one is false. The number of positive integers is infinite. This form of infinity is given a name " aleph_0." $\aleph_0$. Aleph is the first letter in the Hebrew alphabet. Now we shall see the first of the paradoxes of infinity. We considered the positive integers. Now let us look at the negative ones — 1, −2, −3 and so on. The same argument as before tells us that there is no largest negative integer. To each positive integers there is a negative one so their total number is also $\aleph_0$. So the total number is $2\aleph_0$. But if you add $\aleph_0$ to $\aleph_0$ you still get $\aleph_0$. This is one of the properties of infinity. If you add them you get the same infinity and likewise if you multiply them together. This plays a role when we consider fractions — ratios of integers 5/3,3/4,7/8 and so on. Each one of them corresponds to a pair of integers. How many pairs are there? If you think about it there are

$(\aleph_0)^2 = \aleph_0$ pairs. It looks like we can never get away from $\aleph_0$. We will later see that we can. But before we get to that I want to discuss "prime" numbers.

A prime number is only divisible by itself and one. Thirteen is a prime. So is seven. No even number is a prime since they can be divided by two. Some odd numbers are primes and some are not. Nineteen is a prime by twenty one which is 7×3 is not. All numbers can be factored into products of primes. For example $100 = 25 \times 4 = 5^2 \times 2^2$. If you don't have a computer program which you can find on the web finding the prime factorization of large numbers is not easy. I will tell you a story. I used to work from time to time at a big physics laboratory in Geneva called CERN. When I first went there, there was a famous mental calculator named Wim Klein. He could do unbelievable calculations in his head. He loved prime factorization. Once he called me and told me the prime factorization of my phone number 2129827489. It is equal to

59×36098771. If you don't believe me do the multiplication. When I was going to tell this story here I could not find the piece of paper on which Mr. Klein had written down this answer. So I looked up a prime factorization program on the web. There are several. The one I chose found this factorization almost as fast as I could write the phone number. Mr. Klein was beaten by the computer. Try to find the prime factorization of your own phone number.

How many positive prime numbers are there? I will give you an argument which is attributed to Euclid. I do not think that Euclid was a person. There was a think tank of Greeks on an Island in the Aegean. They wore togas and ate grapes and olives. When they decided to go public with their work they made up the name "Euclid" and we have been stuck with it ever since. Here is how the argument goes.

Suppose we could make a finite list of all primes $p_1 \ldots p_m$ where p_m is the biggest one. We could now consider the number

$$p_1 \times p_2 \times \ldots p_m + 1.$$

Thus number is bigger than p_m so it cannot be a prime. Thus it must be divisible by one of the primes in the list. But this is impossible because that prime would divide the product but not the one. Thus there is no such finite list. The positive primes are a subset of all positive integers. We have seen that there are $\aleph_0$ positive integers so this must be the number of primes since $\aleph_0$ is the smallest infinity.

Now I am going to tell you about another discovery due to the Greeks. The story goes that this bothered them so much that they took the man who discovered it out on a boat and pushed him overboard. We have discussed fractions that are ratios of integers such as 3/4 or 7/8. I am now going to show you a number that cannot be written as a ratio of integers. For the Greeks this number appeared in the Pythagorean theorem. Suppose we have a right triangle both of whose sides are 1. Thus 1^2 + 1^2 = 2. Thus there must be a number whose square is 2. This is of course the square root of 2. But what kind of number is this? Can we write it as the ratio of integers like any decent number?

Let us suppose we do it. $\sqrt{2} = p/q$ where p and q are nice decent integers. The first thing we can do is to divide out any common factors in the ratio. We know for example that 8/10 = 4/5 where we have divided out the common factor of 2. If we square the equation above we have $2 = p^2/q^2$. The square of any even number is an even number and the square of any odd number is an odd number. To see this second statement remember that any odd number cam be written as $2n + 1$ so the square is $4(n^2 + n) + 1$ which is again an odd number. So p and q cannot both be odd. In fact $p^2 = 2q^2$ so p^2 is even and then so is p. It is divisible by 2 so that p^2 is divisible by 4. Therefore q^2 must be even and so must q. So the ratio has a common factor of 2 which contradicts the assumption we made that all the common factors were divided out. This is a simple argument to write down but it is tricky so make sure you understand it. Numbers like the square root of 2 are called “irrational” maybe because they drove the Greeks crazy, I next want to turn to the decimal expansion of numbers. I will begin with a few examples.

1/3 = .3333333333.......

1/4 = .2500000000....

1/2 = .50000000......

$\sqrt{2}$ = 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157273501384623091229702492483605585073721264412149709993583141322266592750559275579995050115278206057147010955997160597027453459686201472857

7418640889198609552329230484308714321450839762603627995251407989687253396546331808829640620615258352395054745750287759961729835575220337531857011354374603408498

847160386899970699004815030544027790316454247823068492936918621580578463111596668

871301301561856898723723528850926486124949771542183342042856860601468247207714358

548741556570696776537202264854470158588016207584749226572260020855844665214583988

939443709265918003113882464681570826301005948587040031864803421948972782906410

4507263688131373985525611732204024509122770022694112757362728049573810896750401836

986836845072579936472906076299694138047565482372899718032680247442062926912485

90521810044598421505911202494413417285314781058036033710773091828693147101711116

839165817268894197587165821521282295184884720896946338628915628827659526351405422

67653239........

I am not trying to snow you but to make a point. This decimal expansion goes on forever and there is never a repeating pattern. On the other hand fractions of integers always lead to a repeating pattern if you wait long enough and vice versa. To see how this works take

$$1.063636363\ldots = x$$

Now

$10x = 10.636363$ while $1000x = 1063.636363$. $1000x - 10x = 990x = 1053$.

So $x = 1053/990$. Try it and see that it works.

The ratio of the circumference of a circle to the diameter as you know is called π — pi. The first digits in the expansion are

3.1415926535897932384626433832795028841971693993751058209749445923078164062862

08998628034825342117067982148086513282306647093844609550582231725359408128481

1174502841027019385211055596446229489549303819644288109756659334461284756482337

8678316527120190914564856692346034861045432664821339360726024914127372458700660

6315588174881520920962829254091715364367892590360011330530548820466521384146

9519415116094330572703657595919530921861173819326117931051185480744623799627495

6735188575272489122793818301194912983367336244065664308602139494639522473719070

2179860943702770539217176293176752384674818467669405132000568127145263560827785

7713427577896091736371787214684409012249534301465495853710507922796892589235

4201995611212902196086403441815981362977477130996051870721134999999837297804995

1059731732816096318595024459455346908302642522308253344685035261931188171010

0031378387528865875332083814206171776691473035982534904287554687311595628638823

5378759375195778185778053217122680661300192787661119590921642019893809525720106

5485863278865936153381827968230301952035301852968995773622599413891249721775

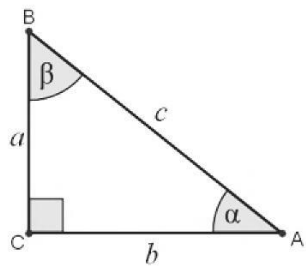
283479131515574857242454150695950829533116861727855889075098381754637464939319255060…..

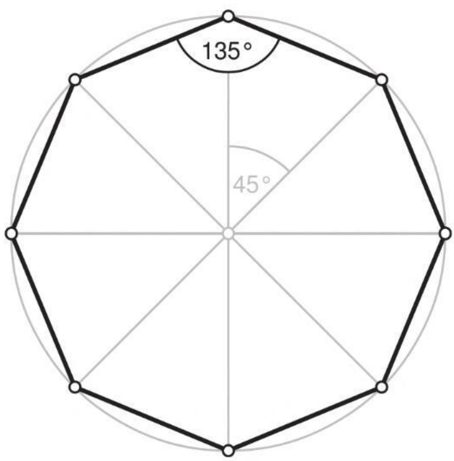
To show that the expansion never repeats requires more sophisticated mathematics than we are doing here. But I will tell you another story. When I was in Geneva I got to know a friend of Klein's named Hans Eberstark. He could do all the mathematical tricks that Klein could do. Sometimes they gave shows together. But he was also a linguist and knew about thirty languages. He worked as an interpreter. The first time I met him is was at an apartment of friend. He had a Philippine cook. Eberstark asked what her language was. It turned out to be a language he did not know. He asked her to give him twelve words and to be sure to write them down so she would remember them. At the end of the evening she came back in and not only did he tell us the twelve words but he was able to say something about them. The word for "apple" was "manzana". Eberstark noted that this was a Spanish word because the Spaniards had imported apples to the Philippines. For the fun of it he memorized 11,944 digits of pi. He chose this number to beat some kind of record.

The next thing I want to tell you about is the infinity of Georg Cantor. Cantor was a German mathematician whose dates were 1845–1918. He did a number of important works in mathematics but I want to tell you about his "diagonal" argument. Suppose all the numbers with decimal expansions could be arrayed as the figure below.

	0	1	2	3	4	5	6	7	8	9	10	
0	0	1	3	6	10	15	21	28	36	45	55	…
1	2	4	7	11	16	22	29	37	46	56	67	…
2	5	8	12	17	23	30	38	47	57	68	80	…
3	9	13	18	24	31	39	48	58	69	81	94	…
4	14	19	25	32	40	49	59	70	82	95	109	…
5	20	26	33	41	50	60	71	83	96	110	125	…
6	27	34	42	51	61	72	84	97	111	126	142	…
7	35	43	52	62	73	85	98	112	127	143	160	…
8	44	53	63	74	86	99	113	128	144	161	179	…
9	54	64	75	87	100	114	129	145	162	180	199	…
10	65	76	88	101	115	130	146	163	181	200	220	…
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

These numbers could then be enumerated and the total would be $\aleph_0$. But there is at least one number missing. You take the first row and replace the first 0 by a 1. Then you take the second row and replace second 4 by a 5 and you go down the diagonal making these replacements. When you get done you have the decimal expansion of a number that was not in our enumeration. The full set of these numbers is a larger infinity than $\aleph_0$. When I first learned about this I thought that it was wonderful and that I wanted to study more mathematics. Maybe you will feel the same. In the meanwhile here is something about pi that you might like. You need to know that if you have a right triangle then sine(α) = a/c. This is trigonometry.





Regular Octagon

A triangle divided into two right triangles.

The first thing we need is not something I can provide a simple proof for. It is actually follows from a theorem of Euclid — him again! Here is a web site where you can find the details.

www.themathpage.com/aTrig/circle.htm

The question we need to answer, and this theorem does it for us, is the following. We know that for any circle the circumference is proportional to the radius. That is C = ar. But is a the same for any circle? The answer — and the theorem helps us to get there — is yes. This is how pi is defined. Pi is a constant that applies to any circle. The equation $C = 2\pi r$ applies to any circle. This is what Euclid understood and this is why pi is so important. Now I want to use an argument that in some form was invented by Archimedes to find the numerical value at least approximately for pi. But I will end up with a formula involving the sine function which is not what Archimedes did as far as I know. He was such a genius that God knows what he ended up with.

The first drawing above is of a regular octagon. Note that it is inscribed inside a circle. Archimedes also studied the case of a regular octagon and other regular polygons drawn outside the circle. I will explain why later. We are only going to discuss in detail the first case. Let me call a side of the octagon S. Then the perimeter of the octagon P is 8S. But P is less than the circumference of the circle. So

$$8S < 2\pi r$$

or

$$\pi > 4S/r.$$

So if we could find S we would have an upper bound for π. Here is where the trigonometry comes in. Look again at the drawing of the octagon with its circumscribed triangles. The angle formed where the sides hit the center of the octagon is 360/8 = 45. But these triangles are not right triangles so I will replace each of them by two right triangles a shown in the next drawing. This splits the angle in two so it is now 22.5. But this gives the base of the triangles S/2. Now we can use our sine formula sine(22.5) = r/S/2 or S = sine(22.5)2r. Thus we have $\pi > 8$sine(22.5). Now we can out in the numbers and find that in the case of the octagon $\pi >$ 3.061467458. This is true but we are still a little far off. You can generalize this to the n-gon and show that $\pi >$ nsine(180/n). Why don't you take n = 96 and see what happens. This is what Archimedes did. How he did the arithmetic no one knows. Maybe he was like Mr. Klein. As n approaches infinity the answer is more and more exact. That is the calculus limit.

Here is a second essay I wrote for the same young lady.

2

Was Einstein Smart?

Some years ago my then New Yorker colleague John McPhee was in the process of writing a profile of the physicist Ted Taylor. Taylor had begun his career as a nuclear weapons designer a Los Alamos. One of his creations was the largest purely fission bomb ever detonated — the so-called Ivy King — which was exploded in November of 1952, in the Pacific. By 1956, he had become disillusioned with working on nuclear weapons for military purposes and he allowed himself to be recruited to lead a new, visionary, indeed incredible, project — the Orion — to design a space ship to be used for planetary exploration that would be powered by a sequence of small nuclear explosions. This enterprise — which ended unsuccessfully in 1965 — was located at the then new General Atomics Company in La Jolla, California. Taylor, in turn, began recruiting other scientists to join him in La Jolla. One of his early, and most important, recruits was Freeman Dyson. Dyson, who was a professor at the Institute for Advanced Study in Princeton, was known throughout the physics community as a mathematical genius. This was McPhee's dilemma.

He had been told by everyone that Dyson was a genius, and he had interviewed him several times, but, as a good reporter, McPhee wanted his own evidence that this was true. That is why he approached me. He knew that I had worked with Dyson on the Orion and that we had remained good friends, so he thought I might have an idea of how he could go about this. The problem was that McPhee did not know any mathematics. He was in the position of someone who wanted to write about Bach, but was tone deaf. Such an individual would have to take the word of others that Bach was a great musician. I gave the matter some thought and finally came up with a suggestion. I would give McPhee a mathematics problem he could understand, one that I thought was pretty tricky. He could then try to solve it and, after in all likelihood, failing, he could go to Dyson and ask for help. He could first find out if Dyson had heard of the problem — hopefully not — then he could give him it to him and watch what happened. The problem I gave McPhee was that of the twelve balls. You have twelve balls that appear to be identical. However one of them — the "guilty" ball — is either lighter or heavier than the others. You also have a balance scale — a scale in which you have platforms on either side of a balance on which you can put some of the balls. For example you might try to balance two balls against two balls. If the scale was, say, unbalanced, you would know the guilty ball was among the four. The problem is to devise a method by which, in at most three weighings, you can find which ball is guilty and whether it is heavier or lighter. I told McPhee my history with this problem. I had heard of it when I was a junior in college. I was up most of the night until I finally solved it. I was very pleased with myself. I had a date the next day to play chess with the most brilliant undergraduate in mathematics and physics in my class. As we were setting up the pieces I gave him the problem. Not only did he solve it before we finished setting up the pieces but he was generalizing it. Suppose you have m balls how many weighings n would it take. You can't do it with two balls and, with three, it takes two weighings, and so on. There is smart, and there is smart.

As far as I know, McPhee never tried this so I don't know what would have happened. Historically speaking, it would have been interesting to see what Einstein would have done with such a problem. I am not aware that he had much interest in puzzles like this. I don't think he played chess and I doubt that he played bridge. Besides, there is always the difference between being smart and seeming smart. For example, Niels Bohr who was after Einstein the greatest physicist of the

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for Advanced Study as was Lee. They won it for a discovery they had made involving elementary particles, but this work was almost a distraction from what they were spending most of their time on — statistical mechanics. This is the discipline that was created in the 19th century that studies the average behavior of systems of particles so numerous that it is a practical impossibility to describe them individually. It was a field that Einstein was particularly fond of. His first papers in physics — the ones that preceded the relativity paper of 1905 — concerned its foundations. This work was never especially recognized because what followed it overshadowed it, and because others did more or less the same thing. But throughout his life Einstein kept returning to statistical mechanics. Lee, who told me about it, and Yang decided that they would call on Einstein and tell him what they had done in that field. Lee was not sure what to expect. This was, although they did not know it, close to the time of Einstein's death. He was in his mid-seventies and Lee was not sure if he still had any interest in the subject. Two things surprised him about the visit. The first was Einstein's hands. To Lee they seemed very large and very powerful. He seemed like a physically strong man. The second thing he did not expect was Einstein's almost instantaneous grasp of what he and Yang had been doing. The subject had evolved tremendously since Einstein's work on it, but he nonetheless understood the new developments and even asked some searching questions.

This persuades *me* that Einstein was smart, but I have the advantage of having studied his physics for decades. I already knew he was smart. To fully convince you, I would have to give you a course in modern physics. I would have to explain to you that Einstein's foot print is everywhere. To give a few examples: in 1916, he published a paper on the emission and absorption of radiation which is the basis of the laser. This was a year after he had published his paper on the general theory of relativity and gravitation which replaced Newton's theory of gravitation. It was the year before he published a codicil to the theory that introduced the idea of a "cosmological constant" which may be the basis of the dark

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with clocks. It is subject to the vagaries of the clocks we have at hand. Newton formulated his physics in terms of absolute time. He published his detailed theory in 1686, in the *Principia,* and for the next two centuries it went pretty much unchallenged. People simply accepted this notion of an absolute time common to all observers. We now come to 1905. Einstein was twenty-six and working in the Swiss National Patent Office in Berne. This was a serious job — examining patents — which he took seriously. The physics he did on the side.

For about a decade he had been puzzling over the following. In Newton's physics there is a principle of relativity. It had been emphasized first by Galileo. As far as I know, it was Professor Frank who introduced the term "Galilean relativity" to describe it. Suppose, Galileo noted, you are on a sailing ship which is at rest with respect to the sea. Furthermore, suppose you drop an object from the mast so that it falls straight down. The object will land at the base of the mast. Now suppose the ship is in motion and that this motion is perfectly uniform — no acceleration — and you perform the same experiment, where will the object land? The answer is in exactly the same place. We might think of this by imagining that the ship is stationary and the sea is somehow being moved underneath it with a uniform motion. In this case we are not surprised at the result above. Galilean relativity, which is built into Newton's laws, is the proposition that we can never distinguish by any mechanical experiment of this kind between a moving ship and a stationary sea and a stationary ship and a moving sea providing that these motions do not involve accelerations. But Einstein realized that there is more to physics than Newtonian mechanics. In particular, there is electromagnetism which included the propagation of light, which is an electromagnetic wave.

He imagined an experiment with light. For this one, we will use a train. Our train is, in the beginning, at rest with respect to the tracks. I want to shave in my compartment, so I rig up a mirror and a bulb behind my head. From its reflection I can see myself. Now we imagine that the train is set into uniform motion and we perform the same activity.

If I emit a sound wave by, say, banging on a drum, I can catch up to it by moving faster than the speed of sound. That is what supersonic airplanes do. Einstein's postulate says that you can never have a superluminous airplane. The speed of light c is the universal speed limit. Once a light wave is emitted you can never catch up to it. At the time that Einstein made this assumption there was no direct experimental evidence to support it. Now we have elementary particles produced in accelerators that move almost at the speed of light so we can observe what happens. But there was no direct evidence against it. It was even suggested by the theory of electricity and magnetism that had been invented in the mid-nineteenth century by the Scottish physicist James Clerk Maxwell. On the other hand, if you believed in Newton's dynamical theory this speed limit was impossible. According to Newton, if you applied a force to an object long enough you could accelerate it to any speed you liked. Thus Einstein had to choose between Newton and Maxwell and he chose Maxwell. Many years ago I had the chance to visit Einstein's house in Princeton. It was soon enough after his death so that his study was about the same as when he used it. He had an etching of Maxwell on the wall. There had been an etching of Newton, but it had come out of its frame and had been replaced by a bit of modern art.

A radical assumption like this must have radical consequences. It does and I would like to explore what it implies about time. The only kind of time that interests me is what Newton called "common" time — the kind that is measured by clocks. In truth, I can make no precise sense out of what Newton meant by "absolute" time. Common time is good enough for me: clock time. A clock is any mechanism that has a periodic behavior. This can be the oscillating of an atom or the beating of a human heart. There are good clocks and bad clocks depending on how dependable the periodic behavior is. Time is measured in terms of the number of these oscillations that occur between events. I am going to analyze in detail a particularly simple clock — a so-called "light clock." We will make use of the figure below.

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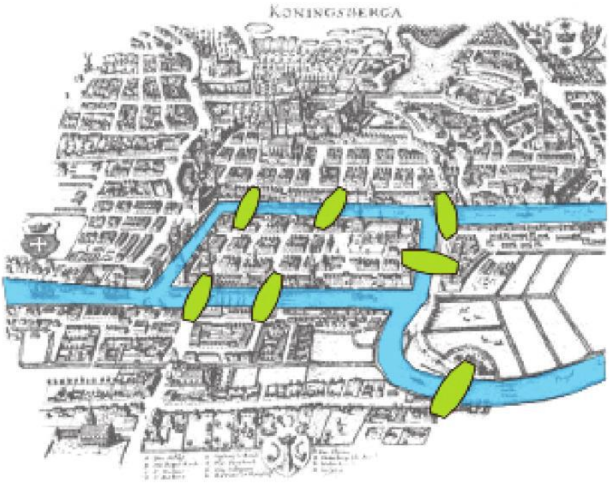
3 A Love Story

The Seven Bridges of Königsberg



Bogdan Giuşcă — Public domain (PD), based on the image

I

There are two very nice seasons in New York — spring and fall. The winters can be very tough. The winds come off the East River and the Hudson and produce a damp cold which is hard to dress for. Sometimes we have ice which makes the sidewalks a sort of malignant ice rink. And the summers are infernal. I gather that in Calcutta a mitigating plea for a crime of violence is that it was committed in the pre-monsoon season. If you have even been there at that time you will