A BRIEF HISTORY OF MATHEMATICAL THOUGHT

LUKE HEATON

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INTRODUCTION

> 'Mathematics is the gate and key of the sciences. … Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.'
>
> Roger Bacon, 1214–1292

The language of mathematics has changed the way we think about the world. Most of our science and technology would have been literally unthinkable without mathematics, and it is also the case that countless artists, architects, musicians, poets and philosophers have insisted that their grasp of the subject was vital to their work. Clearly mathematics is important, and in this book I hope to convey both the poetry of mathematics and the profound cultural influence of various forms of mathematical practice. For better or worse, you can't comprehend the influence of math until you have some understanding of what mathematicians actually do. By way of contrast, you don't need to be an engineer to appreciate the impact of technological change, but it is hard to comprehend the power and influence of mathematical thought without an understanding of the subject on its own terms.

Most people are numerate, and have learned a handful of rules for calculation. Unfortunately the arguments and lines of reasoning behind these techniques are much less widely known, and far too many people mistakenly believe they cannot hope to understand or enjoy the poetry of math. This book is not a training manual in mathematical techniques: it is an informal and poetic guide to a range of mathematical thoughts. I disregard some technicalities along the way, as my primary aim is to show how the language of math has arisen over time, as we attempt to comprehend the patterns of our world. My hope is that by writing about the development of mathematical ideas I can inspire some of my readers, shake up some lazy assumptions about pure and applied mathematics, and show that an understanding of math can help us to arrive at a richer understanding of facts in general.

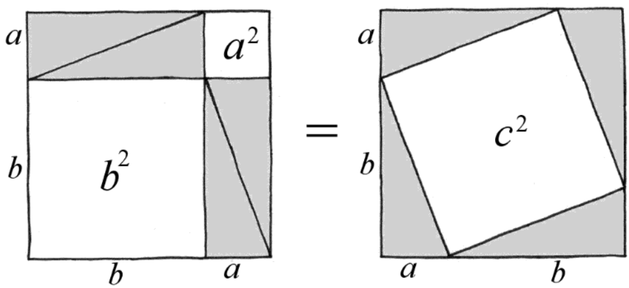
Mathematics is often praised (or ignored) on the grounds that it is far removed from the lives of ordinary people, but that assessment of the subject is utterly mistaken. As G. H. Hardy observed in *A Mathematician's Apology*:

> Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances suggest the contrary, but there are easy explanations. Music can be used to stimulate mass emotion, while mathematics cannot; and musical incapacity is recognized (no doubt rightly) as mildly discreditable, whereas most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity.

The considerable popularity of sudoku is a case in point. These puzzles require nothing but the application of mathematical logic, and yet to avoid scaring people off, they often carry the disclaimer 'no mathematical knowledge required'! The mathematics that we know shapes the way we see the world, not least because mathematics serves as 'the handmaiden of the sciences'. For example, an economist, an engineer or a biologist might measure something several times, and then summarize their measurements by finding the mean or average value. Because we have developed the symbolic techniques for calculating mean values, we can formulate the useful but highly abstract concept of 'the mean value'. We can only do this because we have a mathematical system of symbols. Without those symbols we could not record our data, let alone define the mean.

Mathematicians are interested in concepts and patterns, not just computation. Nevertheless, it should be clear to everyone that computational techniques have been of vital importance for many millennia. For example, most forms of trade are literally inconceivable without the concept of number, and without mathematics you could not organize an empire, or develop modern science. More generally, mathematical ideas are not just practically important: the conceptual tools that we have at our disposal shape the way we approach the world. As the psychologist Abraham Maslow famously remarked, 'If the only tool you have is a hammer, you tend to treat everything as if it were a nail.' Although our ability to count, calculate and measure things in the world is practically and psychologically critical, it is important to emphasize that mathematicians do not spend their time making calculations. The real challenge of mathematics is to construct an *argument*.

Pythagoras' famous Theorem provides an excellent example of how the nature of mathematical thought is widely misunderstood. Most educated people know that given any right-angled triangle, we can use the formula $a^2 + b^2 = c^2$ to find all three lengths, even if we have only been told two of them. As they have been asked to repeatedly perform this kind of calculation, people mistakenly conclude that mathematics is all about applying a given set of rules. Unfortunately, far too few people can give a convincing explanation as to *why* Pythagoras' Theorem must be true, despite the fact that there are literally hundreds of different proofs. One of the simplest arguments for showing that it's true hinges around the following diagram:



Pythagoras: The shapes on either side of the equals sign are contained inside a pair of identical squares, whose sides are $a + b$ units wide. The one on the left contains a square a units wide, a square b units wide plus four right-angled triangles. The one the right contains a square c units wide plus four right-angled triangles. We can convert the picture on the left into the one on the right simply by moving the four triangles, and moving a shape does not change its area. Since the white area is the same in each of the drawings, this demonstrates that $a^2 + b^2 = c^2$ for any right-angled triangle.

Sceptic: How can you be certain that we always get a square on the right-hand side? More specifically, how do you know that your triangles always meet at a point, whatever the values of a and b?

Pythagoras: Both drawings are of equal height ($a + b$ units high). This tells us that the two triangles that are just to the right of the equals sign must touch at a point, because they only just manage to fit inside the containing square. Similarly, the two triangles on the bottom of the right-hand side touch at a point, because the total length along this side is $a + b$, which equals $b + a$ (the width of the containing squares).

Sceptic: OK, but how do you know that the triangles on the right-hand side always meet at right angles? In other words, how do you know that the shape on the right-hand side is really a square?

Pythagoras: You agree that we have four sides of equal length, and all four corners are the same?

Sceptic: Yes. Rotating the picture on the right by 90°, 180° or 270° leaves the diagram unchanged.

Pythagoras: And despite these facts you still aren't convinced that it's a square? No wonder they call you a sceptic!

My aim in writing this book is to show how the language of mathematics has evolved, and to indicate how mathematical arguments relate to the broader human adventure. This book is related to the work of various philosophers (particularly Ludwig Wittgenstein), but it is not a history of non-mathematical ideas, or an attempt to draw battle lines between conflicting 'big pictures' from the philosophy of math. I will have succeeded if my writing provokes thought, but I have also tried to argue against the idea that mathematicians discover facts about abstract objects, just like scientists discover facts about physical objects. Mathematical language does not make sense because abstract objects existed before mathematicians! Contrariwise, we can only become cognisant of abstract objects because mathematical language is something that humans can actually use.

It is fundamental to human understanding that our theories or accounts of the physical world are expressed through language. People make *statements* of fact, and the reflective, systematic study of our ability to make statements leads us into the world of math. Indeed, our understanding of mathematics always begins with a clear, comprehensible case, from which we form a notion of the abstract principles at play. For example, children learn the counting song, and they are then initiated into the practice of counting actual, physical objects. This concrete experience grounds our sense of number, as we abstract away from a particular experience of counting things, justifiably believing that we could set about counting any collection of objects. That is to say, number words become meaningful for an individual as they use those words on some particular occasion, in the presence of actual, countable objects, but once that person has acquired a language, the language itself enables them to think in terms of number, whatever they might wish to count.

Some people mistakenly believe that to do mathematics, we simply need to follow certain rules. I suspect that people arrive at this erroneous position because in order to satisfy their teachers and examiners, all they need to do is apply some rules correctly. In fact, higher mathematics is an essentially creative pursuit that requires imagination. That said, rules are never far behind our creative insights, because in order to contribute to the body of mathematical knowledge, mathematicians need to be able to communicate their ideas. The formal discipline that we require to fully state our arguments is an essential constraint on the shape of mathematical knowledge, but the mathematics that we know also reflects the problems, challenges and cultural concerns that have motivated the various members of the mathematical community.

I hope that reading this book persuades you that mathematicians are explorers of patterns, and formal, logical proofs that can be methodically checked are the ultimate test of mathematical validity. The clarity of a strictly formal proof is a beautiful thing to comprehend, and I think it is fair to say that an argument is only mathematical if it is apparent that it can be formalized. However, while we can gain a sense of understanding by learning to use a particular formal scheme, it is certainly possible to check each step in a formal argument without understanding the subject at hand. Indeed, a computer could do it, even though a computer is no more a mathematician than a photocopier is an artist.

I have taken a more intuitive approach as my aim is not to train the reader in the appropriate formal techniques, but simply to make the heart of each argument as comprehensible as possible. That said, the subject matter of this book is subtle and sophisticated, so there is no escaping the need to take certain arguments carefully and slowly. Mathematics is a subject where you must read the same sentence several times over, and as with poetry, you must read at an appropriate pace.

Over the course of my book I trace out a history of mathematical practice, with a focus on conceptual innovations. I do not claim to have covered all of the key ideas, but I have tried to sketch the major shifts in the popular understanding of math. The book is structured by a combination of historical and thematic considerations, and its thirteen chapters can be grouped into four main sections. I begin by discussing the number concept, from a speculative and rhetorical account of prehistoric rituals to mathematics in the ancient world. I examine the relationship between counting and the continuum of measurement, and try to explain how the rise of algebra has dramatically changed our world.

The first section ends with 'mathematical padlocks' of the modern era, but in the second section I step back in time. More specifically, I discuss the origins of calculus, and the conceptual shift that accompanied the birth of non-Euclidean geometries. In short, I try to explain how modern mathematics grew beyond the science of the Greeks, the Arabs, or other ancient cultures.

In the third section I turn to the most philosophically loaded terms in mathematics: the concept of the infinite, and the fundamentals of formal logic. I also discuss the genius of Alan Turing, and try to elucidate the subtle relationship between truth, proof and computability. In particular, I focus on a proof of the infinite richness of addition and multiplication (as demonstrated by Matiyasevich's Theorem), and examine Kurt Gödel's celebrated theorems on the Incompleteness of Arithmetic.

In the final section I consider the role of mathematics in our attempts to comprehend the world around us. In particular, I describe the importance of models, and the role of mathematics in biology. I conclude by taking a step back from any particular theorem, and try to use what we have learned about mathematical activity to think about thinking in general.

One of the challenges in writing this book was doing justice to the weight of simple, teachable statements. Some statements are like paper darts: you can follow them with a lightness of contemplation, if you know to where they float. If your only guide is to cling to the words themselves, they cannot carry you, as their target has not been spoken. Other statements possess gravitas, as in their accessible simplicity they act like stones, pulling us down to what can and has been said.

Unfortunately, people tend to underestimate the value of simple, understandable statements, as we more often praise ideas by suggesting they are hard to grasp. As the great thinker Blaise Pascal remarked in *The Art of Persuasion*, 'One of the main reasons which puts people off the right way they have to follow is the concept they first encounter that good things are inaccessible by being labelled great, mighty, elevated, sublime. That ruins everything. I would like to call them lowly, commonplace, familiar. These names befit them better. I hate these pompous words ...'

The great edifice of mathematical theorems has a crystalline perfection, and it can seem far removed from the messy and contingent realities of the everyday world. Nevertheless, mathematics is a product of human culture, which has co-evolved with our attempts to comprehend the world. Rather than picturing mathematics as the study of 'abstract' objects, we can describe it as a poetry of patterns, in which our language brings about the truth that it proclaims. The idea that mathematicians bring about the truths that they proclaim may sound rather mysterious, but as a simple example, just think about the game of chess. By describing the rules we can call the game of chess into being, complete with truths that we did not think of when we first invented it. For example, whether or not anyone has ever actually played the game, we can prove that you cannot force a competent player into checkmate if the only pieces at your disposal are a king and a pair of knights. Chess is clearly a human invention, but this fact about chess must be true in any world where the rules of chess are the same, and we cannot imagine a world where we could not decide to keep our familiar rules in place.

Mathematical language and methodology present and represent structures that we can study, and those structures or patterns are as much a human invention as the game of chess. However, mathematics as a whole is much more than an arbitrary game, as the linguistic technologies that we have developed are genuinely fit for human purpose. For example, people (and other animals) mentally gather objects into groups, and we have found that the process of counting really does elucidate the plurality of those groups. Furthermore, the many different branches of mathematics are profoundly interconnected, to art, science and the rest of mathematics.

In short, mathematics is a language, and while we may be astounded that the universe is at all comprehensible, we should not be surprised that science is mathematical. Scientists need to be able to communicate their theories, and when we have a rule-governed understanding, the instructions that a student can follow draw out patterns or structures that the mathematician can then study. When you understand it properly, the purely mathematical is not a distant abstraction – it is as close as the sense that we make of the world: what is seen right there in front of us. In my view, math is not abstract because it has to be, right from the word go. It actually begins with linguistic practice of the simplest and most sensible kind. We only pursue greater levels of abstraction because doing so is a necessary step in achieving the noble goals of modern mathematicians.

In particular, making our mathematical language more abstract means that our conclusions hold more generally, as when children realize that it makes no difference whether they are counting apples, pears or people. From generation to generation, people have found that numbers and other formal systems are deeply compelling: they can shape our imagination, and what is more, they can enable comprehension. The story of math is fascinating in its own right, but in writing this book I hoped to do more than simply sketch a history of mathematical ideas. I am convinced that the history and philosophy of math provide an invaluable perspective on human nature and the nature of facts, and I hope that my book conveys the subject's cultural, aesthetic and philosophical relevance, as well as the compelling drama of mathematical discovery.

Chapter 1:
BEGINNINGS

> 'There can be no doubt that all our knowledge begins with experience. ... But though all our knowledge begins with experience, it does not follow that it all arises out of experience.'
>
> Immanuel Kant, 1724–1804

Language and Purpose

Researchers working with infants and animals have found compelling evidence that we have an innate sense of quantity. More specifically, humans, birds and many other animals can recognize when a small collection has changed in size, even if they do not observe the change taking place. For example, birds can recognize when one of their eggs is missing, even if they did not witness the egg's removal. Similarly, many animals will consistently pick the larger of two collections when they are given a choice. Presumably, this sensitivity to quantity is a necessary precondition for the development of math, and it is interesting to note that some animals are quicker than humans at intuitively sensing differences in quantity. Nevertheless, although such abilities constitute evidence for animal intelligence, it is rather inaccurate to claim that 'birds count their eggs'.

I would argue that 'proto-mathematical' thinking can only begin once we have developed language, and that this kind of understanding is fundamental to many types of human behaviour, not just what we ordinarily think of as 'understanding math'. Of course, any account of the lifestyle of our Stone Age ancestors is bound to be highly speculative, but despite the lack of conclusive evidence, I think it is helpful to imagine how our ancestors first developed rational capabilities, and the enormously complex thing that we call language.

Humans are not the only animals to use tools, and for millions of years our primate ancestors extended their abilities by utilizing what was found at hand. Sticks, stones, fur, leaves, bark and all manner of food stuffs were used in playful ways that we can only guess. Flesh was scraped from fur, sticks were sharpened and adapted to a purpose, and stones were knapped to produce effective butchery kits. Most importantly, about 1.8 million years ago *Homo erectus* started using fire to cook, which reduced the amount of energy needed for digestion, making it possible to grow larger brains and smaller digestive tracts.

As human intelligence evolved, our vocalizations and patterns of interaction developed into something that deserves to be called language. One very plausible speculation is that more intelligent hominids were more successful in making the most from the complex dynamics of their social situation, providing the selective advantage that led to increasing intelligence. In any case, communicative aspects of modern language are common to many animals. For example, many animals can convey a state of panic when they see a predator. It is therefore clear that complex, communicative forms of interaction massively predate the development of the proto-mathematical, or any conception that language might be the thing of interest, rather than the people who were making the sounds.

This idea is worth elaborating, so as an example of how a culture of interactions can evolve to something greater, imagine a woman who lives in a community with a particular culture of responses: *Men who give me tasty food will hear me hum, but those who grab me without giving me tasty food will hear me growl.* If a man was trying to establish a sexual relationship with this woman, he would want to hear the humming sound, because a woman who hums is much more likely to be interested in sex than a woman who growls. Consequently, the man would wilfully do a bit of cooking prior to any sexual advance, acting to establish the circumstances that he associates with the humming sound.

By living in such a social context, we came to feel the sense of our own actions. In other words, we responded to changing social occasions with increasingly sophisticated forms of motivated strategy, and were mindful of complex goals whose achievement required actions beyond those in the immediate present. For example, the occasion of preparing an especially tasty meal is not the same occasion as hearing a woman make the humming sound, but we see that one is motivated by the other.

Social norms and the vocabulary of praise and blame are both potent forces for shaping the imagination. It is absolutely fundamental that we find words with which to judge our actions, and our judgements work with words. An example of this endlessly subtle process can be found in the following conversation:

> 'Let's break into that house.'
> 'I don't know, that seems like a bad idea.'
> 'Go on, don't be chicken.'

We are fearful that our reasoning will compel us to name ourselves cowards, idiots, or many other kinds of undesired utterance. The will to avoid such experience is part of our humanity, as is the compelling nature of the reasons that we find. As Blaise Pascal observed, we are most compelled by the reasons of our own devising, but such complexities can be closely shared and instinctively taught to others. The caveman is compelled by the fact that the woman has established reasons for growling, if he fails to meet the expectations of an established practice. It is a process of judgement that he has a feeling for, and the weight of the utterance is that it is not felt to be arbitrary. Similarly, our potential thief is pulled by the fact that he too can reason himself a coward, and does not wish to do so.

However, it is crucial to note that in each of these examples, the significance of an utterance is inseparable from the fact that another person has decided to say the statement in question. In other words, we reached the point of very sophisticated communication long before we ever considered 'language' as a thing in itself, separate from the people who were speaking.

Human Cognition and the Meaning of Math

The literature of mathematics is largely composed of arguments of the form 'If A and B are true, then it follows that C is also true', and it is worth pausing to wonder how it is that humans developed the capacity for deductive reasoning. We are not the only animals who are alert to the range of possible consequences of our actions, and we might suppose that our grasp of logical consequence is only possible because we have evolved the cognitive abilities needed to predict the practical consequences of the things that we might do. For example, imagine a hungry ape looking at another ape with some food. It might think to itself, 'If I grab the food, that big guy will hit me. I don't want to be hit, so therefore I should restrain myself and not grab the food.'

The fact that we use language fundamentally changes the character of our reasoning, but it is easy to believe that imagining the consequences of potential actions is an ancient ability that confers an evolutionary advantage. However, it is hard to see how the evolution of this kind of 'reasoning' about actions and their consequences could enable abstract thought. After all, the scenario I have described is all about judging the way to behave in a complex context, where any new information might change our prediction of what will happen next, and we ought to be open to noticing further clues. For example, if our ape saw the other ape make a friendly gesture, it might be wise to grab the food instead of letting it go. That is very different from working out logical consequences, where one thing follows from another, regardless of any further information that could plausibly come our way.

Because the social cunning of animals depends on their grasp of entire contexts, where there are always further clues, it is difficult to see how that kind of understanding could provide the cognitive abilities that a mathematician requires. In contrast, our capacity for spatial reasoning is much less open ended, and human beings do not need to be trained to make valid spatial deductions. For example, suppose that there is a jar inside my fridge. Now suppose that there is an olive inside the jar. Is the olive inside the fridge? The answer is yes, of course the olive is inside the fridge, because the olive is in the jar, and the jar is in the fridge. Now imagine that the jar is in the fridge but the olive is not in the fridge. Is the olive in the jar? Of course not, because the jar is in the fridge, and I have just told you that the olive is not in the fridge.

In reasoning about the location of the olive, it is sufficient to bear a thin skeleton of facts in mind. Additional information will not change our thinking, unless it contradicts the facts that form the basis of our deduction. Also note that in order to make these deductions, we do not need to be initiated into some or other method of symbolizing. All humans can reason in this way, so it is plausible to claim that there are innate neural mechanisms that underpin our grasp of the logic of containers. Of course, in order to pose these questions I need to use some words, but humans (and other animals) find it very easy to understand that containers have an inside and an outside, and this kind of understanding provides a structure to our perceptual world.

There is strong empirical evidence that before they learn to speak, and long before they learn mathematics, children start to structure their perceptual world. For example, a child might play with some eggs by putting them in a bowl, and they have some sense that this collection of eggs is in a different spatial region to the things that are outside the bowl. This kind of spatial understanding is a basic cognitive ability, and we do not need symbols to begin to appreciate the sense that we can make of moving something into or out of a container. Furthermore, we can see in an instant the difference between collections containing one, two, three or four eggs. These cognitive capacities enable us to see that when we add an egg to our bowl (moving it from outside to inside), the collection somehow changes, and likewise, taking an egg out of the bowl changes the collection. Even when we have a bowl of sugar, where we cannot see how many grains there might be, small children have some kind of understanding of the process of adding sugar to a bowl, or taking some sugar away. That is to say, we can recognize particular acts of adding sugar to a bowl as being examples of someone 'adding something to a bowl', so the word 'adding' has some grounding in physical experience.

Of course, adding sugar to my cup of tea is not an example of mathematical addition. My point is that our innate cognitive capabilities provide a foundation for our notions of containers, of collections of things, and of adding or taking away from those collections. Furthermore, when we teach the more sophisticated, abstract concepts of addition and subtraction (which are certainly not innate), we do so by referring to those more basic, physically grounded forms of understanding. When we use pen and paper to do some sums we do not literally add objects to a collection, but it is no coincidence that we use the same words for both mathematical addition and the physical case where we literally move some objects. After all, even the greatest of mathematicians first understood mathematical addition by hearing things like 'If you have two apples in a basket and you add three more, how many do you have?'

As the cognitive scientists George Lakoff and Rafael Núñez argue in their thought-provoking and controversial book *Where Mathematics Comes From*, our understanding of mathematical symbols is rooted in our cognitive capabilities. In particular, we have some innate understanding of spatial relations, and we have the ability to construct 'conceptual metaphors', where we understand an idea or conceptual domain by employing the language and patterns of thought that were first developed in some other domain. The use of conceptual metaphor is something that is common to all forms of understanding, and as such it is not characteristic of mathematics in particular. That is simply to say, I take it for granted that new ideas do not descend from on high: they must relate to what we already know, as physically embodied human beings, and we explain new concepts by talking about how they are akin to some other, familiar concept.

Conceptual mappings from one thing to another are fundamental to human understanding, not least because they allow us to reason about unfamiliar or abstract things by using the inferential structure of things that are deeply familiar. For example, when we are asked to think about adding the numbers two and three, we know that this operation is like adding three apples to a basket that already contains two apples, and it is also like taking two steps followed by three steps. Of course, whether we are imagining moving apples into a basket or thinking about an abstract form of addition, we don't actually need to move any objects. Furthermore, we understand that the touch and smell of apples are not part of the facts of addition, as the concepts involved are very general, and can be applied to all manner of situations. Nevertheless, we understand that when we are adding two numbers, the meaning of the symbols entitles us to think in terms of concrete, physical cases, though we are not obliged to do so. Indeed, it may well be true to say that our minds and brains are capable of forming abstract number concepts because we are capable of thinking about particular, concrete cases.

Mathematical reasoning involves rules and definitions, and the fact that computers can add correctly demonstrates that you don't even need to have a brain to correctly employ a specific, notational system. In other words, in a very limited way we can 'do mathematics' without needing to reflect on the significance or meaning of our symbols. However, mathematics isn't only about the proper, rule-governed use of symbols: it is about *ideas* that can be expressed by the rule-governed use of symbols, and it seems that many mathematical ideas are deeply rooted in the structure of the world that we perceive.

Stone Age Rituals and Autonomous Symbols

Mathematicians are interested in ideas, not just the manipulation of 'meaningless' symbols, but the practice of mathematics has always involved the systematic use of symbols. Mathematical symbols do not merely express mathematical ideas: they make mathematics possible. Furthermore, even the greatest mathematicians need to be taught the rules before they can make a contribution of their own. Indeed, the very word mathematics is derived from the Greek for 'teachable knowledge'. The question is, how and why did human cultures develop a system of rules for the use of symbols, and how did those symbols change our lives?

It seems fair to claim that the most fundamental and distinctive feature of human cognition is our boundless imagination. We don't just consider our current situation, we imagine various ways that the future could pan out, and we think about the past and how it could have been different. In general, we inhabit imaginable worlds that follow certain principles, and compared to other animals, our thoughts are not overly constrained by our current situation or perceptions. In particular, we can think about objects that are not ready at hand, and it is reasonable to assume that in the distant past, our ancestors would feel distressed if their desire to act was frustrated by the absence of some object or tool.

As an animal might express the presence of predators, our ancestors would gesture, 'I am missing a flint'. By using their vocal cords, facial expressions and bodily posture, they would express their motivated looking. Fellow primates would respond to this signal in a manner appropriate to the occasion, having an empathic grasp of what it is to search in such a fashion. Over countless generations, our ancestors must have developed ways of conveying a desire for certain objects, even though those objects were currently out of sight. Furthermore, at some point our ancestors must have made the vital step of imbuing those expressive gestures with an essentially mathematical meaning. This remarkable feat was not achieved by the discovery of abstract objects: it was achieved by developing rituals.

Suppose, for example, that there was a pre-existing form of expression that conveyed the speaker's irritation over missing a flint. Now imagine the earliest people running their hands over their treasured tool kit of flints. As a person checked their tools time and time again, they may have expressed their familiarity with these objects by reciting a sequence of names. As each tool was touched in turn, we can imagine our ancestors repeating a distinctive sequence of rhythmic speech, with one word for each tool, like someone saying 'Eeny-meeny-miney-mo'. If this ritual was left unfinished by the time there were no more objects left to touch, Stone Age humans could see that they had a *reason* for making the gesture 'I am missing a flint'.

This is not the same as counting with an abstract concept of number, as whether or not they deserve to be called mathematicians, even the youngest of children will not mistake 'Eeny-meeny-miney' for 'Eeny-meeny-miney-mo'. Sensitivity to the incompleteness of a habitual action is clearly innate, and this is very close to the sing-song voice of baby talk and our natural sense of rhythm. When our ancestors expressed this failure of correspondence between the present tools and the familiar ritual, the other people would also know that something was missing, because they recognized that the ritual had been performed correctly. In other words, it is the *ritual* that tells us that a flint is missing, and not just the individual who performed it.

By possessing such a clear sense of justified speech, people could find grounded meanings in their utterances, and strategically approach the issues that concerned them (namely, 'Is it the case that all the familiar devices are present?'). In this way, expressive gesture came to signify more than an immediate cultural resonance, and the primal gestures that conveyed the sense 'I am irate over a missing flint' become something deeper. The common sense of valid reasoning gave new weight to our communications, as by means of the common practice a statement of fact can be established. In particular, note that the fact's appearance in the world is dependent on the practice itself, not the individual who carried it out. This process of language speaking for itself emphatically does not require the abstract concept of number, and I would argue that proto-mathematical thought had a gradual evolution that predates counting by many tens of thousands of years.

The origin of number words as we understand them today isn't known for certain, but there are some interesting theories supported by linguistic evidence. It may be that practices closely related to counting arose spontaneously throughout the world, more or less independently from place to place. However, the mathematician and historian of science Abraham Seidenberg (1916–1988) proposed that counting was invented just once, and then spread across the globe. Number words are often related to words for body parts, and Seidenberg claimed that the similarities in number words from very distant places constitutes evidence for his theory. He also made the intriguing observation that in almost every numerate culture, there is an ancient association between the odd numbers and the male, while the even numbers are female.

There is certainly plenty of evidence that animals are aware of who is first in the pecking order, who is second, third and so on. Seidenberg suggests that counting originated in rituals based on rank and priority, arguing that counting 'was frequently the central feature of a rite, and that participants in the rite were numbered'. Whether the first numbers or number-like words were applied to an ordered sequence of people, or used to assess the plurality of a collection of tools, it is clear that the human mind has been capable of learning how to count for tens of thousands of years.

It is important to note that mathematics is not a universal human trait, as some cultures have no words for numbers larger than three. Furthermore, some people have a highly cultured sense of quantity even though they cannot really count, as their language has too few number words.

For example, the Vedda tribesman of Sri Lanka used to 'count' coconuts by gathering one twig for each coconut. The people who did this clearly understood that there was a corresponding plurality between the twigs and the coconuts, but if asked how many coconuts a person had collected, they could only point to the pile of twigs and say 'that many'.

The first expressions of quantity are lost deep in the mists of time, but it is surely safe to assume that long before the advent of abstract number words, people had one word for 'hand', and a different word for 'pair of hands'. The move from words that convey quantities of specific physical things to an abstract or universally applicable language of number is an example of logic at work. That is to say, once we have a sequence of words for 'counting' something or other, it is possible to recognize that it is the words themselves that form an ordered, rule-governed sequence, and they need not be bound to any particular thing that people are used to counting.

Making Legible Patterns

As human beings we live in a world of people and things, sights and sounds, tastes and touches. We don't see a pattern on our retina: we see people, trees, windows, cars, and other things of human interest. This relates to the fact that we use language to think about our world, doing things like naming objects, or creating accounts of people or situations. My point is that human beings conceptually structure the perceptual flux in which we live, so our use of symbols, images and words is central to making sense.

For example, imagine a young child drawing a picture of Daddy: a stick-man body with a circle for a head, two dots for eyes and a U-shape for a smile. It is significant that each part of this drawing can be named, as we understand, for example, that two dots can represent the eyes. The art of 30,000 BC was probably somewhat similar to a child's drawing, not because our ancestors were simple minded, but because drawing nameable things is such a basic, human skill. Indeed, we can say that children's drawings are understandable precisely because we can talk our way about them.

Our ancestors decorated caves with vivid illustrations of large mammals, but they also used simpler marks (arrangements of dots, V-shapes, hand prints, etc.). Just as a child might not need to draw ears and a nose before their marks become a face, so the caveman artist may have drawn some tusks and already seen a mammoth. Such stylized, intelligible drawings are not the same as writing, but there is a related logic of meaningful marks, and it is surely safe to assume that our ancestors talked about their drawings. As another example of Stone Age pattern making, archaeologists in central Europe found a shinbone of a wolf marked with fifty-seven deeply cut notches. These marks were arranged in groups of five, and carbon dating indicates that this bone is over 30,000 years old.

People and animals alike are good at spotting patterns. In particular, many birds and mammals are demonstrably sensitive to changes in quantity. Time and time again humans have discovered a basic technique for clearly showing quantity: we group elements together in a regular way, so that a single 'counting' operation is broken into a combination of simpler assessments. For example, we can recognize four as a pair of pairs, or ten as a pair of fives. This means that even before one can count, it is easier to assess the plurality of things if they are arranged in regular groups, rather than scattered in a disordered fashion. Furthermore, once we have words for numbers, this practical idea can lead us to the concept of multiplication. This suggests that by the time that our ancestors had articulated an abstract concept of number, people were counting by dividing their things into regular groups and counting off five, ten, fifteen, twenty (say). In other words, the times tables may be just as old as abstract number words themselves.

Mathematics has been described as the language of patterns, and there is a deep relationship between our innate tendency to recognize patterns, and our cultured sense of shape and number. Long before we developed proper number words, ancient peoples must have recognized the reality of patterns, and explored some formal constraints. Very ancient peoples must have known that triangles can be arranged to produce certain shapes or patterns, but there are some shapes (e.g. a circle) that cannot be made from triangles. People have been exploring patterns for tens of thousands of years, using their material ingenuity (e.g. pottery and basket weaving), music, dance and early verbal art forms. For example, it is an evident truth that if you clap your hands every second heartbeat, and stamp your feet every third one, there necessarily follows one particular combined rhythm and not others.

The meaningfulness of mathematical statements did not appear from nowhere, and we don't need 'proper' mathematics to first be aware of quantity and shape. Before people developed counting or abstract number words, they might have used a phrase like 'as many bison as there are berries on a bush', or shown a quantity with an artistic abundance of marks. After all, our artist ancestors surely strove to be eloquent, and someone must have been keen to show the size of a massive herd. Many generations later, the advent of counting gave birth to the concept of number: a great advance in our ability to conceptually distinguish between different pluralities.

At first our ancestors must have only used their counting words in particular situations, but over time we realized that in order to count any collection of objects, we do not need to keep our eyes on the qualities of the objects themselves. In a sense we can count any collection of objects, so long as we can give each object a name (e.g. by attaching labels). From that point on we can play the counting game simply by reviewing the sequence of names that we ourselves have given, even if our labels become detached from their associated objects.

Many, many social needs require calculation and number, and over the long arc of prehistory mathematics continued to evolve along with the social systems that supported mathematical techniques. In return, more sophisticated mathematics enabled more complex social structures. For example, an inheritance cannot be distributed unless certain facts about division are known, and at a more sophisticated level, tax rates and monetary systems are literally inconceivable without the concept of number.

The development of agriculture revolutionized our ways of life, and according to many ancient historians, geometry (Greek for earth-measuring) came into being as people needed to speak authoritatively and uncontentiously about the size of fields. In particular, every year Egyptian mathematicians needed to replace the property markers that were washed away from the flood plains of the Nile. The story of geometric techniques arising with the need to measure fields is certainly plausible, but there were also prehistoric traditions for communicating specific plans for temples and other buildings, which necessarily involves a language of shape, so this claim may not actually be correct. What is known for certain is that by the third millennium BC, civilizations with sophisticated mathematical practices had developed along the fertile banks of many of the world's great rivers. The Nile, the Tigris and Euphrates, the Indus, the Ganges, the Huang He and the Yangtze all provided ground for these new ways of life. Furthermore, mathematics had a central role to play in the emergence of large-scale civilizations, not least in the development of trade, measured and planned construction, administrative techniques, astronomy and time keeping.

The Storage of Facts

You might guess that the oldest recorded numbers would be fairly small, but in fact archaeologists have found that some of the oldest unambiguously numerical records refer to the many thousands of cattle that Mesopotamian kings had claimed through war. It is also interesting to note that no civilization has ever become literate without first becoming numerate, and almost every numerate civilization is known to have used some kind of counting board or abacus. In other words, tools for recording the counting process are much, much older than tools for recording speech. Greeks and Romans used loose counters, the Chinese had sliding balls on bamboo rods, and the Ancient Hindu mathematicians used dust boards, with erasable marks written in sand.

Because of geographical distance, it is assumed that the development of mathematics in the Americas was completely independent from that of Europe and Asia. It is therefore remarkable to note that around 1,500 years ago, the Maya were employing number symbols much as we do today. The Incas were a more recent civilization than the Maya, and although the Incas never developed a system for making records of the spoken word, they could record information by using a system of knotted cords called *quipus*. These were colour coded to represent the various things that were counted, and scribes would read the clusters of knots by pulling the cord through their hands. Each cluster of knots represented a digit from one to nine, and a zero was represented by a particularly large gap between clusters.

Quipus with as many as 1,800 cords have been found, and as different-coloured threads signified different kinds of information, these fascinating objects demonstrate that sophisticated record keeping is not the exclusive domain of the written word. Very few quipus have survived, and given that they are far more distant than the Incas, it is plausible that many prehistoric civilizations possessed long-lost means of embodying data. It is worth remembering that only a tiny subset of equipment survives the ravages of time, and if people kept records by arranging pebbles or making scratches on bark, we might have no way of knowing. As the mathematical historian Dirk Struik has suggested, the builders of ancient monuments like Stonehenge must have had some idea of what it was that they were building. Many of the regular features of this construction cannot be accidental, and it seems highly unlikely that the builders didn't know what they would do with the stones until they got them on site. Their means of communicating intent may well have involved physical artefacts that embodied data unambiguously. For example, they may have made shadow casting models that showed exactly how many stones were going to be arranged, and their orientation in relation to the sun's path.

The ancient civilizations of Asia used bamboo, bark and eventually paper to keep records of numerical information. Although the origins of their mathematical knowledge remain obscure, certain pieces of Chinese mathematics have been faithfully passed down through hundreds of generations. For example, consider the following 'magic square', known as the *Lo-Shu*. Legend has it that this mathematical pattern emerged from the Yellow River on the back of a giant turtle about 4,000 years ago. We can't really be certain about the true age of the *Lo-Shu*, but we do know that it was considered to be truly ancient knowledge as far back as the Han dynasty (206 BC–AD 220).