From al-Khwarizmi to Emmy Noether

A History of Algebra

Bartel L. van der Waerden

Springer

B.L. van der Waerden

A History of Algebra

From al-Khwārizmī to Emmy Noether

With 28 Figures

Springer-Verlag Berlin Heidelberg GmbH

Prof. Dr. Bartel Leenert van der Waerden
Mathematisches Institut der Universität Zürich

Mathematics Subject Classification (1980):
01-XX, 10-XX, 12-XX, 14-XX, 16-XX, 17-XX, 20-XX, 22-XX

Library of Congress Cataloging in Publication Data
Waerden, B.L. van der (Bartel Leenert), 1903–
A history of algebra.
Bibliography: p. Includes index.
1. Algebra – History. I. Title.
QA151.W343 1985 512'.009 85-2820

ISBN 978-3-642-51601-6 ISBN 978-3-642-51599-6 (eBook)
DOI 10.1007/978-3-642-51599-6

Originally published by Springer-Verlag Berlin Heidelberg in 1985.

2141/3140-543210

Table of Contents

Part Two. Groups

Part Three. Algebras

Part One
Algebraic Equations

Chapter 1
Three Muslimic Authors

It is beyond my competence to write a history of algebra in the Muslimic countries. Every year new publications on the subject appear. I guess the time has not yet come for a comprehensive history of Muslimic mathematics. Therefore I shall restrict myself to three most interesting authors, whose main works are available in modern translations, namely

A. Al-Khwārizmī,
B. Ṯābit ben Qurra,
C. Omar Khayyām.

Part A
Al-Khwārizmī

If we want to form an opinion on the scientific value and the sources of the work of al-Khwārizmī, we have to consider not only his treatise on Algebra, but also his other mathematical, astronomical, and calendaric work. The present section will be divided into twelve subsections.

1. The Man and his Work

An excellent account of the life and work of Muḥammad ben Mūsā al-Khwārizmī has been given by G.J. Toomer in Volume VII of the Dictionary of Scientific Biography, pages 358–365. From this account I quote:

> Only a few details of al-Kwārizmī's life can be gleaned from the brief notices in Islamic bibliographical works and occasional remarks by Islamic historians and geographers. The epithet "al-Khwārizmī" would normally indicate that he came from Khwārizm (Khorezm, corresponding to the modern Khiva and the district surrounding it, south of the Aral Sea in central Asia). But the historian al-Ṭabarī gives him the additional epithet "al-Quṭrubbullī", indicating that he came from Quṭrubull, a district between the Tigris and Euphrates not far from Baghdad, so perhaps his ancestors, rather than he himself, came from Khwārizm; this interpretation is confirmed by some sources which state that his "stock" *(aṣi)* was from Khwārizm....
>
> Under the Caliph al-Ma'mūn (reigned 813–833) al-Khwārizmī became a member of the "House of Wisdom" (Dār al-Ḥikma), a kind of academy of scientists set up at Baghdad, probably by Caliph Harūn al-Rashīd, but owing its preeminence to the interest of al-Ma'mūn, a great

patron of learning and scientific investigation. It was for al-Ma'mūn that al-Khwārizmī composed his astronomical treatise, and his *Algebra* also is dedicated to that ruler.

From now on I shall omit all bars and dots. This simplifies the printing, and it will not give rise to any misunderstanding.

2. *Al-jabr and al-muqabala*

The biographer Haji Khalfa states in his biographical lexicon (ed. Flügel, Vol. 5, p. 67) that al-Khwarizmi was the first Islamic author to write "on the solution of problems by *al-jabr* and *al-muqabala*". What do these two expressions mean?

The usual meaning of *jabr* in mathematical treatises is: adding equal terms to both sides of an equation in order to eliminate negative terms. Another, less frequent meaning is: multiplying both sides of an equation by one and the same number in order to eliminate fractions. See George A. Saliba: The Meaning of al-jabr wa'l muqābalah, Centaurus 17, p. 189–204 (1973).

The usual meaning of *muqabala* is: reduction of positive terms by subtracting equal amounts from both sides of an equation. But al-Karaji also uses the word in the sense: to equate. The literal meaning of the word is: comparing, posing opposite.

The combination of the two words: *al-jabr wal-muqabala* is sometimes used in a more general sense: performing algebraic operations. It can also just mean: The science of algebra.

Let me give some examples of the use of these words in the work of al-Khwarizmi. On page 35 of Rosen's translation of the "Algebra of Mohammed ben Musa", the following problem is posed:

> I have divided ten into two portions. I have multiplied the one of the two portions by the other. After this I have multiplied one of the two by itself, and the product of the multiplication by itself is four times as much as that of one of the portions by the other.

Al-Khwarizmi now calls one of the portions "thing" *(shay)* and the other "ten minus thing". Multipliying the two, he obtains in the translation of Rosen "ten things minus a square".

For the square of the unknown "thing" the author uses the word *mal*, which means something like "wealth" or "property". He finally obtains the equation

"A square, which is equal to forthy things minus four squares".

In modern notation, we may write this equation as

$$x^2 = 40x - 4x^2.$$

Next the author uses the operation *al-jabr*, adding $4x^2$ to both sides, thus obtaining

$$5x^2 = 40x$$

or

$$x^2 = 8x$$

from which he obtains $x = 8$.

Just so, on page 40, al-Khwarizmi has the equation

$$50+x^2=29+10x$$

which is reduced by *al-muqabala* to

$$21+x^2=10x.$$

In the introduction to his treatise the author states that the Imam al-Mamun

"has encouraged me to compose a short work on calculating by Completion and Reduction, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, lawsuits, and trade, and in all their dealings with another, or where the measuring of lands, the digging of channels, geometrical computation, and other objects of various sorts and kinds are concerned...".

The full title of the treatise is "The Compendious Book on Calculation by *al-jabr* and *al-muqabala*". The treatise consists of three parts.

In the first part, al-Khwarizmi explains the solution of six types, to which all linear and quadratic equations can be reduced:

(1) $$ax^2=bx$$

(2) $$ax^2=b$$

(3) $$ax=b$$

(4) $$ax^2+bx=c$$

(5) $$ax^2+c=bx$$

(6) $$ax^2=bx+c,$$

where a, b, and c are given positive numbers.

Al-Khwarizmi gives rules for solving these equations, he presents demonstrations of the rules, and he illustrates them by worked examples. We shall discuss his demonstrations presently.

3. On Mensuration

The second chapter of the "Algebra" is concerned with mensuration. Since Rosen's translation was deemed unsatisfactory, Solomon Gandz published the Arabic text together with a new English translation in his treatise "The Mishnat ha-Middot and the Geometry of Muhammed ibn Musa Al-Khowarizmi", Quellen and Studien zur Geschichte der Mathematik A2 (Springer-Verlag 1932).

The chapter consists mainly of rules for computing areas and volumes. For instance, the area of a circle is found by multiplying half of the diameter by half of the circumference. For finding the circumference, three rules are presented. If the diameter is d and the periphery p, the three rules are

(7) $$p=3\tfrac{1}{7}d,$$

(8) $$p=\sqrt{10d^2},$$

(9) $$p=\tfrac{62832}{20000}d.$$

Note that the rule (7) is due to Archimedes, who proved that p is less than 3 1/7 times d and more than 3 10/71 times d. The same rule (7) is also given by Heron of Alexandria in his "Metrica", and in the Hebrew treatise "Mishnat ha-Middot", edited and translated by Solomon Gandz.

The rule (8) is also found in Chapter XII of the Brahmasphutasiddhanta of Brahmagupta. See H.T. Colebrooke: Algebra with Arithmetic from the Sanskrit of Brahmegupta and Bhascara (London 1817, reprinted 1973 by Martin Sändig, Wiesbaden), p. 308–309.

Most remarkable is the rule (9), which is equivalent to the very accurate estimate

(10) $$\pi \sim 3.1416.$$

Al-Khwarizmi ascribes the rule (9) to "the astronomers", and indeed the same rule is found in the *Aryabhatiya* of the Hindu astronomer Aryabhata (early sixth century AD). Verse II 28 of the Aryabhatiya reads:

> Add 4 to 100, multiply by 8, and add 62000. The result is approximately the circumference of a circle of which the diameter is 20000 (see W.E. Clark: The Aryabhatiya of Aryabhata, p. 28).

In the last chapter of my book "Geometry and Algebra in Ancient Civilizations" (Springer-Verlag 1983) I have shown that the estimate (10) was also known to the Chinese geometer Liu Hui (third century AD). This estimate may well be due to Apollonios of Perge (see p. 196–199 and p. 207–213 of my book).

Al-Khwarizmi states that in every rectangular triangle the two short sides, each multiplied by itself and the products added together, equal the product of the long side multiplied by itself. Thus, if a, b, c are the sides, we have

$$a^2+b^2=c^2.$$

The proof given in the text is valid only in the equilateral case ($a=b$). From this fact we may safely conclude that al-Khwarizmi's main source is not a classical Greek treatise like the "Elements" of Euclid. Aristide Marre, who published a French translation of al-Khwarizmi's chapter on mensuration in Annali di matematica 7 (1866), noted the insufficiency of the proof and added that the author would never have been admitted to the Platonic academy!

An ancient Hebrew treatise exists which is closely connected, in contents and terminology, with Khwarizmi's chapter on Mensuration. The treatise is entitled "Mishnat ha-Middot". It was published, with an English translation and excellent commentary, by Solomon Gandz in Quellen und Studien zur Geschichte der Mathematik A2. By his arguments, Gandz has convinced me that the author of the treatise was *Rabbi Nehemiah*, who lived about AD 150.

The author knew how to compute the periphery of a circle as $3\frac{1}{7}d$. For the area of a circle segment, he presents the same formula as Heron of Alexandria:

$$A=(c+h)\cdot\frac{1}{2}h+\frac{1}{14}\left(\frac{c}{2}\right)^2$$

in which c is the chord and h the height of the segment. This formula is not in al-Khwarizmi's chapter on mensuration, but for the rest there are so many similarities between this chapter and the Mishnat ha-Middot, that one is forced to assume either a direct dependence, as Gandz does, or at least a common source. It is also possible, as Gandz supposes, that al-Khwarizmi used a Persian or Syrian translation of the Mishnat ha-Middot.

4. On the Jewish Calendar

No matter whether one does or does not accept the conclusion of Gandz that al-Khwarizmi's geometry was "verbally taken from the Mishnat ha-Middot", in any case al-Khwarizmi was acquainted with Jewish traditions, for he has written a treatise on the Jewish calendar. This treatise describes the Jewish 19-year cycle and the rules for determining on what weekday the month Tishri begins. It also calculates the interval between the Jewish "era of the creation of Abraham" and the Seleucid era, and it gives rules for determining the mean longitudes of sun and moon. See E.S. Kennedy: Al-Khwārizmī on the Jewish Calendar, Scripta mathematica 27, p. 55–59 (1964).

5. On Legacies

The third and largest part of the Algebra of al-Khwarizmi (p. 86–174 in Rosen's translation) deals with legacies. It consists entirely of problems with solutions. The solutions involve only simple arithmetic or linear equations, but they require considerable understanding of the Islamic law of inheritance. See Solomon Gandz: The Algebra of Inheritance, Osiris 5, p. 319–391 (1938).

6. The Solution of Quadratic Equations

I shall now discuss in somewhat greater detail al-Khwarizmi's solution of the three types of mixed quadratic equations. In al-Khwarizmi's own terminology, the first type reads:

Roots and Squares equal to numbers.

For instance: one square and ten roots of the same amount to thirty-nine dirhems; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

The solution is: you halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four.

Now take the root of this, which is eight, and subtract from it half the number of the roots, which is four. The remainder is three. This is the root of the square you thought for; the square itself is nine.

In modern notation, the equation is

$$x^2+10x=39,$$

which can be transformed into

$$\begin{aligned}(x+5)^2&=39+25=64\\ x+5&=\sqrt{64}\quad=8\\ x&=8-5\quad=3.\end{aligned}$$

Next, al-Khwarizmi presents a demonstration. He draws a square AB, the side of which is the desired root x. On the four sides he constructs rectangles, each having 1/4 of 10, or 2 1/2, as their breadth (see Fig. 1). Now the square

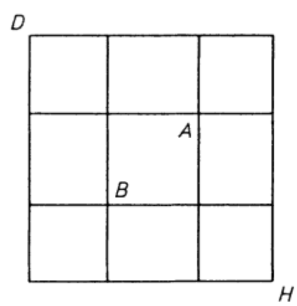

Fig. 1

together with the four rectangles is equal to 39. In order to complete the square DH, we must add four times the square of 2 1/2, that is, 25, says al-Khwarizmi. So the area of the large square is 64, and its side 8. Hence the side of the original square is

$$8-5=3.$$

Al-Khwarizmi next presents another, simpler proof, in which rectangles of breadth 5 are constructed only on two of the sides of the square AB (see Fig. 2). The result is, of course, the same.

Once more, we see that al-Khwarizmi's source is not Euclid, for his first proof is definitely more complicated than Euclid's proof of proposition II 4,

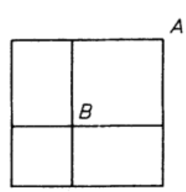

Fig. 2

which says that the square on a line segment $a+b$ is equal to the sum of the squares on a and b and twice the rectangle ab. The second proof of al-Khwarizmi is similar to that of Euclid.

I think this suffices to give the reader an idea of the style of al-Khwārizmī's treatise on *al-jabr* and *al-muqabala*. His treatment of the other types of mixed quadratic equations is quite similar to that of the first type. The other types are:

"Squares and numbers equal to roots",

"Roots and numbers equal to squares".

In each case, the solutions agree with those we learn at school, restricted to positive solutions.

7. The Geography

Besides the Algebra and the treatise on the Jewish calendar, one more treatise is extant in Arabic, namely the *Geography* ("Book of the Form of the Earth"). It consists almost entirely of lists of longitudes and latitudes of cities and localities. The work is a revision of Ptolemy's "Geography". Most probably it was based on a world map made by a commission of learned men (possibly including al-Khwarizmi himself) on the order of Caliph al-Mamun. For more details see Toomer's article al-Khwārizmī in the Dictionary of Scientific Biography VII, p. 361 and 365.

8. On Hindu Numerals

A treatise of al-Khwarizmi on Hindu numerals is extant only in a Latin translation, which was published first by B. Boncompagni under the title "Algoritmi de numero indorum" (Rome, 1857) and next by Kurt Vogel under the title "Mohammed ibn Musa Alchwarizmi's Algorithmus" (Aalen 1963), with a facsimile of the unique manuscript.

9. The Astronomical Tables

Al-Khwarizmi's set of astronomical tables is available only in a Latin translation of a revised version due to Maslama al-Majriti, who lived in Cordova about AD 1000. This version differed from the original text of Khwarizmi in several respects. First, the epoch of the original tables was the era Yazdigerd (16 June 632), whereas al-Majriti used the era Hijra (14 July 622). Also, al-Khwarizmi's table of Sines was based on the radius $R=150$, whereas the extant tables have $R=60$.

The tables have been published, with a German translation and commentary, by Heinrich Suter in Kongelige Dansk Vidensk. Selsk. Hist.-fil. Skrifter III, 1 (Copenhagen 1914). In the same Skrifter IV, 2 (Copenhagen 1962) Otto Neugebauer published an English translation of the introductory chapter and a new, valuable commentary. For additions and corrections to this commentary see C.G. Toomer's review in Centaurus 10, p. 203–212 (1964–65).

If one studies E.S. Kennedy's "Survey of Islamic Astronomical tables", Transactions American Philos. Soc. 46, p. 122–177 (1956), one sees that there are two types of *zijes*, i.e. astronomical table sets: Ptolemaic and Non-Ptolemaic. The Ptolemaic tables are based on Ptolemy's "Almagest" or on his "Handy Tables". The non-Ptolemaic *zijes*, of which al-Khwarizmi's table set is the only extant example, are based on Persian or on Hindu tables or on both. The non-Ptolemaic tables are less accurate, but more convenient than the Ptolemaic ones. This, I think, is the reason why Khwarizmi's tables remained popular long after the better (Ptolemaic) tables were available.

Ibn al-Qifti says in his biography of al-Fazari about al-Khwarizmi:

> He used in his tables the mean motions of the *Sindhind*, but he deviated from it in the equations (of the planets) and in the obliquity (of the ecliptic). He fixed the equations according to the method of the Persians, and the declination of the sun according to the method of Ptolemy.

What does this mean?

Let me begin with the last statement. In the *zij* of al-Khwarizmi there is a table for finding the declination of the sun (Suter's edition, p. 132–136, last column but one). This table is based on the value 23° 51′ of the obliquity of the ecliptic, and it agrees with a table in Ptolemy's "Handy Tables". So here al-Qifti is certainly right: the author of the tables determined the declination of the sun "according to the method of Ptolemy".

Concerning the "equations" of the planets, i.e. the corrections to be added to the mean longitudes, we may note that the maximum values of these corrections in the tables of al-Khwarizmi agree with those adopted in the Persian table set "*Zij-i Shah*". For this table set see E.S. Kennedy: The Sasanian Astronomical Handbook *Zīj-i Shāh*, Journal of the American Oriental Society 78, p. 246–262 (1958). Obviously, when al-Qifti speaks of "the Persians", he has the *Zij-i Shah* in mind, which was still extant in the time of al-Biruni and Ibn al-Qifti.

Thus we may conclude that one of the sources of al-Khwarizmi was the Persian table set "*Zij-i Shah*".

10. The "Sindhind"

I shall now discuss Ibn al-Qifti's first statement: "He used in his tables the mean motions of the Sindhind." The word Sindhind is a corruption of the Sanskrit *Siddhanta*, which is the usual designation of an astronomical textbook. In fact, the mean motions in the tables of al-Khwarizmi are derived from those in the "corrected Brahmasiddhanta" (Brahmasphutasiddhanta) of Brahmagupta. This was proved for the mean longitudes by J.J. Burckhardt, Vierteljahresschrift Naturf. Ges. Zürich 106, p. 213–231 (1961), and for the apogees and nodes by G.J. Toomer in his review of Neugebauer's commentary to al-Khwarizmi's tables (Centaurus 10, p. 207).

Soon after AD 770, a Sanskrit astronomical work called by the Arabs *Sindhind* was brought to the court of Caliph al-Mansur at Baghdad by a man called Kankah (or Mankah?), a member of a political mission from India. This work was translated into Arabic. Based on this translation, Yaqub ben Tariq, who is reported to have been at the court of al-Mansur together with Kankah, composed a table set, which was called *Zij al-Sindhind*. According to the Fihrist of el-Nadim (ed. Flügel, Vol. 1, p. 274) the table set of al-Khwarizmi was also called *Zij al-Sindhind*. It seems that al-Khwarizmi's *Zij* was a revision of an earlier table set based on the *Sindhind*, a revision into which some elements and methods from the *Zij-i Shah* and from Ptolemy's "Handy Tables" were incorporated.

11. The "Method of the Persians"

As we have seen, Ibn al-Qifti says that al-Khwarizmi "fixed the equations according to the Method of the Persians". What was this method?

I shall use the terminology and some notations of E.S. Kennedy's classical "Survey of Islamic Astronomical Tables" (Trans. Amer. Philos. Soc. 46). On pages 148–151 of this survey Kennedy presents an abstract of the tables of al-Khwarizmi, in which al-Khwarizmi's method of finding the true longitudes of the planets is explained.

Let $\bar{\lambda}$ be the mean longitude of any planet. Its true longitude is calculated by the formula

$$\lambda = \bar{\lambda} + e_1 + e_2,$$

where e_1 is the "equation of the centre" and e_2 the "equation of the anomaly". For the sun and the moon we have only one equation e_1 due to the eccentricity of the orbit. In al-Khwarizmi's tables for the sun and the moon, the function $e_1(x)$ is tabulated according to the formula

(12) $$e_1(x) = (\max e_1) \cdot \sin x$$

in which x is the distance of the mean sun or moon from the apogeum of the eccentric orbit:

(13) $$x = \bar{\lambda} - \lambda_{ap}.$$

For the other planets, the calculation is more complicated. One first calculates a preliminary value of the correction e_2, calculated by plane trigonometry from the triangle *EPM* in Fig. 3. In this drawing, the planet is supposed to be carried by an epicycle, which is in turn carried by a concentric circle. The angle e_2 can be tabulated as a function of the angle y (see H. Suter, Tafeln des Muhammed ibn Musa Al-Khwārizmī, pages 136–167, Column 3).

But, says Kennedy, "the inventor of the theory apparently realized that the two equations are not independent". We are required to halve the equation $e_2(y)$ and to add it to x, thus obtaining

(14) $$x' = x + 1/2\, e_2(y).$$

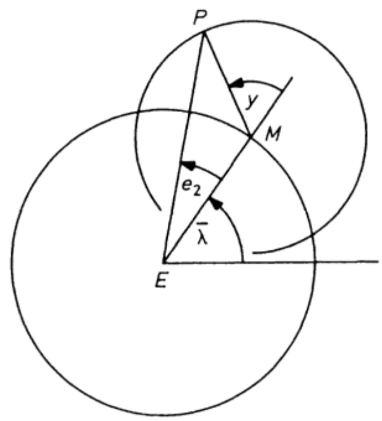

Fig. 3

This x' is used to calculate the correction e_1:

(15) $$e_1(x') = (\max e_1) \cdot \sin x'$$

which is subtracted from y, thus obtaining

(16) $$y' = y - e_1(x').$$

Now the longitude λ can be calculated as

(17) $$\lambda = \bar{\lambda} + e_1(x') + e_2(y').$$

So one has to use the table for e_1 twice, first to find $e_1(x)$ and next $e_1(x')$, and the table for e_2 once to find $e_2(y')$. For the rest, one has to perform only simple additions and subtractions. The procedure is simpler, but less accurate than Ptolemy's method.

As we have seen, al-Khwarizmi used in his tables the "Era Yazdigerd". So we may safely conclude that he learnt the "Method of the Persians" from the latest version of the *Zij-i Shah*, which was composed under the last Sasanid king Yazdigerd III (632–651). See for the history of this version pages 4–5 of a joint paper of J.J. Burckhardt and myself: Das astronomische System der persischen Tafeln, Centaurus 13, p. 1–28 (1968).

In earlier, predominantly Hindu texts we find a related, but slightly more complicated method, which we have called "Method of the Indians". It is based on the formulae

(14) $$x' = x + 1/2\, e_2(y)$$

(15) $$e_1(x') = (\max e_1) \cdot \sin x'$$

(18) $$x'' = x' + 1/2\, e_1(x')$$

(19) $$e_1(x'') = (\max e_1) \cdot \sin x''$$

(20) $$y' = y - e_1(x'')$$

(21) $$\lambda = \bar{\lambda} + e_1(x'') + e_2(y').$$

This method was used by Aryabhata (Aryabhatiya, verses 22–24), by Brahmagupta (Brahmasphutasiddhanta II, 34–38), and by other Hindu astronomers. The difference as compared with the Persian method is that the table for $e_1(x)$ is used twice: once with the argument x' and once with the argument x''. The difference is only small, for $1/2\, e_1(x')$ is in most cases small, so that x'' defined by (18) does not differ much from x'.

In my paper "Ausgleichspunkt, Methode der Perser und indische Planetenrechnung", Archive for History of Exact Sciences 1, p. 107–121 (1961), I have shown that the "Method of the Indians" can be explained as a reasonable approximation, if we suppose that a Greek author before Ptolemy, possibly Apollonios of Perge, started with the model of an epicycle carried by an eccentric circle. I suppose that this author assumed an "equant point" as in Ptolemy's Almagest, such that the motion on the eccenter appeared uniform as seen from the equant point. He invented an approximation which enabled the user of the tables to use only one-entry tables and additions and subtractions. Ptolemy adopted the equant model, but he did not use the approximation. On the other hand, the Hindu authors adopted the simple method of calculation, probably without knowing that it was based on the assumption of an eccenter with equant point.

This seems to be the only hypothesis which explains Ptolemy's equant model, for which Ptolemy himself gives no justification whatever, as well as the very sophisticated "Method of the Indians", for which the Hindu authors give no justification either.

12. Al-Khwarizmi's Sources

We are now in a position to discuss the sources of al-Khwarizmi's work, in particular of his Algebra. Three theories have been proposed. He may have used classical Greek sources, or Hindu sources, or popular mathematical writings belonging to the Hellenistic and post-Hellenistic tradition.

As Toomer notes in his article in the Dictionary of Scientific Biography, both Greek and Hindu algebra had advanced well beyond the elementary stage of al-Khwarizmi's work, and none of the known works in either culture shows much resemblance in presentation to al-Khwarizmi's work. As we have seen, his proofs of the methods of solution of quadratic equations are quite different from the proofs we find in Euclid's Elements. Also, as Toomer notes, al-Khwarizmi's exposition is completely rhetorical, like Sanskrit algebraic works, and unlike the one surviving Greek algebraic treatise, that of Diophantos, which has already developed quite far towards symbolic representation.

I feel that Toomer is right: we may exclude the possibility that al-Khwarizmi's work was much influenced by classical Greek mathematics.

In favour of the Hindu hypothesis it may be argued that al-Khwarizmi did write a treatise on Hindu numerals, that two of his estimates for π are also found in Hindu sources, and that in Chapter 18 of the *Brahmasphutasiddhanta* of Brahmagupta, verse 18, a general rule for the solution of a quadratic equation of type (4) is given. See for this rule H.T. Colebrooke: Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhascara, page 346.

In one case, in the section on Mensuration, al-Khwarizmi gives us a hint concerning his sources. After having mentioned the estimate $3+1/7$ for π, which is "generally followed in practical life, though it is not quite exact", he says:

The mathematicians, however, have two other rules for that. The one of them is: multiply the diameter with itself, then with ten, and then take the root of the product. The root gives the circumference.

The other rule is used by *the astronomers among them* (my italics), and reads: multiply the diameter with sixty-two thousand eight hundred and thirty-two and then divide it by twenty thousand. The quotient gives the circumference.

Note that Aryabhata writes his estimate of π in just the same form as

$$62\,832/20\,000.$$

We know already that al-Khwarizmi used Persian and Hindu sources in composing his astronomical tables. We may suppose that he derived his estimate of π from one of these sources.

After the Greek and the Hindu hypotheses, we may discuss a third hypothesis proposed by Hermann Hankel in his "Geschichte der Mathematik" (Leipzig 1874), p. 259–264, and supported by H. Wiedemann in his article "al-Khwārizmī" in the Encyclopaedia of Islam II, p. 912–913. These authors deny all Greek influence on al-Khwarizmi and assert the prevalence of a native, Syriac-Persian tradition.

In view of the close connection between the Hebrew treatise Mishnat ha-Middot and the geometry of al-Khwarizmi, I feel we should extend the notion "Syriac-Persian" to include Hebrew and other popular traditions as well. We have to admit the existence of a tradition of popular mathematics in Egypt and in the Near East in Hellenistic and post-Hellenistic times. Examples are the mathematical papyri from Egypt discussed on pages 164–170 and 173–177 of my "Geometry and Algebra in Ancient Civilizations", and the "Metrica" of Heron of Alexandria discussed on pages 181–188 of the same book.

The hypothesis of Hankel and Wiedemann was strongly supported by Solomon Gandz, the editor of the "Mishnat ha-Middot". I think I can do no better than quote the final section of his introduction to the Mensuration of al-Khwarizmi:

Al-Khowārizmī, the antagonist of Greek influence

At the university of Baghdad founded by al-Ma'mūn (813–33), the so-called Bayt al-Ḥikma, "the House of Wisdom", where al-Khowārizmī worked under the patronage of the Caliph, there and then flourished also an older contempora of al-Khowārizmī by the name of al-Ḥajjāj ibn Yūsuf ibn Maṭar. This man was the foremost protagonist of the Greek school working for the reception of Greek science by the Arabs. All his life was devoted to the work on Arabic happened before AD 873, for in January 873 Mohammed ben Musa died (see Suter: Die Mathematiker and Astronomen, p. 20).

According to al-Nadim and el-Qifti (see Chwolson I, p. 483 and II, p. 532) Tabit succeeded in establishing at Baghdad a Sabian primate for the whole of Iraq. By this move, the situation of the Sabians was stabilized, and they were respected in the whole country.

Tabit was highly esteemed for his writings in medicine, philosophy, mathematics, astronomy, and astrology. He was also a most competent translator from Greek and Syriac into Arabic. He translated works of Euclid, Archimedes, Apollonios, Autolykos, Ptolemaios, Nikomachos, Proklos, and others (see Chwolson I, p. 553–560).

Barhebraeus reports in his Syrian chronicle that Tabit ben Qurra composed circa 150 works in Syriac. For his works on astronomy and mathematics see H. Suter: Die Mathematiker und Astronomen der Araber (1900), p. 34–38, and Nachtrag, p. 162–163. Here I shall restrict myself to three treatises: one on astronomy, one on algebra, and one on arithmetic.

On the Motion of the Eighth Sphere

Tabit has written a very interesting treatise, which is available only in a Latin translation, entitled "De motu octave spere". The Latin text was published by C.F. Carmody: "The Astronomical Works of Tabit b. Qurra" (Berkeley 1960), p. 84–113. An English translation with commentary was presented by O. Neugebauer in Proceedings of the Amer. Philos. Soc. 106, p. 291–299.

The "eighth sphere" of Tabit is the sphere of the fixed stars. Inside this sphere one has to imagine the seven spheres of the moon, the sun, and the five "star-planets".

In modern astronomy the fixed stars are assumed to be nearly at rest and the equinoxes to have a small *retrograde* motion with respect to the fixed stars: the "precession of the equinoxes". In Ptolemy's theory the equinoxes are fixed, and the stars are supposed to have a slow *forward* motion of 1 degree in 100 years.

Tabit noticed that this small amount is not confirmed by the observations. The motion of the stars with respect to the equinoxes has to be much larger, at least in the time after Ptolemy, if one accepts the very accurate observations made under the reign of al-Mamun. To explain this, Tabit assumed an oscillatory (periodic) motion of the sphere of the fixed stars, the so-called "trepidation".

Another phenomenon which Tabit wanted to explain is an alleged decrease of the obliquity of the ecliptic. The ancient Greeks had used a rough estimate of 24°, Ptolemy had used a slightly smaller estimate due to Eratosthenes, and the observers at Baghdad had found a still smaller obliquity, namely 23° 33′.

Tabit now constructed a model which would explain both phenomena: the alleged trepidation of the fixed stars with respect to the equinoxes, and the alleged decrease of the obliquity. He made the two opposite points "Beginning of Aries" and "Beginning of Libra" on the sphere of the fixed stars move slowly on small circles, whose centres are opposite points of a fixed sphere. For a detailed description of this model I may refer to the paper of Neugebauer just mentioned.

Tabit's treatise ends up with two small tables, from which the motion of the two variable points "Beginning of Aries" and "Beginning of Libra" can be computed.

Geometrical Verification of the Solution of Quadratic Equations

Tabit's short treatise on this subject, entitled "On the Verification of Problems of Algebra by Geometrical Proofs", is preserved in a single manuscript Aya Sophia 2457,3. It was published with a German translation and commentary by P. Luckey in 1941: Berichte über die Verhandlungen der sächs. Akad. Leipzig 93, p. 93–112. I shall now translate parts of Luckey's translation into English. Since the logic of the treatise is perfect, I see no danger in this procedure. The diagrams are not taken from the manuscript, but from Luckey's publication.

There are three fundamental forms (*uṣūl*, roots or elements), to which most problems of algebra can be reduced:

The first basic form is: Wealth (*māl*) and roots are equal to numbers. The way and method of solution by the sixth proposition of Euclid's second book is as I shall describe: We make (Fig. 4) the wealth equal to the square $abgd$, we make bh equal to the same multiple of the unit in which lines are measured as is in the given number of roots, and we complete the area dh. Since the wealth is $abgd$, the root is clearly ab, and in the domain of calculation and number it is equal to the product of ab and the unit, in which the lines are measured.... Now a number of these units equal to the given number of roots is in bh, hence the product of ab and bh is equal to the roots in the domain of calculation and number. But the product of ab and bh is the area dh, because ab is equal to bd. Hence the area dh is in this way equal to the roots of the problem. Hence the whole are gh is equal to the wealth together with the roots.

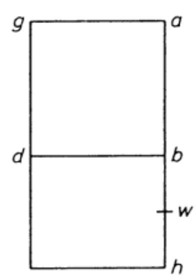

Fig. 4

Tabit's explanation is cumbersome, because he cannot equate an area or line segment with a number. He therefore introduces a unit of length, which I shall denote by e. If the given equation is

$$x^2 + mx = n,$$

in which x is an unknown number, while m and n are given numbers, he translates it into a geometrical equation

$$x^2 + mex = ne^2$$

in which x and e are line segments. He continues:

Now the wealth and the roots together are equal to a known number. So the area gh is known, and it is equal to the product of ah and ab, because ab is equal to ag. So the product of ha and ab is known, and the line bh is known, because its number of units is known. Thus everything is reduced to a well-known geometrical problem, namely: The line bh is known. To it a line ab is added, and the product of ha and ab is known.

Now in proposition 6 of book 2 of the Elements it is proved that, if the line bh is halved at the point w, the product of ha and ab together with the square of bw is equal to the square of aw. But the product of ha and ab is known, and the square of bw is known. So the square of aw is known, hence aw is known, and if the known bw is subtracted, ab results as known, and this is the root. And if we multiply it by itself, the square $abgh$, that is, the wealth, is known, which is what we wanted to prove.

Now comes the most interesting passage in the treatise:

This procedure agrees with the procedure of the people concerned with algebra in their solution of the problem. When they halve the number of roots, this is just so as when we take half of the line bh, and when they multiply it by itself, this is the same as when we take the square of the halved line bh. When they add to the result the (given) number, this is just as when we add the product of ha and ab, in order to obtain the square of the sum of ab and the halved line. Their taking the root of the result is like our saying: The sum of ab and the halved line is known as soon as its square is known.

The next sentence in the text is corrupt. The end of the sentence reads:

... to obtain the residue, just as we obtained ab. They multiplied (the residue) by itself, just as we determined the square of ab, that is, the wealth.

In the same way Tabit treats the second type of equation

$$x^2+b=ax$$

or "wealth and number is equal to roots". He says:

The way and method of solution according to the second book of Euclid by means of proposition 5 is, as I describe it: We make (Fig. 5) the wealth into a square $abgd$ and we make ah equal to such a multiple of the unit in which lines are measured as is in the given number of roots. Obviously, ah is longer than ab, because the roots, which are in the domain of calculation the product of ga and ah, are larger than the wealth. We complete the area gh, and we prove, as before, that it is equal to the roots (that is, equal to the term ax) in the domain of calculation. And if bg, which is the wealth (that is, the term x^2) is subtracted from it, there remains dh equal to the (given) number. So dh is known, and it is equal to the product of ab and bh, and the line ah is known. So now the problem amounts to dividing a given line ah in b in such a way that the product of ab and bh is known.

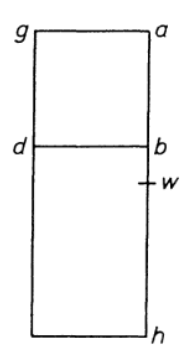

Fig. 5

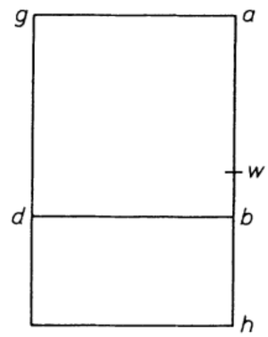

Fig. 6

Now in proposition 5 of the second book of Euclid it is proved that, if ah is halved at w, the product of ab and bh together with the square of bw is equal to the square of aw. But aw is known, and its square is known, and the product of ab and bh is known. So the square of bw is known as a remainder, hence bw is known, and if it is subtracted from aw (Fig. 5) or added to it (Fig. 6), ab results as known, and it is the root. And if we multiply it by itself, $abgd$ is known, and it is the wealth, and this is what we wanted to prove.

This procedure too agrees with the procedure of the algebra people *(ahl al-jabr)* in calculating the problem. For it allows in both ways the application of addition and of subtraction of the line wb.

I think it is not necessary to translate the third part of the text, in which the equation

"Number and Roots are equal to Wealth"

is solved by means of Euclid's proposition II 6, and the agreement with the algebraic solution is proved in the same way as in the other two cases.

In al-Khwarizmi's treatise, the science of algebra is denoted by the double expression "*al-jabr wal muqabala*". Tabit ben Qurra leaves out the second part and refers just to the "solution by al-jabr" as opposed to his own solution by geometry. The algebrists, to which al-Khwarizmi belongs, are called by Tabit "those concerned with algebra" *(aṣḥāb aljabr)* or "the algebra people" *(ahl al-jabr)*. In the text, they are opposed to the geometers, to which Tabit himself belongs.

Tabit judges it necessary to explain in great detail that the algebraic solutions are in full accordance with Euclid's geometrical solution. From this, Luckey concludes that at least for some of his readers this connection between geometry and algebra was new, and he raises the question: Was it new for the "algebra people"? It seems to me that the answer must be "yes", for otherwise the whole treatise of Tabit would be superfluous.

As we have seen in the section on al-Khwarizmi, there were two opposite trends or parties among the mathematicians and astronomers at Baghdad. One of these trends was represented by al-Khwarizmi, who used Indian and Persian sources for his astronomical tables, and who wrote his Algebra, "confining it to what is easiest and most useful in arithmetics, such as men constantly require in cases of inheritance", and so on. On the other hand, we have "the Greek school working for the reception of Greek science by the Arabs", as Gandz puts it. To this Greek school belonged al-Hajjaj, who translated Euclid and Ptolemy, and Tabit ben Qurra.

On Amicable Numbers

Two natural numbers m and n are called *amicable*, if each is equal to the sum of the proper divisors of the other. For instance, the sum of the proper divisors of 284 is 220, and the sum of the proper divisors of 220 is 284. This pair of amicable numbers was known already to the ancient Pythagoreans (see e.g. my "Science Awakening" I, p. 98).

Tabit ben Qurra has written a "Book on the Determination of Amicable Numbers". He proved: If $p=3\cdot 2^{n-1}-1$ and $q=3\cdot 2^n-1$ and $r=9\cdot 2^{2n-1}-1$ are prime, then

$$M=2^n p q \quad \text{and} \quad N=2^n r$$

are amicable numbers.

Tabit's book has been commented upon and partly translated by F. Woepcke: Notice sur une théorie ajoutée par Thābit ben Korra à l'arithmétique spéculative des Grecs, Journal asiatique (4) 20, p. 420–429 (1852).

Tabit's rule for obtaining amicable pairs was rediscovered by *Pierre de Fermat* and *René Descartes.* Besides the well known pair 220 and 284, Fermat found one more pair, namely

$$17296=2^4\times 23\times 47$$
$$18416=2^4\times 1151$$

(Oeuvres II, p. 20–21). No doubt, he derived it by Tabit's rule for $n=4$.

Descartes formulated Tabit's rule explicitely and presented a third example:

$$9363584=2^7\times 191\times 383$$
$$9437056=2^7\times 73727$$

(René Descartes, Oeuvres II, p. 93–94 and p. 148).

Now the question arises: How did Tabit find his rule?

The well known pair 220 and 284 has a factorization of the form

$$2^2 p q \quad \text{and} \quad 2^2 r$$

in which p, q, and r are primes. So let us see whether we can find a pair

$$M=2^n p q \quad \text{and} \quad N=2^n r$$

such that M is the sum of the proper divisors of N and conversely.

I suppose that Tabit knew that the sum of all divisors of N (including N itself) is

$$(1+2+\ldots+2^n)(r+1)$$

and that the sum of all divisors of M is

$$(1+2+\ldots+2^n)(p q+p+q+1).$$

Part C
Omar Khayyam

The Persian poet, philosopher, mathematician, and astronomer Omar ben Ibrahim al-Hayyam, usually called Omar Khayyam, lived in the second half of the eleventh century. His fame in the western world is mainly based on the very free translation of his nearly 600 short poems of four lines each *(Ruba'iyat)* by E. Fitzgerald (1859),

In 1074 Omar Khayyam was called to Isfahan, where a group of outstanding astronomers came together for the foundation of an observatory. "An enormous amount of money was spent for this purpose", says Ibn al Athir. See Aydin Sahili: The Observatory in Islam (Türk Kurumu Basimevi, Ankara 1960).

Here we shall mainly be concerned with Omar Khayyam's treatise "On the Proofs of the Problems of Algebra and Muqabala". My account will be based on the French translation of Franz Woepcke: L'algèbre d'Omar Alkhayyami (Paris 1851). An English translation was published in 1950 by H.J.J. Winter and W. Arafat in Journal R. Asiatic Soc. Bengal. 16, p. 27–77. For an edition of the text with a new French translation and commentary see Roshdi Rashed and Ahmed Djebbar: L'oeuvre algébrique d'Al-Khayyam, University of Aleppo 1981.

In the introduction to his "Algebra" Omar Khayyam explains that "The art of algebra" aims at the determination of *numerical* or *geometrical* unknown quantities. This distinction between *numbers* and *measurable magnitudes* is maintained throughout the treatise. The author mentions four kinds of measurable magnitudes: the *line*, the *surface*, the *solid*, and the *time*. He excludes magnitudes of more than three dimensions such as the "square-square" and the "quadrato-cube", which are used by some algebrists.

The Algebra of Omar Khayyam

The algebra of Omar Khayyam is mainly geometric. He first solves linear and quadratic equations by the geometrical methods explained in Euclid's Elements, and next he shows that cubic equations can be solved by means of intersections of conics.

Omar knows very well that earlier authors sometimes equated geometrical magnitudes with numbers. He avoids this logical inconsistency by a trick, introducing a unit of length. He writes:

> Every time we shall say in this book "a number is equal to a rectangle", we shall understand by the "number" a rectangle of which one side is unity, and the other a line equal in measure to the given number, in such a way that each of the parts by which it is measured is equal to the side we have taken as unity.

In Fig. 7 I have denoted the unity of length by e, and the sides of the rectangle by x and y. The figure illustrates the equation $3=xy$.

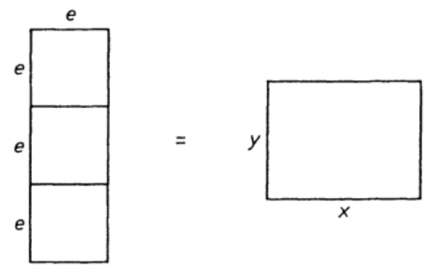

Fig. 7

Omar Khayyam first solves quadratic equations by the usual methods. Next he passes to cubic equations. Some of these, for instance,

(1) $$x^3 + ax^2 = bx$$

can be reduced to quadratic equations. The first type requiring conic sections is

"A number is equal to a cube"

or, in modern notation

(2) $$x^3 = N.$$

Omar first solves an auxiliary problem, namely

"To find two lines between two given lines such that the four lines form a continued proportion".

If the two given lines are called $AB = a$ and $BC = b$, the problem is, to find x and y such that

(3) $$a : x = x : y = y : b.$$

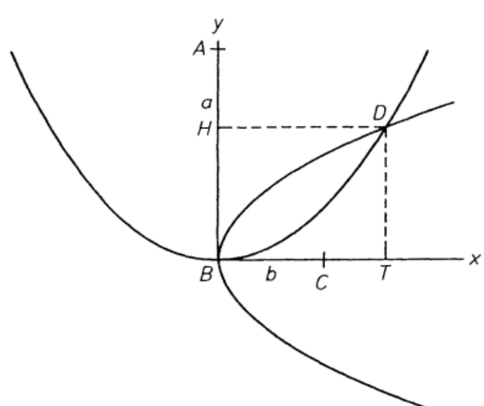

Fig. 8

Omar draws two perpendicular line segments BA and BC, and he constructs two parabolas, both having their summit at B. The first parabola has axis BC and "parameter" BC, the other has axis BA and "parameter" BA. In modern notation, the equations of the two conics are

$$y^2 = bx \quad \text{and} \quad x^2 = ay. \tag{4}$$

Let D be their point of intersection. Then the perpendiculars $x = DH$ and $y = DT$ satisfy (4) and hence (3).

Next, Omar considers the equation (2), in which N is a given number. He constructs a rectangular block with base e^2 and height Ne. Now he has to construct a cube equal to this block. In the case $N = 2$ this is just the well-known Greek problem of "doubling the cube". Hippokrates of Chios had proved that this problem can be reduced to the problem of finding two mean proportionals x and y between two given line segments a and b. Omar Khayyam proceeds just so. He solves the auxiliary problem (3) with $a = e$ and $b = Ne$, and he proves that the first intermediate x is the side of the required cube.

All this is well-known from Greek texts. According to Eutokios, the solution of (3) by means of the intersection of two parabolae is due to Menaichmos.

Next, Omar considers six types of cubic equations in which a binomial is equated to a monomial, namely

$$x^3 + ax = b \tag{5}$$

$$x^3 + b = ax \tag{6}$$

$$x^3 = ax + b \tag{7}$$

$$x^3 + ax^2 = b \tag{8}$$

$$x^3 + b = ax^2 \tag{9}$$

$$x^3 = ax^2 + bx. \tag{10}$$

In Omar's terminology, the equation (5) is written as

"A cube and (a number of) sides are equal to a number".

Omar first constructs a square c^2 equal to the given number b, and next a block with base c^2 and height h equal to the given number b. This means, as he has explained earlier, that the block with sides c, c, and h is made equal to a block with sides e, e, and be, where e is the unity of length and b be the given number on the right hand side of equation (5). Thus, the equation (5) can be written in the homogeneous form

$$x^3 + c^2 x = c^2 h \tag{11}$$

in which $c = AB$ and $h = BC$ are given line segments.

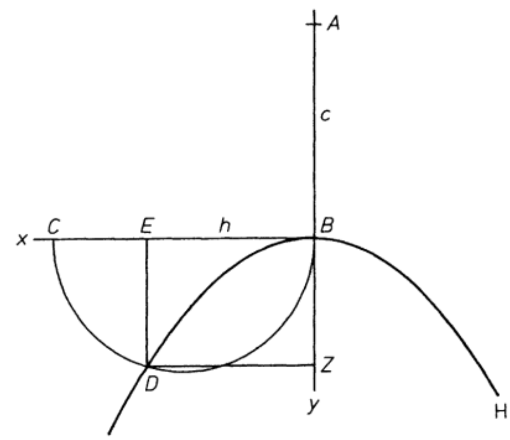

Fig. 9

To solve this equation geometrically, Omar constructs a parabola (see Fig. 9) having its summit at B, its axis being BZ and its "parameter" $AB=c$. Next he describes a semi-circle on the diameter $BC=h$. The semi-circle necessarily has a point of intersection D with the parabola. From D one draws perpendiculars DZ and DE to BZ and BC. Omar now proves that $DZ=x$ solves the equation (11).

In modern terminology, let $x=DZ$ and $y=DE$ be the coordinates of D. The equation of the parabola is

$$x^2=yc, \tag{12}$$

or, in Omar's own words: "The square of DZ will be equal to the product of BZ and AB". The equation of the circle is

$$y^2=x(h-x) \tag{13}$$

which Omar writes as a proportion

"BE is to ED as ED is to EC".

Just so, (12) is written as a proportion:

"AB is to BE as BE is to ED".

From these two proportions Omar concludes that $EB=x$ is a solution, and the only solution of his problem.

Just so, Omar writes the equation (6) in the homogeneous form

$$x^3+c^2h=c^2x \tag{14}$$

and he solves it by intersecting the parabola

$$yc=x^2 \tag{15}$$