A History of Mathematics

THIRD EDITION

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WILEY
John Wiley & Sons, Inc.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

Boyer, Carl B. (Carl Benjamin), 1906–1976.
A history of mathematics / Carl B. Boyer and Uta Merzbach. — 3rd ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-470-52548-7 (pbk.); ISBN 978-0-470-63039-6 (ebk.);
ISBN 978-0-470-63054-9 (ebk.); ISBN 978-0-470-630563 (ebk.)
1. Mathematics—History. I. Merzbach, Uta C., 1933– II. Title.
QA21.B767 2010
510.9—dc22 2010003424

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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Preface to the Third Edition

During the two decades since the appearance of the second edition of this work, there have been substantial changes in the course of mathematics and the treatment of its history. Within mathematics, outstanding results were achieved by a merging of techniques and concepts from previously distinct areas of specialization. The history of mathematics continued to grow quantitatively, as noted in the preface to the second edition; but here, too, there were substantial studies that overcame the polemics of "internal" versus "external" history and combined a fresh approach to the mathematics of the original texts with the appropriate linguistic, sociological, and economic tools of the historian.

In this third edition I have striven again to adhere to Boyer's approach to the history of mathematics. Although the revision this time includes the entire work, changes have more to do with emphasis than original content, the obvious exception being the inclusion of new findings since the appearance of the first edition. For example, the reader will find greater stress placed on the fact that we deal with such a small number of sources from antiquity; this is one of the reasons for condensing three previous chapters dealing with the Hellenic period into one. On the other hand, the chapter dealing with China and India has been split, as content demands. There is greater emphasis on the recurring interplay between pure and applied mathematics as exemplified in chapter 14. Some reorganization is due to an attempt to underline the impact of institutional and personal transmission of ideas; this has affected most of the pre-nineteenth-century chapters. The chapters dealing with the nineteenth century have been altered the least, as I had made substantial changes for some of this material in the second edition. The twentieth-century material has been doubled, and a new final chapter deals with recent trends, including solutions of some longstanding problems and the effect of computers on the nature of proofs.

It is always pleasant to acknowledge those known to us for having had an impact on our work. I am most grateful to Shirley Surrette Duffy for responding judiciously to numerous requests for stylistic advice, even at times when there were more immediate priorities. Peggy Aldrich Kidwell replied with unfailing precision to my inquiry concerning certain photographs in the National Museum of American History. Jeanne LaDuke cheerfully and promptly answered my appeals for help, especially in confirming sources. Judy and Paul Green may not realize that a casual conversation last year led me to rethink some recent material. I have derived special pleasure and knowledge from several recent publications, among them *Klopfer* 2009 and, in a more leisurely fashion, *Szpiro* 2007. Great thanks are due to the editors and production team of John Wiley & Sons who worked with me to make this edition possible: Stephen Power, the senior editor, was unfailingly generous and diplomatic in his counsel; the editorial assistant, Ellen Wright, facilitated my progress through the major steps of manuscript creation; the senior production manager, Marcia Samuels, provided me with clear and concise instructions, warnings, and examples; senior production editors Kimberly Monroe-Hill and John Simko and the copyeditor, Patricia Waldygo, subjected the manuscript to painstakingly meticulous scrutiny. The professionalism of all concerned provides a special kind of encouragement in troubled times.

I should like to pay tribute to two scholars whose influence on others should not be forgotten. The Renaissance historian Marjorie N. Boyer (Mrs. Carl B. Boyer) graciously and knowledgeably complimented a young researcher at the beginning of her career on a talk presented at a Leibniz conference in 1966. The brief conversation with a total stranger did much to influence me in pondering the choice between mathematics and its history.

More recently, the late historian of mathematics Wilbur Knorr set a significant example to a generation of young scholars by refusing to accept the notion that ancient authors had been studied definitively by others. Setting aside the "*magister dixit*," he showed us the wealth of knowledge that emerges from seeking out the texts.

—Uta C. Merzbach
March 2010

Preface to the Second Edition

This edition brings to a new generation and a broader spectrum of readers a book that became a standard for its subject after its initial appearance in 1968. The years since then have been years of renewed interest and vigorous activity in the history of mathematics. This has been demonstrated by the appearance of numerous new publications dealing with topics in the field, by an increase in the number of courses on the history of mathematics, and by a steady growth over the years in the number of popular books devoted to the subject. Lately, growing interest in the history of mathematics has been reflected in other branches of the popular press and in the electronic media. Boyer's contribution to the history of mathematics has left its mark on all of these endeavors.

When one of the editors of John Wiley & Sons first approached me concerning a revision of Boyer's standard work, we quickly agreed that textual modifications should be kept to a minimum and that the changes and additions should be made to conform as much as possible to Boyer's original approach. Accordingly, the first twenty-two chapters have been left virtually unchanged. The chapters dealing with the nineteenth century have been revised; the last chapter has been expanded and split into two. Throughout, an attempt has been made to retain a consistent approach within the volume and to adhere to Boyer's stated aim of giving stronger emphasis on historical elements than is customary in similar works.

The references and general bibliography have been substantially revised. Since this work is aimed at English-speaking readers, many of whom are unable to utilize Boyer's foreign-language chapter references, these have been replaced by recent works in English. Readers are urged to consult the General Bibliography as well, however. Immediately following the chapter references at the end of the book, it contains additional works and further bibliographic references, with less regard to language. The introduction to that bibliography provides some overall guidance for further pleasurable reading and for solving problems.

The initial revision, which appeared two years ago, was designed for classroom use. The exercises found there, and in the original edition, have been dropped in this edition, which is aimed at readers outside the lecture room. Users of this book interested in supplementary exercises are referred to the suggestions in the General Bibliography.

I express my gratitude to Judith V. Grabiner and Albert Lewis for numerous helpful criticisms and suggestions. I am pleased to acknowledge the fine cooperation and assistance of several members of the Wiley editorial staff. I owe immeasurable thanks to Virginia Beets for lending her vision at a critical stage in the preparation of this manuscript. Finally, thanks are due to numerous colleagues and students who have shared their thoughts about the first edition with me. I hope they will find beneficial results in this revision.

—Uta C. Merzbach
Georgetown, Texas
March 1991

Preface to the First Edition

Numerous histories of mathematics have appeared during this century, many of them in the English language. Some are very recent, such as J. F. Scott's *A History of Mathematics*[1]; a new entry in the field, therefore, should have characteristics not already present in the available books. Actually, few of the histories at hand are textbooks, at least not in the American sense of the word, and Scott's *History* is not one of them. It appeared, therefore, that there was room for a new book—one that would meet more satisfactorily my own preferences and possibly those of others.

The two-volume *History of Mathematics* by David Eugene Smith[2] was indeed written "for the purpose of supplying teachers and students with a usable textbook on the history of elementary mathematics," but it covers too wide an area on too low a mathematical level for most modern college courses, and it is lacking in problems of varied types. Florian Cajori's *History of Mathematics*[3] still is a very helpful reference work; but it is not adapted to classroom use, nor is E. T. Bell's admirable *The Development of Mathematics.*[4] The most successful and appropriate textbook today appears to be Howard Eves, *An Introduction to the History of Mathematics*,[5] which I have used with considerable satisfaction in at least a dozen classes since it first appeared in 1953.

[1]London: Taylor and Francis, 1958.
[2]Boston: Ginn and Company, 1923–1925.
[3]New York: Macmillan, 1931, 2nd edition.
[4]New York: McGraw-Hill, 1945, 2nd edition.
[5]New York: Holt, Rinehart and Winston, 1964, revised edition.

illustrations needed in the text; in particular it has been a pleasure to have worked with the staff of John Wiley & Sons. The typing of the final copy, as well as of much of the difficult preliminary manuscript, was done cheerfully and with painstaking care by Mrs. Hazel Stanley of Lawrence, Kansas. Finally, I must express deep gratitude to a very understanding wife. Dr. Marjorie N. Boyer, for her patience in tolerating disruptions occasioned by the development of yet another book within the family.

—Carl B. Boyer
Brooklyn, New York
January 1968

1

Traces

Did you bring me a man who cannot number his fingers?
—From the *Egyptian Book of the Dead*

Concepts and Relationships

Contemporary mathematicians formulate statements about abstract concepts that are subject to verification by proof. For centuries, mathematics was considered to be the science of numbers, magnitudes, and forms. For that reason, those who seek early examples of mathematical activity will point to archaeological remnants that reflect human awareness of operations on numbers, counting, or “geometric” patterns and shapes. Even when these vestiges reflect mathematical activity, they rarely evidence much historical significance. They may be interesting when they show that peoples in different parts of the world conducted certain actions dealing with concepts that have been considered mathematical. For such an action to assume historical significance, however, we look for relationships that indicate this action was known to another individual or group that engaged in a related action. Once such a connection has been established, the door is open to more specifically historical studies, such as those dealing with transmission, tradition, and conceptual change.

Mathematical vestiges are often found in the domain of nonliterate cultures, making the evaluation of their significance even more complex. Rules of operation may exist as part of an oral tradition, often in musical or verse form, or they may be clad in the language of magic or ritual. Sometimes they are found in observations of animal behavior, removing them even further from the realm of the historian. While studies of canine arithmetic or avian geometry belong to the zoologist, of the impact of brain lesions on number sense to the neurologist, and of numerical healing incantations to the anthropologist, all of these studies may prove to be useful to the historian of mathematics without being an overt part of that history.

At first, the notions of number, magnitude, and form may have been related to contrasts rather than likenesses—the difference between one wolf and many, the inequality in size of a minnow and a whale, the unlikeness of the roundness of the moon and the straightness of a pine tree. Gradually, there may have arisen, out of the welter of chaotic experiences, the realization that there are samenesses, and from this awareness of similarities in number and form both science and mathematics were born. The differences themselves seem to point to likenesses, for the contrast between one wolf and many, between one sheep and a herd, between one tree and a forest suggests that one wolf, one sheep, and one tree have something in common—their uniqueness. In the same way it would be noticed that certain other groups, such as pairs, can be put into one-to-one correspondence. The hands can be matched against the feet, the eyes, the ears, or the nostrils. This recognition of an abstract property that certain groups hold in common, and that we call "number," represents a long step toward modern mathematics. It is unlikely to have been the discovery of any one individual or any single tribe; it was more probably a gradual awareness that may have developed as early in man's cultural development as the use of fire, possibly some 300,000 years ago.

That the development of the number concept was a long and gradual process is suggested by the fact that some languages, including Greek, have preserved in their grammar a tripartite distinction between 1 and 2 and more than 2, whereas most languages today make only the dual distinction in "number" between singular and plural. Evidently, our very early ancestors at first counted only to 2, and any set beyond this level was designated as "many." Even today, many people still count objects by arranging them into sets of two each.

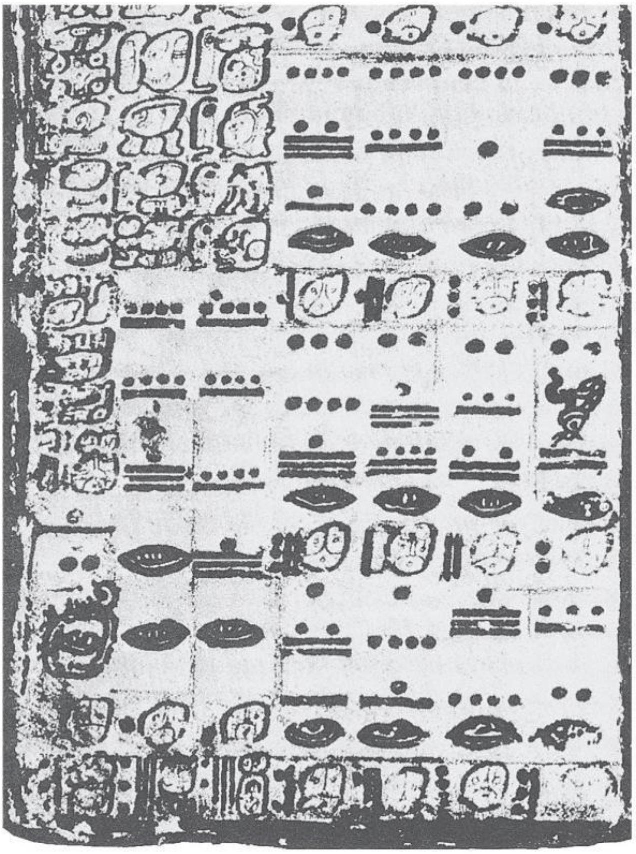
The awareness of number ultimately became sufficiently extended and vivid so that a need was felt to express the property in some way, presumably at first in sign language only. The fingers on a hand can be readily used to indicate a set of two or three or four or five objects, the number 1 generally not being recognized at first as a true "number." By the use of the fingers on both hands, collections containing up to ten elements could be represented; by combining fingers and toes, one could count as high as 20. When the human digits were inadequate, heaps of stones or knotted strings could be used to represent a correspondence with the elements of another set. Where nonliterate peoples used such a scheme of representation, they often piled the stones in groups of five, for they had become familiar with quintuples through observation of the human hand and foot. As Aristotle noted long ago, the widespread use today of the decimal system is but the result of the anatomical accident that most of us are born with ten fingers and ten toes.

Groups of stones are too ephemeral for the preservation of information; hence, prehistoric man sometimes made a number record by cutting notches in a stick or a piece of bone. Few of these records remain today, but in Moravia a bone from a young wolf was found that is deeply incised with fifty-five notches. These are arranged in two series, with twenty-five in the first and thirty in the second: within each series, the notches are arranged in groups of five. It has been dated as being approximately 30,000 years old. Two other prehistoric numerical artifacts were found in Africa: a baboon fibula having twenty-nine notches, dated as being circa 35,000 years old, and the Ishango bone, with its apparent examples of multiplicative entries, initially dated as approximately 8,000 years old but now estimated to be as much as 30,000 years old as well. Such archaeological discoveries provide evidence that the idea of number is far older than previously acknowledged.

Early Number Bases

Historically, finger counting, or the practice of counting by fives and tens, seems to have come later than counter-casting by twos and threes, yet the quinary and decimal systems almost invariably displaced the binary and ternary schemes. A study of several hundred tribes among the American Indians, for example, showed that almost one-third used a decimal base, and about another third had adopted a quinary or a quinary-decimal system; fewer than a third had a binary scheme, and those using a ternary system constituted less than 1 percent of the group. The vigesimal system, with the number 20 as a base, occurred in about 10 percent of the tribes.

An interesting example of a vigesimal system is that used by the Maya of Yucatan and Central America. This was deciphered some time before the rest of the Maya languages could be translated. In their representation of time intervals between dates in their calendar, the Maya used a place value numeration, generally with 20 as the primary base and with 5 as an auxiliary. (See the following illustration.) Units were represented by dots and fives by horizontal bars, so that the number



From the Dresden Codex of the Maya, displaying numbers. The second column on the left, reading down from above, displays the numbers 9, 9, 16, 0, 0, which stand for $9 \times 144{,}000 + 9 \times 7{,}200 + 16 \times 360 + 0 + 0 = 1{,}366{,}560$. In the third column are the numerals 9, 9, 9, 16, 0, representing 1,364,360. The original appears in black and red. (Taken from Morley 1915, p. 266.)

17, for example, would appear as  (that is, as $3(5) + 2$). A vertical positional arrangement was used, with the larger units of time above; hence, the notation  denoted 352 (that is, $17(20) + 12$). Because the system was primarily for counting days within a calendar that had 360 days in a year, the third position usually did not represent multiples of (20)(20), as in a pure vigesimal system, but (18)(20). Beyond this point, however, the base 20 again prevailed. Within this positional notation, the Maya indicated missing positions through the use of a symbol, which appeared in variant forms, somewhat resembling a half-open eye.

propositions in geometry and arithmetic. The design makes it immediately obvious that the areas of triangles are to one another as squares on a side, or, through counting, that the sums of consecutive odd numbers, beginning from unity, are perfect squares. For the prehistoric period there are no documents; hence, it is impossible to trace the evolution of mathematics from a specific design to a familiar theorem. But ideas are like hardy spores, and sometimes the presumed origin of a concept may be only the reappearance of a much more ancient idea that had lain dormant.

The concern of prehistoric humans for spatial designs and relationships may have stemmed from their aesthetic feeling and the enjoyment of beauty of form, motives that often actuate the mathematician of today. We would like to think that at least some of the early geometers pursued their work for the sheer joy of doing mathematics, rather than as a practical aid in mensuration, but there are alternative theories. One of these is that geometry, like counting, had an origin in primitive ritualistic practice. Yet the theory of the origin of geometry in a secularization of ritualistic practice is by no means established. The development of geometry may just as well have been stimulated by the practical needs of construction and surveying or by an aesthetic feeling for design and order.

We can make conjectures about what led people of the Stone Age to count, to measure, and to draw. That the beginnings of mathematics are older than the oldest civilizations is clear. To go further and categorically identify a specific origin in space or time, however, is to mistake conjecture for history. It is best to suspend judgment on this matter and to move on to the safer ground of the history of mathematics as found in the written documents that have come down to us.

2

Ancient Egypt

Sesostris . . . made a division of the soil of Egypt among the inhabitants. . . . If the river carried away any portion of a man's lot, . . . the king sent persons to examine, and determine by measurement the exact extent of the loss. . . . From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece.
—Herodotus

The Era and the Sources

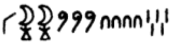
About 450 BCE, Herodotus, the inveterate Greek traveler and narrative historian, visited Egypt. He viewed ancient monuments, interviewed priests, and observed the majesty of the Nile and the achievements of those working along its banks. His resulting account would become a cornerstone for the narrative of Egypt's ancient history. When it came to mathematics, he held that geometry had originated in Egypt, for he believed that the subject had arisen there from the practical need for resurveying after the annual flooding of the river valley. A century later, the philosopher Aristotle speculated on the same subject and attributed the Egyptians' pursuit of geometry to the existence of a priestly leisure class. The debate, extending well beyond the confines of Egypt, about whether to credit progress in mathematics to the practical men (the surveyors, or "rope-stretchers") or to the contemplative elements of society (the priests and the philosophers) has continued to our times. As we shall see, the history of mathematics displays a constant interplay between these two types of contributors.

In attempting to piece together the history of mathematics in ancient Egypt, scholars until the nineteenth century encountered two major obstacles. The first was the inability to read the source materials that existed. The second was the scarcity of such materials. For more than thirty-five centuries, inscriptions used hieroglyphic writing, with variations from purely ideographic to the smoother hieratic and eventually the still more flowing demotic forms. After the third century CE, when they were replaced by Coptic and eventually supplanted by Arabic, knowledge of hieroglyphs faded. The breakthrough that enabled modern scholars to decipher the ancient texts came early in the nineteenth century when the French scholar Jean-François Champollion, working with multilingual tablets, was able to slowly translate a number of hieroglyphs. These studies were supplemented by those of other scholars, including the British physicist Thomas Young, who were intrigued by the Rosetta Stone, a trilingual basalt slab with inscriptions in hieroglyphic, demotic, and Greek writings that had been found by members of Napoleon's Egyptian expedition in 1799. By 1822, Champollion was able to announce a substantive portion of his translations in a famous letter sent to the Academy of Sciences in Paris, and by the time of his death in 1832, he had published a grammar textbook and the beginning of a dictionary.

Although these early studies of hieroglyphic texts shed some light on Egyptian numeration, they still produced no purely mathematical materials. This situation changed in the second half of the nineteenth century. In 1858, the Scottish antiquary Henry Rhind purchased a papyrus roll in Luxor that is about one foot high and some eighteen feet long. Except for a few fragments in the Brooklyn Museum, this papyrus is now in the British Museum. It is known as the Rhind or the Ahmes Papyrus, in honor of the scribe by whose hand it had been copied in about 1650 BCE. The scribe tells us that the material is derived from a prototype from the Middle Kingdom of about 2000 to 1800 BCE. Written in the hieratic script, it became the major source of our knowledge of ancient Egyptian mathematics. Another important papyrus, known as the Golenishchev or Moscow Papyrus, was purchased in 1893 and is now in the Pushkin Museum of Fine Arts in Moscow. It, too, is about eighteen feet long but is only one-fourth as wide as the Ahmes Papyrus. It was written less carefully than the work of Ahmes was, by an unknown scribe of circa. 1890 BCE. It contains twenty-five examples, mostly from practical life and not differing greatly from those of Ahmes, except for two that will be discussed further on. Yet another twelfth-dynasty papyrus, the Kahun, is now in London; a Berlin papyrus is of the same period. Other, somewhat earlier, materials are two wooden tablets from Akhmim of about 2000 BCE and a leather roll containing a list of fractions. Most of this material was deciphered within a hundred years of Champollion's death. There is a striking degree of coincidence between certain aspects of the earliest known inscriptions and the few mathematical texts of the Middle Kingdom that constitute our known source material.

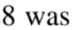
Numbers and Fractions

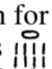
Once Champollion and his contemporaries could decipher inscriptions on tombs and monuments, Egyptian hieroglyphic numeration was easily disclosed. The system, at least as old as the pyramids, dating some 5,000 years ago, was based on the 10 scale. By the use of a simple iterative scheme and of distinctive symbols for each of the first half-dozen powers of 10, numbers greater than a million were carved on stone, wood, and other materials. A single vertical stroke represented a unit, an inverted wicket was used for 10, a snare somewhat resembling a capital C stood for 100, a lotus flower for 1,000, a bent finger for 10,000, a tadpole for 100,000, and a kneeling figure, apparently Heh, the god of the Unending, for 1,000,000. Through repetition of these symbols, the number 12,345, for example, would appear as



Sometimes the smaller digits were placed on the left, and other times the digits were arranged vertically. The symbols themselves were occasionally reversed in orientation, so that the snare might be convex toward either the right or the left.

Egyptian inscriptions indicate familiarity with large numbers at an early date. A museum at Oxford has a royal mace more than 5,000 years old, on which a record of 120,000 prisoners and 1,422,000 captive goats appears. These figures may have been exaggerated, but from other considerations it is clear that the Egyptians were commendably accurate in counting and measuring. The construction of the Egyptian solar calendar is an outstanding early example of observation, measurement, and counting. The pyramids are another famous instance; they exhibit such a high degree of precision in construction and orientation that ill-founded legends have grown up around them.

The more cursive hieratic script used by Ahmes was suitably adapted to the use of pen and ink on prepared papyrus leaves. Numeration remained decimal, but the tedious repetitive principle of hieroglyphic numeration was replaced by the introduction of ciphers or special signs to represent digits and multiples of powers of 10. The number 4, for example, usually was no longer represented by four vertical strokes but by a horizontal bar, and 7 is not written as seven strokes but as a single cipher  resembling a sickle. The hieroglyphic form for the number 28 was ; the hieratic form was simply . Note that the cipher for the smaller digit 8 (or two 4s) appears on the left, rather than on the right. The principle of cipherization, introduced by the Egyptians some 4,000 years ago and used in the Ahmes Papyrus, represented an important contribution to numeration, and it is one of the factors that makes our own system in use today the effective instrument that it is.

Egyptian hieroglyphic inscriptions have a special notation for unit fractions—that is, fractions with unit numerators. The reciprocal of any integer was indicated simply by placing over the notation for the integer an elongated oval sign. The fraction $\frac{1}{8}$ thus appeared as  and $\frac{1}{20}$ was written as . In the hieratic notation, appearing in papyri, the elongated oval is replaced by a dot, which is placed over the cipher for the corresponding integer (or over the right-hand cipher in the case of the reciprocal of a multidigit number). In the Ahmes Papyrus, for example, the fraction $\frac{1}{8}$ appears as , and $\frac{1}{20}$ is written as . Such unit fractions were freely handled in Ahmes's day, but the general fraction seems to have been an enigma to the Egyptians. They felt comfortable with the fraction $\frac{2}{3}$, for which they had a special hieratic sign ; occasionally, they used special signs for fractions of the form $n/(n+1)$, the complements of the unit fractions. To the fraction $\frac{2}{3}$, the Egyptians assigned a special role in arithmetic processes, so that in finding one-third of a number, they first found two-thirds of it and subsequently took half of the result! They knew and used the fact that two-thirds of the unit fraction $1/p$ is the sum of the two unit fractions $1/2p$ and $1/6p$; they were also aware that double the unit fraction $1/2p$ is the unit fraction $1/p$. Yet it looks as though, apart from the fraction $\frac{2}{3}$, the Egyptians regarded the general proper rational fraction of the form m/n not as an elementary "thing" but as part of an uncompleted process. Where today we think of $\frac{3}{5}$ as a single irreducible fraction, Egyptian scribes thought of it as reducible to the sum of three unit fractions, $\frac{1}{3}$ and $\frac{1}{5}$ and $\frac{1}{15}$.

To facilitate the reduction of "mixed" proper fractions to the sum of unit fractions, the Ahmes Papyrus opens with a table expressing $2/n$ as a sum of unit fractions for all odd values of n from 5 to 101. The equivalent of $\frac{2}{5}$ is given as $\frac{1}{3}$ and $\frac{1}{15}$, $\frac{2}{11}$ is written as $\frac{1}{6}$ and $\frac{1}{66}$, and $\frac{2}{15}$ is expressed as $\frac{1}{10}$ and $\frac{1}{30}$. The last item in the table decomposes $\frac{2}{101}$ into $\frac{1}{101}$ and $\frac{1}{202}$ and $\frac{1}{303}$ and $\frac{1}{606}$. It is not clear why one form of decomposition was preferred to another of the indefinitely many that are possible. This last entry certainly exemplifies the Egyptian prepossession for halving and taking a third; it is not at all clear to us why the decomposition $2/n = 1/n + 1/2n + 1/3n + 1/2{\cdot}3{\cdot}n$ is better than $1/n + 1/n$. Perhaps one of the objects of the $2/n$ decomposition was to arrive at unit fractions smaller than $1/n$. Certain passages indicate that the Egyptians had some appreciation of general rules and methods above and beyond the

the desired answer, which was 19. Inasmuch as $8(2+\frac{1}{4}+\frac{1}{8})=19$, one must multiply 7 by $2+\frac{1}{4}+\frac{1}{8}$ to obtain the correct heap; Ahmes found the answer to be $16+\frac{1}{2}+\frac{1}{8}$. Ahmes then "checked" his result by showing that if to $16+\frac{1}{2}+\frac{1}{8}$ one adds $\frac{1}{7}$ of this (which is $2+\frac{1}{4}+\frac{1}{8}$), one does indeed obtain 19. Here we see another significant step in the development of mathematics, for the check is a simple instance of a proof. Although the method of false position was generally used by Ahmes, there is one problem (Problem 30) in which $x+\frac{2}{3}x+\frac{1}{2}x+\frac{1}{7}x=37$ is solved by factoring the left-hand side of the equation and dividing 37 by $1+\frac{2}{3}+\frac{1}{2}+\frac{1}{7}$, the result being $16+\frac{1}{56}+\frac{1}{679}+\frac{1}{776}$.

Many of the "aha" calculations in the Rhind (Ahmes) Papyrus appear to be practice exercises for young students. Although a large proportion of them are of a practical nature, in some places the scribe seemed to have had puzzles or mathematical recreations in mind. Thus, Problem 79 cites only "seven houses, 49 cats, 343 mice, 2401 ears of spelt, 16807 hekats." It is presumed that the scribe was dealing with a problem, perhaps quite well known, where in each of seven houses there are seven cats, each of which eats seven mice, each of which would have eaten seven ears of grain, each of which would have produced seven measures of grain. The problem evidently called not for the practical answer, which would be the number of measures of grain that were saved, but for the impractical sum of the numbers of houses, cats, mice, ears of spelt, and measures of grain. This bit of fun in the Ahmes Papyrus seems to be a forerunner of our familiar nursery rhyme:

> As I was going to St. Ives,
> I met a man with seven wives;
> Every wife had seven sacks,
> Every sack had seven cats,
> Every cat had seven kits,
> Kits, cats, sacks, and wives,
> How many were going to St. Ives?

Geometric Problems

It is often said that the ancient Egyptians were familiar with the Pythagorean theorem, but there is no hint of this in the papyri that have come down to us. There are nevertheless some geometric problems in the Ahmes Papyrus. Problem 51 of Ahmes shows that the area of an isosceles triangle was found by taking half of what we would call the base and multiplying this by the altitude. Ahmes justified his method of finding the area by suggesting that the isosceles triangle can be thought of as two right triangles, one of which can be shifted in position, so that together the two triangles form a rectangle. The isosceles trapezoid is similarly handled in Problem 52, in which the larger base of a trapezoid is 6, the smaller base is 4, and the distance between them is 20. Taking $\frac{1}{2}$ of the sum of the bases, "so as to make a rectangle," Ahmes multiplied this by 20 to find the area. In transformations such as these, in which isosceles triangles and trapezoids are converted into rectangles, we may see the beginnings of a theory of congruence and the idea of proof in geometry, but there is no evidence that the Egyptians carried such work further. Instead, their geometry lacks a clear-cut distinction between relationships that are exact and those that are only approximations.

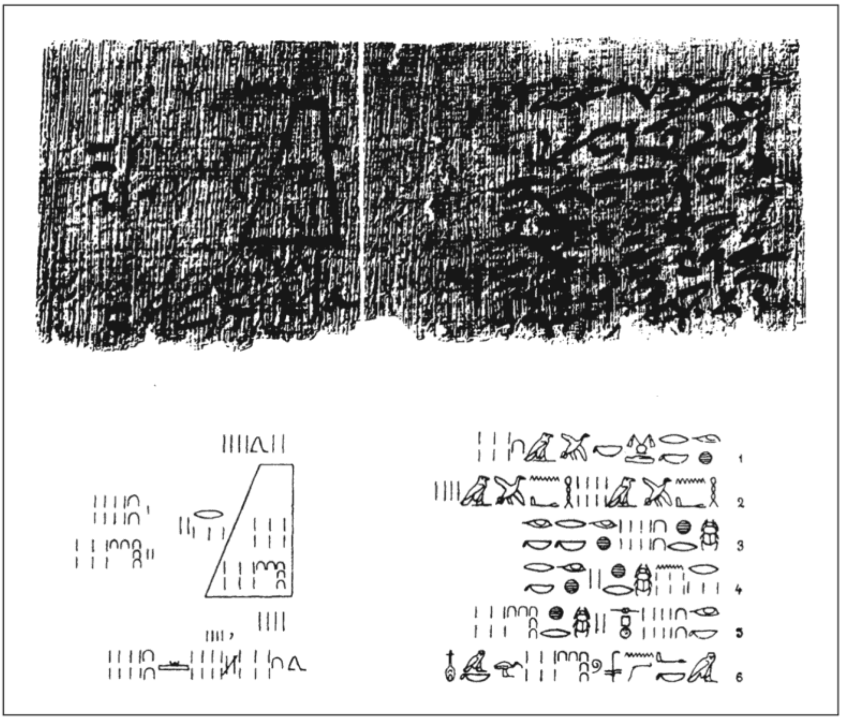
A surviving deed from Edfu, dating from a period some 1,500 years after Ahmes, gives examples of triangles, trapezoids, rectangles, and more general quadrilaterals. The rule for finding the area of the general quadrilateral is to take the product of the arithmetic means of the opposite sides. Inaccurate though the rule is, the author of the deed deduced from it a corollary—that the area of a triangle is half of the sum of two sides multiplied by half of the third side. This is a striking instance of the search for relationships among geometric figures, as well as an early use of the zero concept as a replacement for a magnitude in geometry.

The Egyptian rule for finding the area of a circle has long been regarded as one of the outstanding achievements of the time. In Problem 50, the scribe Ahmes assumed that the area of a circular field with a diameter of 9 units is the same as the area of a square with a side of 8 units. If we compare this assumption with the modern formula $A = \pi r^2$, we find the Egyptian rule to be equivalent to giving π a value of about $3\frac{1}{6}$, a commendably close approximation, but here again we miss any hint that Ahmes was aware that the areas of his circle and square were not exactly equal. It is possible that Problem 48 gives a hint to the way in which the Egyptians were led to their area of the circle. In this problem, the scribe formed an octagon from a square having sides of 9 units by trisecting the sides and cutting off the four corner isosceles triangles, each having an area of $4\frac{1}{2}$ units. The area of the octagon, which does not differ greatly from that of a circle inscribed within the square, is 63 units, which is not far removed from the area of a square with 8 units on a side. That the number $4(8/9)^2$ did indeed play a role comparable to our constant π seems to be confirmed by the Egyptian rule for the circumference of a circle, according to which the ratio of the area of a circle to the circumference is the same as the ratio of the area of the circumscribed square to its perimeter. This observation represents a geometric relationship of far greater precision and mathematical significance than the relatively good approximation for π.

Degree of accuracy in approximation is not a good measure of either mathematical or architectural achievement, and we should not overemphasize this aspect of Egyptian work. Recognition by the Egyptians of interrelationships among geometric figures, on the other hand, has too often been overlooked, and yet it is here that they came closest in attitude to their successors, the Greeks. No theorem or formal proof is known in Egyptian mathematics, but some of the geometric comparisons made in the Nile Valley, such as those on the perimeters and the areas of circles and squares, are among the first exact statements in history concerning curvilinear figures.

The value of $\frac{22}{7}$ is often used today for π; but we must recall that Ahmes's value for π is about $3\frac{1}{6}$, not $3\frac{1}{7}$. That Ahmes's value was also used by other Egyptians is confirmed in a papyrus roll from the twelfth dynasty (the Kahun Papyrus), in which the volume of a cylinder is found by multiplying the height by the area of the base, the base being determined according to Ahmes's rule.

Associated with Problem 14 in the Moscow Papyrus is a figure that looks like an isosceles trapezoid (see Fig. 2.1), but the calculations associated with it indicate that a frustum of a square pyramid is intended. Above and below the figure are signs for 2 and 4, respectively, and within the figure are the hieratic symbols for 6 and 56. The directions



Reproduction (top) of a portion of the Moscow Papyrus, showing the problem of the volume of a frustum of a square pyramid, together with hieroglyphic transcription (below)

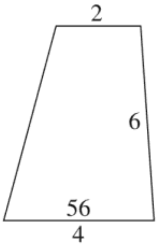


FIG. 2.1

alongside make it clear that the problem calls for the volume of a frustum of a square pyramid 6 units high if the edges of the upper and lower bases are 2 and 4 units, respectively. The scribe directs one to square the numbers 2 and 4 and to add to the sum of these squares the product of 2 and 4, the result being 28. This result is then multiplied by a third of 6, and the scribe concludes with the words "See, it is 56; you have found it correctly." That is, the volume of the frustum has been calculated in accordance with the modern formula $V = h(a^2 + ab + b^2)/3$, where h is the altitude and a and b are the sides of the square bases. Nowhere is this formula written out, but in substance it evidently was known to the Egyptians. If, as in the deed from Edfu, one takes $b = 0$, the formula reduces to the familiar formula, one-third the base times the altitude, for the volume of a pyramid.

How these results were arrived at by the Egyptians is not known. An empirical origin for the rule on the volume of a pyramid seems to be a possibility, but not for the volume of the frustum. For the latter, a theoretical basis seems more likely, and it has been suggested that the Egyptians may have proceeded here as they did in the cases of the isosceles triangle and the isosceles trapezoid—they may mentally have broken the frustum into parallelepipeds, prisms, and pyramids. On replacing the pyramids and the prisms by equal rectangular blocks, a plausible grouping of the blocks leads to the Egyptian formula. One could, for example, have begun with a pyramid having a square base and with the vertex directly over one of the base vertices. An obvious decomposition of the frustum would be to break it into four parts as in Fig. 2.2—a rectangular parallelepiped having a volume b^2h, two triangular prisms, each with a volume of $b(a-b)h/2$, and a pyramid of volume $(a-b)^2h/3$. The prisms can be combined into a rectangular parallelepiped with dimensions b and $a-b$ and h; and the pyramid can be thought of as a rectangular parallelepiped with dimensions $a-b$ and $a-b$ and $h/3$. On cutting up the tallest parallelepipeds so that all altitudes are $h/3$, one can easily arrange the slabs so as to form three layers, each of altitude $h/3$, and having cross-sectional areas of a^2 and ab and b^2, respectively.

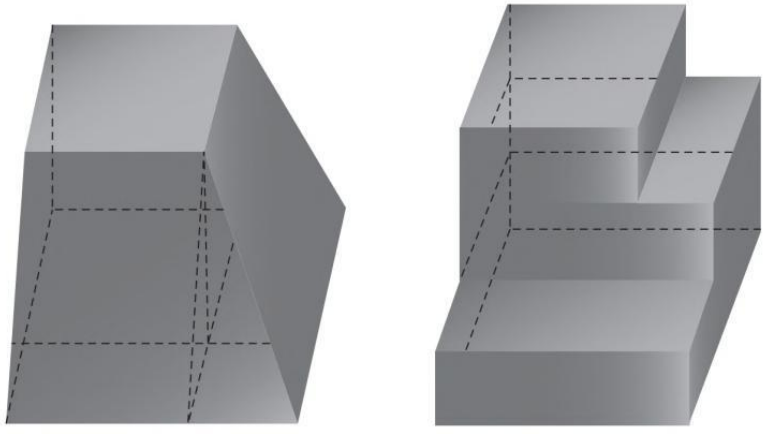


FIG. 2.2

Problem 10 in the Moscow Papyrus presents a more difficult question of interpretation than does Problem 14. Here the scribe asks for the surface area of what looks like a basket with a diameter of $4\frac{1}{2}$. He proceeds as though he were using the equivalent of a formula $S=(1-\frac{1}{9})^2(2x)\cdot x$, where x is $4\frac{1}{2}$, obtaining an answer of 32 units. Inasmuch as $(1-\frac{1}{9})^2$ is the Egyptian approximation of $\pi/4$, the answer 32 would correspond to the surface of a hemisphere of diameter $4\frac{1}{2}$, and this was the interpretation given to the problem in 1930. Such a result, antedating the oldest known calculation of a hemispherical surface by some 1,500 years, would have been amazing, and it seems, in fact, to have been too good to be true. Later analysis indicates that the "basket" may have been a roof—somewhat like that of a Quonset hut in the shape of a half-cylinder of diameter $4\frac{1}{2}$ and length $4\frac{1}{2}$. The calculation in this case calls for nothing beyond knowledge of the length of a semicircle, and the obscurity of the text makes it admissible to offer still more primitive interpretations, including the possibility that the calculation is only a rough estimate of the area of a domelike barn roof. In any case, we seem to have here an early estimation of a curvilinear surface area.

Slope Problems

In the construction of the pyramids, it had been essential to maintain a uniform slope for the faces, and it may have been this concern that led the Egyptians to introduce a concept equivalent to the cotangent of an angle. In modern technology, it is customary to measure the steepness of a straight line through the ratio of the "rise" to the "run." In Egypt, it was

3

Mesopotamia

How much is one god beyond the other god?
—An Old Babylonian astronomical text

The Era and the Sources

The fourth millennium before our era was a period of remarkable cultural development, bringing with it the use of writing, the wheel, and metals. As in Egypt during the first dynasty, which began toward the end of this extraordinary millennium, so also in the Mesopotamian Valley there was at the time a high order of civilization. There the Sumerians had built homes and temples decorated with artistic pottery and mosaics in geometric patterns. Powerful rulers united the local principalities into an empire that completed vast public works, such as a system of canals to irrigate the land and control flooding between the Tigris and Euphrates rivers, where the overflow of the rivers was not predictable, as was the inundation of the Nile Valley. The cuneiform pattern of writing that the Sumerians had developed during the fourth millennium probably antedates the Egyptian hieroglyphic system.

The Mesopotamian civilizations of antiquity are often referred to as Babylonian, although such a designation is not strictly correct. The city of

Babylon was not at first, nor was it always at later periods, the center of the culture associated with the two rivers, but convention has sanctioned the informal use of the name "Babylonian" for the region during the interval from about 2000 to roughly 600 BCE. When in 538 BCE Babylon fell to Cyrus of Persia, the city was spared, but the Babylonian Empire had come to an end. "Babylonian" mathematics, however, continued through the Seleucid period in Syria almost to the dawn of Christianity.

Then, as today, the Land of the Two Rivers was open to invasions from many directions, making the Fertile Crescent a battlefield with frequently changing hegemony. One of the most significant of the invasions was that by the Semitic Akkadians under Sargon I (ca. 2276–2221 BCE), or Sargon the Great. He established an empire that extended from the Persian Gulf in the south to the Black Sea in the north, and from the steppes of Persia in the east to the Mediterranean Sea in the west. Under Sargon, the invaders began a gradual absorption of the indigenous Sumerian culture, including the cuneiform script. Later invasions and revolts brought various racial strains—Ammorites, Kassites, Elamites, Hittites, Assyrians, Medes, Persians, and others—to political power at one time or another in the valley, but there remained in the area a sufficiently high degree of cultural unity to justify referring to the civilization simply as Mesopotamian. In particular, the use of cuneiform script formed a strong bond.

Laws, tax accounts, stories, school lessons, personal letters—these and many other records were impressed on soft clay tablets with styluses, and the tablets were then baked in the hot sun or in ovens. Such written documents were far less vulnerable to the ravages of time than were Egyptian papyri; hence, a much larger body of evidence about Mesopotamian mathematics is available today than exists about the Nilotic system. From one locality alone, the site of ancient Nippur, we have some 50,000 tablets. The university libraries at Columbia, Pennsylvania, and Yale, among others, have large collections of ancient tablets from Mesopotamia, some of them mathematical. Despite the availability of documents, however, it was the Egyptian hieroglyphic, rather than the Babylonian cuneiform, that was first deciphered in modern times. The German philologist F. W. Grotefend had made some progress in the reading of Babylonian script early in the nineteenth century, but only during the second quarter of the twentieth century did substantial accounts of Mesopotamian mathematics begin to appear in histories of antiquity.

Cuneiform Writing

The early use of writing in Mesopotamia is attested to by hundreds of clay tablets found in Uruk and dating from about 5,000 years ago. By this time, picture writing had reached the point where conventionalized stylized forms were used for many things: $\approx$ for water, for eye, and combinations of these to indicate weeping. Gradually, the number of signs became smaller, so that of some 2,000 Sumerian signs originally used, only a third remained by the time of the Akkadian conquest. Primitive drawings gave way to combinations of wedges: water became 𒀀 and eye 𒅆. At first, the scribe wrote from top to bottom in columns from right to left; later, for convenience, the table was rotated counterclockwise through 90°, and the scribe wrote from left to right in horizontal rows from top to bottom. The stylus, which formerly had been a triangular prism, was replaced by a right circular cylinder—or, rather, two cylinders of unequal radius. During the earlier days of the Sumerian civilization, the end of the stylus was pressed into the clay vertically to represent 10 units and obliquely to represent a unit, using the smaller stylus; similarly, an oblique impression with the larger stylus indicated 60 units and a vertical impression indicated 3,600 units. Combinations of these were used to represent intermediate numbers.

Numbers and Fractions: Sexagesimals

As the Akkadians adopted the Sumerian form of writing, lexicons were compiled giving equivalents in the two tongues, and forms of words and numerals became less varied. Thousands of tablets from about the time of the Hammurabi dynasty (ca. 1800–1600 BCE) illustrate a number system that had become well established. The decimal system, common to most civilizations, both ancient and modern, had been submerged in Mesopotamia under a notation that made fundamental the base 60. Much has been written about the motives behind this change; it has been suggested that astronomical considerations may have been instrumental or that the sexagesimal scheme might have been the natural combination of two earlier schemes, one decimal and the other using the base 6. It appears more likely, however, that the base 60 was consciously adopted and legalized in the interests of metrology, for a magnitude of 60 units can be subdivided easily into halves, thirds, fourths, fifths, sixths, tenths, twelfths, fifteenths, twentieths, and thirtieths, thus affording ten possible subdivisions. Whatever the origin, the sexagesimal system of numeration has enjoyed a remarkably long life, for remnants survive, unfortunately for consistency, even to this day in units of time and angle measure, despite the fundamentally decimal form of mathematics in our society.

Positional Numeration

Babylonian cuneiform numeration, for smaller whole numbers, proceeded along the same lines as did the Egyptian hieroglyphic, with repetitions of the symbols for units and tens. Where the Egyptian architect, carving on stone, might write 59 as , the Mesopotamian scribe could similarly represent the same number on a clay tablet through fourteen wedge-shaped marks—five broad sideways wedges or “angle-brackets,” each representing 10 units, and nine thin vertical wedges, each standing for a unit, all juxtaposed in a neat group as . Beyond the number 59, however, the Egyptian and Babylonian systems differed markedly. Perhaps it was the inflexibility of the Mesopotamian writing materials, possibly it was a flash of imaginative insight that made the Babylonians realize that their two symbols for units and tens sufficed for the representation of any integer, however large, without excessive repetitiveness. This was made possible through their invention, some 4,000 years ago, of the positional notation—the same principle that accounts for the effectiveness of our present numeral forms. That is, the ancient Babylonians saw that their symbols could be assigned values that depend on their relative positions in the representation of a number. Our number 222 makes use of the same cipher three times, but with a different meaning each time. Once it represents two units, the second time it means two 10s, and finally it stands for two 100s (that is, twice the square of the base 10). In a precisely analogous way, the Babylonians made multiple use of such a symbol as . When they wrote , clearly separating the three groups of two wedges each, they understood the right-hand group to mean two units, the next group to mean twice their base, 60, and the left-hand group to signify twice the square of their base. This numeral, therefore, denoted $2(60)^2 + 2(60) + 2$ (or 7,322 in our notation).

A wealth of primary material exists concerning Mesopotamian mathematics, but, oddly enough, most of it comes from two periods widely separated in time. There is an abundance of tablets from the first few hundred years of the second millennium BCE (the Old Babylonian age), and many tablets have also been found dating from the last few centuries of the first millennium BCE (the Seleucid period). Most of the important contributions to mathematics will be found to go back to the earlier period, but one contribution is not in evidence until almost 300 BCE. The Babylonians seem at first to have had no clear way in which to indicate an “empty” position—that is, they did not have a zero symbol, although they sometimes left a space where a zero was intended. This meant that their forms for the numbers 122 and 7,202 looked very much alike, for might mean either $2(60) + 2$ or $2(60)^2 + 2$. Context in many cases could be relied on to relieve some of the ambiguity, but the lack of a zero symbol, such as enables us to distinguish at a glance between 22 and 202, must have been quite inconvenient.

By about the time of the conquest by Alexander the Great, however, a special sign, consisting of two small wedges placed obliquely, was invented to serve as a placeholder where a numeral was missing. From that time on, as long as cuneiform was used, the number , or $2(60)^2 + 0(60) + 2$, was readily distinguishable from , or $2(60) + 2$.

The Babylonian zero symbol apparently did not end all ambiguity, for the sign seems to have been used for intermediate empty positions only. There are no extant tablets in which the zero sign appears in a terminal position. This means that the Babylonians in antiquity never achieved an absolute positional system. Position was only relative; hence, the symbol 𒐖 𒐖 could represent $2(60) + 2$ or $2(60)^2 + 2(60)$ or $2(60)^3 + 2(60)^2$ or any one of indefinitely many other numbers in which two successive positions are involved.

Sexagesimal Fractions

Had Mesopotamian mathematics, like that of the Nile Valley, been based on the addition of integers and unit fractions, the invention of the positional notation would not have been greatly significant at the time. It is not much more difficult to write 98,765 in hieroglyphic notation than in cuneiform, and the latter is definitely more difficult to write than the same number in hieratic script. The secret of the superiority of Babylonian mathematics over that of the Egyptians lies in the fact that those who lived "between the two rivers" took the most felicitous step of extending the principle of position to cover fractions as well as whole numbers. That is, the notation 𒐖 𒐖 was used not only for $2(60) + 2$, but also for $2 + 2(60)^{-1}$ or for $2(60)^{-1} + 2(60)^{-2}$ or for other fractional forms involving two successive positions. This meant that the Babylonians had at their command the computational power that the modern decimal fractional notation affords us today. For the Babylonian scholar, as for the modern engineer, the addition or the multiplication of 23.45 and 9.876 was essentially no more difficult than was the addition or the multiplication of the whole numbers 2,345 and 9,876, and the Mesopotamians were quick to exploit this important discovery.

Approximations

An Old Babylonian tablet from the Yale Collection (No. 7289) includes the calculation of the square root of 2 to three sexagesimal places, the answer being written 𒁹 𒎙𒐘 𒐏𒁹 𒌋. In modern characters, this number can be appropriately written as 1;24,51,10, where a semicolon is used to separate the integral and fractional parts, and a comma is used as a separatrix for the sexagesimal positions. This form will generally be used throughout this chapter to designate numbers in sexagesimal notation. Translating this notation into decimal form, we have $1 + 24(60)^{-1} + 51(60)^{-2} + 10(60)^{-3}$. This Babylonian value for $\sqrt{2}$ is equal to approximately 1.414222, differing by about 0.000008 from the true value. Accuracy in approximations was relatively easy for the Babylonians to achieve with their fractional notation, which was rarely equaled until the time of the Renaissance.

seen in a problem text that asks how long it will take money to double at 20 percent annually; the answer given is 3;47,13,20. It seems to be quite clear that the scribe used linear interpolation between the values for $(1;12)^3$ and $(1;12)^4$, following the compound interest formula $a = P(1 + r)^n$, where r is 20 percent, or $\frac{12}{60}$, and reading values from an exponential table with powers of 1;12.

Equations

One table for which the Babylonians found considerable use is a tabulation of the values of $n^3 + n^2$ for integral values of n, a table essential in Babylonian algebra; this subject reached a considerably higher level in Mesopotamia than in Egypt. Many problem texts from the Old Babylonian period show that the solution of the complete three-term quadratic equation afforded the Babylonians no serious difficulty, for flexible algebraic operations had been developed. They could transpose terms in an equation by adding equals to equals, and they could multiply both sides by like quantities to remove fractions or to eliminate factors. By adding $4ab$ to $(a - b)^2$ they could obtain $(a + b)^2$, for they were familiar with many simple forms of factoring. They did not use letters for unknown quantities, for the alphabet had not yet been invented, but words such as "length," "breadth," "area," and "volume" served effectively in this capacity. That these words may well have been used in a very abstract sense is suggested by the fact that the Babylonians had no qualms about adding a "length" to an "area" or an "area" to a "volume."

Egyptian algebra had been much concerned with linear equations, but the Babylonians evidently found these too elementary for much attention. In one problem, the weight x of a stone is called for if $(x + x/7) + \frac{1}{11}(x + x/7)$ is one mina; the answer is simply given as 48;7,30 gin, where 60 gin make a mina. In another problem in an Old Babylonian text, we find two simultaneous linear equations in two unknown quantities, called respectively the "first silver ring" and the "second silver ring." If we call these x and y in our notation, the equations are $x/7 + y/11 = 1$ and $6x/7 = 10y/11$. The answer is expressed laconically in terms of the rule

$$\frac{x}{7} = \frac{11}{7+11} + \frac{1}{72} \quad \text{and} \quad \frac{y}{11} = \frac{7}{7+11} - \frac{1}{72}.$$

In another pair of equations, part of the method of solution is included in the text. Here $\frac{1}{4}$ width + length = 7 hands, and length + width = 10 hands. The solution is first found by replacing each "hand" with 5 "fingers" and then noticing that a width of 20 fingers and a length of 30 fingers will satisfy both equations. Following this, however, the solution is found by an alternative method equivalent to an elimination through combination.

Expressing all dimensions in terms of hands, and letting the length and the width be x and y, respectively, the equations become $y+4x=28$ and $x+y=10$. Subtracting the second equation from the first, one has the result $3x=18$; hence, $x=6$ hands, or 30 fingers, and $y=20$ fingers.

Quadratic Equations

The solution of a three-term quadratic equation seems to have exceeded by far the algebraic capabilities of the Egyptians, but Otto Neugebauer in 1930 disclosed that such equations had been handled effectively by the Babylonians in some of the oldest problem texts. For instance, one problem calls for the side of a square if the area less the side is 14,30. The solution of this problem, equivalent to solving $x^2-x=870$, is expressed as follows:

> Take half of 1, which is 0;30, and multiply 0;30 by 0;30, which is 0;15; add this to 14,30 to get 14,30;15. This is the square of 29;30. Now add 0;30 to 29;30, and the result is 30, the side of the square.

The Babylonian solution is, of course, exactly equivalent to the formula $x=\sqrt{(p/2)^2+q}+p/2$ for a root of the equation $x^2-px=q$, which is the quadratic formula that is familiar to high school students of today. In another text, the equation $11x^2+7x=6;15$ was reduced by the Babylonians to the standard type $x^2+px=q$ by first multiplying through by 11 to obtain $(11x)^2+7(11x)=1,8;45$. This is a quadratic in normal form in the unknown quantity $y=11x$, and the solution for y is easily obtained by the familiar rule $y=\sqrt{(p/2)^2+q}-p/2$, from which the value of x is then determined. This solution is remarkable as an instance of the use of algebraic transformations.

Until modern times, there was no thought of solving a quadratic equation of the form $x^2+px+q=0$, where p and q are positive, for the equation has no positive root. Consequently, quadratic equations in ancient and medieval times—and even in the early modern period—were classified under three types:

1. $x^2+px=q$
2. $x^2=px+q$
3. $x^2+q=px$

All three types are found in Old Babylonian texts of some 4,000 years ago. The first two types are illustrated by the problems given previously; the third type appears frequently in problem texts, where it is treated as equivalent to the simultaneous system $x+y=p$, $xy=q$. So numerous are problems in which one is asked to find two numbers when given their product and either their sum or their difference that these seem to have constituted for the ancients, both Babylonian and Greek, a sort of "normal form" to which quadratics were reduced. Then, by transforming the simultaneous equations $xy = a$ and $x \pm y = b$ into the pair of linear equations $x \pm y = b$ and $x \mp y = \sqrt{b^2 \mp 4a}$, the values of x and y are found through an addition and a subtraction. A Yale cuneiform tablet, for example, asks for the solution of the system $x + y = 6;30$ and $xy = 7;30$. The instructions of the scribe are essentially as follows. First find

$$\frac{x+y}{2} = 3;15$$

and then find

$$\left(\frac{x+y}{2}\right)^2 = 10;33,45.$$

Then,

$$\left(\frac{x+y}{2}\right)^2 - xy = 3;3,45$$

and

$$\sqrt{\left(\frac{x+y}{2}\right)^2 - xy} = 1;45.$$

Hence,

$$\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right) = 3;15 + 1;45$$

and

$$\left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right) = 3;15 - 1;45.$$

From the last two equations, it is obvious that $x = 5$ and $y = 1\frac{1}{2}$. Because the quantities x and y enter symmetrically in the given conditional equations, it is possible to interpret the values of x and y as the two roots of the quadratic equation $x^2 + 7;30 = 6;30x$. Another Babylonian text calls for a number that when added to its reciprocal becomes 2;0,0,33,20. This leads to a quadratic of type 3, and again we have two solutions, 1;0,45 and 0;59,15,33,20.

Cubic Equations

The Babylonian reduction of a quadratic equation of the form $ax^2 + bx = c$ to the normal form $y^2 + by = ac$ through the substitution $y = ax$ shows the extraordinary degree of flexibility in Mesopotamian algebra. There is no record in Egypt of the solution of a cubic equation, but among the Babylonians there are many instances of this.

Pure cubics, such as $x^3 = 0;7,30$, were solved by direct reference to tables of cubes and cube roots, where the solution $x = 0;30$ was read off. Linear interpolation within the tables was used to find approximations for values not listed in the tables. Mixed cubics in the standard form $x^3 + x^2 = a$ were solved similarly by reference to the available tables, which listed values of the combination $n^3 + n^2$ for integral values of n from 1 to 30. With the help of these tables, they easily read off that the solution, for example, of $x^3 + x^2 = 4,12$ is equal to 6. For still more general cases of equations of the third degree, such as $144x^3 + 12x^2 = 21$, the Babylonians used their method of substitution. Multiplying both sides by 12 and using $y = 12x$, the equation becomes $y^3 + y^2 = 4,12$, from which y is found to be equal to 6, hence x is just $\frac{1}{2}$ or 0;30. Cubics of the form $ax^3 + bx^2 = c$ are reducible to the Babylonian normal form by multiplying through by a^2/b^3 to obtain $(ax/b)^3 + (ax/b)^2 = ca^2/b^3$, a cubic of standard type in the unknown quantity ax/b. Reading off from the tables the value of this unknown quantity, the value of x is determined. Whether the Babylonians were able to reduce the general four-term cubic, $ax^3 + bx^2 + ex = d$, to their normal form is not known. It is not too unlikely that they could reduce it, as is indicated by the fact that a solution of a quadratic suffices to carry the four-term equation to the three-term form $px^3 + qx^2 = r$, from which, as we have seen, the normal form is readily obtained. There is, however, no evidence now available to suggest that the Mesopotamian mathematicians actually carried out such a reduction of the general cubic equation.

With modern symbolism, it is a simple matter to see that $(ax)^3 + (ax)^2 = b$ is essentially the same type of equation as $y^3 + y^2 = b$, but to recognize this without our notation is an achievement of far greater significance for the development of mathematics than even the vaunted positional principle in arithmetic that we owe to the same civilization. Babylonian algebra had reached such an extraordinary level of abstraction that the equations $ax^4 + bx^2 = c$ and $ax^8 + bx^4 = c$ were recognized as nothing worse than quadratic equations in disguise—that is, quadratics in x^2 and x^4.

Measurements: Pythagorean Triads

The algebraic achievements of the Babylonians are admirable, but the motives behind this work are not easy to understand. It has commonly been supposed that virtually all pre-Hellenic science and mathematics were purely utilitarian, but what sort of real-life situation in ancient Babylon could possibly lead to problems involving the sum of a number and its reciprocal or a difference between an area and a length? If utility was the motive, then the cult of immediacy was less strong than it is now, for direct connections between purpose and practice in Babylonian mathematics are far from apparent. That there may well have been toleration for, if not encouragement of, mathematics for its own sake is suggested by a tablet (No. 322) in the Plimpton Collection at Columbia University. The tablet dates from the Old Babylonian period (ca. 1900–1600 BCE), and the tabulations it contains could easily be interpreted as a record of business accounts. Analysis, however, shows that it has deep mathematical significance in the theory of numbers and that it was perhaps related to a kind of proto-trigonometry. Plimpton 322 was part of a larger tablet, as is illustrated by the break along the left-hand edge, and the remaining portion contains four columns of numbers arranged in fifteen horizontal rows. The right-hand column contains the digits from 1 to 15, and, evidently, its purpose was simply to identify in order the items in the other three columns, arranged as follows:

1,59,0,15	1,59	2,49	1
1,56,56,58,14,50,6,15	56,7	1,20,25	2
1,55,7,41,15,33,45	1,16,41	1,50,49	3
1,53,10,29,32,52,16	3,31,49	5,9,1	4
1,48,54,1,40	1,5	1,37	5
1,47,6,41,40	5,19	8,1	6
1,43,11,56,28,26,40	38,11	59,1	7
1,41,33,59,3,45	13,19	20,49	8
1,38,33,36,36	8,1	12,49	9
1,35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1,33,45	45,0	1,15,0	11
1,29,21,54,2,15	27,59	48,49	12
1,27,0,3,45	2,41	4,49	13
1,25,48,51,35,6,40	29,31	53,49	14
1,23,13,46,40	56	1,46	15

The tablet is not in such excellent condition that all of the numbers can still be read, but the clearly discernible pattern of construction in the table made it possible to determine from the context the few items that were missing because of small fractures. To understand what the entries in the table probably meant to the Babylonians, consider the right triangle *ABC* (Fig. 3.1). If the numbers in the second and third columns (from left to right) are thought of as the sides a and c, respectively, of the right triangle, then the first, or left-hand, column contains in each case the square of the ratio of c to b. The left-hand column, therefore, is a short table of values of $\sec^2 A$, but we must not assume that the Babylonians were familiar with our secant concept. Neither the Egyptians nor the Babylonians introduced a measure of angles in the modern sense. Nevertheless, the rows of numbers in Plimpton 322 are not arranged in haphazard fashion, as a superficial glance might imply. If the first comma in column one (on the left) is replaced by a semicolon, it is obvious that the numbers in this column decrease steadily from top to bottom. Moreover, the first number is quite close to $\sec^2 45°$, and the last number in the column is approximately $\sec^2 31°$, with the intervening numbers close to in the Plimpton Tablet 322; as in this case, many are still open to multiple interpretations. For instance, in one tablet the geometric progression $1 + 2 + 2^2 + \cdots + 2^9$ is summed, and in another the sum of the series of squares $1^2 + 2^2 + 3^2 + \cdots + 10^2$ is found. One wonders whether the Babylonians knew the general formulas for the sum of a geometric progression and the sum of the first n perfect squares. It is quite possible that they did, and it has been conjectured that they were aware that the sum of the first n perfect cubes is equal to the square of the sum of the first n integers. Nevertheless, it must be borne in mind that tablets from Mesopotamia resemble Egyptian papyri in that only specific cases are given, with no general formulations.

Polygonal Areas

It used to be held that the Babylonians were better in algebra than were the Egyptians, but that they had contributed less to geometry. The first half of this statement is clearly substantiated by what we have learned in previous paragraphs; attempts to bolster the second half of the comparison generally are limited to the measure of the circle or to the volume of the frustum of a pyramid. In the Mesopotamian Valley, the area of a circle was generally found by taking three times the square of the radius, and in accuracy this falls considerably below the Egyptian measure. Yet the counting of decimal places in the approximations for π is scarcely an appropriate measure of the geometric stature of a civilization, and a twentieth-century discovery has effectively nullified even this weak argument.

In 1936, a group of mathematical tablets was unearthed at Susa, a couple of hundred miles from Babylon, and these include significant geometric results. True to the Mesopotamian penchant for making tables and lists, one tablet in the Susa group compares the areas and the squares of the sides of the regular polygons of three, four, five, six, and seven sides. The ratio of the area of the pentagon, for example, to the square on the side of the pentagon is given as 1;40, a value that is correct to two significant figures. For the hexagon and the heptagon, the ratios are expressed as 2;37,30 and 3;41, respectively. In the same tablet, the scribe gives 0;57,36 as the ratio of the perimeter of the regular hexagon to the circumference of the circumscribed circle, and from this, we can readily conclude that the Babylonian scribe had adopted 3;7,30, or $3\frac{1}{8}$, as an approximation for π. This is at least as good as the value adopted in Egypt. Moreover, we see it in a more sophisticated context than in Egypt, for the tablet from Susa is a good example of the systematic comparison of geometric figures. One is almost tempted to see in it the genuine origin of geometry, but it is important to note that it was not so much the geometric context that interested the Babylonians as the numerical approximations that they used in mensuration. Geometry for

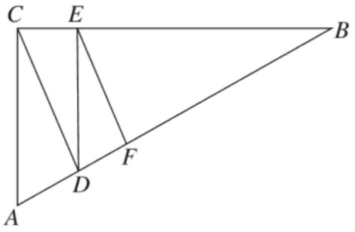


FIG. 3.2

them was not a mathematical discipline in our sense, but a sort of applied algebra or arithmetic in which numbers are attached to figures.

There is some disagreement as to whether the Babylonians were familiar with the concept of similar figures, although this appears to be likely. The similarity of all circles seems to have been taken for granted in Mesopotamia, as it had been in Egypt, and the many problems on triangle measure in cuneiform tablets seem to imply a concept of similarity. A tablet in the Baghdad Museum has a right triangle *ABC* (Fig. 3.2) with sides $a = 60$ and $b = 45$ and $c = 75$, and it is subdivided into four smaller right triangles, *ACD*, *CDE*, *DEF*, and *EFB*. The areas of these four triangles are then given as 8,6 and 5,11;2,24 and 3,19;3,56,9,36 and 5,53;53,39,50,24, respectively. From these values, the scribe computed the length of *AD* as 27, apparently using a sort of "similarity formula" equivalent to our theorem that areas of similar figures are to each other as squares on corresponding sides. The lengths of *CD* and *BD* are found to be 36 and 48, respectively, and through an application of the "similarity formula" to triangles *BCD* and *DCE*, the length of *CE* is found to be 21;36. The text breaks off in the middle of the calculation of *DE*.

Geometry as Applied Arithmetic

Measurement was the keynote of algebraic geometry in the Mesopotamian Valley, but a major flaw, as in Egyptian geometry, was that the distinction between exact and approximate measures was not made clear. The area of a quadrilateral was found by taking the product of the arithmetic means of the pairs of opposite sides, with no warning that this is in most cases only a crude approximation. Again, the volume of a frustum of a cone or a pyramid was sometimes found by taking the arithmetic mean of the upper and lower bases and multiplying by the height; sometimes, for a frustum of a square pyramid with areas a^2 and b^2 for the lower and upper bases, the formula

$$V = \left(\frac{a+b}{2}\right)^2 h$$

was applied. For the latter, however, the Babylonians also used a rule equivalent to

$$V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right],$$

a formula that is correct and reduces to the one used by the Egyptians.

It is not known whether Egyptian and Babylonian results were always independently discovered, but in any case, the latter were definitely more extensive than the former, in both geometry and algebra. The Pythagorean theorem, for example, does not appear in any form in surviving documents from Egypt, but tablets even from the Old Babylonian period show that in Mesopotamia the theorem was widely used. A cuneiform text from the Yale Collection, for example, contains a diagram of a square and its diagonals in which the number 30 is written along one side and the numbers 42;25,35 and 1;24,51,10 appear along a diagonal. The last number obviously is the ratio of the lengths of the diagonal and a side, and this is so accurately expressed that it agrees with $\sqrt{2}$ to within about a millionth. The accuracy of the result was made possible by knowledge of the Pythagorean theorem. Sometimes, in less precise computations, the Babylonians used 1;25 as a rough-and-ready approximation to this ratio. Of more significance than the precision of the values, however, is the implication that the diagonal of any square could be found by multiplying the side by $\sqrt{2}$. Thus, there seems to have been some awareness of general principles, despite the fact that these are exclusively expressed in special cases.

Babylonian recognition of the Pythagorean theorem was by no means limited to the case of a right isosceles triangle. In one Old Babylonian problem text, a ladder or a beam of length 0;30 stands against a wall; the question is, how far will the lower end move out from the wall if the upper end slips down a distance of 0;6 units? The answer is correctly found by use of the Pythagorean theorem. Fifteen hundred years later, similar problems, some with new twists, were still being solved in the Mesopotamian Valley. A Seleucid tablet, for example, proposes the following problem. A reed stands against a wall. If the top slides down 3 units when the lower end slides away 9 units, how long is the reed? The answer is given correctly as 15 units.

Ancient cuneiform problem texts provide a wealth of exercises in what we might call geometry, but which the Babylonians probably thought of as applied arithmetic. A typical inheritance problem calls for the partition of a right-triangular property among six brothers. The area is given as 11,22,30 and one of the sides is 6,30; the dividing lines are to be equidistant and parallel to the other side of the triangle. One is asked to find the difference in the allotments. Another text gives the bases of an isosceles trapezoid as 50 and 40 units and the length of the sides as 30; the altitude and the area are required (van der Waerden 1963, pp. 76–77).

The ancient Babylonians were aware of other important geometric relationships. Like the Egyptians, they knew that the altitude in an isosceles triangle bisects the base. Hence, given the length of a chord in a circle of known radius, they were able to find the apothem. Unlike the Egyptians, they were familiar with the fact that an angle inscribed in a semicircle is a right angle, a proposition generally known as the Theorem of Thales, despite the fact that Thales lived more than a millennium after the Babylonians had begun to use it. This misnaming of a well-known theorem in geometry is symptomatic of the difficulty in assessing the influence of pre-Hellenic mathematics on later cultures. Cuneiform tablets had a permanence that could not be matched by documents from other civilizations, for papyrus and parchment do not so easily survive the ravages of time. Moreover, cuneiform texts continued to be recorded down to the dawn of the Christian era, but were they read by neighboring civilizations, especially the Greeks? The center of mathematical development was shifting from the Mesopotamian Valley to the Greek world half a dozen centuries before the beginning of our era, but reconstructions of early Greek mathematics are rendered hazardous by the fact that there are virtually no extant mathematical documents from the pre-Hellenistic period. It is important, therefore, to keep in mind the general characteristics of Egyptian and Babylonian mathematics so as to be able to make at least plausible conjectures concerning analogies that may be apparent between pre-Hellenic contributions and the activities and attitudes of later peoples.

There is a lack of explicit statements of rules and of clear-cut distinctions between exact and approximate results. The omission in the tables of cases involving irregular sexagesimals seems to imply some recognition of such distinctions, but neither the Egyptians nor the Babylonians appear to have raised the question of when the area of a quadrilateral (or of a circle) is found exactly and when only approximately. Questions about the solvability or unsolvability of a problem do not seem to have been raised, nor was there any investigation into the nature of proof. The word "proof" means various things at different levels and ages; hence, it is hazardous to assert categorically that pre-Hellenic peoples had no concept of proof, nor any feeling of the need for proof. There are hints that these people were occasionally aware that certain area and volume methods could be justified through a reduction to simpler area and volume problems. Moreover, pre-Hellenic scribes not infrequently checked or "proved" their divisions by multiplication; occasionally, they verified the procedure in a problem through a substitution that verified the correctness of the answer. Nevertheless, there are no explicit statements from the pre-Hellenic period that would indicate a felt need for proofs or a concern for questions of logical principles. In Mesopotamian problems, the words "length" and "width" should perhaps be interpreted much as we interpret the letters *x* and *y*, for the writers of cuneiform tablets may well have moved on from specific instances to general abstractions. How else does one explain the addition of a length to an area? In Egypt also, the use of the word for quantity is not incompatible with an abstract interpretation such as we read into it today. In addition, there were in Egypt and Babylonia problems that have the earmarks of recreational mathematics. If a problem calls for a sum of cats and measures of grain, or of a length and an area, one cannot deny to the perpetrator either a modicum of levity or a feeling for abstraction. Of course, much of pre-Hellenic mathematics was practical, but surely not all of it. In the practice of computation, which stretched over a couple of millennia, the schools of scribes used plenty of exercise material, often, perhaps, simply as good clean fun.

refer to this period as the "Heroic Age of Mathematics," for seldom, either before or since, have men with so little to work with tackled mathematical problems of such fundamental significance. No longer was mathematical activity centered almost entirely in two regions nearly at opposite ends of the Greek world; it flourished all around the Mediterranean. In what is now southern Italy, there were Archytas of Tarentum (born ca. 428 BCE) and Hippasus of Metapontum (fl. ca. 400 BCE); at Abdera in Thrace, we find Democritus (born ca. 460 BCE); nearer the center of the Greek world, on the Attic peninsula, there was Hippias of Elis (born ca. 460 BCE); and in nearby Athens, there lived at various times during the pivotal last half of the fifth century BCE three scholars from other regions: Hippocrates of Chios (fl. ca. 430 BCE), Anaxagoras of Clazomenae (fl. 428 BCE), and Zeno of Elea (fl. ca. 450 BCE). Through the work of these seven men, we shall describe the fundamental changes in mathematics that took place a little before the year 400 BCE. Again, we must remember that although the histories of Herodotus and Thucydides and the plays of Aeschylus, Euripides, and Aristophanes have in some measure survived, scarcely a line is extant of what was written by mathematicians of the time.

Firsthand mathematical sources from the fourth century BCE are almost as scarce, but this inadequacy is made up for in large measure by accounts written by philosophers who were au courant with the mathematics of their day. We have most of what Plato wrote and about half of the work of Aristotle; with the writings of these intellectual leaders of the fourth century BCE as a guide, we can give a far more dependable account of what happened in their day than we could about the Heroic Age.

Thales and Pythagoras

Accounts of the origins of Greek mathematics center on the so-called Ionian and Pythagorean schools and the chief representative of each—Thales and Pythagoras—although, as just noted, reconstructions of their thought rest on fragmentary reports and traditions built up during later centuries. The Greek world had its center between the Aegean and Ionian seas for many centuries, but Hellenic civilization was far from localized there. By about 600 BCE, Greek settlements were scattered along the borders of most of the Black Sea and the Mediterranean Sea, and it was in these outskirts that a new surge in mathematics developed. In this respect, the sea-bordering colonists, especially in Ionia, had two advantages: they had the bold and imaginative spirit typical of pioneers, and they were in closer proximity to the two chief river valleys where knowledge thrived. Thales of Miletus (ca. 624–548 BCE) and Pythagoras of Samos (ca. 580–500 BCE) had a further advantage: they were in a position to travel to centers of ancient learning and there acquire firsthand information on astronomy and mathematics. In Egypt, they are said to have learned geometry; in Babylon, under the enlightened Chaldean ruler Nebuchadnezzar, Thales may have come in touch with astronomical tables and instruments. Tradition has it that in 585 BCE, Thales amazed his countrymen by predicting the solar eclipse of that year. The historicity of this tradition is very much open to question, however.

What is really known about the life and work of Thales is very little indeed. Ancient opinion is unanimous in regarding Thales as an unusually clever man and the first philosopher—by general agreement, the first of the Seven Wise Men. He was regarded as "a pupil of the Egyptians and the Chaldeans," an assumption that appears plausible. The proposition now known as the theorem of Thales—that an angle inscribed in a semicircle is a right angle—may well have been learned by Thales during his travels to Babylon. Tradition goes further, however, and attributes to him some sort of demonstration of the theorem. For this reason, Thales has frequently been hailed as the first true mathematician—as the originator of the deductive organization of geometry. This report, or legend, was embellished by adding to this theorem four others that Thales is said to have proved:

1. A circle is bisected by a diameter.
2. The base angles of an isosceles triangle are equal.
3. The pairs of vertical angles formed by two intersecting lines are equal.
4. If two triangles are such that two angles and a side of one are equal, respectively, to two angles and a side of the other, then the triangles are congruent.

There is no document from antiquity that can be pointed to as evidence of this achievement, yet the tradition has been persistent. About the nearest one can come to reliable evidence on this point is derived from a source a thousand years after the time of Thales. A student of Aristotle's by the name of Eudemus of Rhodes (fl. ca. 320 BCE) wrote a history of mathematics. This has been lost, but before it disappeared, someone had summarized at least part of the history. The original of this summary has also been lost, but during the fifth century of our era, information from the summary was incorporated by the Neoplatonic philosopher Proclus (410–485) into the early pages of his *Commentary on the First Book of Euclid's Elements*.

Designations of Thales as the first mathematician largely hinge on the remarks of Proclus. Later in his *Commentary*, Proclus—again depending on Eudemus—attributes to Thales the four theorems mentioned previously. There are other scattered references to Thales in ancient sources, but most of these describe his more practical activities. They do not establish the bold conjecture that Thales created demonstrative geometry, but in any case, Thales is the first man in history to whom specific mathematical discoveries have been attributed.

That it was the Greeks who added the element of logical structure to geometry is virtually universally admitted today, but the big question remains whether this crucial step was taken by Thales or by others later—perhaps as much as two centuries later. On this point, we must suspend final judgment until there is additional evidence on the development of Greek mathematics.

Pythagoras is scarcely less controversial a figure than Thales, for he has been more thoroughly enmeshed in legend and apotheosis. Thales had been a man of practical affairs, but Pythagoras was a prophet and a mystic, born at Samos, one of the Dodecanese islands not far from Miletus, the birthplace of Thales. Although some accounts picture Pythagoras as having studied under Thales, this is rendered unlikely by the half-century difference in their ages. Some similarity in their interests can readily be accounted for by the fact that Pythagoras also traveled to Egypt and Babylon—possibly even to India. During his peregrinations, he evidently absorbed not only mathematical and astronomical information but also much religious lore. Pythagoras was, incidentally, virtually a contemporary of Buddha, Confucius, and Laozi (Lao-tzu); the century was a crucial time in the development of religion, as well as of mathematics. When Pythagoras returned to the Greek world, he settled at Croton on the southeastern coast of what is now Italy, but at that time was known as Magna Graecia. There he established a secret society that somewhat resembled an Orphic cult, except for its mathematical and philosophical basis.

That Pythagoras remains a very obscure figure is due in part to the loss of documents from that age. Several biographies of Pythagoras were written in antiquity, including one by Aristotle, but these have not survived. A further difficulty in clearly identifying the figure of Pythagoras lies in the fact that the order he established was communal as well as secret. Knowledge and property were held in common, hence attribution of discoveries was not to be made to a specific member of the school. It is best, consequently, not to speak of the work of Pythagoras, but rather of the contributions of the Pythagoreans, although in antiquity it was customary to give all credit to the master.

Perhaps the most striking characteristic of the Pythagorean order was the confidence it maintained in the pursuit of philosophical and mathematical studies as a moral basis for the conduct of life. The very words "philosophy" (or "love of wisdom") and "mathematics" (or "that which is learned") are supposed to have been coined by Pythagoras himself to describe his intellectual activities.

It is evident that the Pythagoreans played an important role in the history of mathematics. In Egypt and Mesopotamia, the elements of arithmetic and geometry were primarily exercises in the application of numerical procedures to specific problems, whether concerned with beer or pyramids or the inheritance of land; we find nothing resembling a philosophical discussion of principles. Thales is generally regarded as having made a beginning in this direction, although tradition supports the view of Eudemus and Proclus that the new emphasis in mathematics was due primarily to the Pythagoreans. With them, mathematics was more closely related to a love of wisdom than to the exigencies of practical life. That Pythagoras was one of the most influential figures in history is difficult to deny, for his followers, whether deluded or inspired, spread their beliefs throughout most of the Greek world. The harmonies and mysteries of philosophy and mathematics were essential parts of the Pythagorean rituals. Never before or since has mathematics played so large a role in life and religion as it did among the Pythagoreans.

The motto of the Pythagorean school is said to have been "All is number." Recalling that the Babylonians had attached numerical measures to things around them, from the motions of the heavens to the values of their slaves, we may perceive in the Pythagorean motto a strong Mesopotamian affinity. The very theorem to which the name of Pythagoras still clings quite likely was derived from the Babylonians. It has been suggested, as justification for calling it the Theorem of Pythagoras, that the Pythagoreans first provided a demonstration, but this conjecture cannot be verified. It is reasonable to assume that the earliest members of the Pythagorean school were familiar with geometric properties known to the Babylonians, but when the Eudemus-Proclus summary ascribes to them the construction of the "cosmic figures" (that is, the regular solids), there is room for doubt. The cube, the octahedron, and the dodecahedron could perhaps have been observed in crystals, such as those of pyrite (iron disulfide), but a scholium in Euclid's *Elements* XIII reports that the Pythagoreans knew only three of the regular polyhedra: the tetrahedron, the cube, and the dodecahedron. Familiarity with the last figure is rendered plausible by the discovery near Padua of an Etruscan dodecahedron of stone dating from before 500 BCE. It is not improbable, therefore, that even if the Pythagoreans did not know of the octahedron and the icosahedron, they knew of some of the properties of the regular pentagon. The figure of a five-pointed star (which is formed by drawing the five diagonals of a pentagonal face of a regular dodecahedron) is said to have been the special symbol of the Pythagorean school. The star pentagon had appeared earlier in Babylonian art, and it is possible that here, too, we find a connecting link between pre-Hellenic and Pythagorean mathematics.

One of the tantalizing questions in Pythagorean geometry concerns the construction of a pentagram or a star pentagon. If we begin with a regular polygon *ABCDE* (Fig. 4.1) and draw the five diagonals, these diagonals intersect in points $A'B'C'D'E'$, which form another regular pentagon.

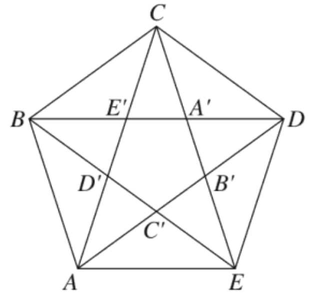


FIG. 4.1

Noting that the triangle BCD', for example, is similar to the isosceles triangle BCE, and noting also the many pairs of congruent triangles in the diagram, it is not difficult to see that the diagonal points $A'B'C'D'E'$ divide the diagonals in a striking manner. In each case, a diagonal point divides a diagonal into two unequal segments such that the ratio of the whole diagonal is to the larger segment as this segment is to the smaller segment. This subdivision of a diagonal is the well-known "golden section" of a line segment, but this name was not used until a couple of thousand years later—just about the time when Johannes Kepler wrote lyrically:

> Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.

To the ancient Greeks, this type of subdivision soon became so familiar that no need was felt for a special descriptive name; hence, the longer designation "the division of a segment in mean and extreme ratio" generally was replaced by the simple words "the section."

One important property of "the section" is that it is, so to speak, self-propagating. If a point P_1 divides a segment RS (Fig. 4.2) in mean and extreme ratio, with RP_1 the longer segment, and if on this larger segment we mark off a point P_2 such that $RP_2 = P_1S$, then segment RP_1 will in turn be subdivided in mean and extreme ratio at point P_2. Again, on marking off on RP_2 point P_3 such that $RP_3 = P_2P_1$, segment RP_2 will be divided in mean and extreme ratio at P_3. This iterative procedure can be carried out as many times as desired, the result being an ever smaller segment RP_n divided in mean and extreme ratio by point P_{n+1}. Whether the earlier Pythagoreans noticed this

FIG. 4.2

discipline as well as a technique, and a transition to such an outlook seems to have been nurtured in the Pythagorean school.

If tradition is to be trusted, the Pythagoreans not only established arithmetic as a branch of philosophy; they seem to have made it the basis of a unification of all aspects of the world around them. Through patterns of points, or unextended units, they associated number with geometric extension; this in turn led them to an arithmetic of the heavens. Philolaus (died ca. 390 BCE), a later Pythagorean who shared the veneration of the tetractys or decad, wrote that it was "great, all-powerful and all-producing, the beginning and the guide of the divine as of the terrestrial life." This view of the number 10 as the perfect number, the symbol of health and harmony, seems to have provided the inspiration for the earliest nongeocentric astronomical system. Philolaus postulated that at the center of the universe, there was a central fire about which the earth and the seven planets (including the sun and the moon) revolved uniformly. Inasmuch as this brought to only nine the number of heavenly bodies (other than the sphere of fixed stars), the Philolaic system assumed the existence of a tenth body—a "counterearth" collinear with the earth and the central fire—having the same period as the earth in its daily revolution about the central fire. The sun revolved about the fire once a year, and the fixed stars were stationary. The earth in its motion maintained the same uninhabited face toward the central fire, hence neither the fire nor the counterearth was ever seen. The postulate of uniform circular motion that the Pythagoreans adopted was to dominate astronomical thought for more than 2,000 years. Copernicus, almost 2,000 years later, accepted this assumption without question, and it was the Pythagoreans to whom Copernicus referred to show that his doctrine of a moving earth was not so new or revolutionary.

The thoroughness with which the Pythagoreans wove number into their thought is well illustrated by their concern for figurate numbers. Although no triangle can be formed by fewer than three points, it is possible to have triangles of a larger number of points, such as six, ten, or fifteen (see Fig. 4.4). Numbers such as 3, 6, 10, and 15 or, in general, numbers given by the formula

$$N = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

were called triangular, and the triangular pattern for the number 10, the holy tetractys, vied with the pentagon for veneration in Pythagorean number theory. There were, of course, indefinitely many other categories of privileged numbers. Successive square numbers are formed from the sequence $1 + 3 + 5 + 7 + \cdots + (2n - 1)$, where each odd number in turn was looked on as a pattern of dots resembling a gnomon (the Babylonian shadow clock) placed around two sides of the preceding square pattern of dots (see Fig. 4.4). Hence, the word "gnomon" (related to the word for "knowing") came to be attached to the odd numbers themselves.

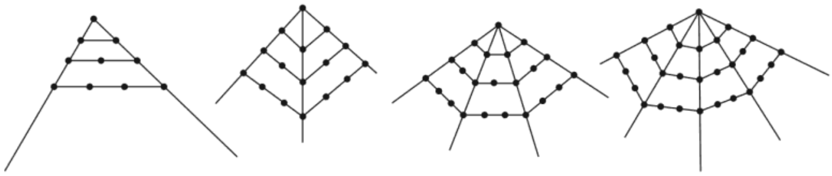


FIG. 4.4

The sequence of even numbers, $2+4+6+\cdots+2n=n(n+1)$, produces what the Greeks called "oblong numbers," each of which is double a triangular number. Pentagonal patterns of points illustrated the pentagonal numbers given by the sequence

$$N=1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$$

and hexagonal numbers were derived from the sequence

$$1+5+9+\cdots+(4n-3)=2n^2-n.$$

In a similar manner, polygonal numbers of all orders are designated; the process, of course, is easily extended to three-dimensional space, where one deals with polyhedral numbers. Emboldened by such views, Philolaus is reported to have maintained that

> All things which can be known have number; for it is not possible that without number anything can be either conceived or known.

The dictum of Philolaus seems to have been a tenet of the Pythagorean school; hence, stories arose about the discovery by Pythagoras of some simple laws of music. Pythagoras is reputed to have noticed that when the lengths of vibrating strings are expressible as ratios of simple whole numbers, such as 2 to 3 (for the fifth) or as 3 to 4 (for the fourth), the tones will be harmonious. If, in other words, a string sounds the note C when plucked, then a similar string twice as long will sound the note C an octave below, and tones between these two notes are emitted by strings whose lengths are given by intermediate ratios: 16 : 9 for D, 8 : 5 for E, 3 : 2 for F, 4 : 3 for G, 6 : 5 for A, and 16 : 15 for B, in ascending order. Here we have perhaps the earliest quantitative laws of acoustics—possibly the oldest of all quantitative physical laws. So boldly imaginative were the early Pythagoreans that they hastily extrapolated to conclude that the heavenly bodies in their motions similarly emitted harmonious tones, the "harmony of the spheres." Pythagorean science, like Pythagorean mathematics, seems to have been an odd congeries of sober thought and fanciful speculation. The doctrine of a spherical earth is often ascribed to

Pythagoras, but it is not known whether this conclusion was based on observation (perhaps of new constellations as Pythagoras traveled southward) or on imagination. The very idea that the universe is a "cosmos," or a harmoniously ordered whole, seems to be a related Pythagorean contribution—one that at the time had little basis in direct observation but that has been enormously fruitful in the development of astronomy. As we smile at ancient number fancies, we should at the same time be aware of the impulse these gave to the development of both mathematics and science. The Pythagoreans were among the earliest people to believe that the operations of nature could be understood through mathematics.

Proportions

Proclus, quoting perhaps from Eudemus, ascribed to Pythagoras two specific mathematical discoveries: (1) the construction of the regular solids and (2) the theory of proportionals. Although there is question about the extent to which this is to be taken literally, there is every likelihood that the statement correctly reflects the direction of Pythagorean thought. The theory of proportions clearly fits into the pattern of early Greek mathematical interests, and it is not difficult to find a likely source of inspiration. It is reported that Pythagoras learned in Mesopotamia of three means—the arithmetic, the geometric, and the subcontrary (later called the harmonic)—and of the "golden proportion" relating two of these: the first of two numbers is to their arithmetic mean as their harmonic mean is to the second of the numbers. This relationship is the essence of the Babylonian square-root algorithm; hence, the report is at least plausible. At some stage, however, the Pythagoreans generalized this work by adding seven new means to make ten in all. If b is the mean of a and c, where $a < c$, then the three quantities are related according to one of the following ten equations:

$$(1)\ \frac{b-a}{c-b} = \frac{a}{a} \qquad (6)\ \frac{b-a}{c-b} = \frac{c}{b}$$

$$(2)\ \frac{b-a}{c-b} = \frac{a}{b} \qquad (7)\ \frac{c-a}{b-a} = \frac{c}{a}$$

$$(3)\ \frac{b-a}{c-b} = \frac{a}{c} \qquad (8)\ \frac{c-a}{c-b} = \frac{c}{a}$$

$$(4)\ \frac{b-a}{c-b} = \frac{c}{a} \qquad (9)\ \frac{c-a}{b-a} = \frac{b}{a}$$

$$(5)\ \frac{b-a}{c-b} = \frac{b}{a} \qquad (10)\ \frac{c-a}{c-b} = \frac{b}{a}.$$

The first three equations are, of course, the equations for the arithmetic, the geometric, and the harmonic means, respectively.

It is difficult to assign a date to the Pythagorean study of means, and similar problems arise with respect to the classification of numbers. The study of proportions or the equality of ratios presumably formed at first a part of Pythagorean arithmetic or theory of numbers. Later, the quantities a, b, and c entering in such proportions were more likely to be regarded as geometric magnitudes, but the period in which the change took place is not clear. In addition to the polygonal numbers mentioned previously and the distinction between odd and even, the Pythagoreans at some stage spoke of odd-odd and even-odd numbers, based on whether the number in question was the product of two odd numbers or of an odd and an even number, so that sometimes the name "even number" was reserved for integral powers of two. By the time of Philolaus, the distinction between prime and composite numbers seems to have become important. Speusippus, a nephew of Plato and his successor as head of the Academy, asserted that 10 was "perfect" for the Pythagoreans because, among other things, it is the smallest integer n for which there are just as many primes between 1 and n as nonprimes. (Occasionally, prime numbers were called linear, inasmuch as they are usually represented by dots in one dimension only.) Neopythagoreans sometimes excluded 2 from the list of primes on the ground that 1 and 2 are not true numbers, but the generators of the odd and even numbers. The primacy of the odd numbers was assumed to be established by the fact that odd + odd is even, whereas even + even remains even.

To the Pythagoreans has been attributed the rule for Pythagorean triads given by $(m^2 - 1)/2$, m, $(m^2 + 1)/2$, where m is an odd integer, but inasmuch as this rule is so closely related to the Babylonian examples, it is perhaps not an independent discovery. Also ascribed to the Pythagoreans, with doubt as to the period in question, are the definitions of perfect, abundant, and deficient numbers, based on whether the sum of the proper divisors of the number is equal to, greater than, or less than the number itself. According to this definition, 6 is the smallest perfect number, with 28 next. That this view was probably a later development in Pythagorean thought is suggested by the early veneration of 10 rather than 6. Hence, the related doctrine of "amicable" numbers is also likely to have been a later notion. Two integers a and b are said to be "amicable" if a is the sum of the proper divisors of b and if b is the sum of the proper divisors of a. The smallest such pair are the integers 220 and 284.

Numeration

The Hellenes were celebrated as shrewd traders and businessmen, and there must have been a lower level of arithmetic or computation that satisfied the needs of the vast majority of Greek citizens. Number activities of this type would have been beneath the notice of philosophers, and recorded accounts of practical arithmetic were unlikely to find their way into the libraries of scholars. If, then, there are not even fragments surviving of the more sophisticated Pythagorean works, it is clear that it would be unreasonable to expect manuals of trade mathematics to survive the ravages of time. Hence, it is not possible to tell at this distance how the ordinary processes of arithmetic were carried out in Greece 2,500 years ago. About the best one can do is to describe the systems of numeration that appear to have been in use.

In general, there seem to have been two chief systems of numeration in Greece: one, probably the earlier, is known as the Attic (or Herodianic) notation; the other is called the Ionian (or alphabetic) system. Both systems are, for integers, based on the 10 scale, but the former is the more primitive, being based on a simple iterative scheme found in the earlier Egyptian hieroglyphic numeration and in the later Roman numerals. In the Attic system, the numbers from 1 to 4 were represented by repeated vertical strokes. For the number 5 a new symbol—the first letter Π (or Γ) of the word for five, "pente"—was adopted. (Only capital letters were used at the time, both in literary works and in mathematics, lowercase letters being an invention of the later ancient or early medieval period.) For numbers from 6 through 9, the Attic system combined the symbol Γ with unit strokes, so that 8, for example, was written as ΓIII. For positive integral powers of the base (10), the initial letters of the corresponding number words were adopted—Δ for deka (10), H for hekaton (100), X for khilioi (1,000), and M for myrioi (10,000). Except for the forms of the symbols, the Attic system is much like the Roman, but it had one advantage. Where the Latin word adopted distinctive symbols for 50 and 500, the Greeks wrote these numbers by combining letters for 5, 10, and 100, using 𐅄 (or 5 times 10) for 50, and 𐅅 (or 5 times 100) for 500. In the same way, they wrote 𐅆 for 5,000 and 𐅇 for 50,000. In Attic script, the number 45,678, for example, would appear as

MMMM𐅆𐅅H𐅄ΔΔΓIII

The Attic system of notation (also known as Herodianic, inasmuch as it was described in a fragment attributed to Herodian, a grammarian of the second century) appears in inscriptions at various dates from 454 to 95 BCE, but by the early Alexandrian Age, at about the time of Ptolemy Philadelphius, it was being displaced by the Ionian or alphabetic numerals. Similar alphabetic schemes were used at one time or another by various Semitic peoples, including the Hebrews, the Syrians, the Aramaeans, and the Arabs—as well as by other cultures, such as the Gothic—but these would seem to have been borrowed from the Greek notation. The Ionian effectively that they could calculate in terms of integral multiples of the subdivisions. This undoubtedly is the explanation for the popularity in antiquity of duodecimal and sexagesimal subdivisions, for the decimal system here is at a severe disadvantage. Decimal fractions were rarely used, either by the Greeks or by other Western peoples, before the period of the Renaissance. The abacus can be readily adapted to any system of numeration or to any combination of systems; it is likely that the widespread use of the abacus accounts at least in part for the amazingly late development of a consistent positional system of notation for integers and fractions. In this respect, the Pythagorean Age contributed little if anything.

The point of view of the Pythagoreans seems to have been so overwhelmingly philosophical and abstract that technical details in computation were relegated to a separate discipline, called logistic. This dealt with the numbering of things, rather than with the essence and properties of number as such, matters of concern in arithmetic. That is, the ancient Greeks made a clear distinction between mere calculation, on the one hand, and what today is known as the theory of numbers, on the other. Whether such a sharp distinction was a disadvantage to the historical development of mathematics may be a moot point, but it is not easy to deny to the early Ionian and Pythagorean mathematicians the primary role in establishing mathematics as a rational and liberal discipline. It is obvious that tradition can be quite inaccurate, but it is seldom entirely misdirected.

Fifth-Century Athens

The fifth century BCE was a crucial period in the history of Western civilization, for it opened with the defeat of the Persian invaders and closed with the surrender of Athens to Sparta. Between these two events lay the great Age of Pericles, with its accomplishments in literature and art. The prosperity and intellectual atmosphere of Athens during the century attracted scholars from all parts of the Greek world, and a synthesis of diverse aspects was achieved. From Ionia came men such as Anaxagoras, with a practical turn of mind; from southern Italy came others, such as Zeno, with stronger metaphysical inclinations. Democritus of Abdera espoused a materialistic view of the world, while Pythagoras in Italy held idealistic attitudes in science and philosophy. In Athens, one found eager devotees of old and new branches of learning, from cosmology to ethics. There was a bold spirit of free inquiry that sometimes came into conflict with established mores.

In particular, Anaxagoras was imprisoned in Athens for impiety in asserting that the sun was not a deity but a huge red-hot stone as big as the whole Peloponnesus, and that the moon was an inhabited earth that borrowed its light from the sun. He well represents the spirit of rational inquiry, for he regarded as the aim of his life the study of the nature of the universe—a purposefulness that he derived from the Ionian tradition of which Thales had been a founder. The intellectual enthusiasm of Anaxagoras was shared with his countrymen through the first scientific best-seller—a book *On Nature*—which could be bought in Athens for only a drachma. Anaxagoras was a teacher of Pericles, who saw to it that his mentor was ultimately released from prison. Socrates was at first attracted to the scientific ideas of Anaxagoras but found the naturalistic Ionian view less satisfying than the search for ethical verities. Greek science had been rooted in a highly intellectual curiosity that is often contrasted with the utilitarian immediacy of pre-Hellenic thought; Anaxagoras clearly represented the typical Greek motive—the desire to know. In mathematics also, the Greek attitude differed sharply from that of the earlier potamic cultures. The contrast was clear in the contributions generally attributed to Thales and Pythagoras, and it continues to show through in the more reliable reports about what went on in Athens during the Heroic Age. Anaxagoras was primarily a natural philosopher, rather than a mathematician, but his inquiring mind led him to share in the pursuit of mathematical problems.

Three Classical Problems

We are told by Plutarch that while Anaxagoras was in prison, he occupied himself with an attempt to square the circle. Here we have the first mention of a problem that was to fascinate mathematicians for more than 2,000 years. There are no further details concerning the origin of the problem or the rules governing it. At a later date, it came to be understood that the required square, exactly equal in area to the circle, was to be constructed by the use of a compass and a straightedge alone. Here we see a type of mathematics that is quite unlike that of the Egyptians and the Babylonians. It is not the practical application of a science of number to a facet of life experience, but a theoretical question involving a nice distinction between accuracy in approximation and exactitude in thought.

Anaxagoras died in 428 BCE, the year that Archytas was born, just one year before Plato's birth and one year after Pericles' death. It is said that Pericles died of the plague that carried off perhaps a quarter of the Athenian population, and the deep impression that this catastrophe created is perhaps the origin of a second famous mathematical problem. It is reported that a delegation had been sent to the oracle of Apollo at Delos to inquire how the plague could be averted, and the oracle had replied that the cubical altar to Apollo must be doubled. The Athenians are said to have dutifully doubled the dimensions of the altar, but this was of no avail in curbing the plague. The altar had, of course, been increased eightfold in volume, rather than twofold. Here, according to the legend, was the origin of the "duplication of the cube" problem, one that henceforth was usually referred to as the "Delian problem"—given the edge of a cube, construct with compasses and straightedge alone the edge of a second cube having double the volume of the first.

At about the same time, there circulated in Athens still a third celebrated problem: given an arbitrary angle, construct by means of compasses and straightedge alone an angle one-third as large as the given angle. These three problems—the squaring of the circle, the duplication of the cube, and the trisection of the angle—have since been known as the "three famous (or classical) problems" of antiquity. More than 2,200 years later, it was to be proved that all three of the problems were unsolvable by means of straightedge and compass alone. Nevertheless, the better part of Greek mathematics and of much later mathematical thought was suggested by efforts to achieve the impossible—or, failing this, to modify the rules. The Heroic Age failed in its immediate objective, under the rules, but the efforts were crowned with brilliant success in other respects.

Quadrature of Lunes

Somewhat younger than Anaxagoras and coming originally from about the same part of the Greek world was Hippocrates of Chios. He should not be confused with his still more celebrated contemporary, the physician Hippocrates of Cos. Both Cos and Chios are islands in the Dodecanese group, but in about 430 BCE, Hippocrates of Chios left his native land for Athens in his capacity as a merchant. Aristotle reported that Hippocrates was less shrewd than Thales and that he lost his money in Byzantium through fraud; others say that he was beset by pirates. In any case, the incident was never regretted by the victim, for he counted this his good fortune, in that as a consequence he turned to the study of geometry, in which he achieved remarkable success—a story typical of the Heroic Age. Proclus wrote that Hippocrates composed an "Elements of Geometry," anticipating by more than a century the better-known *Elements* of Euclid. Yet the textbook of Hippocrates—as well as another reported to have been written by Leon, a later associate of the Platonic school—has been lost, although it was known to Aristotle. In fact, no mathematical treatise from the fifth century has survived, but we do have a fragment concerning Hippocrates that Simplicius (fl. ca. 520 CE) claims to have copied literally from the *History of Mathematics* (now lost) by Eudemus. This brief statement, the nearest thing we have to an original source on the mathematics of the time, describes a portion of the work of Hippocrates dealing with the quadrature of lunes. A lune is a figure bounded by two circular arcs of unequal radii; the problem of the quadrature of lunes undoubtedly arose from that of squaring the circle. The Eudemian fragment attributes to Hippocrates the following theorem:

> Similar segments of circles are in the same ratio as the squares on their bases.

The Eudemian account reports that Hippocrates demonstrated this by first showing that the areas of two circles are to each other as the squares on their diameters. Here Hippocrates adopted the language and the concept of proportion that played so large a role in Pythagorean thought. In fact, it is thought by some that Hippocrates became a Pythagorean. The Pythagorean school in Croton had been suppressed (possibly because of its secrecy, perhaps because of its conservative political tendencies), but the scattering of its adherents throughout the Greek world served only to broaden the influence of the school. This influence undoubtedly was felt, directly or indirectly, by Hippocrates.

The theorem of Hippocrates on the areas of circles seems to be the earliest precise statement on curvilinear mensuration in the Greek world. Eudemus believed that Hippocrates gave a proof of the theorem, but a rigorous demonstration at that time (say, about 430 BCE) would appear to be unlikely. The theory of proportions at that stage probably was established only for commensurable magnitudes. The proof as given in Euclid XII.2 comes from Eudoxus, a man who lived halfway in time between Hippocrates and Euclid. Just as much of the material in the first two books of Euclid seems to stem from the Pythagoreans, however, so it would appear reasonable to assume that the formulations, at least, of much of Books III and IV of the *Elements* came from the work of Hippocrates. Moreover, if Hippocrates did give a demonstration of this theorem on the areas of circles, he may have been responsible for the introduction into mathematics of the indirect method of proof. That is, the ratio of the areas of two circles is equal to the ratio of the squares on the diameters or it is not. By a reductio ad absurdum from the second of the two possibilities, the proof of the only alternative is established.

From this theorem on the areas of circles, Hippocrates readily found the first rigorous quadrature of a curvilinear area in the history of mathematics. He began with a semicircle circumscribed about an isosceles right triangle, and on the base (hypotenuse) he constructed a segment similar to the circular segments on the sides of the right triangle (Fig. 4.5). Because the segments are to each other as squares on their bases and from the Pythagorean theorem as applied to the right triangle, the sum of the two small circular segments is equal to the larger circular segment. Hence, the difference between the semicircle on AC and the segment $ADCE$ equals triangle ABC. Therefore, the lune $ABCD$ is

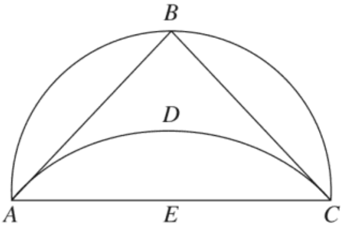


FIG. 4.5

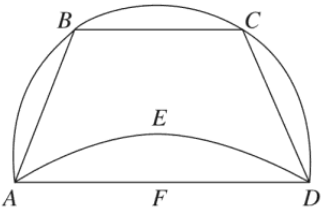


FIG. 4.6

precisely equal to triangle *ABC*, and because triangle *ABC* is equal to the square on half of *AC*, the quadrature of the lune has been found.

Eudemus also described a Hippocratean lune quadrature based on an isosceles trapezoid, *ABCD*, inscribed in a circle so that the square on the longest side (base), *AD*, is equal to the sum of the squares on the three equal shorter sides, *AB* and *BC* and *CD* (Fig. 4.6). Then, if on side *AD* one constructs a circular segment, *AEDF*, similar to those on the three equal sides, lune *ABCDE* is equal to trapezoid *ABCDF*.

That we are on relatively firm ground historically in describing the quadrature of lunes by Hippocrates is indicated by the fact that scholars other than Simplicius also refer to this work. Simplicius lived in the sixth century, but he depended not only on Eudemus (fl. ca. 320 BCE) but also on Alexander of Aphrodisias (fl. ca. 200 CE), one of the chief commentators on Aristotle. Alexander described two quadratures other than those given previously. (1) If on the hypotenuse and the sides of an isosceles right triangle one constructs semicircles (Fig. 4.7), then the lunes created on the smaller sides together equal the triangle. (2) If on a diameter of a semicircle one constructs an isosceles trapezoid with three equal sides (Fig. 4.8), and if on the three equal sides semicircles are constructed, then the trapezoid is equal in area to the sum of four curvilinear areas: the three equal lunes and a semicircle on one of the equal sides of the trapezoid. From the second of these quadratures, it would follow that if the lunes can be squared, the semicircle—hence, the circle—can also be squared. This conclusion seems to have encouraged

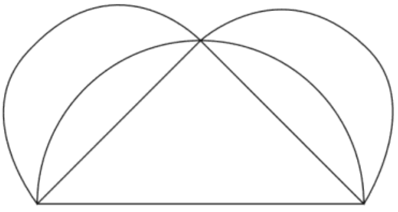


FIG. 4.7

The curve of Hippias is generally known as the quadratrix, because it can be used to square the circle. Whether Hippias himself was aware of this application cannot now be determined. It has been conjectured that Hippias knew of this method of quadrature but that he was unable to justify it. Since the quadrature through Hippias's curve was specifically given later by Dinostratus, we shall describe this work below.

Hippias lived at least as late as Socrates (d. 399 BCE), and from the pen of Plato we have an unflattering account of him as a typical Sophist—vain, boastful, and acquisitive. Socrates is reported to have described Hippias as handsome and learned but boastful and shallow. Plato's dialogue on Hippias satirizes his show of knowledge, and Xenophon's *Memorabilia* includes an unflattering account of Hippias as one who regarded himself an expert in everything from history and literature to handicrafts and science. In judging such accounts, however, we must remember that Plato and Xenophon were uncompromisingly opposed to the Sophists in general. It is also well to bear in mind that both Protagoras, the "founding father of the Sophists," and Socrates, the arch opponent of the movement, were antagonistic to mathematics and the sciences. With respect to character, Plato contrasts Hippias with Socrates, but one can bring out much the same contrast by comparing Hippias with another contemporary—the Pythagorean mathematician Archytas of Tarentum.

Philolaus and Archytas of Tarentum

Pythagoras is said to have retired to Metapontum toward the end of his life and to have died there about 500 BCE. Tradition holds that he left no written works, but his ideas were carried on by a large number of eager disciples. The center at Croton was abandoned when a rival political group from Sybaris surprised and murdered many of the leaders, but those who escaped the massacre carried the doctrines of the school to other parts of the Greek world. Among those who received instruction from the refugees was Philolaus of Tarentum, and he is said to have written the first account of Pythagoreanism—permission having been granted, so the story goes, to repair his damaged fortunes. Apparently, it was this book from which Plato derived his knowledge of the Pythagorean order. The number fanaticism that was so characteristic of the brotherhood evidently was shared by Philolaus, and it was from his account that much of the mystical lore concerning the tetractys was derived, as well as knowledge of the Pythagorean cosmology. The Philolaean cosmic scheme is said to have been modified by two later Pythagoreans, Ecphantus and Hicetas, who abandoned the central fire and the counterearth and explained day and night by placing a rotating earth at the center of the universe. The extremes of Philolaean number worship also seem to have undergone some modification, more especially at the hands of Archytas, a student of Philolaus's at Tarentum.

The Pythagorean sect had exerted a strong intellectual influence throughout Magna Graecia, with political overtones that may be described as a sort of "reactionary international," or perhaps better as a cross between Orphism and Freemasonry. At Croton, political aspects were especially noticeable, but at outlying Pythagorean centers, such as Tarentum, the impact was primarily intellectual. Archytas believed firmly in the efficacy of number; his rule of the city, which allotted him autocratic powers, was just and restrained, for he regarded reason as a force working toward social amelioration. For many years in succession, he was elected general, and he was never defeated, yet he was kind and a lover of children, for whom he is reported to have invented "Archytas's rattle." Possibly also the mechanical dove, which he is said to have fashioned of wood, was built to amuse the young folk.

Archytas continued the Pythagorean tradition in placing arithmetic above geometry, but his enthusiasm for number had less of the religious and mystical admixture found earlier in Philolaus. He wrote on the application of the arithmetic, geometric, and subcontrary means to music, and it was probably either Philolaus or Archytas who was responsible for changing the name of the last one to "harmonic mean." Among his statements in this connection was the observation that between two whole numbers in the ratio $n{:}(n+1)$, there could be no integer that is a geometric mean. Archytas gave more attention to music than had his predecessors, and he felt that this subject should play a greater role than literature in the education of children. Among his conjectures was one that attributed differences in pitch to varying rates of motion resulting from the flow that caused the sound. Archytas seems to have paid considerable attention to the role of mathematics in the curriculum, and to him has been ascribed the designation of the four branches in the mathematical quadrivium—arithmetic (or numbers at rest), geometry (or magnitudes at rest), music (or numbers in motion), and astronomy (or magnitudes in motion). These subjects, together with the trivium consisting of grammar, rhetoric, and dialectics (which Aristotle traced back to Zeno), later constituted the seven liberal arts; hence, the prominent role that mathematics has played in education is in no small measure due to Archytas.

It is likely that Archytas had access to an earlier treatise on the elements of mathematics, and the iterative square-root process often known as Archytas's had been used long before in Mesopotamia. Nevertheless, Archytas was a contributor of original mathematical results. The most striking contribution was a three-dimensional solution of the Delian problem, which may be most easily described, somewhat anachronistically, in the modern language of analytic geometry. Let a be the edge of the cube to be doubled, and let the point $(a, 0, 0)$ be the center of three mutually perpendicular circles of radius a and each lying in a plane perpendicular to a coordinate axis. Through the circle perpendicular to the x-axis, construct a right circular cone with vertex (0, 0, 0); through the circle in the xy-plane, pass a right circular cylinder; and let the circle in the xz-plane be revolved about the z-axis to generate a torus. The equations of these three surfaces are, respectively, $x^2 = y^2 + z^2$ and $2ax = x^2 + y^2$ and $(x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)$. These three surfaces intersect in a point whose x-coordinate is $a\sqrt[3]{12}$; hence, the length of this line segment is the edge of the cube desired.

The achievement of Archytas is the more impressive when we recall that his solution was worked out synthetically without the aid of coordinates. Nevertheless, Archytas's most important contribution to mathematics may have been his intervention with the tyrant Dionysius to save the life of his friend Plato. The latter remained to the end of his life deeply committed to the Pythagorean veneration of number and geometry, and the supremacy of Athens in the mathematical world of the fourth century BCE resulted primarily from the enthusiasm of Plato, the "maker of mathematicians." Before taking up the role of Plato in mathematics, however, it is necessary to discuss the work of an earlier Pythagorean—an apostate by the name of Hippasus.

Hippasus of Metapontum (or Croton), roughly contemporaneous with Philolaus, is reported to have originally been a Pythagorean but to have been expelled from the brotherhood. One account has it that the Pythagoreans erected a tombstone to him, as though he were dead; another story reports that his apostasy was punished by death at sea in a shipwreck. The exact cause of the break is unknown, in part because of the rule of secrecy, but there are three suggested possibilities. According to one, Hippasus was expelled for political insubordination, having headed a democratic movement against the conservative Pythagorean rule. A second tradition attributes the expulsion to disclosures concerning the geometry of the pentagon or the dodecahedron—perhaps a construction of one of the figures. A third explanation holds that the expulsion was coupled with the disclosure of a mathematical discovery of devastating significance for Pythagorean philosophy—the existence of incommensurable magnitudes.

Incommensurability

It had been a fundamental tenet of Pythagoreanism that the essence of all things, in geometry as well as in the practical and theoretical affairs of man, is explainable in terms of *arithmos*, or intrinsic properties of whole numbers or their ratios. The dialogues of Plato show, however, that the Greek mathematical community had been stunned by a disclosure that virtually demolished the basis for the Pythagorean faith in whole numbers. This was the discovery that within geometry itself, the whole numbers and their ratios are inadequate to account for even simple fundamental properties. They do not suffice, for example, to compare the diagonal of a square or a cube or a pentagon with its side. The line segments are incommensurable, no matter how small a unit of measure is chosen.

The circumstances surrounding the earliest recognition of incommensurable line segments are as uncertain as is the time of the discovery. Ordinarily, it is assumed that the recognition came in connection with the application of the Pythagorean theorem to the isosceles right triangle. Aristotle referred to a proof of the incommensurability of the diagonal of a square with respect to a side, indicating that it was based on the distinction between odd and even. Such a proof is easy to construct. Let d and s be the diagonal and the side of a square, and assume that they are commensurable—that is, that the ratio d / s is rational and equal to p / q, where p and q are integers with no common factor. Now, from the Pythagorean theorem it is known that $d^2 = s^2 + s^2$; hence, $(d / s)^2 = p^2 / q^2 = 2$, or $p^2 = 2q^2$. Therefore, p^2 must be even; hence, p must be even. Consequently, q must be odd. Letting $p = 2r$ and substituting in the equation $p^2 = 2q^2$, we have $4r^2 = 2q^2$, or $q^2 = 2r^2$. Then q^2 must be even; hence, q must be even. Yet q was previously shown to be odd, and an integer cannot be both odd and even. It follows, therefore, by the indirect method, that the assumption that d and s are commensurable must be false.

In this proof, the degree of abstraction is so high that the possibility that it was the basis for the original discovery of incommensurability has been questioned. There are, however, other ways in which the discovery could have come about. Among these is the simple observation that when the five diagonals of a regular pentagon are drawn, these diagonals form a smaller regular pentagon (Fig. 4.10), and the diagonals of the second pentagon in turn form a third regular pentagon, which is still smaller. This process can be continued indefinitely, resulting in pentagons that are as small as desired and leading to the conclusion that the ratio of a diagonal to a side in a regular pentagon is not rational. The irrationality of this ratio is, in fact, a consequence of the argument presented in connection with Fig. 4.2, in which the golden section was shown to repeat itself over and over again. Was it perhaps this property that led to the disclosure, possibly by Hippasus, of incommensurability? There is no surviving document to resolve the question, but the suggestion is at least a plausible one. In this case, it would not have been $\sqrt{2}$ but $\sqrt{5}$ that first disclosed the existence of incommensurable magnitudes, for the solution of the equation $a : x = x : (a - x)$ leads to $(\sqrt{5} - 1)/2$ as the ratio of the side of a regular pentagon to a diagonal. The ratio of the diagonal of a cube to an edge is $\sqrt{3}$, and here, too, the specter of the incommensurable rears its ugly head.

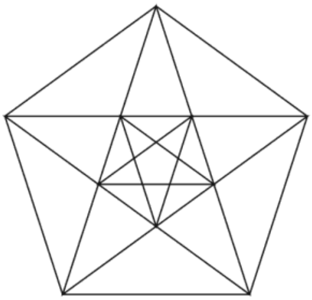


FIG. 4.10

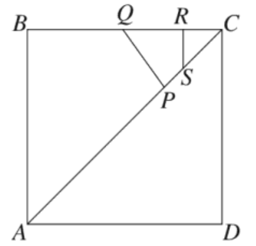


FIG. 4.11

A geometric proof somewhat analogous to that for the ratio of the diagonal of a pentagon to its side can also be provided for the ratio of the diagonal of a square to its side. If in the square *ABCD* (Fig. 4.11) one lays off on the diagonal *AC* the segment $AP = AB$ and at *P* erects the perpendicular *PQ*, the ratio of *CQ* to *PC* will be the same as the ratio of *AC* to *AB*. Again, if on *CQ* one lays off $QR = QP$ and constructs *RS* perpendicular to *CR*, the ratio of hypotenuse to side again will be what it was before. This process, too, can be continued indefinitely, thus affording a proof that no unit of length, however small, can be found so that the hypotenuse and a side will be commensurable.

Paradoxes of Zeno

The Pythagorean doctrine that "Numbers constitute the entire heaven" was now faced with a very serious problem indeed, but it was not the only one, for the school was also confronted with arguments propounded by the neighboring Eleatics, a rival philosophical movement. Ionian philosophers of Asia Minor had sought to identify a first principle for all things. Thales had thought to find this in water, but others preferred to think of air or fire as the basic element. The Pythagoreans had taken a more abstract direction, postulating that number in all of its plurality was the basic stuff behind phenomena; this numerical atomism, beautifully illustrated in the geometry of figurate numbers, had come under attack by the followers of Parmenides of Elea (fl. ca. 450 BCE). The fundamental tenet of the Eleatics was the unity and permanence of being, a view that contrasted with the Pythagorean ideas of multiplicity and change. Of Parmenides' disciples, the best known was Zeno the Eleatic (fl. ca. 450 BCE), who propounded arguments to prove the inconsistency in the concepts of multiplicity and divisibility. The method Zeno adopted was dialectical, anticipating Socrates in this

Deductive Reasoning

There are several conjectures as to the causes leading to the conversion of the mathematical prescriptions of pre-Hellenic peoples into the deductive structure that appears in Greece. Some have suggested that Thales in his travels had noted discrepancies in pre-Hellenic mathematics—such as the Egyptian and Babylonian rules for the area of a circle—and that he and his early successors therefore saw the need for a strict rational method. Others, more conservative, would place the deductive form much later—perhaps even as late as the early fourth century, following the discovery of the incommensurable. Other suggestions find the cause outside mathematics. One, for example is that deduction may have come out of logic, in attempts to convince an opponent of a conclusion by looking for premises from which the conclusion necessarily follows.

Whether deduction came into mathematics in the sixth century BCE or the fourth and whether incommensurability was discovered before or after 400 BCE, there can be no doubt that Greek mathematics had undergone drastic changes by the time of Plato. The dichotomy between number and continuous magnitude required a new approach to the Babylonian algebra that the Pythagoreans had inherited. The old problems in which, given the sum and the product of the sides of a rectangle, the dimensions were required had to be dealt with differently from the numerical algorithms of the Babylonians. A "geometric algebra" had to take the place of the older "arithmetic algebra," and in this new algebra there could be no adding of lines to areas or adding of areas to volumes. From now on, there had to be a strict homogeneity of terms in equations, and the Mesopotamian normal forms, $xy = A$, $x \pm y = b$, were to be interpreted geometrically. The obvious conclusion, which the reader can arrive at by eliminating y, is that one must construct on a given line b a rectangle whose unknown width x must be such that the area of the rectangle exceeds the given area A by the square x^2 or (in the case of the minus sign) falls short of the area A by the square x^2 (Fig. 4.12). In this way, the Greeks built up the solution of quadratic equations by their process known as "the application of areas," a portion of geometric algebra that is fully covered by Euclid's *Elements*. Moreover, the uneasiness resulting from incommensurable magnitudes led to an avoidance of ratios, insofar as possible, in elementary mathematics. The

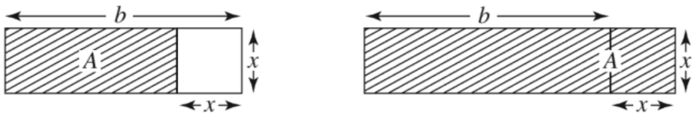


FIG. 4.12

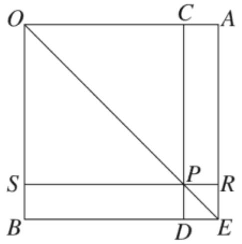


FIG. 4.13

linear equation $ax = bc$, for example, was looked on as an equality of the areas ax and bc, rather than as a proportion—an equality between the two ratios $a : b$ and $c : x$. Consequently, in constructing the fourth proportion, x in this case, it was usual to construct a rectangle *OCDB* with sides $b = OB$ and $c = OC$ (Fig. 4.13) and then along *OC* to lay off $OA = a$. One completes rectangle *OAEB* and draws the diagonal *OE* cutting *CD* at *P*. It is now clear that *CP* is the desired line x, for rectangle *OARS* is equal in area to rectangle *OCDB*. Not until Book V of the *Elements* did Euclid take up the difficult matter of proportionality.

Greek geometric algebra strikes the modern reader as excessively artificial and difficult; to those who used it and became adept at handling its operations, however, it probably appeared to be a convenient tool. The distributive law $a(b + c + d) = ab + ac + ad$ undoubtedly was far more obvious to a Greek scholar than to the beginning student of algebra today, for the former could easily picture the areas of the rectangles in this theorem, which simply says that the rectangle on a and the sum of segments b, c, d is equal to the sum of the rectangles on a and each of the lines b, c, d taken separately (Fig. 4.14). Again, the identity $(a + b)^2 = a^2 + 2ab + b^2$ becomes obvious from a diagram that shows the three squares and the two equal rectangles in the identity (Fig. 4.15); and a difference of two squares $a^2 - b^2 = (a + b)(a - b)$ can be pictured in a similar fashion (Fig. 4.16). Sums, differences, products, and quotients of

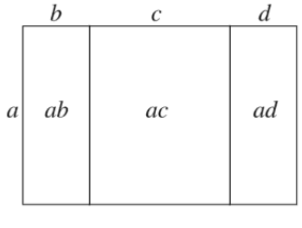


FIG. 4.14

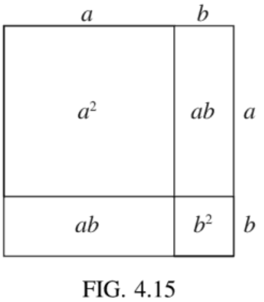


FIG. 4.15

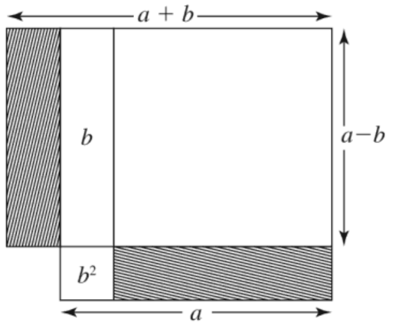


FIG. 4.16

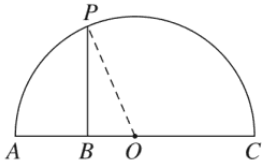


FIG. 4.17

line segments can easily be constructed with a straightedge and a compass. Square roots also afford no difficulty in geometric algebra. If one wishes to find a line x such that $x^2 = ab$, one simply follows the procedure found in elementary geometry textbooks today. One lays off on a straight line the segment ABC, where $AB = a$ and $BC = b$ (Fig. 4.17). With AC as the diameter, one constructs a semicircle (with center O) and at B erects the perpendicular BP, which is the segment x desired. It is interesting that here, too, the proof as given by Euclid, probably following the earlier avoidance of ratios, makes use of areas rather than proportions. If in our figure we let $PO = AO = CO = r$ and $BO = s$, Euclid would say essentially that $x^2 = r^2 - s^2 = (r - s)(r + s) = ab$.

Democritus of Abdera

The Heroic Age in mathematics produced half a dozen great figures, and among them must be included a man who is better known as a chemical philosopher. Democritus of Abdera (ca. 460−370 BCE) is today celebrated as a proponent of a materialistic atomic doctrine, but in his time he had also acquired a reputation as a geometer. He is reported to have traveled more widely than anyone of his day—to Athens, Egypt, Mesopotamia, and possibly India—acquiring what learning he could, but his own achievements in mathematics were such that he boasted that not even the "rope-stretchers" in Egypt excelled him. He wrote a number of mathematical works, not one of which is extant today.

The key to the mathematics of Democritus is to be found in his physical doctrine of atomism. All phenomena were to be explained, he argued, in terms of indefinitely small and infinitely varied (in size and shape), impenetrably hard atoms moving about ceaselessly in empty space. The physical atomism of Leucippus and Democritus may have been suggested by the geometric atomism of the Pythagoreans, and it is not surprising that the mathematical problems with which Democritus was chiefly concerned were those that demand some sort of infinitesimal approach. The Egyptians, for example, were aware that the volume of a pyramid is one-third the product of the base and the altitude, but a proof of this fact almost certainly was beyond their capabilities, for it requires a point of view equivalent to the calculus. Archimedes later wrote that this result was due to Democritus but that the latter did not prove it rigorously. This creates a puzzle, for if Democritus added anything to the Egyptian knowledge here, it must have been some sort of demonstration, albeit inadequate. Perhaps Democritus showed that a triangular prism can be divided into three triangular pyramids that are equal in height and area of the base and then deduced, from the assumption that pyramids of the same height and equal bases are equal, the familiar Egyptian theorem.

This assumption can be justified only by the application of infinitesimal techniques. If, for example, one thinks of two pyramids of equal bases and the same height as composed of indefinitely many infinitely thin equal cross-sections in one-to-one correspondence (a device usually known as Cavalieri's principle, in deference to the seventeenth-century geometer), the assumption appears to be justified. Such a fuzzy geometric atomism might have been at the base of Democritus's thought, although this has not been established. In any case, following the paradoxes of Zeno and the awareness of incommensurables, such arguments based on an infinity of infinitesimals were not acceptable. Archimedes consequently could well hold that Democritus had not given a rigorous proof. The same judgment would be true with respect to the theorem, also attributed by Archimedes to Democritus, that the volume of a cone is one-third the volume of the circumscribing cylinder. This result was probably looked on by Democritus as a corollary to the theorem on the pyramid, for the cone is essentially a pyramid whose base is a regular polygon of infinitely many sides.

Democritean geometric atomism was immediately confronted with certain problems. If the pyramid or the cone, for example, is made up of indefinitely many infinitely thin triangular or circular sections parallel to the base, a consideration of any two adjacent laminae creates a paradox. If the adjacent sections are equal in area, then, because all sections are equal, the totality will be a prism or a cylinder and not a pyramid or a cone. If, on the other hand, adjacent sections are unequal, the totality will be a step pyramid or a step cone and not the smooth-surfaced figure one has in mind. This problem is not unlike the difficulties with the incommensurable and with the paradoxes of motion. Perhaps, in his *On the Irrational*, Democritus analyzed the difficulties here encountered, but there is no way of knowing what direction his attempts may have taken. His extreme unpopularity in the two dominant philosophical schools of the next century, those of Plato and Aristotle, may have encouraged the disregard of Democritean ideas. Nevertheless, the chief mathematical legacy of the Heroic Age can be summed up in six problems: the squaring of the circle, the duplication of the cube, the trisection of the angle, the ratio of incommensurable magnitudes, the paradoxes on motion, and the validity of infinitesimal methods. To some extent, these can be associated, although not exclusively, with men considered in this chapter: Hippocrates, Archytas, Hippias, Hippasus, Zeno, and Democritus. Other ages were to produce a comparable array of talent, but perhaps never again was any age to make so bold an attack on so many fundamental mathematical problems with such inadequate methodological resources. It is for this reason that we have called the period from Anaxagoras to Archytas the Heroic Age.

Mathematics and the Liberal Arts

We included Archytas among the mathematicians of the Heroic Age, but in a sense he really is a transition figure in mathematics during Plato's time. Archytas was among the last of the Pythagoreans, both literally and figuratively. He could still believe that number was all-important in life and in mathematics, but the wave of the future was to elevate geometry to the ascendancy, largely because of the problem of incommensurability. On the other hand, Archytas is reported to have established the quadrivium—arithmetic, geometry, music, and astronomy—as the core of a liberal education, and here his views were to dominate much of pedagogical thought to our day. The seven liberal arts, which remained a shibboleth for almost two millennia, were made up of Archytas's quadrivium and the trivium of grammar, rhetoric, and Zeno's dialectic. Consequently, one may with some justice hold that the mathematicians of the Heroic Age were responsible for much of the direction in Western educational traditions, especially as transmitted through the philosophers of the fourth century BCE.

The Academy

The fourth century BCE had opened with the death of Socrates, a scholar who adopted the dialectic method of Zeno and repudiated the Pythagoreanism of Archytas. Socrates admitted that in his youth, he had been attracted by such questions as why the sum $2+2$ was the same as the product 2×2, as well as by the natural philosophy of Anaxagoras, but on realizing that neither mathematics nor science could satisfy his desire to know the essence of things, he gave himself up to his characteristic search for the good.

The dialogue that Plato composed in memory of his friend Theaetetus contains information on another mathematician whom Plato admired and who contributed to the early development of the theory of incommensurable magnitudes. Reporting on the then recent discovery of what we call the irrationality of $\sqrt{2}$, Plato in the *Theaetetus* says that his teacher, Theodorus of Cyrene—of whom Theaetetus was also a pupil—was the first to prove the irrationality of the square roots of the nonsquare integers from 3 to 17 inclusive. It is not known how he did this or why he stopped with $\sqrt{17}$. The proof, in any case, would have been constructed along the lines of that for $\sqrt{2}$ as given by Aristotle and interpolated in later versions of Book X of the *Elements*. References in ancient historical works indicate that Theodorus made discoveries in elementary geometry that later were incorporated into Euclid's *Elements*, but the works of Theodorus are lost.

Plato is important in the history of mathematics largely for his role as inspirer and director of others, and perhaps to him is due the sharp distinction in ancient Greece between arithmetic (in the sense of the theory of numbers) and logistic (the technique of computation). Plato regarded logistic as appropriate for the businessman and for the man of war, who "must learn the art of numbers or he will not know how to array his troops." The philosopher, on the other hand, must be an arithmetician "because he has to arise out of the sea of change and lay hold of true being." Moreover, Plato says in the *Republic*, "Arithmetic has a very great and elevating effect, compelling the mind to reason about abstract number." So elevating are Plato's thoughts concerning numbers that they reach the realm of mysticism and apparent fantasy. In the last book of the *Republic*, he refers to a number that he calls "the lord of better and worse births." There has been much speculation concerning this "Platonic number," and one theory is that it is the number $60^4 = 12{,}960{,}000$—important in Babylonian numerology and possibly transmitted to Plato through the Pythagoreans. In the *Laws*, the number of citizens in the ideal state is given as 5040 (that is, $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$). This is sometimes referred to as the Platonic nuptial number, and various theories have been advanced to suggest what Plato had in mind.

As in arithmetic, where Plato saw a gulf separating the theoretical and the computational aspects, so also in geometry he espoused the cause of pure mathematics as against the materialistic views of the artisan or the technician. Plutarch, in his *Life of Marcellus*, speaks of Plato's indignation at the use of mechanical contrivances in geometry. Apparently, Plato regarded such use as "the mere corruption and annihilation of the one good of geometry, which was thus shamefully turning its back upon the unembodied objects of pure intelligence." Plato may consequently have been largely responsible for the prevalent restriction in Greek geometric constructions to those that can be effected by straightedge and compasses alone. The reason for the limitation is not likely to have been the simplicity of the instruments used in constructing lines and circles, but rather the symmetry of the configurations. Any one of the infinitely many diameters of a circle is a line of symmetry of the figure; any point on an infinitely extended straight line can be thought of as a center of symmetry, just as any line perpendicular to the given line is a line with respect to which the given line is symmetric. Platonic philosophy, with its apotheosization of ideas, would quite naturally find a favored role for the line and the circle among geometric figures. In a somewhat similar manner, Plato glorified the triangle. The faces of the five regular solids in Plato's view were not simple triangles, squares, and pentagons. Each of the four faces of the tetrahedron, for example, is made up of six smaller right triangles, formed by altitudes of the equilateral triangular faces. The regular tetrahedron he therefore thought of as made up of twenty-four scalene right triangles in which the hypotenuse is double one side; the regular octahedron contains 8×6 or 48 such triangles, and the icosahedron is made up of 20×6 or 120 triangles. In a similar way the hexahedron (or cube) is constructed of twenty-four isosceles right triangles, for each of the six square faces contains four right triangles when the diagonals of the squares are drawn.

To the dodecahedron, Plato had assigned a special role as representative of the universe, cryptically saying that "God used it for the whole" (*Timaeus* 55C). Plato looked on the dodecahedron as composed of 360 scalene right triangles, for when the five diagonals and the five medians are drawn in each of the pentagonal faces, each of the twelve faces will contain thirty right triangles. The association of the first four regular solids with the traditional four universal elements provided Plato in the *Timaeus* with a beautifully unified theory of matter, according to which everything was constructed of ideal right triangles. The whole of physiology, as well as the sciences of inert matter, is based in the *Timaeus* on these triangles.

Pythagoras is reputed to have established mathematics as a liberal subject, but Plato was influential in making the subject an essential part of the curriculum for the education of statesmen. Influenced perhaps by Archytas, Plato would add to the original subjects in the quadrivium a new subject, stereometry, for he believed that solid geometry had not been sufficiently emphasized. Plato also discussed the foundations of mathematics, clarified some of the definitions, and reorganized the assumptions. He emphasized that the reasoning used in geometry does not refer to the visible figures that are drawn but to the absolute ideas that they represent. The Pythagoreans had defined a point as "unity having position," but Plato would rather think of it as the beginning of a line. The definition of a line as "breadthless length" seems to have originated in the school of Plato, as well as the idea that a line "lies evenly with the points on it." In arithmetic, Plato emphasized not only the distinction between odd and even numbers, but also the categories "even times even," "odd times even," and "odd times odd." Although we are told that Plato added to the axioms of mathematics, we do not have an account of his premises.

Few specific mathematical contributions are attributed to Plato. A formula for Pythagorean triples—$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2$, where n is any natural number—bears Plato's name, but this is merely a slightly modified version of a result known to the Babylonians and the Pythagoreans. Perhaps more genuinely significant is the ascription to Plato of the so-called analytic method. In demonstrative mathematics one begins with what is given, either generally in the axioms and the postulates or more specifically in the problems at hand. Proceeding step by step, one then arrives at the statement that was to have been proved. Plato seems to have pointed out that often it is pedagogically convenient, when a chain of reasoning from premises to conclusion is not obvious, to reverse the process. One might begin with the proposition that is to be proved and from it deduce a conclusion that is known to hold. If, then, one can reverse the steps in this chain of reasoning, the result is a legitimate proof of the proposition. It is unlikely that Plato was the first to note the efficacy in the analytic point of view, for any preliminary investigation of a problem is tantamount to this. What Plato is likely to have done is to formalize this procedure or perhaps to give it a name.

The role of Plato in the history of mathematics is still bitterly disputed. Some regard him as an exceptionally profound and incisive thinker; others picture him as a mathematical pied piper who lured men away from problems that concerned the world's work and who encouraged idle speculation. In any case, few would deny that Plato had a tremendous effect on the development of mathematics. The Platonic Academy in Athens became the mathematical center of the world, and it was from this school that the leading teachers and research workers came during the middle of the fourth century. Of these, the greatest was Eudoxus of Cnidus (408?–335? BCE), a man who was at one time a pupil of Plato and who became the most renowned mathematician and astronomer of his day.

Eudoxus

We sometimes read of the "Platonic reform" in mathematics, and although the phrase tends to exaggerate the changes taking place, the work of Eudoxus was so significant that the word "reform" is not inappropriate. In Plato's youth, the discovery of the incommensurable had caused a veritable logical scandal, for it had raised havoc with theorems involving proportions. Two quantities, such as the diagonal and the side of a square,

are incommensurable when they do not have a ratio such as a (whole) number has to a (whole) number. How, then, is one to compare ratios of incommensurable magnitudes? If Hippocrates really did prove that the areas of circles are to each other as squares on their diameters, he must have had some way of handling proportions or the equality of ratios. We do not know how he proceeded or whether to some extent he anticipated Eudoxus, who gave a new and generally accepted definition of equal ratios. Apparently, the Greeks had made use of the idea that four quantities are in proportion, $a:b=c:d$, if the two ratios $a:b$ and $c:d$ have the same mutual subtraction. That is, the smaller in each ratio can be laid off on the larger the same integral number of times, and the remainder in each case can be laid off on the smaller the same integral number of times, and the new remainder can be laid off on the former remainder the same integral number of times, and so on. Such a definition would be awkward to use, and it was a brilliant achievement of Eudoxus to discover the theory of proportion used in Book V of Euclid's *Elements*.

The word "ratio" essentially denoted an undefined concept in Greek mathematics, for Euclid's "definition" of ratio as a kind of relation in size between two magnitudes of the same type is quite inadequate. More significant is Euclid's statement that magnitudes are said to have a ratio to one another if a multiple of either can be found to exceed the other. This is essentially a statement of the so-called axiom of Archimedes—a property that Archimedes himself attributed to Eudoxus. The Eudoxian concept of ratio consequently excludes zero and clarifies what is meant by magnitudes of the same kind. A line segment, for example, is not to be compared, in terms of ratio, with an area; nor is an area to be compared with a volume.

Following these preliminary remarks on ratios, Euclid gives in Definition 5 of Book V the celebrated formulation by Eudoxus:

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or are alike less than, the latter equimultiples taken in corresponding order (Heath 1981, vol. 2, p. 114).

That is, $a/b=c/d$ if and only if given integers m and n, whenever $ma<nb$, then $mc<nd$, or if $ma=nb$, then $mc=nd$, or if $ma>nb$, then $mc>nd$.

The Eudoxian definition of equality of ratios is not unlike the process of cross-multiplication that is used today for fractions—$a/b=c/d$ according as $ad=bc$—a process equivalent to a reduction to a common denominator. To show that $\frac{3}{6}$ is equal to $\frac{4}{8}$, for example, we multiply 3 and 6 by 4, to obtain 12 and 24, and we multiply 4 and 8 by 3, obtaining the