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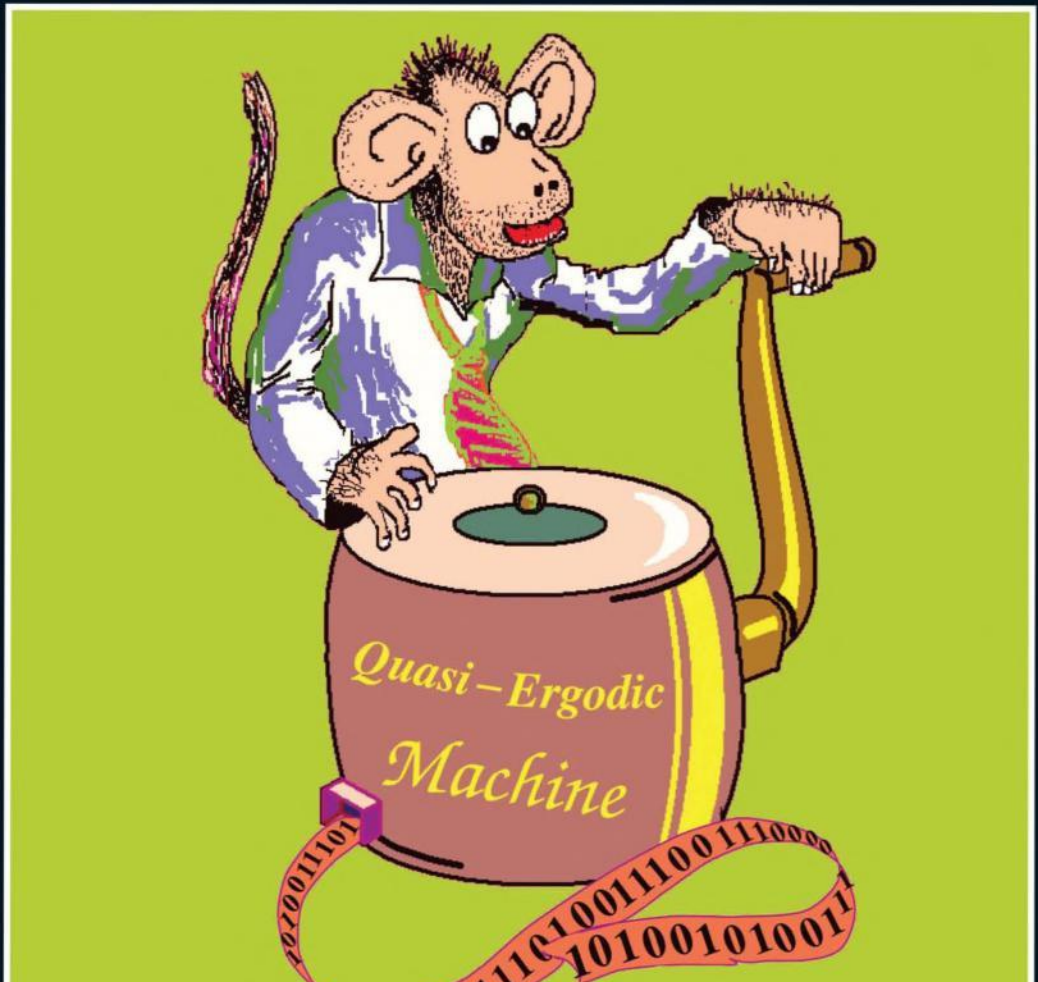
A Nonlinear Dynamics Perspective of Wolfram's New Kind of Science

Volume IV

Leon O Chua



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Volume IV

Leon O Chua
University of California at Berkeley, USA

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AN ODE TO THE UNKNOWABLE

Mirroring Dicken's *A Tale of Two Cities*, this volume depicts the extreme contrast between the *simplest*, and the *unknowable*, local rules of one-dimensional cellular automata. The simplest class (*Group 1*) consists of *67*, out of 256, local rules where almost all initial configurations tend to *period-1 attractors*. The unknowable class (*Group 5* and *Group 6*) embodies *40* local rules where almost all initial configurations tend to *unpredictable*, dubbed *quasi-ergodic*, *attractors* bearing the telltale fingerprints of *Gödel's incompleteness theorem*. While despairing over the human frailty to decipher God's forbidden secrets, CA affecionados can rejoice at the prospect of never running out of challenges of teasing out partial truths, however meager, hidden within the *18* globally-equivalent *unknowable local rules*, including *rule 137*, the prototypic *universal Turing machine*.

Chapter 1

QUASI-ERGODICITY

Our scientific odyssey through the theory of 1-D cellular automata is enriched by the definition of *quasi-ergodicity*, a new empirical property discovered by analyzing the time-1 return maps of local rules. *Quasi-ergodicity* plays a key role in the classification of rules into six groups: in fact, it is an exclusive characteristic of *complex* and *hyper Bernoulli-shift* rules. Besides introducing *quasi-ergodicity*, this paper answers several questions posed in the previous chapters of our quest. To start with, we offer a rigorous explanation of the fractal behavior of the time-1 characteristic functions, finding the equations that describe this phenomenon. Then, we propose a classification of rules according to the presence of *Isles of Eden*, and prove that only 28 local rules out of 256 do not have any of them; this result sheds light on the importance of *Isles of Eden*. A section of this paper is devoted to the characterization of *Bernoulli basin-tree* diagrams through modular arithmetic; the formulas obtained allow us to shorten drastically the number of cases to take into consideration during numerical simulations. Last but not least, we present some theorems about additive rules, including an analytical explanation of their scale-free property.

Keywords: Cellular automata; quasi-ergodicity; ergodicity; nonlinear dynamics; attractors; Isles of Eden; Bernoulli shift; shift maps; basin tree diagram; Bernoulli velocity; Bernoulli return time; complex Bernoulli shifts; hyper Bernoulli shifts; Binomial series; scale-free phenomena; Rule 45; Rule 60; Rule 90; Rule 105; Rule 150; Rule 154; additive rules; permutive rules; dissipative rules; conservative rules; fractals; basin-tree generation formula.

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1. Remembrance of Things Past

This exposition continues our saga on a nonlinear dynamics perspective of the 256 elementary cellular automata rules, as featured in eight *tutorial-review* papers: Part I [Chua *et al.*, 2002], Part II [Chua *et al.*, 2003], Part III [Chua *et al.*, 2004], Part IV [Chua *et al.*, 2005a], Part V [Chua *et al.*, 2005b], Part VI [Chua *et al.*, 2006], Part VII [Chua *et al.*, 2007a], and Part VIII [Chua *et al.*, 2007b].[1] In this paper, we examine the 18 yet *untamed* rules listed in Tables 11 and 12 of [Chua *et al.*, 2007a]; namely, the ten *complex Bernoulli-shift* rules $\{$ 18 , 22 , 54 ,

[1] Parts I to VI have been collected into two recent edited volumes [Chua, 2006] and [Chua, 2007], respectively. Part VII and VIII will appear in a future edited volume III.

73 , 90 , 105 , 122 , 126 , 146 , 150 } and the eight *hyper Bernoulli-shift* rules { 26 , 30 , 41 , 45 , 60 , 106 , 110 , 154 }.

Remarkably, we have observed *empirically* that *all complex* and *hyper* Bernoulli-shift rules exhibit an *Ergodic-like* dynamics, which we christened *quasi-ergodicity*. Our main goal of this paper is to describe and characterize this unifying *empirical* phenomenon.

We will revisit our fabled *Isles of Eden* from Parts VII and VIII and offer an alternate perspective of such rare gems. We will show that all local rules harbored a few precious Isles of Eden, except for 28 rules, which we will prove *analytically* to be devoid of *Isles of Eden*; these are the God forsaken rules!

We will also revisit the *scale-free phenomenon* reported in Parts VII and VIII for *additive* rules, and prove these empirical observations are in fact fundamental properties possessed by such rules. In particular, we will present and prove several *analytical* theorems for rules 60 , 90 , 105 and 150 .

For the reader's convenience, let us briefly review some highlights from our earlier adventures, henceforth referred to collectively as *Age*-1 *Episodes*, *a la* Tolkien's "*The Lord of the Rings*".

We are concerned exclusively with our tiny universe of 256 one-dimensional binary *cellular automata*, with a periodic boundary condition, as depicted in Fig. 1(a). Each "ring" has $L \triangleq I + 1$ *cells*, labeled consecutively from $i = 0$ to $i = I$. Each cell "i" has two states $x_i \in \{0, 1\}$, where we usually code the states "0" and "1" by the color "*blue*" and "*red*", respectively. A clock sets the pace in *discrete* times, dubbed "*iterations*" by the mathematical community, or "generations" by the life science community. The state x_i^{t+1} of all "i" at time $t + 1$ (i.e. the next generation) is determined by the state of its nearest neighbors x_{i-1}^t, and x_{i+1}^t, and itself x_i^t, at *time* t [Fig. 1(c)], in accordance with a prescribed Boolean truth table of *eight* distinct 3-input patterns [Fig. 1(d)].[2]

1.1. *Boolean cube representation*

We have found it extremely useful to map these eight 3-input patterns into the eight vertices of the "*cube*" shown in Fig. 1(b), henceforth called a *Boolean cube*. The rationale for identifying which vertex corresponds to which pattern was presented in [Chua *et al.*, 2002], in order to provide the *genesis* of the truth tables from a nonlinear physical system perspective, namely, *cellular neural networks* (CNN) [Chua, 1996], thereby providing a bridge between *nonlinear dynamics* and *cellular automata*. For readers who have not been exposed to the *age*-1 *episodes* alluded to above, it is not necessary to read the cited literature. Simply map the *output* of each prescribed Boolean function (i.e. the 8-bit, yet unspecified, binary string in Fig. 1(d)) onto the corresponding colors (*red* for 1, *blue* for 0) at the vertices of the Boolean cube. Since there are $2^8 = 256$ distinct combinations of eight bits, there are exactly 256 Boolean cubes with distinct vertex color combinations, one for each Boolean function, as displayed in Table 1.

1.2. *Index of complexity*

A careful examination of these 256 Boolean cubes shows that it is possible to separate, and *segregate*, all *red* vertices of each Boolean cube from the *blue* vertices by $\kappa = 1$, 2, or 3 *parallel* planes. An example illustrating this separation is shown in Fig. 2 for rules 170 , 110 and 184 , respectively. The integer κ is called the *index of complexity* of rule N . We will always use the color *red* for $\kappa = 1$, *blue* for $\kappa = 2$, and *green* for $\kappa = 3$ to code the rule number N of each of the 256 Boolean cubes, as printed at the bottom of each Boolean cube in Table 1.

It is natural to associate the 8-bit pattern of each Boolean function with a *decimal* number N representing the corresponding 8-bit word; namely,

$$N = \beta_7 \bullet 2^7 + \beta_6 \bullet 2^6 + \beta_5 \bullet 2^5 + \beta_4 \bullet 2^4 + \beta_3 \bullet 2^3 + \beta_2 \bullet 2^2 + \beta_1 \bullet 2^1 + \beta_0 \bullet 2^0, \quad \beta \in \{0, 1\}.$$

Observe that since $\beta_i = 0$ for each *blue* vertex in Fig. 1(b), N is simply obtained by adding the *weights* (indicated next to each pattern in Fig. 1(b)) associated with all *red* vertices. For example, for the Boolean cube shown in Fig. 3(b), we have

$$\begin{aligned} N &= 0 \bullet 2^7 + 1 \bullet 2^6 + 1 \bullet 2^5 + 0 \bullet 2^4 + 1 \bullet 2^3 \\ &\quad + 1 \bullet 2^2 + 1 \bullet 2^1 + 1 \bullet 2^0 \\ &= 2^6 + 2^5 + 2^3 + 2^2 + 2^1 \\ &= 110 \end{aligned}$$

[2]Throughout the paper we intentionally use both t and n to indicate the time (or iterations) to stress the equivalence between a discrete Cellular Automaton and a continuous nonlinear system.

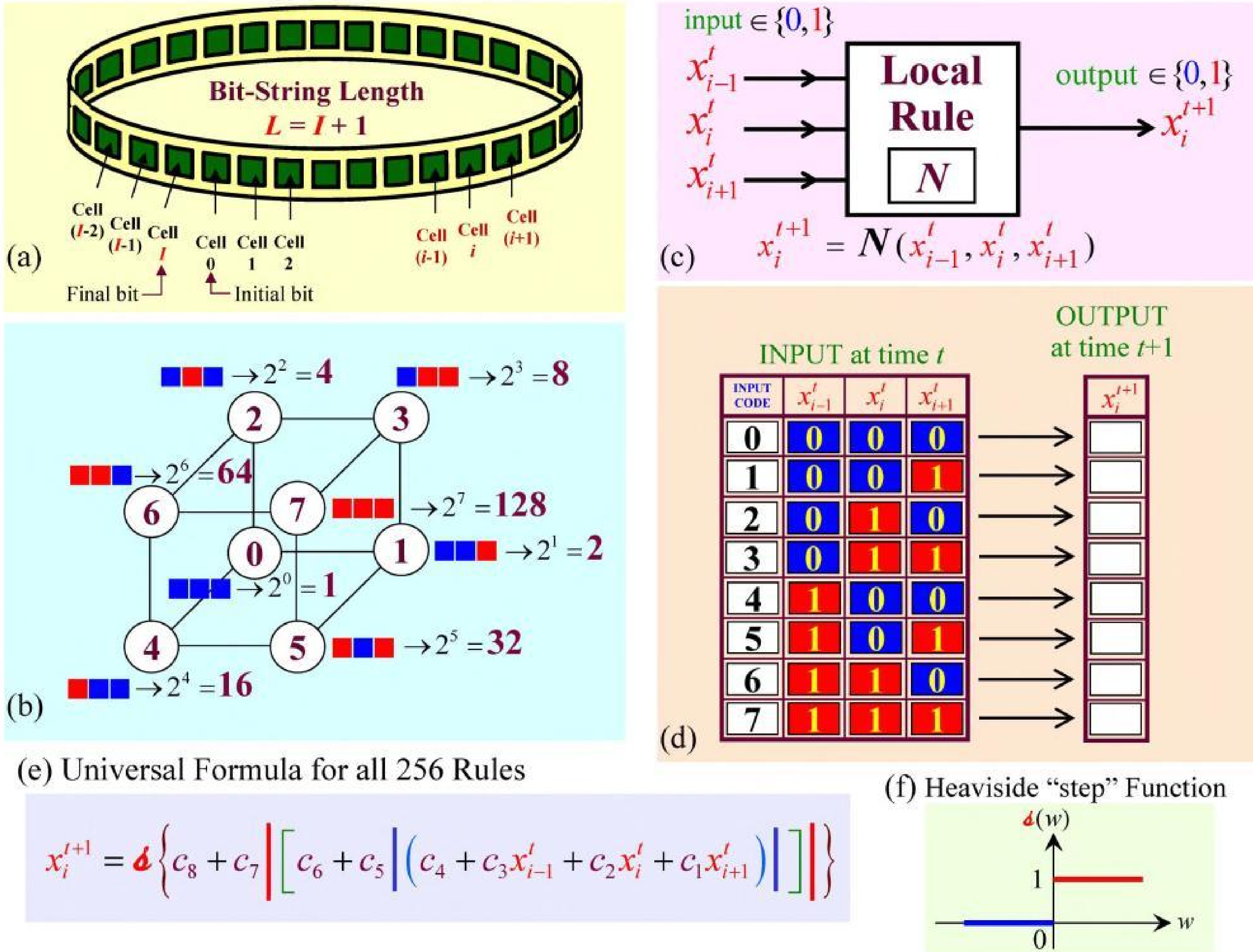


Fig. 1. Notations, symbols, and universal formula for local rule N .

Table 1. Boolean cubes defining 256 CA rules. Each vertex (k) codes for a 3-bit input pattern.

(0) ≜ 0 0 0 (1) ≜ 0 0 1 (2) ≜ 0 1 0 (3) ≜ 0 1 1

(4) ≜ 1 0 0 (5) ≜ 1 0 1 (6) ≜ 1 1 0 (7) ≜ 1 1 1

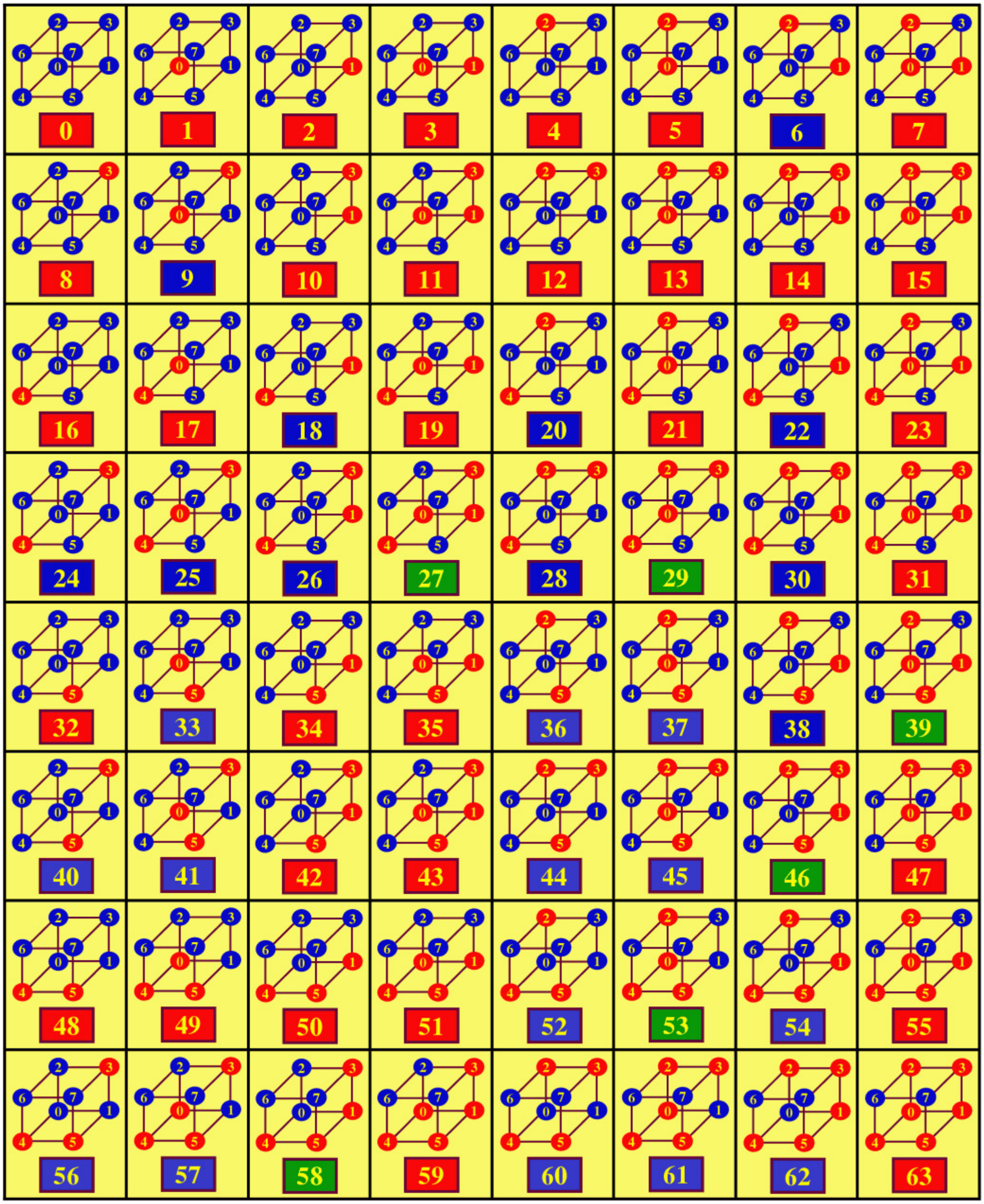


Table 1. (*Continued*)

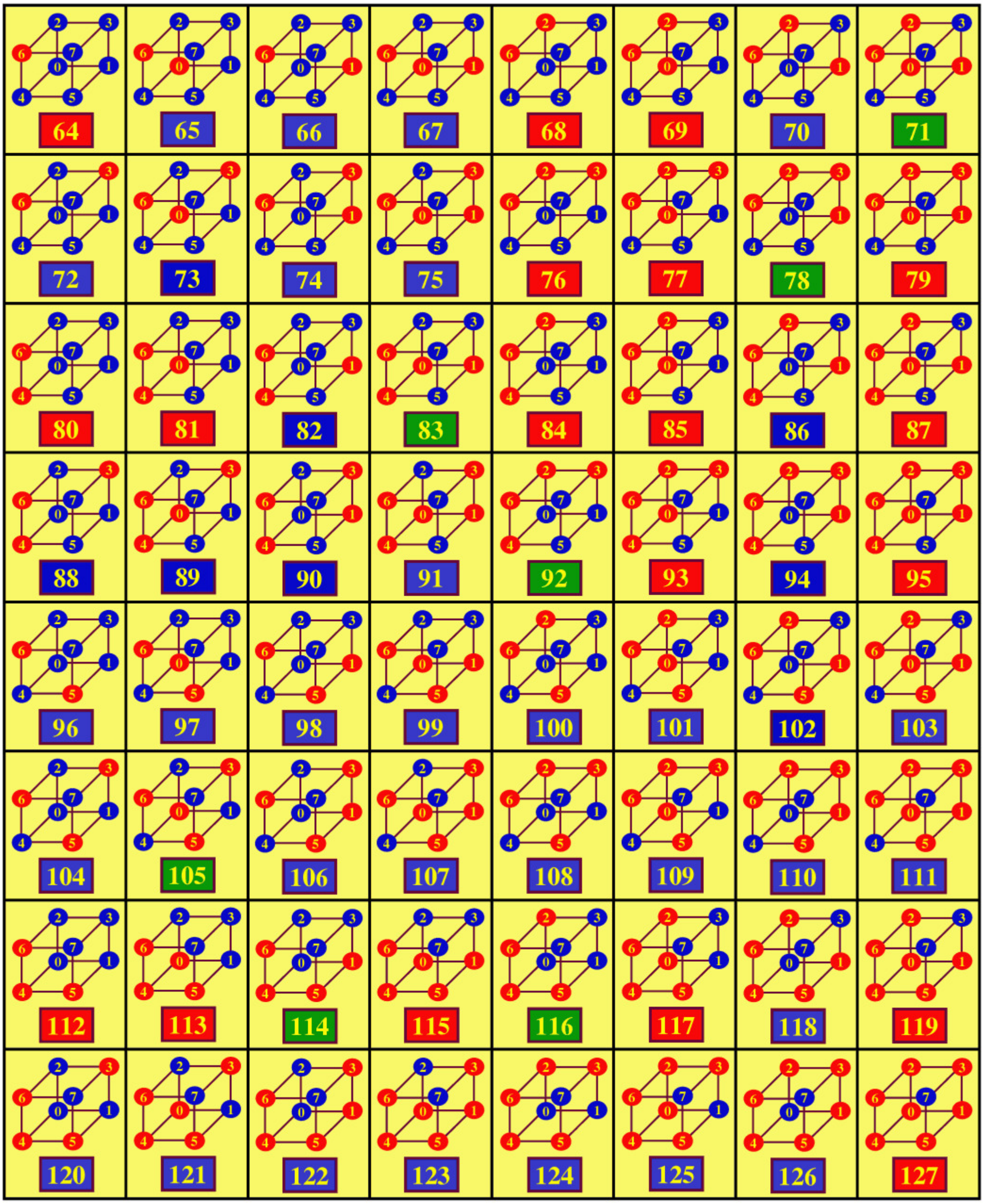
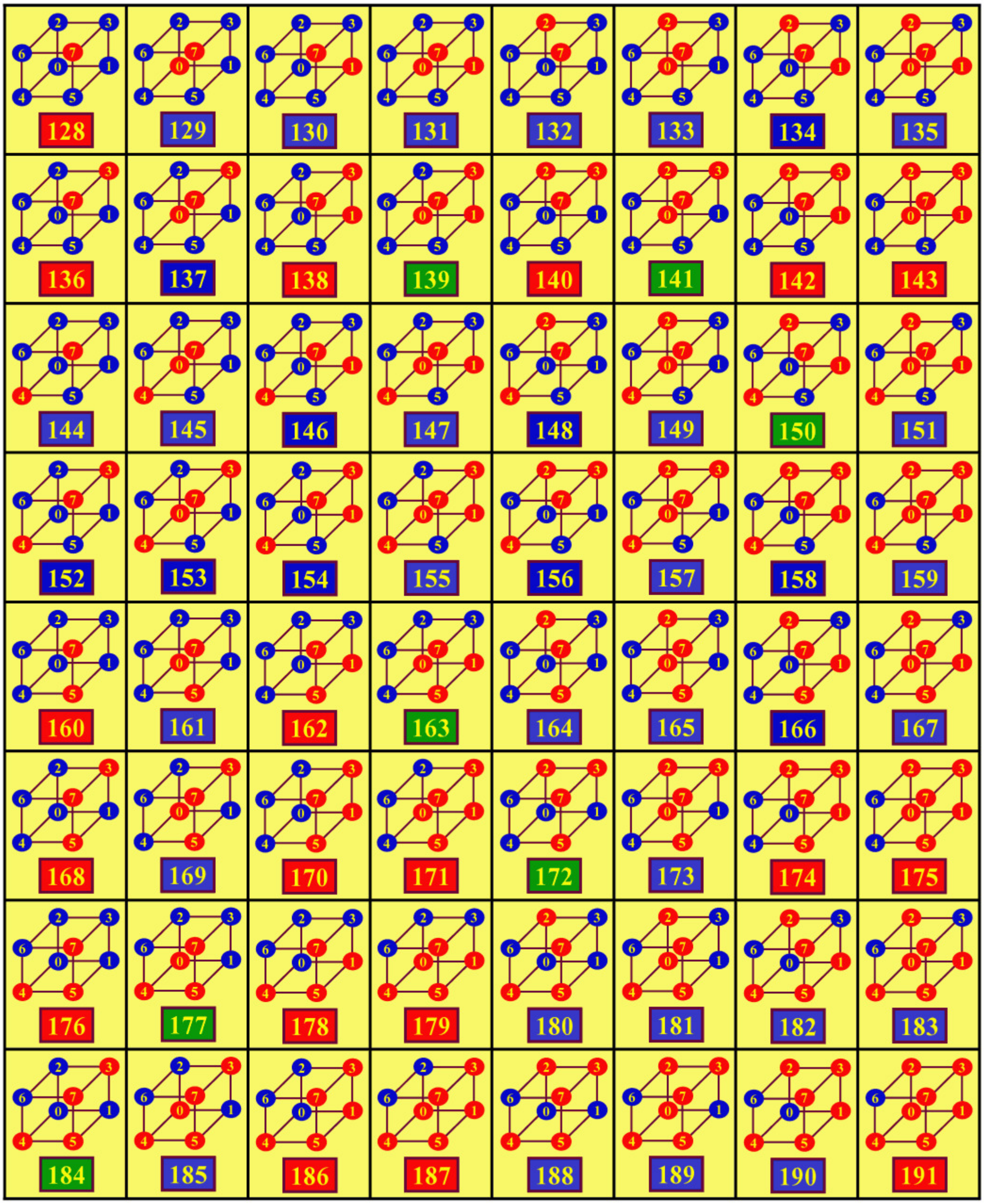


Table 1. (*Continued*)



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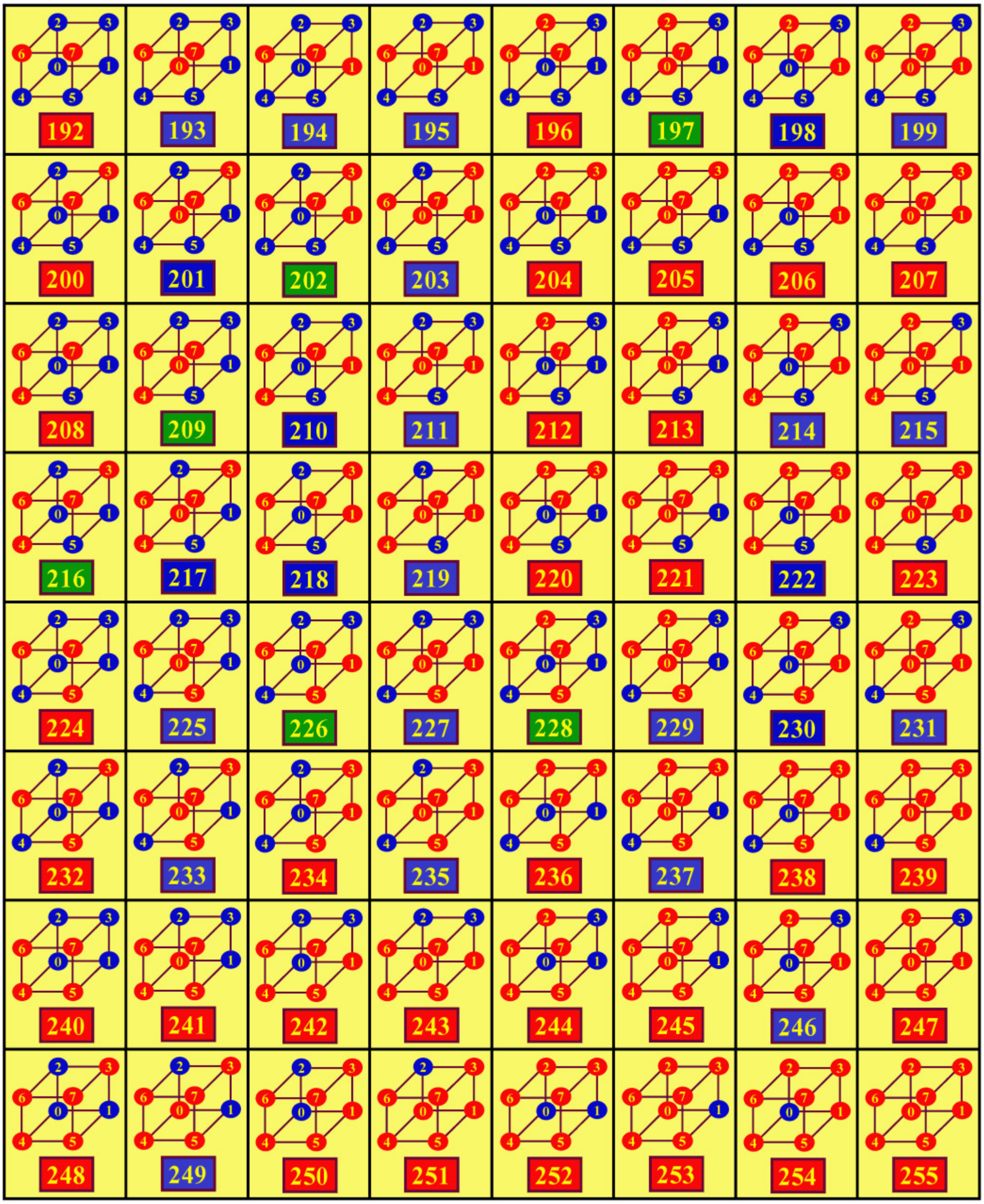
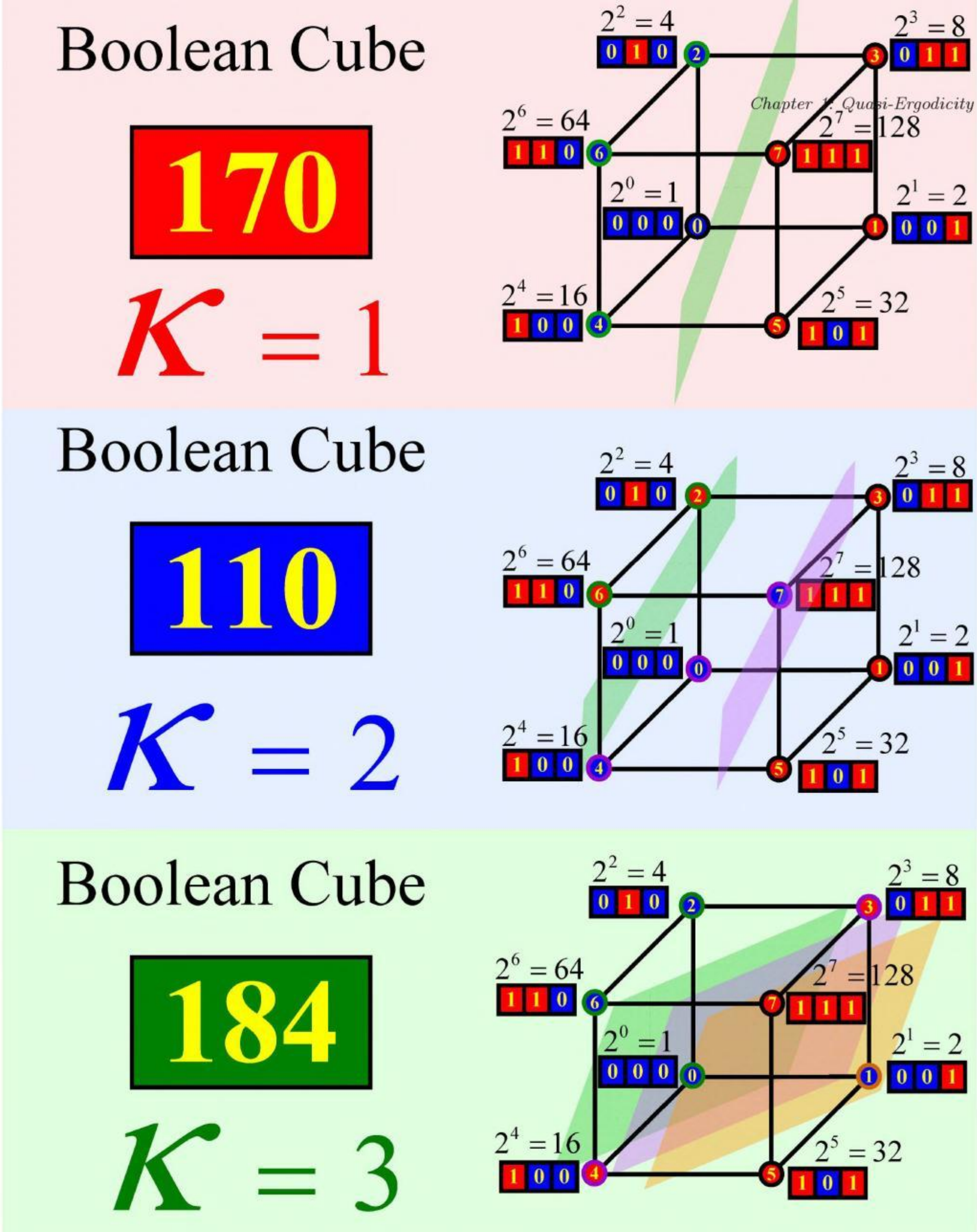


192	193	194	195	196	197	198	199
200	201	202	203	204	205	206	207
208	209	210	211	212	213	214	215
216	217	218	219	220	221	222	223
224	225	226	227	228	229	230	231
232	233	234	235	236	237	238	239
240	241	242	243	244	245	246	247
248	249	250	251	252	253	254	255



□

Fig. 2. The number κ of parallel planes which *separate* all vertices having one color from those having a different color on the other side is called the index of complexity of rule N .

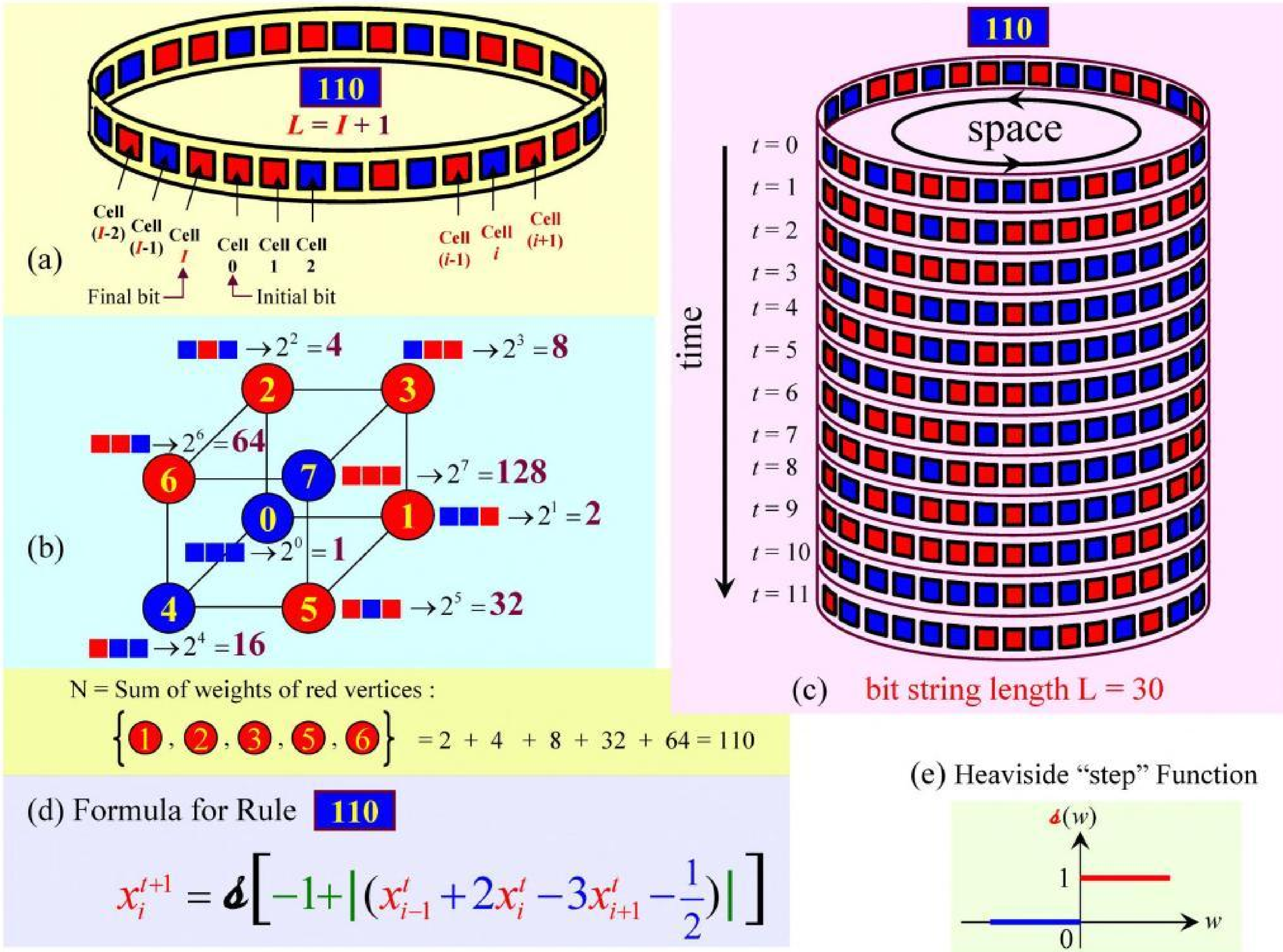


Fig. 3. An example (Rule 110) illustrating the notations, symbols and universal formula from Fig. 1.

Consequently, the Boolean function defined by the Boolean cube in Fig. 3(b) is identified as $N = 110$.

1.3. *One formula specifies all 256 rules*

While the usual procedure for specifying a Boolean function is to give the *truth table*, as God-given laws, we have discovered the following *nonlinear difference equation*, with eight parameters $\{c_1, c_2, \ldots, c_8\}$, which is capable of generating any of the 256 Boolean cubes in Table 1, by merely assigning eight *real numbers* to these eight parameters:

$$x_i^{t+1} = \{c_8 + c_7|[c_6 + c_5|(c_4 + c_3 x_{i-1}^t + c_2 x_i^t + c_1 x_{i+1}^t)|]|\} \qquad (1)$$

The *difference equation* (1) is extremely robust in the sense that a very large set of real numbers can be chosen to generate each Boolean cube, as depicted in the parameter space $\mathbb{R}^8$ in Fig. 4. One such set of numbers is given for each of the 256 rules in Table A-3 of [Chua *et al.*, 2006].

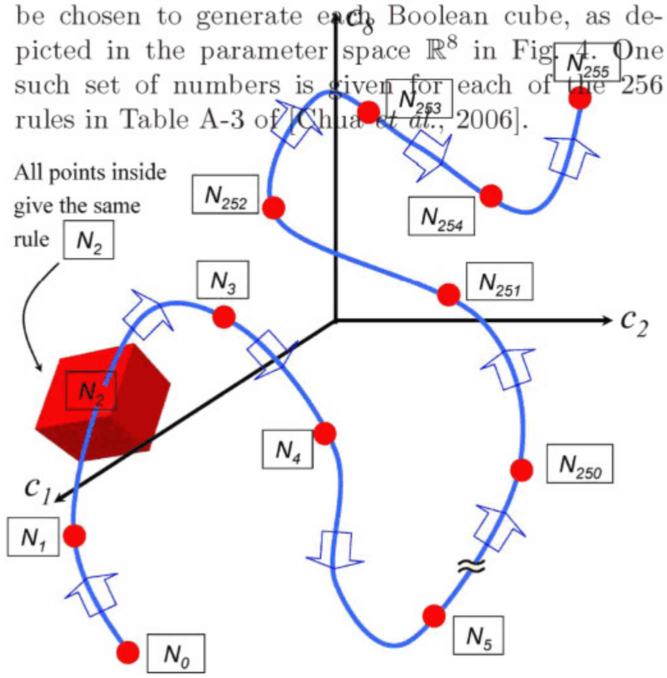


Fig. 4. An abstraction showing a curve meandering through 8-dimensional parameter space, showing all 256 local rules, not necessary in consecutive order. The *robustness* of the universal formula is depicted by an *open set* of parameter points (surrounding a typical parameter vector for rule N_2) all of which would generate the same truth table as N_2 .

For example, for rule 110 , we read off the following values from page 1363 of [Chua *et al.*, 2006]:

$$\begin{aligned} &c_1 = -3, \quad c_2 = 2, \quad c_3 = 1, \quad c_4 = -\frac{1}{2}, \\ &c_5 = 1, \quad c_6 = -1, \quad c_7 = 0, \quad c_8 = 0 \end{aligned} \qquad (2)$$

Substituting the eight real numbers from Eq. (2) into Eq. (1), we obtain the following difference equation for executing rule 110 :

$$x_i^{t+1} = \left[-1 + \left|\left(x_{i-1}^t + 2x_i^t - 3x_{i+1}^t - \frac{1}{2}\right)\right|\right] \qquad (3)$$

To verify that Eq. (3) can indeed generate the truth table for rule 110 , let us substitute the eight input patterns listed on the left side of Fig. 1(d):

Input code ⓪: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (0, 0, 0)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(0 + 2 \bullet 0 - 3 \bullet 0 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 0.5] \\ &= 0 \end{aligned} \qquad (4a)$$

Input code ①: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (0, 0, 1)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(0 + 2 \bullet 0 - 3 \bullet 1 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 3.5] \\ &= 1 \end{aligned} \qquad (4b)$$

Input code ②: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (0, 1, 0)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(0 + 2 \bullet 1 - 3 \bullet 0 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 1.5] \\ &= 1 \end{aligned} \qquad (4c)$$

Input code ③: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (0, 1, 1)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(0 + 2 \bullet 1 - 3 \bullet 1 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 1.5] \\ &= 1 \end{aligned} \qquad (4d)$$

Input code ④: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (1, 0, 0)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(1 + 2 \bullet 0 - 3 \bullet 0 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 0.5] \\ &= 0 \end{aligned} \qquad (4e)$$

Input code ⑤: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (1, 0, 1)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(1 + 2 \bullet 0 - 3 \bullet 1 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 4.5] \\ &= 1 \end{aligned} \tag{4f}$$

Input code ⑥: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (1, 1, 0)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(1 + 2 \bullet 1 - 3 \bullet 0 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 2.5] \\ &= 1 \end{aligned} \tag{4g}$$

Input code ⑦: $(x_{i-1}^t, x_i^t, x_{i+1}^t) = (1, 1, 1)$

$$\begin{aligned} x_i^{t+1} &= \left[-1 + \left|\left(1 + 2 \bullet 1 - 3 \bullet 1 - \frac{1}{2}\right)\right|\right] \\ &= [-1 + 0.5] \\ &= 0 \end{aligned} \tag{4h}$$

Mapping each vertex ⓚ *red* if $x_i^{t+1} = 1$, and *blue* if $x_i^{t+1} = 0$, onto the "*blank*" vertices in the Boolean cube in Fig. 1(b), we obtain the Boolean cube for 110 in Fig. 3(b), which indeed defines rule 110 , as expected.

For ease of future reference, we have enshrined the "shot-gun" Eq. (1) in Fig. 1(e). Observe that in the most general case, this universal formula has eight *non-zero* parameters and two "*nested*" *absolute value* functions. The analytical derivation of this formula in [Chua *et al.*, 2002] shows that the number of absolute value functions required for each rule N is precisely equal to $\kappa - 1$, where κ is the *index of complexity* of N , which in turn is defined to be the number of *parallel planes* needed to separate the "red" vertices from the "blue" vertices. In the case of rule 110 , we recall from Fig. 3 that $\kappa_{110} = 2$ because we only need two parallel planes. Consequently, we expect the two parameters $c_7 = c_8 = 0$ so that *only one* absolute value function is needed for 110 .[3] For the readers convenience, Table 2 gives the *explicit* formula for each of the 256 local rules defined by the Boolean cubes displayed in Table 1.

We end this subsection by emphasizing that the significance of the *universal difference equation* enshrined in Fig. 1(e) should not be construed merely as an elegant mathematical formula, but rather as a mathematical bridge *essential* for deriving and proving *analytical* results and *theorems*, as demonstrated in the rigorous derivation of the Bernoulli shift formulas for rules 170 , 240 , 15 , 85 , and 184 for *finite* L in [Chua *et al.*, 2005a]. Such a feat would not have been possible without exploiting this universal formula in an essential way.

1.4. *Space-time pattern and time-τ return maps*

Given any initial binary bit-string configuration at time $t = 0$, the local rule N is used to update the state x_i^{t+1} of each cell "i" at time $t+1$, using the states x_{i-1}^t, x_i^t, and x_{i+1}^t of the three neighboring cells $i-1$, i, and $i+1$, centered at location "i", respectively. The *space-time pattern* for the initial state shown in Fig. 3(a) is shown in Fig. 3(c) for $t = 0, 1, 2, \ldots, 11$. For simplicity, space-time patterns are generally plotted by displaying a *line* of L cells, with the implicit understanding that the *leftmost* bit is "*glued*" to the *rightmost* bit. Such space-time patterns are useful if they are *T-periodic* with a small period T and a relatively short *transient* regime. For large period T, such as the space-time patterns generated from the 18 *complex* and *hyper Bernoulli* rules to be studied in this paper, it is much more revealing to recast space-time patterns into a *time-τ return map* [Chua *et al.*, 2005a]

$$\rho_\tau[N]\colon \phi_{n-\tau} \mapsto \phi_n \tag{5}$$

where

$$\phi \triangleq \sum_{i=0}^{I} 2^{-(i+1)} x_i \tag{6}$$

is the *decimal* equivalent of the binary bit string

$$\mathbf{x^n} = (x_0^n \quad x_1^n \quad x_2^n \quad \cdots \quad x_{L-1}^n) \tag{7}$$

and τ is an integer. In most cases, such as the *quasi-ergodic* space-time patterns to be presented in Sec. 2, we choose $\tau = 1$ and will be concerned usually with the time-1 map $\rho[N] \triangleq \rho_1[N]$, where we drop the subscript "1" to avoid clutter. Observe

[3] We caution the reader that to avoid ambiguity, Eq. (1) should actually be written as three *separate* "telescoping" equations, as in Eq. (A.3) of [Chua *et al.*, 2006]. Hence, $c_6 = c_7 = 0$ in Eq. (1) should be interpreted to mean the *deletion* of the *outer absolute value function* associated with c_6 and c_7.

Table 2. Formula defining *the truth table* of all 256 local rules.

N	Formula for Local Rule	N	Formula for Local Rule
0	$x_i^{t+1} = \delta\left[-\frac{1}{2}\right] = 0 \Rightarrow$ **All neurons quenched**	**8**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
1	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2}\right]$	**9**	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|(2x_{i-1}^t + x_i^t - x_{i+1}^t)\right\|\right]$
2	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{1}{2}\right]$	**10**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_{i+1}^t - \frac{1}{2}\right]$
3	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + \frac{1}{2}\right]$	**11**	$x_i^{t+1} = \delta\left[-2x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{1}{2}\right]$
4	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{1}{2}\right]$	**12**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t - \frac{1}{2}\right]$
5	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_{i+1}^t + \frac{1}{2}\right]$	**13**	$x_i^{t+1} = \delta\left[-2x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
6	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|(2x_{i-1}^t - x_i^t - x_{i+1}^t + 1)\right\|\right]$	**14**	$x_i^{t+1} = \delta\left[-2x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{1}{2}\right]$
7	$x_i^{t+1} = \delta\left[-2x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{3}{2}\right]$	**15**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + \frac{1}{2}\right] = \overline{x_{i-1}^t} \Rightarrow$ **Complemented Right shift**

Table 2. (*Continued*)

N	Formula for Local Rule
16	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t - x_{i+1}^t - \frac{1}{2}\right]$
17	$x_i^{t+1} = \delta\left[-x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
18	$x_i^{t+1} = \delta\left[\frac{1}{2} - \lvert(x_{i-1}^t + 2x_i^t + x_{i+1}^t - 1)\rvert\right]$
19	$x_i^{t+1} = \delta\left[-x_{i-1}^t - 2x_i^t - x_{i+1}^t + \frac{3}{2}\right]$
20	$x_i^{t+1} = \delta\left[\frac{1}{2} - \lvert(-2x_{i-1}^t - 2x_i^t + x_{i+1}^t + 2)\rvert\right]$
21	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t - 2x_{i+1}^t + \frac{3}{2}\right]$
22	$x_i^{t+1} = \delta\left[\frac{1}{2} - \lvert(x_{i-1}^t + x_i^t + x_{i+1}^t - 1)\rvert\right]$
23	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{3}{2}\right]$
24	$x_i^{t+1} = \delta\left[-1 + \lvert(x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2})\rvert\right]$
25	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(x_{i-1}^t - 2x_i^t + 4x_{i+1}^t - 1)\rvert\right]$
26	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(2x_{i-1}^t - x_i^t + 4x_{i+1}^t - 3)\rvert\right]$
27	$x_i^{t+1} = \delta\left[1 - \lvert(-\frac{3}{2} + \lvert(-2x_{i-1}^t - 4x_i^t + x_{i+1}^t + 1)\rvert)\rvert\right]$
28	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(2x_{i-1}^t + 4x_i^t - x_{i+1}^t - 3)\rvert\right]$
29	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(-2x_{i-1}^t + x_i^t - 4x_{i+1}^t + 1)\rvert)\rvert\right]$
30	$x_i^{t+1} = \delta\left[1 - \lvert(-2x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{3}{2})\rvert\right]$
31	$x_i^{t+1} = \delta\left[-3x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{7}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
32	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{3}{2}\right]$	40	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\lvert(-2x_{i-1}^t - 2x_i^t + x_{i+1}^t + 1)\right\rvert\right]$
33	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\lvert(-2x_{i-1}^t + x_i^t + 2x_{i+1}^t)\right\rvert\right]$	41	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\lvert(-x_{i-1}^t - x_i^t + x_{i+1}^t)\right\rvert\right]$
34	$x_i^{t+1} = \delta\left[-x_i^t + x_{i+1}^t - \frac{1}{2}\right]$	42	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + 2x_{i+1}^t - \frac{1}{2}\right]$
35	$x_i^{t+1} = \delta\left[-x_{i-1}^t - 2x_i^t + x_{i+1}^t + \frac{1}{2}\right]$	43	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{1}{2}\right]$
36	$x_i^{t+1} = \delta\left[-1 + \left\lvert(x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{1}{2})\right\rvert\right]$	44	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\lvert(-4x_{i-1}^t - 2x_i^t + x_{i+1}^t + 2)\right\rvert\right]$
37	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\lvert(2x_{i-1}^t - x_i^t - 4x_{i+1}^t + 1)\right\rvert\right]$	45	$x_i^{t+1} = \delta\left[1 - \left\lvert(-2x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{1}{2})\right\rvert\right]$
38	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\lvert(-x_{i-1}^t + 2x_i^t + 4x_{i+1}^t - 3)\right\rvert\right]$	46	$x_i^{t+1} = \delta\left[1 - \left\lvert\left(\frac{3}{2} - \left\lvert(x_{i-1}^t - 2x_i^t - 3x_{i+1}^t + 4)\right\rvert\right)\right\rvert\right]$
39	$x_i^{t+1} = \delta\left[1 - \left\lvert\left(-\frac{3}{2} + \left\lvert(-4x_{i-1}^t - 2x_i^t + x_{i+1}^t + 1)\right\rvert\right)\right\rvert\right]$	47	$x_i^{t+1} = \delta\left[-3x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{5}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
48	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + -\frac{1}{2}\right]$	56	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(2x_{i-1}^t + 4x_i^t - x_{i+1}^t - 2)\rvert\right]$
49	$x_i^{t+1} = \delta\left[x_{i-1}^t - 2x_i^t - x_{i+1}^t + \frac{1}{2}\right]$	57	$x_i^{t+1} = \delta\left[1 - \lvert(-x_{i-1}^t - 2x_i^t + x_{i+1}^t + \frac{1}{2})\rvert\right]$
50	$x_i^{t+1} = \delta\left[x_{i-1}^t - 2x_i^t + x_{i+1}^t - \frac{1}{2}\right]$	58	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(-x_{i-1}^t + 2x_i^t - 4x_{i+1}^t + 3)\rvert)\rvert\right]$
51	$x_i^{t+1} = \delta\left[-x_i^t + \frac{1}{2}\right] = \overline{x_i^t} \Rightarrow$ **Self Complement**	59	$x_i^{t+1} = \delta\left[-x_{i-1}^t - 3x_{i+1}^t + x_{i+1}^t + \frac{5}{2}\right]$
52	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t - 4x_i^t - x_{i+1}^t + 3)\rvert\right]$	60	$x_i^{t+1} = \delta\left[\frac{1}{2} - \lvert(x_{i-1}^t + x_i^t - 1)\rvert\right]$
53	$x_i^{t+1} = \delta\left[1 - \lvert(-\frac{3}{2} + \lvert(4x_{i-1}^t + x_i^t - 3x_{i+1}^t - 2)\rvert)\rvert\right]$	61	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t - 2x_i^t + x_{i+1}^t + 1)\rvert\right]$
54	$x_i^{t+1} = \delta\left[1 - \lvert(-x_{i-1}^t - 2x_i^t - x_{i+1}^t + \frac{3}{2})\rvert\right]$	62	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t - 2x_i^t - x_{i+1}^t + 2)\rvert\right]$
55	$x_i^{t+1} = \delta\left[-x_{i-1}^t - 3x_i^t - x_{i+1}^t + \frac{7}{2}\right]$	63	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + \frac{3}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
64	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{3}{2}\right]$	72	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|(-2x_{i-1}^t + x_i^t - 2x_{i+1}^t + 1)\right\|\right]$
65	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|(2x_{i-1}^t - 2x_i^t + x_{i+1}^t)\right\|\right]$	73	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|(x_{i-1}^t - x_i^t + x_{i+1}^t)\right\|\right]$
66	$x_i^{t+1} = \delta\left[-1 + \left\|(-x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{1}{2})\right\|\right]$	74	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|(2x_{i-1}^t + x_i^t + 4x_{i+1}^t - 4)\right\|\right]$
67	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|(2x_{i-1}^t - 4x_i^t - x_{i+1}^t + 1)\right\|\right]$	75	$x_i^{t+1} = \delta\left[1 - \left\|(2x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{1}{2})\right\|\right]$
68	$x_i^{t+1} = \delta\left[x_i^t - x_{i+1}^t - \frac{1}{2}\right]$	76	$x_i^{t+1} = \delta\left[-x_{i-1}^t + 2x_i^t - x_{i+1}^t - \frac{1}{2}\right]$
69	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t - 2x_{i+1}^t + \frac{1}{2}\right]$	77	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
70	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|(-x_{i-1}^t + 4x_i^t + 2x_{i+1}^t - 3)\right\|\right]$	78	$x_i^{t+1} = \delta\left[1 - \left\|\left(-\frac{3}{2} + \left\|(3x_{i-1}^t - x_i^t + 2x_{i+1}^t)\right\|\right)\right\|\right]$
71	$x_i^{t+1} = \delta\left[1 - \left\|\left(\frac{3}{2} - \left\|(-4x_{i-1}^t + x_i^t - 2x_{i+1}^t + 1)\right\|\right)\right\|\right]$	79	$x_i^{t+1} = \delta\left[-3x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{5}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
80	$x_i^{t+1} = \mathrm{sgn}\left[x_{i-1}^t - x_{i+1}^t - \frac{1}{2}\right]$	88	$x_i^{t+1} = \mathrm{sgn}\left[\frac{3}{2} - \left\|(-4x_{i-1}^t - x_i^t - 2x_{i+1}^t + 4)\right\|\right]$
81	$x_i^{t+1} = \mathrm{sgn}\left[x_{i-1}^t - x_i^t - 2x_{i+1}^t + \frac{1}{2}\right]$	89	$x_i^{t+1} = \mathrm{sgn}\left[1 - \left\|(x_{i-1}^t - x_i^t + 2x_{i+1}^t - \frac{1}{2})\right\|\right]$
82	$x_i^{t+1} = \mathrm{sgn}\left[\frac{3}{2} - \left\|(-2x_{i-1}^t - x_i^t - 4x_{i+1}^t + 3)\right\|\right]$	90	$x_i^{t+1} = \mathrm{sgn}\left[\frac{1}{2} - \left\|(x_{i-1}^t + x_{i+1}^t - 1)\right\|\right]$
83	$x_i^{t+1} = \mathrm{sgn}\left[1 - \left\|\left(\frac{3}{2} - \left\|(-3x_{i-1}^t + 4x_i^t - 2x_{i+1}^t + 1)\right\|\right)\right\|\right]$	91	$x_i^{t+1} = \mathrm{sgn}\left[\frac{3}{2} - \left\|(-2x_{i-1}^t + x_i^t - 2x_{i+1}^t + 1)\right\|\right]$
84	$x_i^{t+1} = \mathrm{sgn}\left[x_{i-1}^t + x_i^t - 2x_{i+1}^t - \frac{1}{2}\right]$	92	$x_i^{t+1} = \mathrm{sgn}\left[1 - \left\|\left(\frac{3}{2} - \left\|(x_{i-1}^t - 2x_i^t + 4x_{i+1}^t)\right\|\right)\right\|\right]$
85	$x_i^{t+1} = \mathrm{sgn}\left[-x_{i+1}^t + \frac{1}{2}\right] = \overline{x_{i+1}^t} \Rightarrow$ **Complemented Left shift**	93	$x_i^{t+1} = \mathrm{sgn}\left[-x_{i-1}^t + x_i^t - 3x_{i+1}^t + \frac{5}{2}\right]$
86	$x_i^{t+1} = \mathrm{sgn}\left[1 - \left\|(-x_{i-1}^t - x_i^t - 2x_{i+1}^t + \frac{3}{2})\right\|\right]$	94	$x_i^{t+1} = \mathrm{sgn}\left[\frac{3}{2} - \left\|(-2x_{i-1}^t - x_i^t - 2x_{i+1}^t + 2)\right\|\right]$
87	$x_i^{t+1} = \mathrm{sgn}\left[-x_{i-1}^t - x_i^t - 3x_{i+1}^t + \frac{7}{2}\right]$	95	$x_i^{t+1} = \mathrm{sgn}\left[-x_{i-1}^t - x_{i+1}^t + \frac{3}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
96	$x_i^{t+1} = \eth\left[\frac{1}{2} - \left|(-2x_{i-1}^t + x_i^t + x_{i+1}^t + 1)\right|\right]$	**104**	$x_i^{t+1} = \eth\left[\frac{1}{2} - \left|(-x_{i-1}^t - x_i^t - x_{i+1}^t + 2)\right|\right]$
97	$x_i^{t+1} = \eth\left[\frac{1}{2} - \left|(-x_{i-1}^t + x_i^t + x_{i+1}^t)\right|\right]$	**105**	$x_i^{t+1} = \eth\left[\frac{1}{2} - \left|\left(1 - \left|(-x_{i-1}^t + x_i^t - x_{i+1}^t + 1)\right|\right)\right|\right]$
98	$x_i^{t+1} = \eth\left[\frac{3}{2} - \left|(-x_{i-1}^t - 2x_i^t - 4x_{i+1}^t + 4)\right|\right]$	**106**	$x_i^{t+1} = \eth\left[1 - \left|(x_{i-1}^t + x_i^t + 2x_{i+1}^t - \frac{5}{2})\right|\right]$
99	$x_i^{t+1} = \eth\left[1 - \left|(-x_{i-1}^t + 2x_i^t + x_{i+1}^t - \frac{1}{2})\right|\right]$	**107**	$x_i^{t+1} = \eth\left[-\frac{1}{2} + \left|(-x_{i-1}^t - x_i^t + x_{i+1}^t + 1)\right|\right]$
100	$x_i^{t+1} = \eth\left[\frac{3}{2} - \left|(-x_{i-1}^t - 4x_i^t - 2x_{i+1}^t + 4)\right|\right]$	**108**	$x_i^{t+1} = \eth\left[1 - \left|(-x_{i-1}^t - 2x_i^t - x_{i+1}^t + \frac{5}{2})\right|\right]$
101	$x_i^{t+1} = \eth\left[1 - \left|(x_{i-1}^t - x_i^t - 2x_{i+1}^t + \frac{1}{2})\right|\right]$	**109**	$x_i^{t+1} = \eth\left[-\frac{1}{2} + \left|(-x_{i-1}^t + x_i^t - x_{i+1}^t + 1)\right|\right]$
102	$x_i^{t+1} = \eth\left[\frac{1}{2} - \left|(x_i^t + x_{i+1}^t - 1)\right|\right]$	**110**	$x_i^{t+1} = \eth\left[-1 + \left|(x_{i-1}^t + 2x_i^t - 3x_{i+1}^t - \frac{1}{2})\right|\right]$
103	$x_i^{t+1} = \eth\left[\frac{3}{2} - \left|(-x_{i-1}^t + 2x_i^t + 2x_{i+1}^t - 1)\right|\right]$	**111**	$x_i^{t+1} = \eth\left[-\frac{1}{2} + \left|(x_{i-1}^t - 2x_i^t + 2x_{i+1}^t - 1)\right|\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
112	$x_i^{t+1} = \delta\left[2x_{i-1}^t - x_i^t - x_{i+1}^t - \frac{1}{2}\right]$	120	$x_i^{t+1} = \delta\left[1 - \left|(2x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{5}{2})\right|\right]$
113	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2}\right]$	121	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(-x_{i-1}^t + x_i^t + x_{i+1}^t - 1)\right|\right]$
114	$x_i^{t+1} = \delta\left[1 - \left|(\frac{3}{2} - \left|(-4x_{i-1}^t + 2x_i^t - x_{i+1}^t + 3)\right|)\right|\right]$	122	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(-2x_{i-1}^t - x_i^t - 2x_{i+1}^t + 3)\right|\right]$
115	$x_i^{t+1} = \delta\left[x_{i-1}^t - 3x_i^t - x_{i+1}^t + \frac{5}{2}\right]$	123	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(2x_{i-1}^t + x_i^t - 2x_{i+1}^t - 1)\right|\right]$
116	$x_i^{t+1} = \delta\left[1 - \left|(\frac{3}{2} - \left|(-4x_{i-1}^t - x_i^t + 2x_{i+1}^t + 3)\right|)\right|\right]$	124	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(2x_{i-1}^t + 2x_i^t + x_{i+1}^t - 3)\right|\right]$
117	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t - 3x_{i+1}^t + \frac{5}{2}\right]$	125	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(-x_{i-1}^t + x_i^t + 2x_{i+1}^t - 2)\right|\right]$
118	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(-x_{i-1}^t - 2x_i^t - 2x_{i+1}^t + 2)\right|\right]$	126	$x_i^{t+1} = \delta\left[1 - \left|(-x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{3}{2})\right|\right]$
119	$x_i^{t+1} = \delta\left[-x_i^t - x_{i+1}^t + \frac{3}{2}\right]$	127	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{5}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
128	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{5}{2}\right]$	136	$x_i^{t+1} = \delta\left[x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
129	$x_i^{t+1} = \delta\left[-1 + \left\|\left(x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{3}{2}\right)\right\|\right]$	137	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(-x_{i-1}^t - 2x_i^t + 4x_{i+1}^t - 1\right)\right\|\right]$
130	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|\left(-x_{i-1}^t + x_i^t - 2x_{i+1}^t + 2\right)\right\|\right]$	138	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + 2x_{i+1}^t - \frac{3}{2}\right]$
131	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(4x_{i-1}^t - 2x_i^t - x_{i+1}^t\right)\right\|\right]$	139	$x_i^{t+1} = \delta\left[1 - \left\|\left(-\frac{3}{2} + \left\|\left(-x_{i-1}^t - 2x_i^t + 3x_{i+1}^t - 2\right)\right\|\right)\right\|\right]$
132	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|\left(2x_{i-1}^t - x_i^t - 2x_{i+1}^t + 1\right)\right\|\right]$	140	$x_i^{t+1} = \delta\left[-x_{i-1}^t + 2x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
133	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(-2x_{i-1}^t - x_i^t + 4x_{i+1}^t\right)\right\|\right]$	141	$x_i^{t+1} = \delta\left[1 - \left\|\left(\frac{3}{2} - \left\|\left(-2x_{i-1}^t - 4x_i^t + 3x_{i+1}^t + 2\right)\right\|\right)\right\|\right]$
134	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\|\left(x_{i-1}^t - x_i^t - x_{i+1}^t + 1\right)\right\|\right]$	142	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{1}{2}\right]$
135	$x_i^{t+1} = \delta\left[1 - \left\|\left(2x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2}\right)\right\|\right]$	143	$x_i^{t+1} = \delta\left[-3x_{i-1}^t + x_i^t + x_{i+1}^t + \frac{3}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
144	$x_i^{t+1}=\delta\left[\frac{1}{2}-\lvert(-2x_{i-1}^t+x_i^t-x_{i+1}^t+2)\rvert\right]$	**152**	$x_i^{t+1}=\delta\left[\frac{3}{2}-\lvert(x_{i-1}^t+4x_i^t-2x_{i+1}^t-2)\rvert\right]$
145	$x_i^{t+1}=\delta\left[\frac{3}{2}-\lvert(-x_{i-1}^t-2x_i^t+4x_{i+1}^t)\rvert\right]$	**153**	$x_i^{t+1}=\delta\left[\frac{1}{2}-\lvert(-x_i^t+x_{i+1}^t)\rvert\right]$
146	$x_i^{t+1}=\delta\left[\frac{1}{2}-\lvert(x_{i-1}^t-x_i^t+x_{i+1}^t-1)\rvert\right]$	**154**	$x_i^{t+1}=\delta\left[1-\lvert(x_{i-1}^t-x_i^t+2x_{i+1}^t-\frac{3}{2})\rvert\right]$
147	$x_i^{t+1}=\delta\left[1-\lvert(x_{i-1}^t-2x_i^t+x_{i+1}^t-\frac{1}{2})\rvert\right]$	**155**	$x_i^{t+1}=\delta\left[\frac{3}{2}-\lvert(-x_{i-1}^t+2x_i^t-2x_{i+1}^t+1)\rvert\right]$
148	$x_i^{t+1}=\delta\left[\frac{1}{2}-\lvert(-x_{i-1}^t-x_i^t+x_{i+1}^t+1)\rvert\right]$	**156**	$x_i^{t+1}=\delta\left[1-\lvert(x_{i-1}^t+2x_i^t-x_{i+1}^t-\frac{3}{2})\rvert\right]$
149	$x_i^{t+1}=\delta\left[1-\lvert(-x_{i-1}^t-x_i^t+2x_{i+1}^t+\frac{1}{2})\rvert\right]$	**157**	$x_i^{t+1}=\delta\left[\frac{3}{2}-\lvert(x_{i-1}^t+2x_i^t-2x_{i+1}^t-1)\rvert\right]$
150	$x_i^{t+1}=\delta\left[\frac{3}{2}-\lvert(\frac{7}{2}-\lvert(-4x_{i-1}^t-2x_i^t+4x_{i+1}^t-\frac{1}{2})\rvert)\rvert\right]$	**158**	$x_i^{t+1}=\delta\left[-\frac{1}{2}+\lvert(x_{i-1}^t-x_i^t-x_{i+1}^t)\rvert\right]$
151	$x_i^{t+1}=\delta\left[-\frac{1}{2}+\lvert(x_{i-1}^t+x_i^t+x_{i+1}^t-2)\rvert\right]$	**159**	$x_i^{t+1}=\delta\left[-\frac{1}{2}+\lvert(-2x_{i-1}^t+x_i^t+x_{i+1}^t+1)\rvert\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
160	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_{i+1}^t - \frac{3}{2}\right]$	**168**	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t + 2x_{i+1}^t - \frac{5}{2}\right]$
161	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(2x_{i-1}^t + x_i^t - 4x_{i+1}^t + 1)\rvert\right]$	**169**	$x_i^{t+1} = \delta\left[1 - \lvert(x_{i-1}^t + x_i^t - 2x_{i+1}^t + \frac{1}{2})\rvert\right]$
162	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + 2x_{i+1}^t - \frac{3}{2}\right]$	**170**	$x_i^{t+1} = \delta\left[x_{i+1}^t - \frac{1}{2}\right] = x_{i+1}^t \Rightarrow$ **Left Shift**
163	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(3x_{i-1}^t - 4x_i^t - 2x_{i+1}^t + 1)\rvert)\rvert\right]$	**171**	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + 2x_{i+1}^t + \frac{1}{2}\right]$
164	$x_i^{t+1} = \delta\left[-\frac{3}{2} + \lvert(-2x_{i-1}^t + x_i^t - 2x_{i+1}^t + 1)\rvert\right]$	**172**	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(-3x_{i-1}^t - 4x_i^t + 2x_{i+1}^t + 3)\rvert)\rvert\right]$
165	$x_i^{t+1} = \delta\left[\frac{1}{2} - \lvert(-x_{i-1}^t + x_{i+1}^t)\rvert\right]$	**173**	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t - x_i^t + 2x_{i+1}^t)\rvert\right]$
166	$x_i^{t+1} = \delta\left[1 - \lvert(x_{i-1}^t - x_i^t - 2x_{i+1}^t + \frac{3}{2})\rvert\right]$	**174**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + 2x_{i+1}^t - \frac{1}{2}\right]$
167	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t + x_i^t + 2x_{i+1}^t - 1)\rvert\right]$	**175**	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_{i+1}^t + \frac{1}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
176	$x_i^{t+1} = \delta\left[2x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{3}{2}\right]$	184	$x_i^{t+1} = \delta\left[1 - \left\|\left(-\frac{3}{2} + \left\|\left(x_{i-1}^t - 2x_i^t + 3x_{i+1}^t - 3\right)\right\|\right)\right\|\right]$
177	$x_i^{t+1} = \delta\left[1 - \left\|\left(-\frac{3}{2} + \left\|\left(4x_{i-1}^t - 2x_i^t - x_{i+1}^t - 2\right)\right\|\right)\right\|\right]$	185	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(-x_{i-1}^t - 2x_i^t + 2x_{i+1}^t\right)\right\|\right]$
178	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{1}{2}\right]$	186	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + 2x_{i+1}^t - \frac{1}{2}\right]$
179	$x_i^{t+1} = \delta\left[x_{i-1}^t - 3x_i^t + x_{i+1}^t + \frac{3}{2}\right]$	187	$x_i^{t+1} = \delta\left[-x_i^t + x_{i+1}^t + \frac{1}{2}\right]$
180	$x_i^{t+1} = \delta\left[1 - \left\|\left(-2x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{3}{2}\right)\right\|\right]$	188	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(2x_{i-1}^t + 2x_i^t - x_{i+1}^t - 2\right)\right\|\right]$
181	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\|\left(2x_{i-1}^t + x_i^t - 2x_{i+1}^t - 1\right)\right\|\right]$	189	$x_i^{t+1} = \delta\left[1 - \left\|\left(x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{1}{2}\right)\right\|\right]$
182	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left\|\left(-x_{i-1}^t + x_i^t - x_{i+1}^t\right)\right\|\right]$	190	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left\|\left(-2x_{i-1}^t + 2x_i^t + x_{i+1}^t\right)\right\|\right]$
183	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left\|\left(x_{i-1}^t - 2x_i^t + x_{i+1}^t + 1\right)\right\|\right]$	191	$x_i^{t+1} = \delta\left[-x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{3}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule
192	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - \frac{3}{2}\right]$
193	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\vert(4x_{i-1}^t - 2x_i^t - x_{i+1}^t - 1)\right\vert\right]$
194	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\vert(-2x_{i-1}^t + 4x_i^t + x_{i+1}^t - 2)\right\vert\right]$
195	$x_i^{t+1} = \delta\left[\frac{1}{2} - \left\vert(-x_{i-1}^t + x_i^t)\right\vert\right]$
196	$x_i^{t+1} = \delta\left[x_{i-1}^t + 2x_i^t - x_{i+1}^t - \frac{3}{2}\right]$
197	$x_i^{t+1} = \delta\left[1 - \left\vert\left(\frac{3}{2} - \left\vert(-3x_{i-1}^t + 4x_i^t + 2x_{i+1}^t - 2)\right\vert\right)\right\vert\right]$
198	$x_i^{t+1} = \delta\left[1 - \left\vert\left(x_{i-1}^t - 2x_i^t - x_{i+1}^t + \frac{3}{2}\right)\right\vert\right]$
199	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\vert(2x_{i-1}^t - 2x_i^t - x_{i+1}^t + 1)\right\vert\right]$
200	$x_i^{t+1} = \delta\left[x_{i-1}^t + 2x_i^t + x_{i+1}^t - \frac{5}{2}\right]$
201	$x_i^{t+1} = \delta\left[1 - \left\vert\left(x_{i-1}^t - 2x_i^t + x_{i+1}^t + \frac{1}{2}\right)\right\vert\right]$
202	$x_i^{t+1} = \delta\left[1 - \left\vert\left(\frac{3}{2} - \left\vert(3x_{i-1}^t - 4x_i^t + 2x_{i+1}^t)\right\vert\right)\right\vert\right]$
203	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left\vert(-2x_{i-1}^t + 2x_i^t - x_{i+1}^t)\right\vert\right]$
204	$x_i^{t+1} = \delta\left[x_i^t - \frac{1}{2}\right] = x_i^t \Rightarrow$ **Identity**
205	$x_i^{t+1} = \delta\left[-x_{i-1}^t + 2x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
206	$x_i^{t+1} = \delta\left[-x_{i-1}^t + 2x_i^t + x_{i+1}^t - \frac{1}{2}\right]$
207	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + \frac{1}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
208	$x_i^{t+1} = \delta\left[2x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{3}{2}\right]$	216	$x_i^{t+1} = \delta\left[1 - \left|\left(-\frac{3}{2} + \left|(-3x_{i-1}^t + x_i^t - 4x_{i+1}^t + 4)\right|\right)\right|\right]$
209	$x_i^{t+1} = \delta\left[1 - \left|\left(\frac{3}{2} - \left|(2x_{i-1}^t + x_i^t - 4x_{i+1}^t - 1)\right|\right)\right|\right]$	217	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(-x_{i-1}^t + 2x_i^t - 2x_{i+1}^t)\right|\right]$
210	$x_i^{t+1} = \delta\left[1 - \left|(-2x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{3}{2})\right|\right]$	218	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(2x_{i-1}^t - x_i^t + 2x_{i+1}^t - 2)\right|\right]$
211	$x_i^{t+1} = \delta\left[\frac{3}{2} - \left|(-2x_{i-1}^t + 2x_i^t - x_{i+1}^t + 1)\right|\right]$	219	$x_i^{t+1} = \delta\left[1 - \left|(-x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{1}{2})\right|\right]$
212	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{1}{2}\right]$	220	$x_i^{t+1} = \delta\left[x_{i-1}^t + 2x_i^t - x_{i+1}^t - \frac{1}{2}\right]$
213	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - 3x_{i+1}^t + \frac{3}{2}\right]$	221	$x_i^{t+1} = \delta\left[x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
214	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(x_{i-1}^t + x_i^t - x_{i+1}^t)\right|\right]$	222	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(-x_{i-1}^t - 2x_i^t + x_{i+1}^t)\right|\right]$
215	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \left|(2x_{i-1}^t + 2x_i^t + x_{i+1}^t - 3)\right|\right]$	223	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{3}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
224	$x_i^{t+1} = \delta\left[2x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{5}{2}\right]$	232	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
225	$x_i^{t+1} = \delta\left[1 - \lvert(-2x_{i-1}^t + x_i^t + x_{i+1}^t + \frac{1}{2})\rvert\right]$	233	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \lvert(x_{i-1}^t + x_i^t + x_{i+1}^t - 1)\rvert\right]$
226	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(-2x_{i-1}^t - x_i^t - 4x_{i+1}^t + 5)\rvert)\rvert\right]$	234	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t + 2x_{i+1}^t - \frac{3}{2}\right]$
227	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(2x_{i-1}^t - 2x_i^t - x_{i+1}^t)\rvert\right]$	235	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \lvert(2x_{i-1}^t + 2x_i^t + x_{i+1}^t - 2)\rvert\right]$
228	$x_i^{t+1} = \delta\left[1 - \lvert(\frac{3}{2} - \lvert(2x_{i-1}^t + 4x_i^t + x_{i+1}^t - 5)\rvert)\rvert\right]$	236	$x_i^{t+1} = \delta\left[x_{i-1}^t + 2x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
229	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-2x_{i-1}^t + x_i^t + 2x_{i+1}^t)\rvert\right]$	237	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \lvert(2x_{i-1}^t - x_i^t + 2x_{i+1}^t - 2)\rvert\right]$
230	$x_i^{t+1} = \delta\left[\frac{3}{2} - \lvert(-x_{i-1}^t + 2x_i^t + 2x_{i+1}^t - 2)\rvert\right]$	238	$x_i^{t+1} = \delta\left[x_i^t + x_{i+1}^t - \frac{1}{2}\right]$
231	$x_i^{t+1} = \delta\left[1 - \lvert(x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2})\rvert\right]$	239	$x_i^{t+1} = \delta\left[-x_{i-1}^t + x_i^t + x_{i+1}^t + \frac{1}{2}\right]$

Table 2. (*Continued*)

N	Formula for Local Rule	N	Formula for Local Rule
240	$x_i^{t+1} = \delta\left[x_{i-1}^t - \frac{1}{2}\right] = x_{i-1}^t \Rightarrow$ **Right Shift**	**248**	$x_i^{t+1} = \delta\left[2x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{3}{2}\right]$
241	$x_i^{t+1} = \delta\left[2x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{1}{2}\right]$	**249**	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \lvert(2x_{i-1}^t - x_i^t - x_{i+1}^t + 1)\rvert\right]$
242	$x_i^{t+1} = \delta\left[2x_{i-1}^t - x_i^t + x_{i+1}^t - \frac{1}{2}\right]$	**250**	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_{i+1}^t - \frac{1}{2}\right]$
243	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + \frac{1}{2}\right]$	**251**	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t + x_{i+1}^t + \frac{1}{2}\right]$
244	$x_i^{t+1} = \delta\left[2x_{i-1}^t + x_i^t - x_{i+1}^t - \frac{1}{2}\right]$	**252**	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - \frac{1}{2}\right]$
245	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_{i+1}^t + \frac{1}{2}\right]$	**253**	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t - x_{i+1}^t + \frac{1}{2}\right]$
246	$x_i^{t+1} = \delta\left[-\frac{1}{2} + \lvert(-x_{i-1}^t + 2x_i^t - 2x_{i+1}^t)\rvert\right]$	**254**	$x_i^{t+1} = \delta\left[x_{i-1}^t + x_i^t + x_{i+1}^t - \frac{1}{2}\right]$
247	$x_i^{t+1} = \delta\left[x_{i-1}^t - x_i^t - x_{i+1}^t + \frac{3}{2}\right]$	**255**	$x_i^{t+1} = \delta\left[\frac{1}{2}\right] = 1 \Rightarrow$ **All neurons Firing**

that $0 \leq \phi < 1$ for finite I, and when $I \to \infty$, we have a *time-1 map* over the unit interval

$$\rho\colon [0,1) \to [0,1) \tag{8}$$

It is important to remember that each *time-1 map* is uniquely associated with one space-time pattern, or "*orbit*", from *one* initial bit-string configuration.

1.5. *We only need to study 88 rules!*

Although there are 256 local rules, only 88 rules are *globally independent* [Chua *et al.*, 2004] *from each other*. All other rules are equivalent to one of the 88 rules listed in Table 4 of [Chua *et al.*, 2007a]. These 88 rules are listed[4] in Table 3 along with an integer code $M \in \{1,2,3,4,5,6\}$, where M denotes one of the following six distinct *qualitative dynamics* exhibited by a particular local rule N [Chua *et al.*, 2007a], corresponding to *random initial configurations*:

Group 1 Rules

Almost all space-time patterns converge to a *period-1 orbit*. The *time-1* map corresponding to each period-1 orbit would consist of a *single point attractor*, or an *Isle of Eden*,[5] on the main diagonal line, after deleting points belonging to the transient regime.

Group 2 Rules

Almost all space-time patterns converge to a *period-2* orbit. The *time-1* map of each period-2 *attractor*, or *Isle of Eden*, consists of *two points*, symmetrical with respect to the main diagonal line.

Group 3 Rules

Almost all space-time patterns converge to a *period-3* orbit. The time-1 map of the period-3 *attractor*, or *Isle of Eden*, consists of *three points*.

Group 4 Rules

Almost all space-time patterns converge to a *Bernoulli σ_τ-shift attractor*, or *Isle of Eden*, where $|\sigma| \in \{1,2,3\}$ and $|\tau| \in \{1,2,3,4,5\}$. □

We stress that the above *qualitative behaviors* do *not* depend on the *length* L of the bit strings, and do *not* depend on the initial configurations, even though there may exist several Bernoulli attractors with different σ and τ, each with its *basin of attraction*.

Group 5 and Group 6 Rules

The space-time patterns typically have very long transients and converge to a period-T attractor with a very large period T. Moreover, the asymptotic behavior depends not only on the initial configuration, but also on the *length* L of the bit string. One difference between a *group 5* rule and a *group 6* rule is that the former is bilateral (and hence has only *one* globally-equivalent rule), whereas the latter is *non-bilateral* (and hence has *three* other globally-equivalent rules).

The classification of each of the 256 local rules is given in Tables 7–9, 11, and 12 in [Chua *et al.*, 2007a].

Given any rule not among those listed in the 88 globally-equivalence classes in Table 3, one can easily look up Table 4 from [Chua *et al.*, 2007a], or Table 3 from [Chua *et al.*, 2007b], to identify its equivalent rule, and then look up its *complexity index* κ (red, blue or green), and group M $(1, 2, \ldots$ or $6)$ from Table 3. For future reference, the complexity index κ and class M of all 256 rules are listed in Table 4. Counting the number of globally equivalent rules from each class from Tables 3 and 4, respectively, we summarize their distributions in Figs. 5 and 6, respectively.

1.6. *The "Magic" rule spaces*

In [Cattaneo & Quaranta Vogliotti, 1997], a subset of 104, among 256, local rules have been derived and shown to exhibit "neural-like" behaviors. The authors' approach is based on an exhaustive mathematical analysis on a *bi-infinite* sequence space, consuming more than 20 printed pages. The authors were so perplexed by their discovery that they dubbed these rules "*magic*".

A cursory inspection of the 256 Boolean cubes listed in Table 1 would extract, in a few minutes, 104 local rules with a *complexity index* $\kappa = 1$, namely, those Boolean cubes whose *red* vertices can be separated from the *blue* vertices by no more than

[4]This list is not unique in the sense that one can pick many other groups containing 88 independent rules. Our choice is obtained by scanning the 256 rules from $N = 0$ to $N = 255$, and deleting any rule that is equivalent to a previously listed rule.

[5]Robust Isles of Eden can be observed only for those rules endowed with *dense* Isles of Eden orbits [Chua *et al.*, 2007a, 2007b].

Table 3. List of 88 globally independent rules. Color surrounding rule number N corresponds to complexity index $\kappa = 1$ (red), 2 (blue) or 3 (green). The integer on the lower right corner identifies the characteristic property of the rule, as specified in the color legend.

0 $_1$	1 $_2$	2 $_4$	3 $_4$	4 $_1$	5 $_2$	6 $_4$	7 $_4$
8 $_1$	9 $_4$	10 $_4$	11 $_4$	12 $_1$	13 $_1$	14 $_4$	15 $_4$
18 $_5$	19 $_2$	22 $_5$	23 $_2$	24 $_4$	25 $_4$	26 $_6$	27 $_4$
28 $_2$	29 $_2$	30 $_6$	32 $_1$	33 $_2$	34 $_4$	35 $_4$	36 $_1$
37 $_2$	38 $_4$	40 $_1$	41 $_6$	42 $_4$	43 $_4$	44 $_1$	45 $_6$
46 $_4$	50 $_2$	51 $_2$	54 $_5$	56 $_4$	57 $_4$	58 $_4$	60 $_6$
62 $_3$	72 $_1$	73 $_5$	74 $_4$	76 $_1$	77 $_1$	78 $_1$	90 $_5$
94 $_1$	104 $_1$	105 $_5$	106 $_6$	108 $_2$	110 $_6$	122 $_5$	126 $_5$
128 $_1$	130 $_4$	132 $_1$	134 $_4$	136 $_1$	138 $_4$	140 $_1$	142 $_4$
146 $_5$	150 $_5$	152 $_4$	154 $_6$	156 $_2$	160 $_1$	162 $_4$	164 $_1$
168 $_1$	170 $_4$	172 $_1$	178 $_2$	184 $_4$	200 $_1$	204 $_1$	232 $_1$

Color Legend

N_1	N_2	N_3	N_4	N_5	N_6
Period-1	Period-2	Period-3	Bernoulli	Complex	Hyper

Table 4. List of 256 local rules. Color surrounding each rule number N corresponds to the complexity index $\kappa = 1$ (red), 2 (blue) or 3 (green) of N. The integer in the lower right corner identifies the characteristic property of the rule, as specified in Table 3.

0 1	1 2	2 4	3 4	4 1	5 2	6 4	7 4	8 1	9 4	10 4	11 4	12 1	13 1	14 4	15 4
16 4	17 4	18 5	19 2	20 4	21 4	22 5	23 2	24 4	25 4	26 6	27 4	28 2	29 2	30 6	31 4
32 1	33 2	34 4	35 4	36 1	37 2	38 4	39 4	40 1	41 6	42 4	43 4	44 1	45 6	46 4	47 4
48 4	49 4	50 2	51 2	52 4	53 4	54 5	55 2	56 4	57 4	58 4	59 4	60 6	61 4	62 3	63 4
64 1	65 4	66 4	67 4	68 1	69 1	70 2	71 2	72 1	73 5	74 4	75 6	76 1	77 1	78 1	79 1
80 4	81 4	82 6	83 4	84 4	85 4	86 6	87 4	88 4	89 6	90 5	91 2	92 1	93 1	94 1	95 2
96 1	97 6	98 4	99 4	100 1	101 6	102 6	103 4	104 1	105 5	106 6	107 6	108 2	109 5	110 6	111 4
112 4	113 4	114 4	115 4	116 4	117 4	118 3	119 4	120 6	121 6	122 5	123 5	124 4	125 4	126 5	127 2
128 1	129 5	130 4	131 3	132 1	133 1	134 4	135 6	136 1	137 6	138 4	139 4	140 1	141 1	142 4	143 4
144 4	145 3	146 5	147 5	148 4	149 6	150 5	151 5	152 4	153 6	154 6	155 4	156 2	157 2	158 4	159 4
160 1	161 5	162 4	163 4	164 1	165 5	166 6	167 6	168 1	169 6	170 4	171 4	172 1	173 4	174 4	175 4
176 4	177 4	178 2	179 2	180 6	181 6	182 5	183 5	184 4	185 4	186 4	187 4	188 4	189 4	190 4	191 4
192 1	193 6	194 4	195 6	196 1	197 1	198 2	199 2	200 1	201 2	202 1	203 1	204 1	205 1	206 1	207 1
208 4	209 4	210 6	211 4	212 4	213 4	214 4	215 4	216 1	217 1	218 1	219 1	220 1	221 1	222 1	223 1
224 1	225 6	226 4	227 4	228 1	229 4	230 4	231 4	232 1	233 1	234 1	235 1	236 1	237 1	238 1	239 1
240 4	241 4	242 4	243 4	244 4	245 4	246 4	247 4	248 1	249 1	250 1	251 1	252 1	253 1	254 1	255 1

$\kappa = 1$ (Red) 104 rules, $\kappa = 2$ (Blue) 126 rules, $\kappa = 3$ (Green) 26 rules

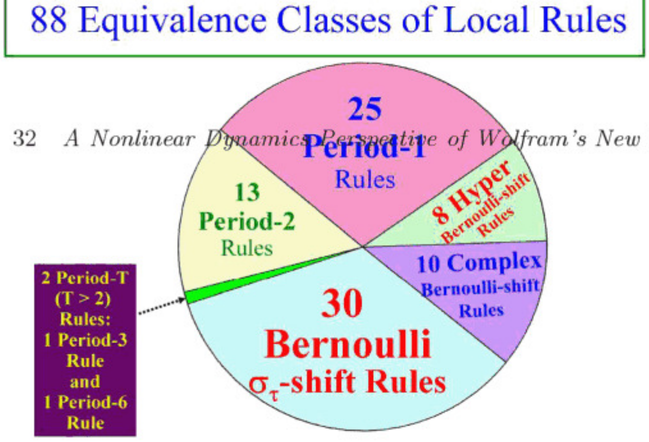


Fig. 5. Partitioning of the 88 globally-independent rules into 6 classes.

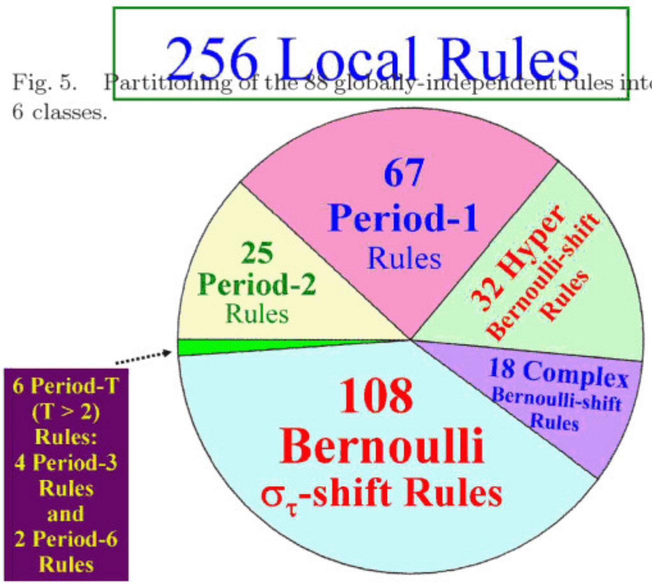


Fig. 6. Partitioning of the 256 local rules into 6 classes.

one plane.[6] These 104 rules are listed in Table 5 along with their classification number M, extracted from Table 4. A comparison of the 104 $\kappa = 1$ rules in Table 5 with those derived in [Cattaneo & Quaranta Vogliotti, 1997] shows that they are identical.[7] From our Boolean cube perspective, the "magic" connotation is perhaps a bit of an anti-climax.

1.7. *Symmetries among Boolean cubes*

Many of the 256 Boolean cubes in Table 1 share interesting symmetrical features which give rise to important predictable dynamics and applications. We will briefly recall some of these symmetries and present new interpretations.

1.7.1. *Local complementation* T^c

We define the *local complementation*

$$N \xrightarrow{T^c} N^c \tag{9}$$

of a Boolean cube N to be the Boolean cube N^c obtained by *complementing* the color of each vertex, i.e. $0 \rightarrow 1$ and $1 \rightarrow 0$. Table 6 shows ten Boolean cubes and their local complements. This transformation is called *local* to emphasize that the space-time patterns of N and N^c, with the *same initial configuration*, are *not* the complement of each other because the complementation is valid only for *one* iteration. This observation is demonstrated in Fig. 5 of [Chua *et al.*, 2004], where

$$110 \xrightarrow{T^c} 145 \tag{10}$$

Observe that the space-time patterns of 110 and 145 (for the same initial configuration) are *not* the complement of each other, except for the first iteration.[8]

1.7.2. *Three equivalence transformations* $T^\dagger$, $\bar{T}$, *and* T^*

There are exactly three global transformations that hold for all iterations, and for all initial configurations. They are the *Left-Right Transformation* $T^\dagger$, the *Global Complementation* $\bar{T}$, and the *Left-Right Complementation* T^* [Chua *et al.*, 2004]. These three transformations, along with the *identity* transformation, have been shown in [Chua *et al.*, 2004] to form an Abelian group known as Klein's *Vierergruppe*.

Given any local rule N, the transformed rules $N^\dagger \triangleq T^\dagger(N)$, $\bar{N} \triangleq \bar{T}(N)$, and $N^* \triangleq T^*(N)$ can be derived by inspection via the simple geometrical operations illustrated in Fig. 7.

[6] Since no *plane* is needed for rules 0 and 255, these two rules may be reclassified with a complexity index $\kappa = 0$.

[7] Except for rule "36" listed in Tables 4 and 8 of [Cattaneo & Quaranta Vogliotti, 1997], which we believe is a typo that should be rectified to rule 136, as correctly reported in Fig. 12 of the same paper.

[8] There are some rules, however, where the *local* complementation $T^c(N)$ coincides with the *global* complementation $\bar{T}(N)$ to be defined below. In such cases, the space-time patterns of N and N^c are also complements of each other for all t.

Table 5. A gallery of 104 *linearly-separable* local rules. The *red* color engulfing each rule number N implies a *complexity index* $\kappa = 1$ for all 104 rules.

: List of 104 Linearly-Separable Boolean Function Rules.

0	1	2	3	4	5	7	8	10	11	12	13	14
1	2	4	4	1	2	4	1	4	4	1	1	4
15	16	17	19	21	23	31	32	34	35	42	43	47
4	4	4	2	4	2	4	1	4	4	4	4	4
48	49	50	51	55	59	63	64	68	69	76	77	79
4	4	2	2	2	4	4	1	1	1	1	1	1
80	81	84	85	87	93	95	112	113	115	117	119	127
4	4	4	4	4	1	2	4	4	4	4	4	2
128	136	138	140	142	143	160	162	168	170	171	174	175
1	1	4	1	4	4	1	4	1	4	4	4	4
176	178	179	186	187	191	192	196	200	204	205	206	207
4	2	2	4	4	4	1	1	1	1	1	1	1
208	212	213	220	221	223	224	232	234	236	238	239	240
4	4	4	1	1	1	1	1	1	1	1	1	4
241	242	243	244	245	247	248	250	251	252	253	254	255
4	4	4	4	4	4	1	1	1	1	1	1	1

Color Legend Period-1 Rule N_1 Period-2 Rule N_2 Bernoulli Rule N_4

To derive the *left-right transformation* $N^\dagger \triangleq T^\dagger(N)$ of any local rule N, simply obtain the *mirror image* of the Boolean cube N about the *main diagonal* plane (shown shaded in Fig. 7(a)) passing through the vertex (0). Note that this operation can be implemented by identifying each pair of symmetrically located vertices with *opposite* colors, and then complementing the colors. Hence $124 = T^\dagger(110)$, and $110 = T^\dagger(124)$.

To derive the *global complementation* $\bar{N} \triangleq T(N)$ of any local rule N, simply identify each pair of *diagonally opposite* vertices that have the *same* color (either both red or both blue), and then change the color. For the example illustrated in Fig. 7(b), we have $N = 110$. Among the four pairs of diagonally opposite vertices of 110, we find that only three pairs of vertices {(0), (7)}, {(2), (5)}, and {(1), (6)} have the same colors. Changing *only* the color of these three pairs, respectively, we obtain its *global complement* $137 = T(110)$, and conversely, $110 = T(137)$.

To derive the *left-right complementation* $N^* \triangleq T^*(N)$ of any local rule N, simply take the *global complementation* first, followed by the *left-right transformation*, or vice-versa, i.e.

$$N^* \triangleq T^*(N) = T^\dagger(T(N)) = T(T^\dagger(N)) \tag{11}$$

For example, consider taking first the global complementation of 110 to obtain $137 = T(110)$ in Fig. 7(b). If we follow this operation by applying the left-right transformation to 137, we would obtain $193 = T^\dagger(137)$ by reflecting the Boolean cube 137 in Fig. 7(b) about the main diagonal, as shown in Fig. 7(c), to obtain

$$193 = T^\dagger(137) = T^\dagger(T(110)) \tag{12}$$

Table 6. Some Boolean cubes and their local complements.

N	N^c	N	N^c
30	225	110	145
45	210	150	105
60	195	154	101
90	165	170	85
105	150	184	71

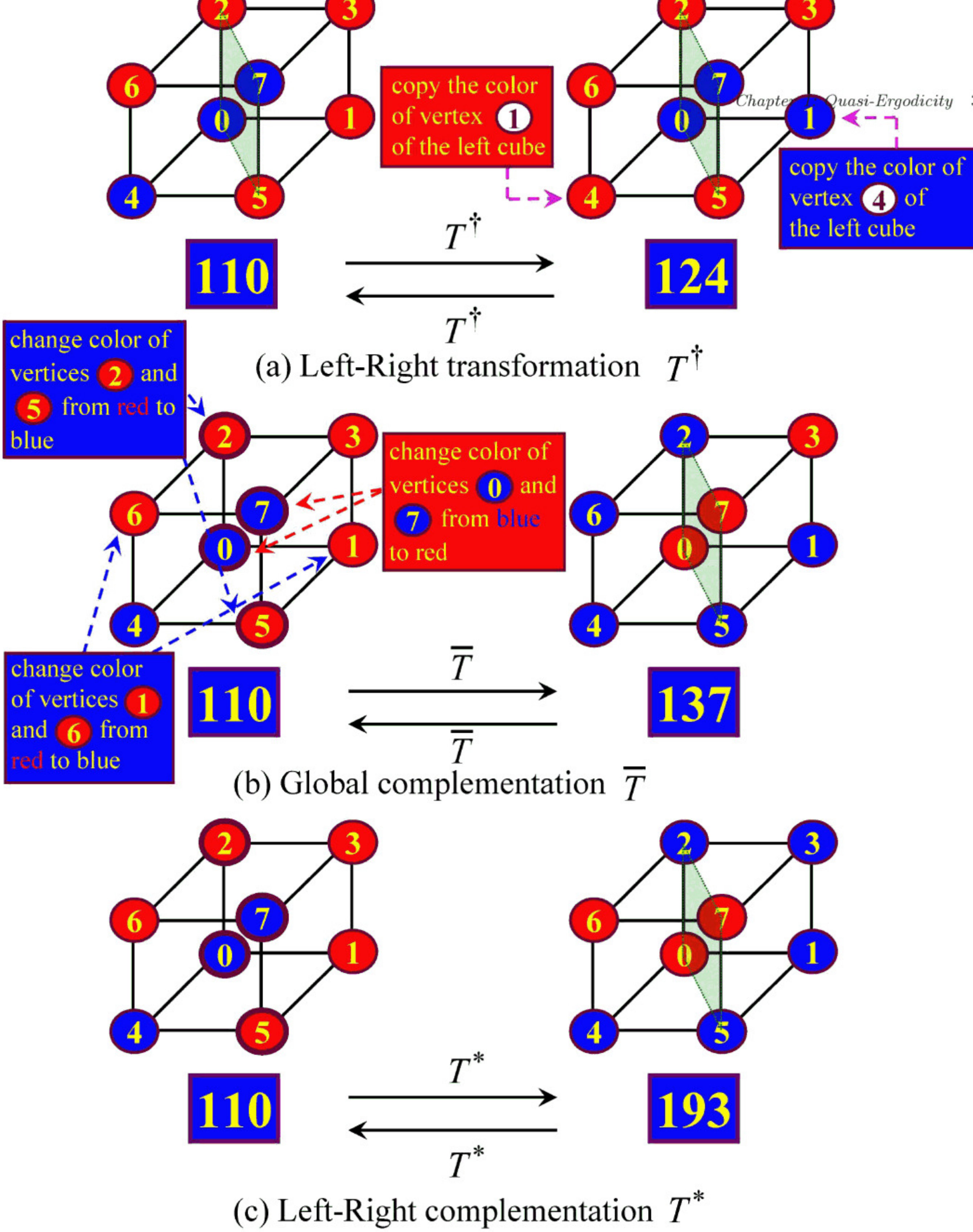
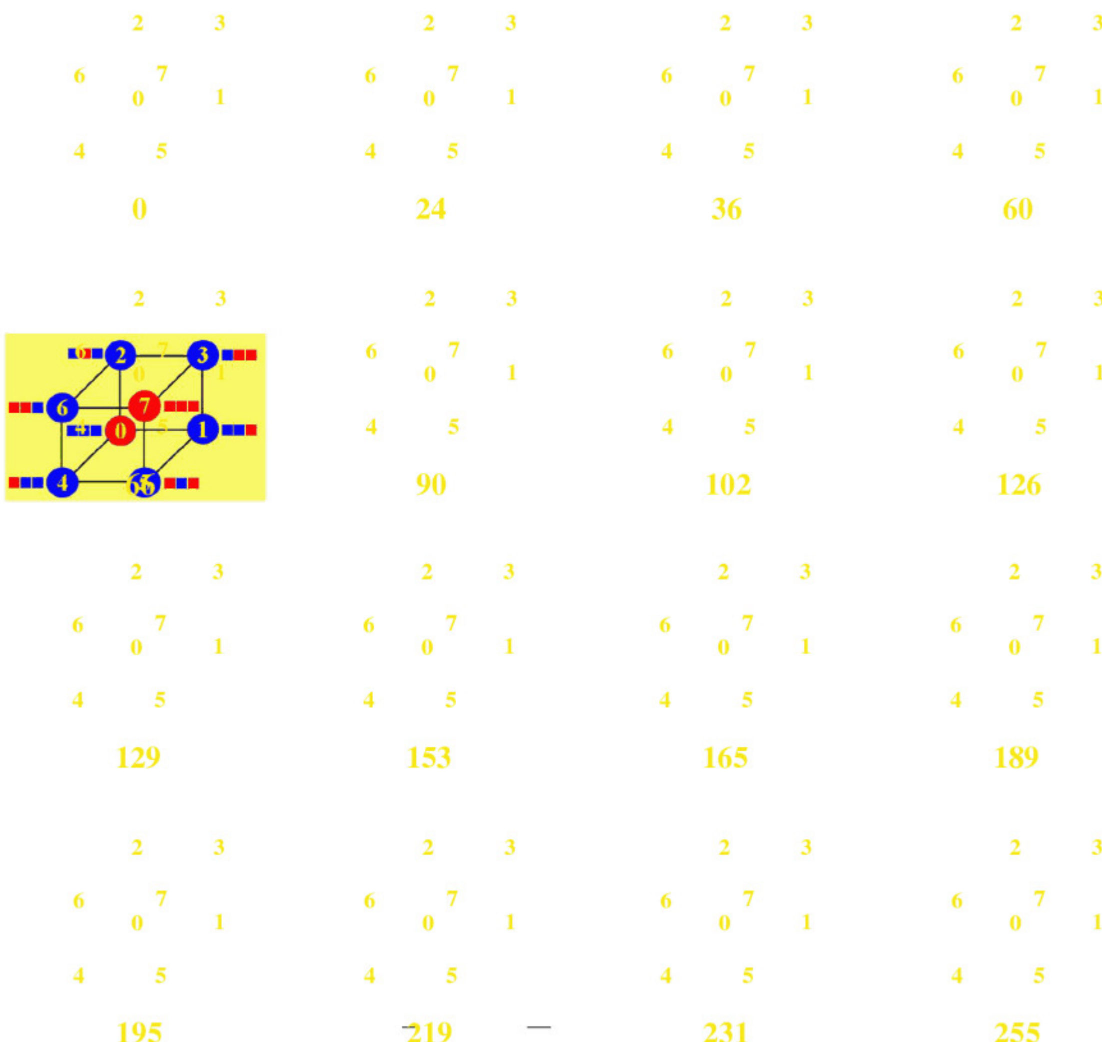


Fig. 7. Geometrical constructions for deriving (a) $N^\dagger \triangleq T^\dagger(N)$, (b) $\bar{N} \triangleq \bar{T}(N)$, and (c) $N^* \triangleq T^*(N)$.

Table 7. A gallery of 16 centrally-symmetric local rules.



1.7.3. *Perfect complementary rules*

It follows from the geometrical construction of the global complementary transformation $T(N) \mapsto N$ in Fig. 7(b) that if all four pairs of diagonally opposite vertices {(⓪, ⑦), (②, ⑤), (③, ④), (①, ⑥)} of N have identical colors, respectively, then

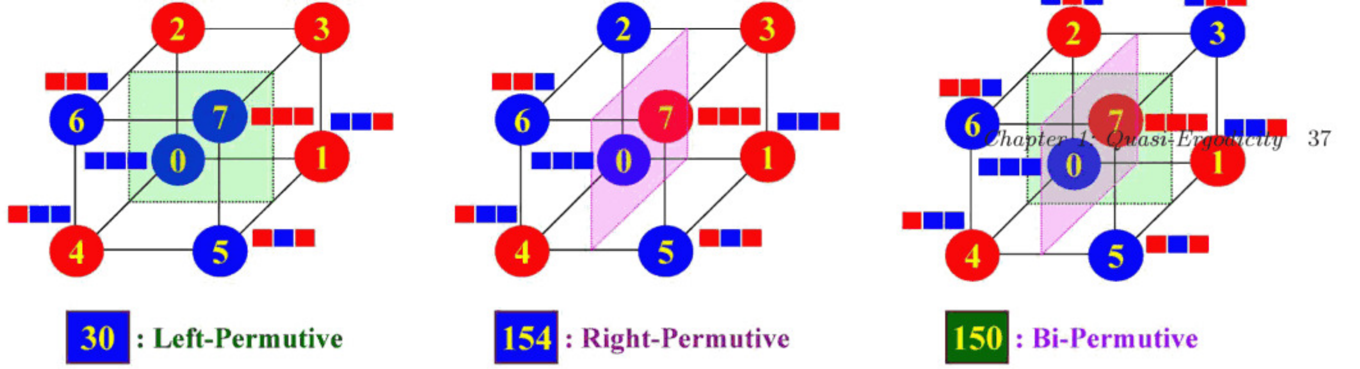
$$N \triangleq T(N) = T^C(N) \triangleq N^C \tag{13}$$

We will henceforth call such perfectly symmetrical transformations (relative to the complementation transformation) *perfect complementations*. The set of all perfect complementary rules are listed in Table 7. Since these Boolean cubes exhibit perfect symmetry in color with respect to the origin located at the *center* of the cube, we will henceforth call them *centrally-symmetric* local rules. Clearly, the local complementary space-time patterns of all centrally-symmetric local rules hold for all times.

1.7.4. *Permutive rules*[9]

There are 28 local rules whose Boolean cubes exhibit an *anti-symmetry* with respect to some vertical plane through the center of the cube, as illustrated in Fig. 8.

[9]Permutive rules are originally defined by [Hedlund, 1969] on a formal topological setting. We have opted for an equivalent but geometrical definition via the Boolean cubes for pedagogical reasons. These two representations are equivalent, as shown in Appendix C.



(a) (b) (c)

Fig. 8. Geometrical illustrations of *permutive* rules. (a) Rule 30 is *Left-Permutive* because the colors of the vertices $\{\textcircled{0}, \textcircled{1}, \textcircled{2}, \textcircled{3}\}$ in the back face are the complement of the colors of the vertices $\{\textcircled{4}, \textcircled{5}, \textcircled{6}, \textcircled{7}\}$ in the front face. (b) Rule 154 is *Right-Permutive* because the colors of the vertices $\{\textcircled{1}, \textcircled{3}, \textcircled{5}, \textcircled{7}\}$ in the right face are the complement of the colors of the vertices $\{\textcircled{0}, \textcircled{2}, \textcircled{4}, \textcircled{6}\}$ on the left face. (c) Rule 150 is *Bi-Permutive* because it is both Left and Right Permutive.

A local rule N is said to be *Left-Permutive, iff* the vertical symmetry plane is parallel to the paper, as depicted by the "*green*" plane in Fig. 8(a). It is said to be *Right-Permutive, iff* the vertical symmetry plane is perpendicular to the paper, as depicted by the "*pink*" plane in Fig. 8(b). It is said to be *Bi-Permutive, iff* it is both *Left-* and *Right-Permutive* as depicted by the "*green*" and "*pink*" vertical symmetry planes, respectively. A local rule N is said to be *Permutive iff* N is *Left* and/or *Right-Permutive.*

An examination of the 256 Boolean cubes in Table 1 shows that there are only 16 Left-Permutive rules, 16 Right-Permutive rules, and 4 Bi-Permutive rules, as displayed in Tables 8–10, respectively. The *union* of all these local rules gives only 28 *distinct Permutive rules*, as exhibited in Table 11. We will show in the following sections that *Permutive rules* possess some remarkable properties.

1.7.5. *Superposition of local rules*

The eight Boolean cubes exhibited in Table 12 are *independent* in the sense that it is impossible to decompose any of them into the "*union*" of two or more simpler Boolean cubes by taking the logic "OR" operation between the colors of corresponding vertices, where "*red*" is coded "1" and "*blue*" is coded "0", respectively. Since each of these eight Boolean cubes contains *one, and only one*, red vertex, together they constitute a *basis function* where the "union" of two or more such rules can generate any of the remaining $256 - 8 = 248$ local rules, as depicted in Table 13. It is easy to prove that each rule N in Table 13 has a *unique* decomposition via the eight Boolean cubes basis functions in Table 12.

1.7.6. *Rules with explicit period-1 and/or period-2 orbits*

Recall from Fig. 6 that among the 256 local rules, 69 are endowed with robust *period*-1 (attractor or Isle-of-Eden) orbits, and another 25 rules are endowed with *period*-2 orbits. It is generally impossible to predict the bit-string pattern of such period-1 or period-2 orbits without actually evolving the rule from some initial state.

The purpose of this subsection is to prove a surprising and quite remarkable result asserting that the *period*-1 and *period*-2 bit strings of a large number of local rules can be *predicted* without carrying out any simulations. Such period-k bit string ($k = 1$ or $k = 2$) patterns are endowed upon those Boolean cubes whose *main-diagonal vertices* $\{\textcircled{0}, \textcircled{2}, \textcircled{5}, \textcircled{7}\}$ exhibit certain color combinations.

Explicit period-$(1, 2)$ *pattern theorem*

There are ten distinct color combinations among the four vertices $\{\textcircled{0}, \textcircled{2}, \textcircled{5}, \textcircled{7}\}$ on the *main-diagonal* plane of the Boolean cubes, labeled Type A, B, ..., J in Tables 14(A), (B), ..., (J) for which the corresponding local rules have an *explicit* period-1 and/or period-2 bit-string pattern, regardless of the colors of the remaining *nondiagonal* vertices $\{\textcircled{1}, \textcircled{3}, \textcircled{4}, \textcircled{6}\}$.