A PANORAMA OF PURE MATHEMATICS

JEAN DIEUDONNÉ

A Panorama of Pure Mathematics

As Seen by N. Bourbaki

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Introduction

This book is addressed to readers whose mathematical knowledge extends at least as far as the first two years of a university honors course. Its aim is to provide an *extremely sketchy* survey of a rather large area of modern mathematics, and a guide to the literature for those who wish to embark on a more serious study of any of the subjects surveyed.

By "Bourbaki mathematics" I mean, with very few exceptions, the set of topics covered in the exposés of the Séminaire Bourbaki. Since the beginning of their collective work, the collaborators of N. Bourbaki have taken a definite view of mathematics, inherited from the tradition of H. Poincaré and E. Cartan in France, and Dedekind and Hilbert in Germany. The "Eléments de Mathématique" have been written in order to provide solid foundations and convenient access to this aspect of mathematics, in a form sufficiently general for use in as many contexts as possible.

From 1948 onward, the Bourbaki group has organized a seminar, consisting in principle of 18 lectures each year. The purpose of these lectures is to describe those recent results that appear to the organizers to be of most interest and importance. These lectures, almost all of which have been published, now exceed 500 in number, and collectively constitute a veritable encyclopedia of these mathematical theories.

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No publication under the name of N. Bourbaki has ever described how the topics for exposition in the seminar have been chosen. One can therefore only attempt to discern common features by examining these choices from outside, and their relation to the totality of the mathematical literature of our age. I wish to make it clear that the conclusions I have drawn from this examination are my own, and do not claim in any way to represent the opinions of the collaborators of N. Bourbaki.

The history of mathematics shows that a theory almost always originates in efforts to solve a specific problem (for example, the duplication of the cube in Greek mathematics). It may happen that these efforts are fruitless, and we have our first category of problems:

(I) Stillborn problems (examples: the determination of Fermat primes, or the irrationality of Euler's constant).

A second possibility is that the problem is solved but does not lead to progress on any other problem. This gives a second class:

(II) Problems without issue (this class includes many problems arising from "combinatorics").

A more favorable situation is one in which an examination of the techniques used to solve the original problem enables one to apply them (perhaps by making them considerably more complicated) to other similar or more difficult problems, without necessarily feeling that one really understands why they work. We may call these

(III) Problems that beget a *method* (analytic number theory and the theory of finite groups provide many examples).

In a few rather rare cases the study of the problem ultimately (and perhaps only after a long time) reveals the existence of unsuspected underlying structures that not only illuminate the original question but also provide powerful general methods for elucidating a host of other problems in other areas; thus we have

(IV) Problems that belong to an active and fertile general *theory* (the theory of Lie groups and algebraic topology are typical examples at the present time).

However, as Hilbert emphasized, a mathematical theory cannot flourish without a constant influx of new problems. It has often happened that once the problems that are of the greatest importance for their consequences and their connections with other branches of mathematics have been solved, the theory tends to concentrate more and more on special and isolated questions (possibly very difficult ones). Hence we have yet another category:

(V) Theories in decline (at least for the time being: invariant theory, for example, has passed through this phase several times).

Finally, if a happy choice of axioms, motivated by specific problems, has led to the development of techniques of great efficacy in many areas of mathematics, it may happen that attempts are made with no apparent motive to modify these axioms somewhat arbitrarily, in the hope of repeating the success of the original theory. This hope is usually in vain, and thus we have, in the phrase of Pólya and Szegö†

† G. Pólya and G. Szegö, "Problems and Theorems in Analysis," Springer-Verlag, Berlin and New York, 1972.

(VI) Theories in a state of *dilution* (following the example of these authors, we shall cite no instances of this).

In terms of this classification, it appears to me that the majority of the topics expounded in the Séminaire Bourbaki belong to category (IV) and (to a lesser extent) category (III). This is, I believe, as objective an opinion as I can form, and I shall abstain from further comment.

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Since the number and variety of the lectures in the Séminaire make them difficult to use, I have grouped them into sections under a fairly small number of headings, each of which contains a closely related group of subjects. One of the characteristics of Bourbaki mathematics is its extraordinary *unity*: there is hardly any idea in one theory that does not have notable repercussions in several others, and it would therefore be absurd, and contrary to the very spirit of our science, to attempt to compartmentalize it with rigid boundaries, in the manner of the traditional division into algebra, analysis, geometry, etc. now completely obsolete. The reader should therefore attach no importance to this grouping, which is purely a matter of convenience; its aim is to provide a clear overall view, halfway between the chaos of the chronological order of the lectures, and fragmentation into a dust-cloud of minitheories. At the beginning of each section I have inserted an "organization chart" designed to illustrate graphically its connections with the others, with arrows to indicate the direction of influence.

Each section contains, to the extent that it is feasible, a rapid didactic exposition of the main questions to be considered. With a few exceptions, only those are mentioned that have been covered in the Séminaire Bourbaki; the order followed is not in general the historical order, and the infrequent historical indications make no pretence of being systematic. At the end of each section will be found a list of the mathematicians who have made significant contributions to the theories described, and a brief mention of the connections (where they exist) between these theories and the natural sciences.

Each section or heading is designated by a boldface capital letter followed by a Roman numeral. This designation refers to the place occupied by the heading in the *Table of subjects* (p. 5), the capital letter indicating the level at which the heading is placed. These levels range from top to bottom, roughly speaking in decreasing order of what might be called their "Bourbaki density," that is to say (without pretension to numerical accuracy, which would be absurd), the proportion of the topics covered by the Séminaire Bourbaki to the total mathematical literature relating to the heading concerned.

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The references have been organized in such a way as to serve as a guideline to readers who wish to learn more. References to the Séminaire Bourbaki are indicated by the letter B followed by the number of the exposé. They are augmented by references to:

(i) the Séminaires H. Cartan, denoted by the letter C followed by the year;

(ii) the expository lectures organized by the American Mathematical Society and published in its *Bulletin*; these are indicated by the letters BAMS followed by the volume number of the *Bulletin* and the name of the lecturer;

(iii) the *Symposia* organized by the American Mathematical Society, denoted by the letters SAMS followed by a roman numeral and (sometimes) the author's name;

(iv) the lectures given at the recent International Congresses of Mathematicians at Stockholm (1962), Nice (1970), and Vancouver (1974); these are indicated by the name of one of these cities and the lecturer's name (in the case of the Nice Congress, the figure I indicates a one-hour lecture, and an indication of the section of the Congress a half-hour lecture);

(v) the "Lecture Notes in Mathematics" published by Springer-Verlag, denoted by the letters LN followed by a number (and by an author's name, in the case of a colloquium or symposium);

(vi) various articles and books, denoted by the letter or the number in brackets under which they are listed in the bibliography.

No reference is given for mathematical terms currently used in the first two years of a university honors course. For others, either a brief explicit definition is given, or a reference to a textbook in the bibliography.

The headings at level **D** in the table of subjects are those of Bourbaki density zero. They refer to theories that have in part been fixed for a considerable time, and constitute, in the etymological sense of the word, the *classical* part of mathematics, which serves as a basis for the rest of the edifice. The reader will find these theories expounded in the volumes of the "Eléments de Mathématique" that have already been published. Research still continues in these various theories, about which I shall say nothing except to remark on the curious historical phenomenon of a science divided into two parts that in practice ignore each other, without apparently causing the least impediment to their respective developments.

I wish to thank readers whose comments enabled me to correct certain errors and omissions in the second (French) edition. At the end of each section I have appended a list of references given in the text, together with some additional ones for the reader's benefit.

TABLE OF SUBJECTS

Levels							
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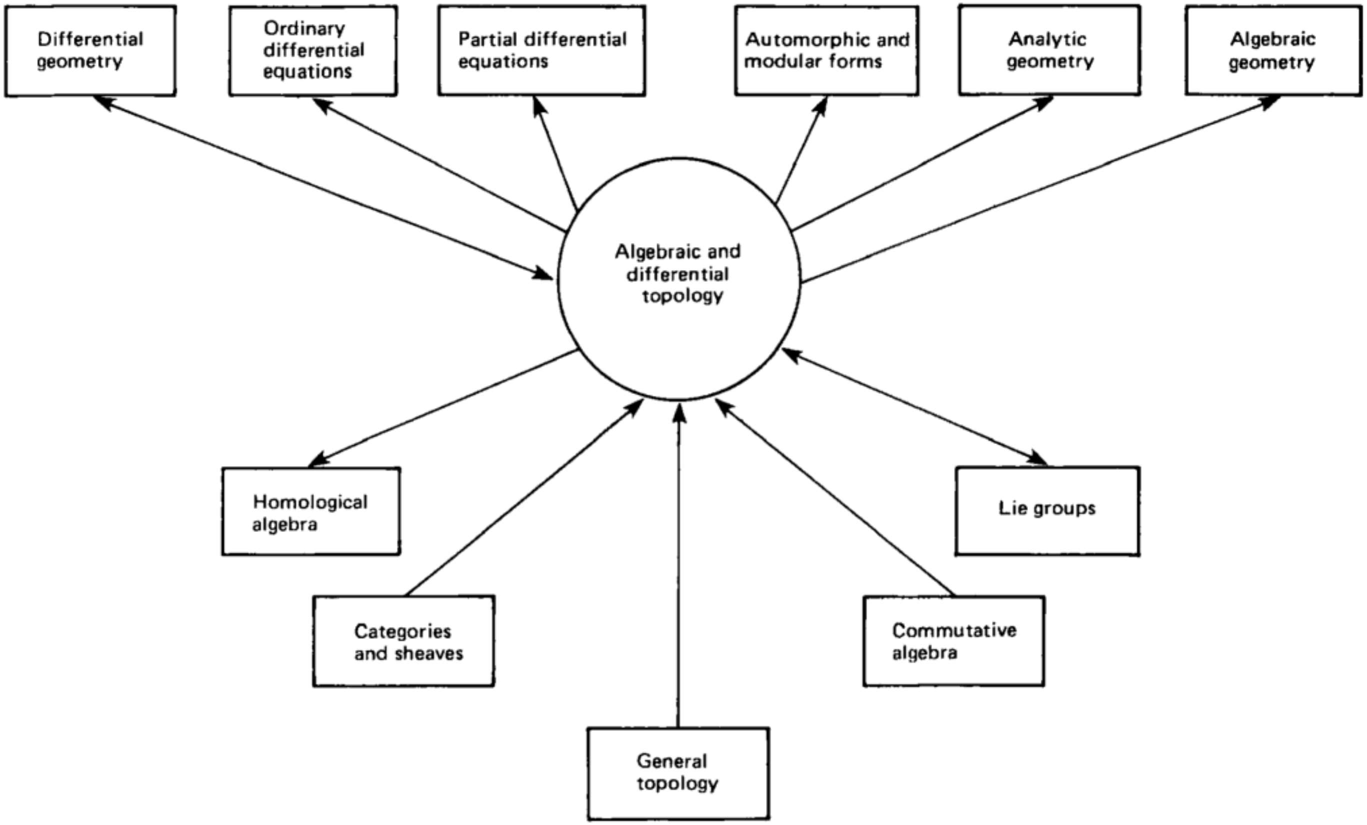
A I



Differential geometry
Ordinary differential equations
Partial differential equations
Automorphic and modular forms
Analytic geometry
Algebraic geometry
Algebraic and differential topology
Homological algebra
Lie groups
Categories and sheaves
Commutative algebra
General topology

A I

Algebraic and differential topology

It may already be predicted without great likelihood of error that the 20th century will come to be known in the history of mathematics as the century of *topology*, and more precisely of what used to be called "combinatorial" topology, and which has developed in recent times into *algebraic topology* and *differential topology*. These disciplines were created in the last years of the 19th century by H. Poincaré, in order to provide a firm mathematical basis for the intuitive ideas of Riemann. At first they developed rather slowly, and it was not until the 1930s that they took wing. Since then they have multiplied, diversified, and refined their methods, and have progressively infiltrated all other parts of mathematics; and there is as yet no indication of any slowing down of this conquering march.

1. Techniques

The initial problem of algebraic topology, roughly speaking, is to "classify" topological spaces: two spaces are to be put in the same "class" if they are homeomorphic. The general idea is to attach to each topological space "invariants," which may be numbers, or objects endowed with algebraic structures (such as groups, rings, modules, etc.) in such a way that homeomorphic spaces have the same "invariants" (up to isomorphism, in the case of algebraic structures). The ideal would be to have enough "invariants" to be able to *characterize* a "class" of homeomorphic spaces, but this ambition has been realized in only a very small number of cases (for recent progress, see Vancouver (Sullivan) and T. Price, *Math. Chronicle* **7** (1978)).

This original problem may be reformulated as the study of continuous mappings that are bijective and bicontinuous. In this form it is merely one of a whole series of problems of existence of continuous mappings subjected to other conditions, such as to be injective, or surjective, or to be sections or retractions of given continuous mappings, or extensions of given continuous mappings, etc. [171 *bis*]. All these problems are amenable to the methods of algebraic topology.

The idea of homeomorphism is related to, but distinct from, the more intuitive notion of "deformation." In order to formulate mathematically the idea that a subspace Y_1 of a topological space X can be "deformed" into another subspace Y_2, one is led to the following definition: denoting by I the interval [0, 1] in **R**, there exists a continuous mapping $(y, t) \mapsto F(y, t)$ of $Y_1 \times I$ into X such that (i) $F(y, 0) = y$ for all $y \in Y_1$, (ii) for each $t \in I$, the mapping $y \mapsto F(y, t)$ is a homeomorphism of Y_1 onto a subspace of X, and (iii) when $t = 1$, this subspace is Y_2. The mapping F is said to be an *isotopy* of Y_1 onto Y_2. The notion of isotopy is thus a *strengthening* of the notion of homeomorphism. The study of isotopy is difficult and has only recently led to substantial results (**B** 157, 245, 373; [86]).

Homotopy (C 1949, 1954; [50], [78], [170]). The notion that has become the most important in topology is a *weakening* of the notion of isotopy. Two continuous mappings g, h of a space X into a space Y are said to be *homotopic* if there exists a continuous mapping $F : X \times I \to Y$ such that $F(x, 0) = g(x)$ and $F(x, 1) = h(x)$, but with no conditions imposed on the mapping $x \mapsto F(x, t)$ for $t \neq 0, 1$. F is called a *homotopy* from g to h. The property of being homotopic is an equivalence relation on the set $\mathscr{C}(X, Y)$ of all continuous mappings of X into Y, and the set [X, Y] of classes of homotopic mappings is evidently an "invariant" of the two spaces X, Y. It is *functorial* (**C** I) in X and Y: if $\alpha : X_1 \to X$ (resp. $\beta : Y \to Y_1$) is a continuous mapping, and if $g, h \in \mathscr{C}(X, Y)$ are homotopic, then so also are $g \circ \alpha$ and $h \circ \alpha$ (resp. $\beta \circ g$ and $\beta \circ h$); whence we have a mapping $\alpha^* : [X, Y] \to [X_1, Y]$ (resp. $\beta_* : [X, Y] \to [X, Y_1]$).

The notion of homotopy leads to a "classification" of topological spaces that is coarser than classification by homeomorphism, but is much easier to handle. A continuous mapping $f : X \to Y$ is called a *homotopy equivalence* if there exists a continuous mapping $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to the identity mapping of X and $f \circ g : Y \to Y$ homotopic to the identity mapping of Y. If there exists a homotopy equivalence $f : X \to Y$, the spaces X and Y are said to have the *same homotopy type*. Most of the "invariants" of algebraic topology are invariants of homotopy type (and not merely invariants under homeomorphisms). For example, $\mathbf{R}^n$ (or more generally any topological vector space over **R**) and a space consisting of a single point have the same homotopy type (spaces having the homotopy type of a single point are said to be *contractible*).

Besides the general notion of homotopy, there are more restrictive notions, such as the *simple* homotopy equivalence of J. H. C. Whitehead for spaces endowed with a "cellular" subdivision (such spaces are called CW-*complexes* or *cell-complexes* [170]; they are generalizations of polyhedra (B 392; LN 48; BAMS 72 (Milnor)). Another variant is to consider homotopies $(x, t) \mapsto F(x, t)$ that are *independent of t in a given subspace* A of X; this leads to the notion of homotopy relative to a subspace. The case in which A consists of a single point is the most common. It is convenient to define a new category (**C** I) in which the objects (called "pointed spaces") are pairs (X, x_0) consisting of a topological space X and a point $x_0 \in X$, the morphisms $(X, x_0) \to (Y, y_0)$ being continuous mappings $f: X \to Y$ such that $f(x_0) = y_0$. A homotopy $(x, t) \mapsto F(x, t)$ between two such morphisms is then required to satisfy $F(x_0, t) = y_0$ for all $t \in I$. In this way we obtain an equivalence relation, for which the set of equivalence classes is again denoted by $[X, Y]$ if there is no risk of confusion.

Historically speaking, algebraic topology was at first mainly preoccupied with finite-dimensional spaces such as subspaces of $\mathbf{R}^n$. These are the spaces that arise most frequently in applications to other branches of mathematics. However, it is better to make no restrictive hypotheses on the dimension, because it is then possible to use with great advantage constructions that, when applied to finite-dimensional spaces, lead in general to spaces of infinite dimension: for example, for two spaces X and Y, the space $\mathscr{C}(X, Y)$ of all continuous mappings of X into Y, endowed with the "compact-open" topology (for each compact subset $K \subset X$ and each open subset U of Y, the sets $W(K, U) = \{f \in \mathscr{C}(X, Y): f(K) \subset U\}$ form a basis of open sets for this topology). An important special case is the *space of paths* $\mathscr{C}(I, X)$. If (X, x_0) is a pointed space, $\mathscr{C}(I, X)$ is also a pointed space, the distinguished point being the constant mapping $\tilde{x}_0: I \to x_0$. The *loop-space* of (X, x_0) is the pointed space $(\Omega(X, x_0), \tilde{x}_0)$ consisting of the paths $\gamma: I \to X$ such that $\gamma(0) = \gamma(1) = x_0$; it is usually denoted by ΩX if there is no risk of confusion.

All these definitions are functorial (**C** I). The functor $\mathscr{C}_0(X, Y)$ of morphisms $(X, x_0) \to (Y, y_0)$ of pointed spaces is the analog in this category of the Hom functor for modules (**C** I). There is also a construction that gives an analog of the tensor product: in the product space $X \times Y$, we consider the subspace $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$, and the quotient space $X \wedge Y = (X \times Y)/(X \vee Y)$, obtained by identifying all the points of $X \vee Y$ to a single point (which is the distinguished point of $X \wedge Y$). In particular, for each pointed space X, $\mathbf{S}_1 \wedge X$ is called the *suspension* of X, written SX. If X is a sphere $\mathbf{S}_n$, then $\mathbf{S}_1 \wedge \mathbf{S}_n$ is homeomorphic to $\mathbf{S}_{n+1}$.

In the category of pointed spaces, Ω and S are *adjoint functors* (**C** I); that is to say there exists a canonical functorial bijection

$$\mathscr{C}_0(SX, Y) \simeq \mathscr{C}_0(X, \Omega Y) \tag{1}$$

and therefore also a canonical bijection for the homotopy classes

$$[SX, Y] \simeq [X, \Omega Y]. \tag{2}$$

Homotopy groups. Juxtaposition of loops in the space ΩY is a law of composition, which gives rise to a group structure when we pass to the quotient by considering the homotopy classes of these loops. It follows that the set $[X, \Omega Y]$ (and hence also $[SX, Y]$) is canonically endowed with a group structure. The groups $[X, \Omega^k Y]$ are *commutative* for $k \geqslant 2$. For $n \geqslant 1$, the group $[S_0, \Omega^n X]$ is called the *nth homotopy group* of X and is denoted by $\pi_n(X)$. By virtue of (2), it can also be written $[S_k, \Omega^{n-k} X]$ for $0 \leqslant k \leqslant n$, and in particular as $[S_n, X]$. The group $\pi_1(X)$, which is the group of homotopy classes of loops on X, is also called the *fundamental group* of X; in general it is not commutative, whereas the $\pi_n(X)$ for $n \geqslant 2$ are commutative.

If we take $Y = SX$ in (1), the identity mapping $SX \to SX$ determines a canonical mapping $X \to \Omega SX$, hence a canonical mapping (the *suspension mapping*) $[X, Y] \to [X, \Omega SY] \simeq [SX, SY]$. By iteration we obtain a sequence of mappings

$$[X, Y] \to [SX, SY] \to [S^2X, S^2Y] \to \cdots,$$

all of which except for the first are group homomorphisms. If Y is a polyhedron (see later) and X is finite-dimensional, these homomorphisms are isomorphisms from a certain point onward. In particular, we have suspension homomorphisms $\pi_n(X) \to \pi_{n+1}(SX)$.

Homotopy groups are difficult to calculate. The case most intensively studied is that of the homotopy groups of spheres $\pi_m(S_n)$. We have $\pi_m(S_n) = 0$ for $m < n$, and $\pi_n(S_n) = \mathbf{Z}$; but the groups $\pi_m(S_n)$ for $m > n$ are far from being completely known. In the sequence of suspension homomorphisms

$$\pi_{n+k}(S_n) \to \pi_{n+k+1}(S_{n+1}) \to \cdots,$$

the groups end by being isomorphic to $\pi_{2k+2}(S_{k+2})$ (called the *stable* groups). The groups $\pi_m(S_n)$ are known to be *finite* for $m > n$, with the single exception of the groups $\pi_{2n-1}(S_n)$, n even (Serre's theorem). The stable groups are known explicitly for the first 60 or so values of k, and so far do not appear to satisfy any simple general laws; on the other hand, there are general results for certain p-primary components of these groups (B 44; C 1954–5; [182]).

By contrast, for the homotopy groups of the (compact) *classical groups*, there is better information. For the unitary group $\mathbf{U}(n)$, the groups $\pi_i(\mathbf{U}(n))$ are known for $i \leqslant 2n + 2$; in particular, for $i < 2n$ we have $\pi_i(\mathbf{U}(n)) = \mathbf{Z}$ for i odd, and $\pi_i(\mathbf{U}(n)) = 0$ for i even (Bott's periodicity theorem). For the orthogonal group $\mathbf{O}(n)$, there are analogous results (with period 8) (B 172, 215, 259; C 1959–60).

Homotopy and cohomology [50]. Let n be an integer ≥ 1 and let G be a group (commutative if $n \geq 2$). A space X is said to be an *Eilenberg–MacLane space* $K(G, n)$ if $\pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) = G$. Such spaces exist for all (G, n). If G, G_1 are two groups, $[K(G, n), K(G_1, n)]$ is in canonical one–one correspondence with the set $\mathrm{Hom}(G, G_1)$ of homomorphisms of G into G_1.

In particular, all the spaces K(G, n) for given G and n have the same homotopy type. For $n \geqslant 2$, ΩK(G, n) is a K(G, $n-1$).

For a space X, the set [X, K(G, n)] is naturally endowed with a commutative group structure, by virtue of the homotopy equivalence

$$K(G, n) \to \Omega^m K(G, m+n).$$

This group is called the *nth cohomology group of* X *with coefficients in* G, and is denoted by $H^n(X, G)$.

There are canonical isomorphisms

$$[X, K(G, n)] \simeq [X, \Omega^m K(G, m+n)] \simeq [S^m X, K(G, m+n)].$$

Since the mapping $K(G, n) \to \Omega^m K(G, m+n)$ corresponds canonically to a mapping $S^m K(G, n) \to K(G, m+n)$, the isomorphism above is also the composition

$$[X, K(G, n)] \to [S^m X, S^m K(G, n)] \to [S^m X, K(G, m+n)].$$

This leads to a generalization of the groups $H^n(X, G)$. A *spectrum of spaces* is a sequence $\mathbf{B} = (B_m)_{m \in \mathbf{Z}}$ of pointed spaces and continuous mappings of pointed spaces $SB_m \to B_{m+1}$. For each space X, we have therefore a sequence of homomorphisms of commutative groups

$$\cdots \to [S^m X, B_{m+n}] \to [S^{m+1} X, B_{m+n+1}] \to \cdots$$

and the direct limit (**C** I) $H^n(X, \mathbf{B})$ of this sequence is called the *nth (generalized) cohomology group of* X relative to the spectrum **B** (LN 28, 99). The most important generalized cohomology groups come from K-theory (**B** I).

Homology and cohomology. The cohomology groups $H^n(X, \mathbf{Z})$ have an earlier history, and were originally defined in terms of other groups, the *homology groups* $H_n(X, \mathbf{Z})$. A space X is said to be a (generalized) *polyhedron* if it is homeomorphic to the geometric realization of a simplicial set (**B** I), and it is regarded as endowed with the additional structure consisting of the "n-simplexes" of this polyhedron, i.e., the images s_n in X of the $\{x_n\} \times \Delta(n)$, where x_n is an n-simplex of the simplicial object of which X is the geometric realization (n being any integer $\geqslant 0$). The classical notion is that of a *finite* polyhedron, the geometric realization of a simplicial set having only a finite number of simplexes. If X is a polyhedron and A is a commutative ring, we may consider for each integer $n \geqslant 0$ the A-module C_n of formal linear combinations of n-simplexes of X with coefficients in A. It is immediate that the C_n form a chain complex (**B** I) with respect to the boundary operator d_n defined by

$$d_n s_n = \sum_{i=0}^{n} (-1)^i F_i^n s_n,$$

where $F_i^n s_n$ is the image of $\{F_i^n x_n\} \times \Delta(n-1)$ in X. The *nth homology module* $H_n(X, A)$ is by definition the nth homology A-module of this complex of A-modules.

For an arbitrary topological space X, we define C_n to be the A-module of formal linear combinations of continuous mappings $\Delta(n) \to X$, and hence we obtain the *singular homology* A-modules $H_n(X, A)$. A space is said to be *triangulable* if it is homeomorphic to a polyhedron, and the images of the simplexes of this polyhedron are said to form a *triangulation* of the space. For a topological space homeomorphic to a finite polyhedron, the singular homology of the space is isomorphic to that of the polyhedron (defined in terms of the simplexes of the latter). Recently, Sullivan has shown how the homotopy of a polyhedron may be studied by generalizing the notion of differential form (B 475).

If we now consider, for a finite polyhedron X, the cochain complex (C_n^*) obtained by duality from (C_n) (**B** I), then the nth cohomology object of this complex is isomorphic to the A-module $H^n(X, A)$ defined above by means of homotopy.

The reason for these isomorphisms is to be found in the axiomatic characterization of cohomology (Eilenberg–Steenrod): the $H^n(X, G)$ satisfy a small number of properties that characterize them, in the sense that two systems of groups that satisfy these properties for finite polyhedra are necessarily isomorphic (C 1948–9; LN 12; [52], [170]). We remark also that the generalized cohomology groups $H^n(X, \mathbf{B})$ defined by a spectrum of spaces **B** satisfy the Eilenberg–Steenrod axioms, with the exception of the "dimension axiom," which fixes the cohomology of a space consisting of a single point (LN 28, 99). We can also define (generalized) homology groups relative to a spectrum **B**: for we have a sequence of homomorphisms of commutative groups

$$\cdots \to \pi_{n+k}(B_k \wedge X) \to \pi_{n+k+1}(B_{k+1} \wedge X) \to \cdots$$

arising from the mappings $SB_k \to B_{k+1}$, and $H_n(X, \mathbf{B})$ is defined to be the direct limit (**C** I) of this sequence. The groups $H^n(X, \mathbf{B})$ and $H_n(X, \mathbf{B})$ are related by *duality* properties that generalize the relations indicated above for the classical homology and cohomology of finite polyhedra (LN 28, 99; Nice (Mischenko); [194], [195]).

Cohomology and homology rings. For a space X and a commutative ring A, $H^\cdot(X, A) = \bigoplus_{n \geq 0} H^n(X, A)$ is a *graded* A-module (**C** II), and $X \mapsto H^\cdot(X, A)$ is a contravariant functor from the category of topological spaces into the category of graded A-modules. The diagonal mapping $\delta : X \to X \times X$ therefore defines a homomorphism of graded A-modules

$$H^\cdot(X \times X, A) \to H^\cdot(X, A).$$

On the other hand, under fairly weak conditions on X, Y, and A, $H^{\bullet}(X \times Y, A)$ is isomorphic to the graded tensor product A-module $H^{\bullet}(X, A) \otimes_A H^{\bullet}(Y, A)$ (Künneth theorem). The preceding homomorphism therefore defines on $H^{\bullet}(X, A)$ a structure of a graded A-*algebra*, which is *anticommutative* (i.e., $x_p x_q = (-1)^{pq} x_q x_p$ for $x_p \in H^p(X, A)$, $x_q \in H^q(X, A)$).

Likewise, we may consider the graded A-module $H_{\bullet}(X, A) = \bigoplus_{n \geqslant 0} H_n(X, A)$; but this time the functor $X \mapsto H_{\bullet}(X, A)$ is covariant, and we cannot define a "homology ring" in the same way as before. However, if X is a compact connected triangulable *manifold* of dimension n, and if C′ is a p-chain and C″ a q-chain (not necessarily belonging to the same triangulation), it is possible under certain conditions of "general position" to define an "intersection $(p + q - n)$-chain" $C' \cdot C''$ (provided that $p + q \geqslant n$) in such a way that $C' \cdot C''$ is a $(p + q - n)$-cycle if C′ and C″ are cycles, and that in this case the homology class of $C' \cdot C''$ depends only on those of C′ and C″. In this way we obtain on $H_{\bullet}(X, A)$ a structure of a graded anticommutative A-algebra, by reason of the fact that for any two given homology classes, it is always possible to find cycles in these classes that are in general position [152]. For a 0-cycle $C = \sum_j n_j P_j$, where the P_j are distinct points of X and the n_j are integers (of either sign), the number $\sum_j n_j$ is called the *degree* of C, denoted by deg(C); it depends only on the homology class of C. If C′ (resp. C″) is a p-cycle (resp. q-cycle) with $p + q = n$, and C′, C″ are in general position, the number $\deg(C' \cdot C'')$ is called the *intersection number* of C′ and C″, denoted by $(C' \cdot C'')$; it depends only on the homology classes of C′ and C″.

In particular, the intersection product determines a canonical bilinear mapping $H_p(X, \mathbf{R}) \times H_{n-p}(X, \mathbf{R}) \to \mathbf{R}$ that, for a *compact* manifold, puts $H_p(X, \mathbf{R})$ and $H_{n-p}(X, \mathbf{R})$ in duality ("Poincaré duality"). Hence we have a canonical isomorphism $H^p(X, \mathbf{R}) \simeq H_{n-p}(X, \mathbf{R})$ (which, however, is *not* valid for an arbitrary finite polyhedron).

Fibrations. Let $p : X \to B$ be a continuous mapping and let F be a topological space. The space X is said to be a *locally trivial fiber bundle* with *base* B, *fiber* F, and *projection p* if, for each point $b \in B$, there exists an open neighborhood U of b and a homeomorphism $\varphi : U \times F \to p^{-1}(U)$ such that $p(\varphi(y, z)) = y$ for all $y \in U$ and $z \in F$ (in other words, X is "locally" (over B) a *product*). For each $y \in U$, the "fibers" $p^{-1}(y)$ are all homeomorphic to F.

A *covering* X of B is a locally trivial bundle over B with discrete fibers. A *vector bundle* is such that in the above definition F and each fiber $p^{-1}(y)$ is a vector space over **R**, and for each $y \in U$ the mapping $z \mapsto \varphi(y, z)$ is a linear bijection of F onto $p^{-1}(y)$. The classic example is the tangent bundle T(B) of a differential manifold B, in which the fibers are the tangent spaces at the points of B [D, Chapter 16]. A *principal bundle* is a fiber bundle X with projection $p : X \to B$, endowed with the additional structure consisting of the action of a topological group G on X, such that this action is continuous, the orbits of G are the fibers $p^{-1}(y)$, and G acts simply transitively on each fiber ([D, Chapter 16], [87], [170], [171]).

When G acts continuously on a space E, we can associate canonically to a principal G-bundle X a bundle over B with fibers homeomorphic to E. The bundles obtained in this way are called *bundles with structure group* G. For example, a vector bundle over a differential manifold B may be regarded as a bundle with structure group the orthogonal group $\mathbf{O}(n)$, where n is the dimension of the fibers ([D, Chapter 16], [87]).

These definitions can be transposed into other categories, for example, categories of manifolds of various types (see below): we have simply to replace the continuous mappings by morphisms of the category in question.

An important property of fiber bundles is the *homotopy lifting property*: if P is a polyhedron, $g : P \to X$ a continuous mapping of P into a bundle X with base B and projection p, and if $F : P \times I \to B$ is a homotopy from $f = p \circ g$ to the mapping $z \mapsto F(z, 1)$, then there exists a homotopy $G : P \times I \to X$ such that $p \circ G = F$. More generally, a mapping $p : X \to B$ is called a *Serre fibration* (or simply a *fibration*) if it satisfies the homotopy lifting property. A typical example is the mapping $p : E(B) \to B$ where, for a pointed space (B, b_0), E(B) is the space of paths $I \to B$ with origin b_0, and p maps each path to its endpoint, so that $p^{-1}(b_0) = \Omega B$. It can be shown that every continuous mapping can be factorized into the composition of a fibration and a homotopy equivalence: this result often makes it possible to reduce the study of an arbitrary continuous mapping to that of a fibration.

If X is a fiber bundle with base B and projection p, and if $f : B' \to B$ is a continuous mapping, we define a fiber bundle X′ over B′ by taking X′ to be the set of points $(b', x) \in B' \times X$ such that $f(b') = p(x)$. The restriction $p' : X' \to B'$ of the first projection defines X′ as a fiber bundle over B′; this bundle is denoted by $f^*(X)$ and is called the *inverse image* of X by f. At each point $b' \in B'$, the fiber $p'^{-1}(b')$ is canonically homeomorphic to $p^{-1}(f(b'))$. If X is a vector bundle (resp. a principal G-bundle), then so is X′. There is an analogous definition for Serre fibrations.

This construction leads in particular to a *classification* of principal bundles with given group G over the most familiar types of space. It can be shown that there exists a "classifying space" BG and a principal bundle E with base BG and group G, which is contractible and such that every principal bundle with group G and base B is isomorphic to a bundle $f^*(E)$ for some continuous mapping $f : B \to BG$; moreover, two such bundles are isomorphic if and only if the corresponding mappings of B into BG are *homotopic*. There is an analogous property for the classification of bundles with structure group G. This leads to the definition of cohomological invariants attached to the isomorphism classes of bundles over B: the mapping $f : \mathrm{B} \to \mathrm{BG}$ defines a homomorphism of cohomology rings

$$f^* : \mathrm{H}^{\cdot}(\mathrm{BG}, \mathrm{A}) \to \mathrm{H}^{\cdot}(\mathrm{B}, \mathrm{A}).$$

The elements of the image of f^* are called *characteristic classes* of the bundle $f^*(\mathrm{E})$; since they do not vary when f is replaced by a homotopic mapping, they are invariants of the isomorphism class of $f^*(\mathrm{E})$, which play a large role in numerous questions of differential topology, differential geometry, and global analysis (BAMS 75 (F. Peterson)). The most important are the Stiefel–Whitney classes, Pontrjagin classes, and Chern classes; the first two correspond to orthogonal groups, the third to the unitary group [126].

The notion of fibration also enables us to characterize homotopy types by a system of invariants. Given a sequence of groups $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_n, \ldots$, commutative for $n \geqslant 2$, we define a sequence of spaces $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n, \ldots$, where $\mathrm{X}_1 = \mathrm{K}(\mathrm{G}_1, 1)$ and X_n for $n \geqslant 2$ is a bundle with base X_{n-1} and fiber $\mathrm{K}(\mathrm{G}_n, n)$. The inverse limit (**C** I) X of the sequence (X_n) is such that $\pi_n(\mathrm{X}) = \mathrm{G}_n$ for all n, and every space Y has the same homotopy type as such an inverse limit; this homotopy type is characterized by the G_n and, for each $n \geqslant 2$, the isomorphism class of the bundle X_n with base X_{n-1}; it can be shown that these isomorphism classes are in one–one correspondence with cohomology classes in $\mathrm{H}^{n+1}(\mathrm{X}_{n-1}, \mathrm{G}_n)$ (Postnikov's construction).

If $p : \mathrm{X} \to \mathrm{B}$ is a fibration of pointed spaces, and $\mathrm{F} = p^{-1}(b_0)$, where b_0 is the distinguished point of B, there is an exact sequence (**C** I) of homotopy groups

$$\pi_1(\mathrm{B}) \leftarrow \pi_1(\mathrm{X}) \leftarrow \pi_1(\mathrm{F}) \leftarrow \pi_2(\mathrm{B}) \leftarrow \cdots \leftarrow \pi_n(\mathrm{B}) \leftarrow \pi_n(\mathrm{X}) \leftarrow \pi_n(\mathrm{F}) \leftarrow \pi_{n+1}(\mathrm{B}) \leftarrow \cdots.$$

For cohomology, the relations between the cohomology groups of B, X, and F are more complex and are expressed by the *spectral sequence of a fibration* (B 44; C 1958–9; LN 2).

We remark that a space E (even if "very good," for example, a homogeneous space of a Lie group) may admit no "nontrivial" fibration (i.e., in which neither the base nor the fiber consists of a single point) (B 472).

These are the most fundamental basic notions of algebraic and differential topology. In addition there are a considerable number of auxiliary notions and various geometrical or topological constructions, in which all the techniques of homological algebra may be brought into play (B 54; C 1954–5, 1958–9; BAMS 74 (Heller), 77 (M. Curtis); SAMS XXII; LN 2, 12, 13, 157, 161, 168, 368) and more recently techniques inspired by commutative algebra and group theory, such as Galois theory, the theory of nilpotent groups, localization and completion of rings (Nice C 2 (Sullivan); LN 304, 418; [79]), or the theory of formal groups (B 408).

There exists also a purely combinatorial version of the notions of homotopy and of fibration, in which there is no longer any mention of continuous mappings, but only simplicial sets (**B** I); often it is more convenient to work in this category, and then pass back to topology by consideration of "geometric realizations" of these sets (Kan theory: B 199; C 1954–5, 1956–7; LN 43, 252, 271; [119]).

2. Results

We shall encounter applications of algebraic or differential topology in almost all the great mathematical theories of the present age. Here we shall restrict our survey to problems whose initial formulation has obvious topological aspects.

As a general rule, a *positive* solution of such a problem usually consists of an effective construction of the solution by geometrico-topological methods; on the other hand, a *negative* answer is generally obtained by showing that a *positive* solution, if it existed, would imply certain relations between topological invariants, and then by showing that these relations cannot be satisfied.

The different sorts of "manifolds." Riemann and Poincaré were led to develop topological notions in the context of the spaces most frequently encountered in classical analysis and geometry, namely, "manifolds." A *topological manifold* of dimension n is a metrizable space X in which each point admits a neighborhood U that is homeomorphic to an open subset of $\mathbf{R}^n$; an *atlas* of X is a family of such homeomorphisms $\varphi_\alpha : \mathrm{U}_\alpha \to \varphi_\alpha(\mathrm{U}_\alpha) \subset \mathbf{R}^n$, where the U_α form an open covering of X (observe that this definition makes sense only by virtue of the celebrated theorem of Brouwer on the *invariance of dimension*, namely, that there exists no homeomorphism of an open subset of $\mathbf{R}^m$ onto an open subset of $\mathbf{R}^n$ if $m \neq n$). We may impose additional structure on X by requiring the existence of an atlas with supplementary conditions on the *transition homeomorphisms* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathrm{U}_\alpha \cap \mathrm{U}_\beta) \to \varphi_\beta(\mathrm{U}_\alpha \cap \mathrm{U}_\beta)$ for each pair of indices such that $\mathrm{U}_\alpha \cap \mathrm{U}_\beta \neq \varnothing$. In particular we define in this way the notions of *piecewise-linear manifold*, *differential manifold*, *real-analytic manifold*, and *complex-analytic manifold* by requiring the $\varphi_\beta \circ \varphi_\alpha^{-1}$ to be respectively piecewise-linear, of class C^∞, real-analytic, or complex-analytic (in which case n must be even and $\mathbf{R}^n$ identified with $\mathbf{C}^{n/2}$). The topological manifolds form a category TOP, in which the morphisms are continuous mappings. The other types of manifold also form categories in which the morphisms are continuous mappings which, relative to the distinguished "charts" for the type of manifold in question, are "locally" piecewise-linear, of class C^∞, real-analytic, or complex-analytic, respectively; the first two of these categories are denoted by PL and DIFF. We shall see in (**A** II) and (**A** VIII) the consequences, for the topology of a space, of the existence of a differential or analytic manifold structure.

After the introduction of simplicial methods by Poincaré, the question naturally arose of whether a topological manifold necessarily admits a PL-manifold structure, and whether such a structure is unique ("Hauptvermutung"). Again, it can be shown that every DIFF-manifold can be "triangulated" and hence endowed with an essentially unique PL-structure; conversely, the question arises of whether every PL-manifold admits a DIFF-manifold structure, and whether such a structure is unique. Finally, there is the question of classifying TOP (or PL, or DIFF) manifolds having the same homotopy type, in terms of "concordance" (a weakened version of isotopy that takes account of the structures of manifold under consideration) [188].

These problems have been almost completely resolved.

The general idea is to work in the tangent bundle (for the category DIFF) or analogous constructs (microbundles, Spivak bundles) for the other categories. This introduces classifying spaces: for the category DIFF, it is the classifying bundle BO of the direct limit (**C** I) of the orthogonal groups $\mathbf{O}(n)$ as $n \to \infty$; for the other two categories, there are analogous spaces BPL, BTOP, with fibrations

$$\mathrm{BO} \to \mathrm{BPL} \to \mathrm{BTOP}$$

in which the fibers of the distinguished points are denoted by PL/O and TOP/PL. To a DIFF (resp. PL, TOP) structure on M there corresponds therefore a continuous mapping f of M into BO (resp. BPL, BTOP). For a PL-manifold to admit a DIFF-structure, it is necessary and sufficient that the mapping $f : \mathrm{M} \to \mathrm{BPL}$ should factorize into $\mathrm{M} \to \mathrm{BO} \to \mathrm{BPL}$, and the concordance classes of the DIFF-structures on M are then in one–one correspondence with [M, PL/O]. The first particular cases of these theorems were the celebrated example (Milnor) of an "exotic" DIFF-structure on the sphere $\mathbf{S}_7$ (not isomorphic to the usual structure), and an example due to Kervaire of a PL-manifold of dimension 10 that does not admit a DIFF-structure. When $\mathrm{M} = \mathbf{S}_n$, the set [M, PL/O] has a natural structure of a finite group Θ_n for $n > 4$ (Kervaire–Milnor): for example, Θ_{11} is cyclic of order 992. The passage from TOP structures to PL structures is more complex, but the set [M, TOP/PL] plays a preponderant role; an essential fact (Kirby–Siebenmann) is that TOP/PL has the homotopy type of the Eilenberg–MacLane space $\mathrm{K}(\mathbf{Z}/2\mathbf{Z}, 3)$. It follows that if M is a compact topological manifold of dimension $\geqslant 5$, there is an "obstruction" to the existence of a PL-structure on M that is a cohomology class in $\mathrm{H}^4(\mathrm{M}, \mathbf{Z}/2\mathbf{Z})$; if this class is zero, the classes of possible PL-structures are in one–one correspondence with $H^3(M, \mathbf{Z}/2\mathbf{Z})$. There are explicit examples of five-dimensional manifolds that have no PL-structure, and others having several nonisomorphic PL-structures (B 263, 280, 362; Nice I (W. Browder, C. T. C. Wall); Nice C 2 (Siebenmann); LN 197; [210]).

These results are the culmination of a whole series of researches pursued over a decade by many mathematicians. Besides the general techniques of algebraic topology, great use is made of cobordism and the theory of immersions (see below), and especially of the technique called "surgery," which comes from Morse theory. If $\mathbf{D}_n$ denotes the closed unit ball in $\mathbf{R}^n$, $\mathbf{D}_n$ is homeomorphic to $\mathbf{D}_k \times \mathbf{D}_{n-k}$, and its boundary is therefore the union of $\mathbf{S}_{k-1} \times \mathbf{D}_{n-k}$ and $\mathbf{D}_k \times \mathbf{S}_{n-k-1}$. A *handle* of type k in an n-dimensional manifold M is a closed subset A of M homeomorphic to $\mathbf{D}_k \times \mathbf{D}_{n-k}$, the intersection of A and $\overline{M - A}$ being the portion $\mathbf{S}_{k-1} \times \mathbf{D}_{n-k}$ of the boundary of A (the terminology is justified only when $k = 1$). Surgery on such a handle is a geometrical operation that results in replacing the k-handle by an $(n - k)$-handle (B 230, 397; BAMS 68 (A. Wallace); SAMS III (Milnor); [26], [188]).

Finally, we remark that a good proportion of the results on finite-dimensional manifolds become simpler for manifolds of infinite dimension (in which the "models" $\mathbf{R}^n$ are replaced by a Hilbert space); it is remarkable that this theory has led to a proof of a conjecture of J. H. C. Whitehead on *finite* cell-complexes, namely, that every homeomorphism of such complexes is a simple homotopy equivalence (B 428).

The Poincaré conjecture. Elementary algebraic topology shows that every compact, orientable, simply connected surface is homeomorphic to the sphere $\mathbf{S}_2$. Poincaré conjectured that the same is true for manifolds of dimension 3 and $\mathbf{S}_3$: at present, neither proof nor counterexample is known. The conjecture may be generalized to any dimension: if a compact, simply connected manifold of dimension n has the same homology (or cohomology) as $\mathbf{S}_n$, is it homeomorphic to $\mathbf{S}_n$? Surprisingly, this question has been answered affirmatively for $n \geqslant 5$, first by Smale for DIFF-manifolds, then by Stallings for PL-manifolds, and by M. H. A. Newman for TOP-manifolds; for $n = 4$, as for $n = 3$, the problem remains open (B 208, 230; BAMS 69 (Smale); C 1961–2).

Cobordism. The modern flowering of differential topology can be dated from Thom's solution (1954) of two problems posed earlier by Steenrod: in a differential manifold M, when is a homology class "represented" by a submanifold, and when is an n-dimensional manifold the boundary of an $(n + 1)$-dimensional manifold? Thom's principal idea was to reduce these problems to problems of homotopy of mappings into a "Thom complex" constructed from a ball bundle associated with a principal bundle over a classifying space BO(N), for N sufficiently large. Another of Thom's ideas was to introduce an equivalence relation in the set of oriented manifolds: two manifolds V, V′ are *cobordant* if the oriented manifold $V' - V$, the disjoint union of V′ and the manifold $-V$ with the opposite orientation to that of V, is the boundary of a manifold W. The set Ω^n of "cobordism classes" of dimension n is naturally endowed with a commutative group structure, the group operation being defined by disjoint union (B 78, 89, 180, 188). A remarkable fact is that certain invariants of the DIFF structure are also invariants for the relation of cobordism; and a knowledge of Ω^n leads to unsuspected relations between these invariants (B 88, [180]).

These ideas have been considerably developed and diversified in several directions (B 408; LN 178; [174]). An important variant is the notion of *h-cobordism*, which requires that in the definition above the injections $V \to W$ and $V' \to W$ should be homotopy equivalences. Smale deduced his theorem on the Poincaré conjecture from a fundamental result on h-cobordism: if $\dim V \geqslant 5$ and if V and V′ are simply connected, an h-cobordism W of V with V′ is diffeomorphic to $V \times I$. His method of proof consists of considering W as obtained by "attaching handles" to the manifold $V \times I$, and then showing that, under the given hypotheses, the handles can be removed one by one without changing W (up to diffeomorphism).

When V and V′ are no longer assumed to be simply connected, the h-cobordism theorem is no longer true; a supplementary condition is needed, which is related to the notion of "Whitehead torsion" (B 392; LN 48).

Let us note at this point a problem in some sense opposite to Steenrod's problem: given a noncompact manifold V without boundary, does there exist a manifold W with boundary such that V is the interior of W? This problem has been solved by Siebenmann by means of K-theory (B 304).

Immersions, embeddings, and knot theory. An *immersion* of a differential manifold M of dimension m in a differential manifold N of dimension $n > m$ is a C^∞-mapping $f: M \to N$ whose tangent mapping is everywhere injective. The mapping f itself need not be injective; an injective immersion is called an *embedding*. The classification problem for embeddings is the determination of the classes of embeddings for the following equivalence relation: "f and g are isotopic under a differential isotopy." For immersions, we must (since immersions are not in general injective) replace "isotopy" by "regular homotopy," which means a homotopy $(x, t) \mapsto F(x, t)$ such that $x \mapsto F(x, t)$ is an immersion for each $t \in I$.

The most interesting case is that in which $N = \mathbf{R}^n$ and M is compact; the classification of immersions was first achieved by Smale for $M = \mathbf{S}_m$, and then by M. Hirsch in the general case. The idea is to reduce to a homotopy problem by passing to the tangent bundles of M and N: for example, the immersions of $\mathbf{S}_m$ in $\mathbf{R}^n$ are classified by the elements of the homotopy group $\pi_m(\mathbf{S}_{n,m})$ of the Stiefel manifold $\mathbf{S}_{n,m}$ [D, Chapter 16] of orthogonal m-frames in $\mathbf{R}^n$. For embeddings, again the classification problem has been reduced to a problem of homotopy, provided that $n > 3(m+1)/2$ (Haefliger) (B 157, 245; BAMS 69 (Smale)).

The classical theory of *knots* is the particular case of the classification of embeddings for $M = \mathbf{S}_1$ and $N = \mathbf{R}^3$; it is far from complete, so that the inequality $n > 3(m+1)/2$ is essential (B 485). When $n > 3(m+1)/2$ there are no "knotted m-spheres" in $\mathbf{R}^n$, and all embeddings of $\mathbf{S}_m$ in $\mathbf{R}^n$ are regularly isotopic; if $n \leqslant 3(m+1)/2$, the theory of "knotted spheres" has hardly begun (B 280; [37]).

The whole of the preceding theory has its analogs in the categories PL and TOP ([71], [86]; SAMS XXII (Lashof)). But there are some rather surprising differences: for example, there are no knotted m-spheres in $\mathbf{R}^n$ (*in the* PL *sense*) as soon as $n \geqslant m + 3$, whereas for $3(m+1)/2 \geqslant n \geqslant m + 3$ there may be m-spheres that are knotted in the DIFF sense but not in the PL sense.

For the case $M = \mathbf{S}_{n-1}$, $N = \mathbf{R}^n$, a problem which goes back to Jordan and Schoenflies is whether an embedding of $\mathbf{S}_{n-1}$ in $\mathbf{R}^n$ can be extended to an embedding of the ball $\mathbf{D}_n$ in $\mathbf{R}^n$; this is true in the category DIFF, and for $n = 2$ in the category TOP, but (in TOP) there is a counterexample of Alexander (the "horned sphere") when $n = 3$. Mazur and Morton Brown have proved that a homeomorphism $f : \mathbf{S}_{n-1} \to \mathbf{R}^n$ of $\mathbf{S}_{n-1}$ onto a closed subspace of $\mathbf{R}^n$ can be extended to a homeomorphism of $\mathbf{D}_n$ onto a closed subspace of $\mathbf{R}^n$, provided that f can be extended to a homeomorphism of an open neighborhood of $\mathbf{S}_{n-1}$ in $\mathbf{R}^n$ onto an open subset of $\mathbf{R}^n$ (B 205).

Finally, on the question of the *existence* of embeddings or immersions, a classical result of Whitney (for the category DIFF) is that there always exists an immersion of M in $\mathbf{R}^{2m-1}$ and an embedding in $\mathbf{R}^{2m}$; but it can be asked whether it is not possible in certain cases to reduce the number n. The theory of characteristic classes and K-theory provide answers to this question. The case in which M is a projective space $\mathbf{P}_m(\mathbf{R})$ has been studied the most. For example, it is known that Whitney's results are the best possible when $m = 2^r$; on the other hand, if $m = 2^r + 2$, we may take $n = 2m - 4$ for immersions and $n = 2m - 3$ for embeddings (LN 279; SAMS XXII (Gitler)).

Fixed points; spaces with group action. The property, for a continuous mapping $f : X \to X$ of a space into itself, of having a *fixed point*, i.e., a point $x \in X$ such that $f(x) = x$, is fundamental in existence proofs in functional analysis. One of the most famous theorems from the beginnings of algebraic topology is Brouwer's theorem, to the effect that every continuous mapping f of the closed ball $\mathbf{D}_n$ into itself has at least one fixed point. Another capital result is the *Lefschetz trace formula*, which, under certain conditions, expresses the number of fixed points of f in terms of cohomology: if X is a finite oriented polyhedron of dimension n, the mapping f determines endomorphisms f^i of the cohomology vector spaces $H^i(X, \mathbf{R})$ for $0 \leqslant i \leqslant n$; if $\mathrm{Tr}(f^i)$ is the trace of the endomorphism f^i, the Lefschetz number $L(f) = \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(f^i)$ is equal to the sum, over all the fixed points of f, of the "indices" $\sigma(x)$, provided that the fixed points are isolated (and hence finite in number) and such that at each of them the diagonal of $X \times X$ and the graph of f intersect "transversally," so that their "intersection number" $\sigma(x) = \pm 1$ at this point is defined. (See (**A** V) for a generalization of this formula.)

If $f: X \to X$ is a homeomorphism, the positive and negative powers f^n form a group G acting on X, and the fixed points of f are the orbits of G that consist of a single point. We are thus led to the general study, from the topological point of view, of the orbits and the orbit space of a topological group G acting continuously on a space X. This study has many applications and ramifications, in diverse domains, such as the existence of fixed points or the topology of Lie groups and their homogeneous spaces (see **B** II) (B 45, 251; BAMS 66 (P. Smith, Conner-Floyd), 76 (Fadell); LN 34, 36, 46, 73, 298, 299; [20]).

3. Connections with the natural sciences

The majority are *indirect*, via other mathematical theories in which topology plays a part. Doubtless the reason for this is to be found in the fact that the theorems of algebraic topology are *qualitative* in nature, and affirm for example the existence (or the nonexistence) of an object, without in general providing any means of determining it explicitly. However, there is a very recent application of the calculation of homotopy groups of certain homogeneous spaces to the classification of "defects" of crystalline structures and liquid crystals (Poenaru, Toulouze, L. Michel, Bouligand).

4. The originators

The principal ideas in algebraic and differential topology are due to the following mathematicians:

Homology and cohomology. B. Riemann (1826–1866), H. Poincaré (1854–1912), L. E. J. Brouwer (1881–1966), S. Lefschetz (1884–1972), E. Noether (1882–1935), J. Alexander (1888–1971), H. Hopf (1894–1971), H. Whitney, H. Cartan, N. Steenrod (1910–1971), M. Atiyah, F. Hirzebruch, J. F. Adams, D. Sullivan.

Homotopy. W. Hurewicz (1904–1956), H. Hopf (1894–1971), J. H. C. Whitehead (1904–1960), S. Eilenberg, S. MacLane, H. Cartan, J.-P. Serre, D. Kan.

Fiber bundles, characteristic classes. H. Whitney, H. Hopf (1894–1971), S. Chern, L. Pontrjagin, N. Steenrod (1910–1971), J. Leray, A. Borel, F. Hirzebruch, J.-P. Serre, J. Milnor, D. Kan, S. Novikov.

Topology of manifolds. J. Milnor, M. Kervaire, S. Smale, J. Stallings, D. Sullivan, C. T. C. Wall, W. Browder, R. Kirby, L. Siebenmann, T. Chapman.

Cobordism. R. Thom, J. Milnor, C. T. C. Wall, D. Quillen, S. Novikov.

Immersions, embeddings, knots. C. Jordan (1838–1922), J. Alexander (1888–1971), H. Whitney, S. Smale, B. Mazur.

Fixed points, transformation groups P. Smith (1900–1980), A. Borel.

Topology of Lie groups and homogeneous spaces. E. Cartan (1869–1951), H. Hopf (1894–1971), L. Pontrjagin, J. Leray, A. Weil, A. Borel, R. Bott.

Topology in dimensions $\leqslant 3$. R. Bing, E. Moise, C. Papakyriakopoulos (1914–1976), W. Thurston.

The following have also made substantial contributions to these theories: J. Adem, P. Alexandroff, D. Barden, M. Barratt, J. Boardman, M. Bockstein, K. Borsuk, G. Bredon, E. H. Brown, Morton Brown, A. Bousfield, G. Brumfiel, S. Cairns, E. Čech (1893–1960), J. Cerf, A. Černavskii, P. Conner, M. Dehn (1878–1952), A. Dold, E. Dyer, B. Eckmann, R. D. Edwards, C. Ehresmann (1905–1979), F. Farrell, J. Feldbau (1914–1945), E. Floyd, R. Fox (1913–1973), H. Freudenthal, T. Ganea (1923–1971), H. Gluck, W. Gysin, A. Haefliger, P. Heegard (1871–1948), P. Hilton, G. Hirsch, M. Hirsch, W. C. Hsiang, W. Y. Hsiang, J. Hudson, I. James, H. Künneth (1892–1974), P. Landweber, R. Lashof, H. Lebesgue (1875–1941), J. Lees, J. Levine, E. Lima, A. Liulevicius, G. Livesay, L. Lusternik, M. Mahowald, W. Massey, P. May, R. Milgram, E. Mischenko, D. Montgomery, J. C. Moore, C. Morlet, J. Munkres, M. H. A. Newman, F. Peterson, V. Poenaru, M. Postnikov, D. Puppe, K. Reidemeister (1893–1971), V. Rohlin, J. Roitberg, M. Rothenberg, H. Samelson, L. Schnirelmann (1905–1938), A. Schoenflies (1853–1928), G. Segal, H. Seifert, J. Shaneson, A. Shapiro (1921–1962), W. Shih, L. Smith, E. Spanier, M. Spivak, J. Stasheff, E. Stiefel (1909–1978), A. Svarč, E. Thomas,

H. Tietze (1880–1964), H. Toda, T. tom Dieck, E. van Kampen (1908–1942), L. Vietoris, F. Waldhausen, A. Wallace, H. C. Wang (1919–1978), J. West, G. W. Whitehead, W. T. Wu, C. Yang, C. Zeeman, J. Zilber, W. Meeks, S. Yau.

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B: 44, 45, 54, 78, 88, 89, 157, 172, 180, 188, 199, 205, 208, 215, 230, 245, 251, 259, 263, 280, 304, 362, 373, 392, 397, 408, 428, 472, 475, 485, 497, 509, 515, 516, 527, 574, 578.

LN: 2, 12, 13, 28, 34, 36, 43, 46, 48, 73, 99, 157, 161, 168, 178, 197, 249, 252, 271, 279, 298, 299, 304, 368, 422, 438, 473, 533, 540, 542, 557, 577, 591, 628, 657, 658, 664, 673, 722, 741, 763.

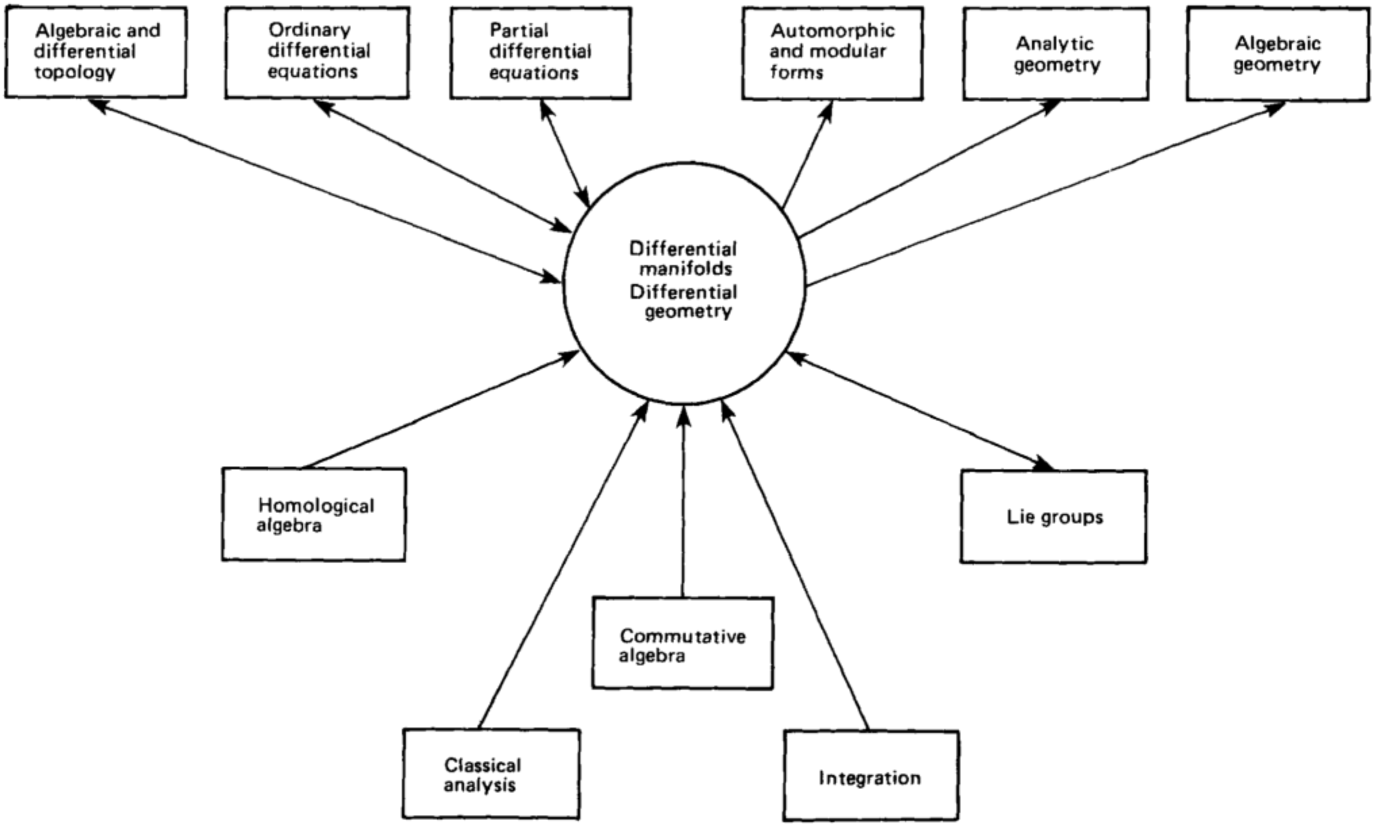
BAMS: 66 (Smith, Conner, Floyd), 68 (Wallace), 69 (Smale), 72 (Milnor), 74 (Heller), 75 (Peterson), 76 (Fadell), 81 (Burgess), 83 (Conner-Raymond, May, Lacher).

SAMS: III, XXII, XXXII.

Astérisque: 6, 12, 26, 32–33, 45.

[20], [26], [37], [50], [52], [71], [78], [79], [80], [86], [87], [119], [126], [152], [170], [171], [171 *bis*], [174], [181], [182], [188], [194], [195], [210], [212], [217], [226].

A II



Algebraic and differential topology
Ordinary differential equations
Partial differential equations
Automorphic and modular forms
Analytic geometry
Algebraic geometry
Differential manifolds
Differential geometry
Homological algebra
Lie groups
Commutative algebra
Classical analysis
Integration

A II

Differential manifolds. Differential geometry

The study of problems in analysis such as the behavior of solutions of differential equations, partial differential equations, integral equations, etc., leads naturally to formulating these problems not only on open sets in $\mathbf{R}^n$, but on *differential manifolds*; this is particularly true of the problems of this nature that have arisen from mechanics or physics ever since the classical epoch, and they are joined nowadays by all those that inevitably arise in the theory of Lie groups and their homogeneous spaces (**B** II). Modern analysis, when it goes beyond "local" results (valid only in an unspecified neighborhood of a point or a subset) is therefore analysis on differential manifolds, also called *global* analysis: its methods and its principal results are described in (**A** III), (**A** IV), and (**A** V).

Of course, analysis on manifolds deals only with notions defined *intrinsically*, that is to say independently of all choices of charts (**A** I). The study of these notions constitutes the *general theory* of differential manifolds. On the other hand, differential manifolds can be endowed with richer structures, involving additional data (G-structures, connections, etc.) that originate from problems of geometry or mechanics, and the problems concerning these structures form the province of *differential geometry*. In both cases, the problems envisaged almost always involve the underlying topological properties of the manifolds under consideration.

1. The general theory

The main subjects studied concern the singularities of differentiable mappings and of vector fields.

Singularities of differentiable mappings. If f is a real-valued C^∞-function on a differential manifold V, the *critical points* of f are the points $x \in V$ at which the differential df vanishes. Simple examples show that the set of critical points of f can be an arbitrary closed set in V, and it appears therefore unrealistic to attempt a classification of C^∞-functions based on the nature of their critical points. A critical point is said to be *nondegenerate* if the polynomial formed by the second-degree terms in the Taylor expansion of f in a neighborhood of this point (with respect to any chart) is a nondegenerate quadratic form; the index of this form is by definition the *index* of the critical point. Functions that have only nondegenerate critical points and take distinct values at these points (which are necessarily isolated) are very special, being closely related to the topology of V (they are functions that determine a "presentation by handles" of V; see (**A** I)); but the remarkable fact is that, with respect to a suitable topology on the space $\mathscr{E}(V)$ of C^∞-mappings of V into **R**, these "Morse functions" form a dense open set in $\mathscr{E}(V)$.

These results, which constitute the beginnings of Morse theory (B 36; BAMS 64 (Pitcher); [124]), have provided the starting point of a vast program of study of the C^∞-mappings of a compact differential manifold M into a differential manifold N, inaugurated in about 1955 by Whitney and Thom. The fundamental ideas introduced by these authors are the following:

(a) Only "generic" mappings are considered: these are mappings characterized by conditions on the "jets"† of a certain order (depending on the dimensions m, n of M and N).

(b) Introduction of two equivalence relations on the set $\mathscr{E}(M, N)$ of C^∞-mappings $f : M \to N$: *differential equivalence*, which means that f and f' are equivalent if there exists a diffeomorphism g (resp. h) of M (resp. N) onto itself such that $f' = h \circ f \circ g$; and *topological equivalence*, in which g and h are only required to be homeomorphisms.

(c) Introduction of a natural topology on the set $\mathscr{E}(M, N)$. A mapping $f \in \mathscr{E}(M, N)$ is said to be differentiably (resp. topologically) *stable* if the mappings differentiably (resp. topologically) equivalent to f form a neighborhood of f.

These definitions make it plausible that a "generic" mapping should be stable in one or other of the two senses just defined, and that the generic mappings should be *dense* in $\mathscr{E}(M, N)$.

(d) A *singular* point $x \in M$ for $f : M \to N$ is a point at which the tangent linear mapping of f does not have maximum rank. For a generic mapping f, one expects that the singular points will form a submanifold $S(f)$; when restricted to $S(f)$, the mapping f will have a submanifold $S(S(f))$ of singular points, and so on. Moreover, from the homological point of view, the singular

† A *jet* of order k is an equivalence class of C^k-mappings that, at a point, have the same derivatives up to and including order k (with respect to any charts). The jets of order 1 may be identified with tangent linear mappings [D, Chapter 16].

loci S(f), S(S(f)), etc., are related to the Stiefel–Whitney classes of M (**A** I) and to the images under f^* of those of N, by universal polynomial formulas ("Thom polynomials") (B 134; C 1956–7).

The realization of this program is far advanced, primarily by the work of J. Mather. It is established that the topologically stable mappings always form a dense open subset of $\mathscr{E}$(M, N), but the same statement for differential stability is true only for certain explicitly determined pairs of dimensions (m, n) (the "good dimensions"). Generic mappings are always topologically stable, and in good dimensions, generic mappings are identical with differentiably stable mappings. There are regular methods for determining explicitly the germs of generic mappings, up to equivalence, for given m and n; finally, regular methods are now beginning to be developed for the calculation of the Thom polynomials. The techniques of proof consist in reducing questions of differential stability to analogous questions about the jets of the mappings under consideration, and then using the instruments of the theory of commutative *local rings* (**C** II), by virtue of a key result, namely, Malgrange's generalization to C^∞-functions of the Weierstrass "preparation theorem" (B 336, 424; BAMS 75 (Thom); LN 192, 197, 209, 371, 373; Vancouver (Arnol'd); [51], [204]).

Vector fields on differential manifolds. On a differential manifold M, the *critical points* at which a vector field X [D, Chapter 16] of class C^r $(r \geqslant 1)$ vanishes play an important role in the study of the integral curves of the field, of which they are the singular points (**A** III). H. Poincaré was the first to discover a relation between the critical points of a vector field on a surface and the topological invariants of the surface, and the general form of this relation was given by H. Hopf. Suppose that M is compact, and that the critical points of X are finite in number; to each of these points there is intrinsically associated an integer, called the *index* of the point, and the sum of the indices (also called the index of X) is equal to the Euler–Poincaré characteristic of M. If $X_1, \ldots, X_k$ are k vector fields on M, the *singular points* of this system are the points $x \in$ M at which the vectors $X_1(x), \ldots, X_k(x)$ are linearly dependent; the notion of index can be generalized to such systems, and there are results on its relations with the homology of M when $k = 2$. A much studied problem is the determination of the largest integer k for which there exist k vector fields $X_1, \ldots, X_k$ with no singular points. If $k = n =$ dim(M), the manifold M is said to be *parallelizable*. This problem is completely solved in the case of spheres $\mathbf{S}_n$ (J. F. Adams): write $n + 1$ in the form $(2a + 1)2^{c+4d}$, where a, c, d are integers $\geqslant 0$, and $c \leqslant 3$; then it can be shown with the help of generalized cohomology based on K-theory that the number k is equal to $2^c + 8d - 1$ (B 233, BAMS 75 (Thomas), 76 (Baum)). In particular, the only parallelizable spheres are $\mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_7$.

2. G-structures

The method of the moving trihedron was invented in the 19th century by Ribaucour and Darboux for the differential study of surfaces. E. Cartan was the first to perceive the much greater range of this method, and he applied it with virtuosity to many questions of differential geometry and the general theory of partial differential equations. C. Ehresmann clarified and systematized the ideas of E. Cartan, by setting them in the context of the theory of fiber bundles. With the tangent bundle T(M) of a manifold M of dimension n there is naturally associated a principal bundle R(M), called the *bundle of frames* of M. Its fiber at each point $x \in \mathrm{M}$ is the set of bases of the tangent space $\mathrm{T}_x(\mathrm{M})$, and the structure group is the general linear group $\mathbf{GL}(n, \mathbf{R})$; moreover, T(M) may be considered as the associated vector bundle with fiber $\mathbf{R}^n$ [D, Chapter 20]. If G is a closed subgroup of $\mathbf{GL}(n, \mathbf{R})$, a G-*structure* on M is a subspace $\mathrm{S}_\mathrm{G}(\mathrm{M})$ of R(M) that is a principal G-bundle over M (the action of G on the fibers being the restriction of the action of $\mathbf{GL}(n, \mathbf{R})$) [D, Chapter 20]. Then T(M) appears as a vector bundle with G as structure group, and the relation between $\mathrm{S}_\mathrm{G}(\mathrm{M})$ and T(M) is very analogous to that which exists between a group G and a homogeneous space G/H; and just as in this latter case it is better to work in G rather than in G/H, in order to benefit from the richer group structure, in the same way the essential idea of E. Cartan's "method of moving frames" consists in calculating in $\mathrm{S}_\mathrm{G}(\mathrm{M})$ rather than in T(M).

The most important cases are: (1) $\mathrm{G} = \mathbf{O}(n)$, the orthogonal group, in which case the G-structures are called *Riemannian structures*; (2) $n = 2m$ is even, $\mathrm{G} = \mathbf{Sp}(2m, \mathbf{R})$, the symplectic group, to which correspond the *almost-Hamiltonian* (or *symplectic*) structures; (3) $n = 2m$ is even, $\mathrm{G} = \mathbf{GL}(m, \mathbf{C})$, the complex general linear group, in which case the G-structures are called *almost-complex* structures. Every complex-analytic manifold is canonically endowed with an almost-complex structure, but the converse is false: an almost-complex structure does not necessarily come from a complex-analytic structure on M, unless a supplementary condition of "integrability" is satisfied (B 166).

On any manifold it is always possible (in infinitely many ways) to define a Riemannian structure. On the other hand, for other subgroups G of $\mathbf{GL}(n, \mathbf{R})$, the existence of a G-structure on M implies relations between the topological invariants of M, notably the characteristic classes. For example, it can be shown (Borel–Serre) that the only spheres that admit an almost-complex structure are $\mathbf{S}_2$ and $\mathbf{S}_6$. In the other direction, the same differential manifold may be subjacent to several nonisomorphic complex structures: for example, there are infinitely many nonisomorphic complex structures on $\mathbf{S}_2 \times \mathbf{S}_2$ (B 35; BAMS 72 (Chern)).

The general notion of a *connection* on a manifold endowed with a G-structure was also introduced in substance by E. Cartan. Essentially this is a structure that allows one to "compare" the tangent spaces at two infinitely near points. In order to define it we require at each point $r \in S_G(M)$ a "horizontal" supplement, in the tangent space to $S_G(M)$ at r, to the subspace tangent to the fiber at r; it is necessary that this horizontal subspace should vary differentiably with r, and that the set of horizontal subspaces should be stable under G (B 24, 101; [D, Chapters 17 and 20]; BAMS 72 (Chern); [100]). Given a connection, it is possible to define the *parallel transport* of a frame along a path γ in M: it is enough to lift γ to a path in $S_G(M)$ for which the tangent at each point is "horizontal." The *geodesics* of a connection are the curves for which a tangent vector remains tangent under parallel transport along the curve.

The presence of a connection also enables one to define the derivative of a tensor field in the direction of a tangent vector at a point (the *covariant derivative* relative to the connection). Furthermore, to each connection there are intrinsically attached two tensors, the *curvature* tensor and the *torsion* tensor. For a Riemannian structure, there is a distinguished connection, called the *Levi-Civita connection*, characterized by the fact that its torsion is zero [D, Chapters 17 and 20].

Riemannian manifolds. The assignment of a Riemannian structure on M is equivalent to the assignment of a ds^2, a tensor field that on each tangent space is a positive-definite quadratic form. If $\dim(M) = n$, this tensor field gives rise canonically to a "p-dimensional element of area" for $1 \leqslant p \leqslant n$, which is a positive measure on each p-dimensional submanifold. In particular, when $p = 1$, the length of a curve in M is defined, and the geodesics of the Levi-Civita connection are precisely the extremal curves with respect to this length. The global study of the geodesics of a Riemannian manifold has been pursued unremittingly since the time of Jacobi (see [D, Chapter 20]; LN 55), and has given rise to Morse theory [124]. A problem studied first by Poincaré is that of the existence and number of distinct closed geodesics on a compact manifold; this is related to the topology of the manifold, and is still not completely solved (B 364, 406; [144], [223]).

The problem of extremal submanifolds with respect to the p-dimensional area can be posed not only for $p = 1$, but also for $1 < p \leqslant n - 1$, and leads to a system of nonlinear partial differential equations of the second order. Up to now, this problem has been considered mainly in the case $p = n - 1$, in which the extremals are called "minimal hypersurfaces." When $n = 3$, it goes back to Lagrange, and the study of minimal surfaces in $\mathbf{R}^3$ was the subject of much work throughout the 19th century (Weierstrass, Schwarz, etc.), and today still presents many unanswered questions (BAMS 71 on E. The remarkable fact is that the theory of these manifolds is much simpler than that of manifolds of finite dimension; all Hilbert manifolds are diffeomorphic to open subsets of E, and if two open subsets of E have the same homotopy type, then they are diffeomorphic (B 284, 378, 540; BAMS 72 (Eells); Nice C 1 (R. D. Anderson), C 2 (Kuiper), C 4 (Ebin–Marsden, Eells–Elworthy); LN 259, 282; SAMS XV).

5. Connections with the natural sciences

The fundamental postulate of the theory of Relativity is that space-time is a differential manifold endowed with a *pseudo-Riemannian* structure, i.e., a G-structure where G is the Lorentz group, which leaves invariant the quadratic form $x_0^2 - x_1^2 - x_2^2 - x_3^2$ on $\mathbf{R}^4$. The theory of geodesics for such a structure and the theory of singularities of differentiable mappings therefore play an important part in relativistic theories of cosmology, in particular in the study of singularities of space-time ("black holes") ([141]; BAMS 83 (R. Sacks-H. Wu); LN 209).

About ten years ago, R. Thom developed some extremely original ideas on the possibility of applying the theory of singularities of differentiable mappings to the qualitative study of physicochemical and biological phenomena, and even to linguistics: this is what he calls *catastrophe theory*, which has aroused considerable interest in many places, and some controversy [180].

6. The originators

The principal ideas in the theory of differential manifolds and G-structures are due to the following mathematicians:

The notion of a differential manifold. C. F. Gauss (1777–1855), B. Riemann (1826–1866), H. Weyl (1885–1955).

Singularities of differentiable mappings. H. Whitney, R. Thom, B. Malgrange, J. Mather, D. Sullivan.

Vector fields. A. Poincaré (1854–1912), H. Hopf (1894–1971), J. F. Adams, M. Atiyah.

G-structures, connections. E. Cartan (1869–1951), T. Levi-Civita (1873–1941), S. Chern.

Riemannian manifolds. B. Riemann (1826–1866), E. Cartan (1869–1951), J. Nash.

Topology of differential manifolds. E. Cartan (1869–1951), G. de Rham, S. Chern, A. Weil, W. Thurston.

Infinite-dimensional manifolds. M. Morse (1892–1977), S. Smale, V. Arnol'd.

The following have also made substantial contributions to these theories: C. Allendoerfer (1911–1974), W. Ambrose, R. Anderson, L. Auslander, A. Avez, F. Almgren, E. Beltrami (1835–1899), M. Berger, S. Bernstein (1880–1968), C. Bessaga, L. Bianchi (1856–1928), W. Blaschke (1885–1962), J. Boardman, S. Bochner, E. Bombieri, R. Bonič, O. Bonnet (1819–1892), D. Burghelea, E. Calabi, J. Cheeger, E. Christoffel (1829–1900), S. Cohn-Vossen (1902–1936), G. Darboux (1842–1917), E. De Giorgi, A. Douady, D. Ebin, B. Eckmann, J. Eells, C. Ehresmann (1905–1979), H. Eliasson, K. Elworthy, C. Fefferman, W. Fenchel, A. Fet, J. Frampton, E. Giusti, D. Gromoll, M. Gromov, J. Hadamard (1865–1963). A. Haefliger, P. Hartman, D. Henderson, D. Hilbert (1862–1943), W. Y. Hsiang, I. James, H. Karcher, W. Klingenberg, U. Koschorke, J. L. Koszul, N. Kuiper, S. Kobayashi, R. Lashof, H. Lawson, J. Levine, H. Lewy, A. Lichnerowicz, W. Liebmann (1874–1939), R. Lipschitz (1832–1903), P. Libermann, L. Lusternik, J. Marsden, W. Meyer, J. Milnor, G. Mostow, N. Moulis, K. Mukherjea, S. Myers (1910–1955), A. Nijenhuis, L. Nirenberg, J. Nitsche, K. Nomizu, R. Osserman, R. Palais, A. Pelczynski, R. Penrose, W. Pohl, I. Porteous, M. Rauch, A. Ribaucour (1845–1893), G. Ricci (1853–1925), W. Rinow, E. Ruh, L. Schnirelmann (1905–1938), F. Schur (1856–1932), H. A. Schwarz (1843–1921), G. Segal, J. Simons, I. Singer, N. Steenrod (1910–1971), E. Stiefel (1909–1978), E. Thomas, V. Topogonov, J. Tougeron, A. Tromba, K. Weierstrass (1815–1897), J. West, G. W. Whitehead, H. Yamabe (1923–1960), K. Yano, S. Yau, R. Böhme, J. Douglas, H. Federer, W. Fleming, W. Meeks, F. Morgan, F. Tomi.

References

B: 24, 35, 36, 38, 101, 134, 147, 166, 193, 233, 237, 284, 336, 353, 364, 378, 406, 410, 424, 440, 526, 527, 529, 540, 573, 579.

LN: 55, 66, 67, 192, 197, 209, 259, 282, 335, 371, 373, 484, 520, 525, 535, 552, 570, 588, 597, 610, 640, 678.

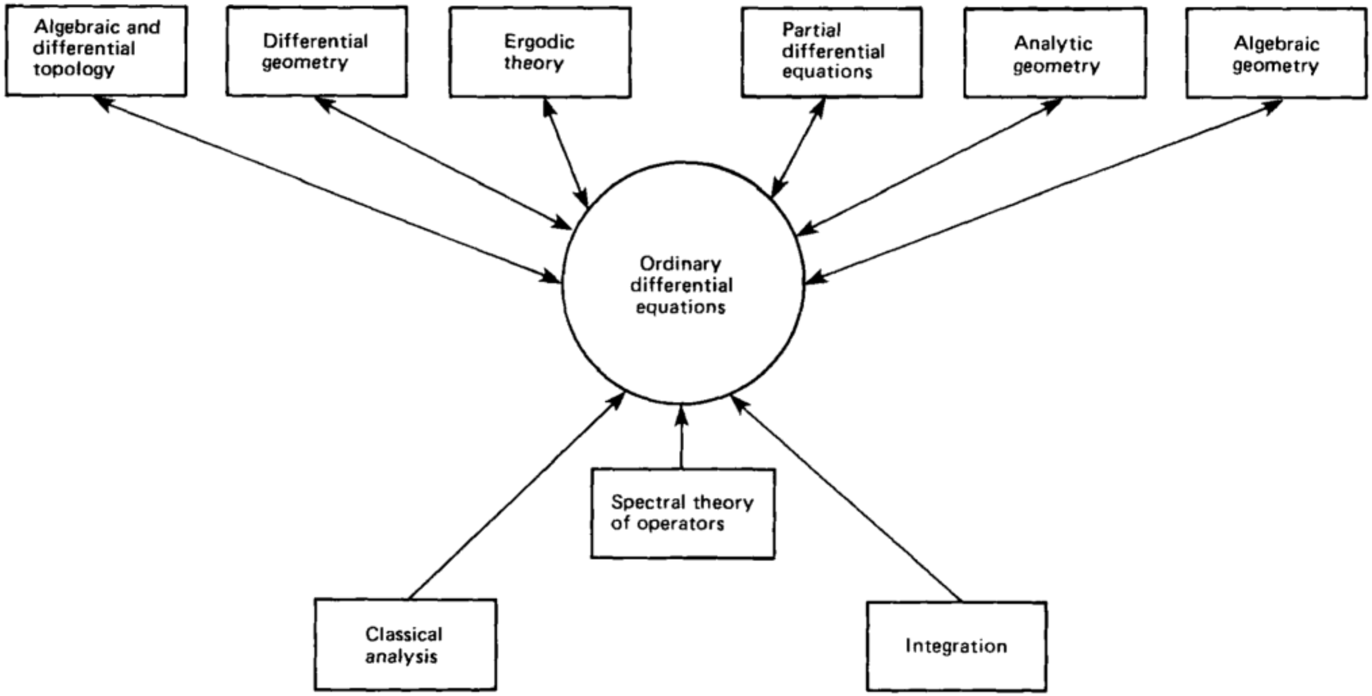
BAMS: 64 (Pitcher), 71 (Nitsche), 72 (Chern, Eells), 75 (Thom, Thomas, Osserman) 83 (K. Chen, R. Gardner, R. Sacks-H. Wu), 84 (Osserman).

SAMS: XV

Astérisque: 32–33, 45, 58.

[100], [124], [141], [144], [180], [204], [223].

A III



Algebraic and differential topology
Differential geometry
Ergodic theory
Partial differential equations
Analytic geometry
Algebraic geometry
Ordinary differential equations
Spectral theory of operators
Classical analysis
Integration

A III

Ordinary differential equations

For 300 years, the theory of ordinary differential equations has been one of the most intensively studied branches of mathematics, and a great variety of methods have been devised to attack the innumerable problems it has raised.

1. The algebraic theory

In the 18th century, the study of differential equations was directed primarily toward obtaining "general" solutions by operations considered as "simple," such as "quadratures" and series expansions. This period of empiricism generated a heterogeneous collection of results and was followed in the 19th century by an effort of reflection on the methods used, analogous to that which, between 1770 and 1830, led to the theory of Galois for algebraic equations. And indeed these investigations culminated in a "Galois theory," in an almost completely algebraised framework, first for linear differential equations (Picard–Vessiot theory), and then for algebraic differential equations, in ever closer liaison with the modern theory of algebraic groups (**A** IX) (B 17; [101]).

2. Ordinary differential equations in the complex domain

After the creation by Cauchy of the theory of holomorphic functions, the analysts of the 19th century embarked on the study of singular points, in the complex domain, of solutions of analytic differential equations. This study culminated in the spectacular results of Painlevé on second-order equations all of whose integrals are meromorphic [89]. Recently, this subject has again become an area of active research, in the current of ideas flowing from the general theory of singularities in analytic geometry (**A** VIII).

3. The qualitative study of ordinary differential equations

Around 1880, Liapunov and H. Poincaré gave a new direction to the theory by initiating a global geometrical study of the family of integral curves of a differential equation. Liapunov was primarily interested in problems of *stability* of an integral curve under small variations of the initial conditions; the methods he invented in order to determine the stability of an integral curve have been applied and developed in a great many investigations ([111]; LN 35), and even the notion of stability is capable of many variations. Poincaré started with a first-order differential equation, written in the form $\frac{dx}{X} = \frac{dy}{Y}$, where X and Y are polynomials in x and y with no common factor, and straightaway set himself the most general problem, namely, the "qualitative" description of all the integral curves of the equation. In order to be able to deal with infinite branches, he projected the plane Oxy onto a sphere (with center outside the plane) from the center of the sphere, and this led him to the study of integral curves of a field of tangent vectors to the sphere. A little later, in order to deal with first-order differential equations $F(x, y, y') = 0$ not solved for y', he considered the problem as being equivalent to the determination, on the surface $F(x, y, p) = 0$, of the integral curves of the equation $dy - p\,dx = 0$; again this is a particular case of integral curves of a vector field on a surface. Generally speaking, problems relative to differential equations of higher order, or to systems of differential equations, can be thought of in terms of the integral curves of a vector field on a differential manifold M of arbitrary dimension. The most frequently studied case is that in which M is a compact manifold and X is a C^∞-vector field: if we denote by $F_X(x, t)$ the value at $t \in \mathbf{R}$ of the integral v such that $v(0) = x$, then $F_X(x, t)$ is defined for *all* t, and the mapping $(t, x) \mapsto F_X(x, t)$ is a C^∞-action of the additive group $\mathbf{R}$ on the manifold M (**B** II). A whole chapter of the qualitative theory of differential equations is thus included as a particular case in the study of such actions by an arbitrary topological group G ([67], [133], [173]).

The fundamental notions in the study of a vector field X and its integral curves are those of critical point (**A** II) and closed integral curve. At a critical point a, the characteristic multipliers of X are by definition the eigenvalues of the Jacobian matrix of the components of the field X at the point a (relative to any system of local coordinates). A closed integral curve $\gamma : t \mapsto \gamma(t)$ by definition contains no critical point and is periodic; if $\tau > 0$ is the smallest period, then for each point $x \in \gamma$, the mapping $y \mapsto F_X(y, \tau)$ is a diffeomorphism of a neighborhood of x onto a neighborhood of x, and its tangent linear mapping at x is an automorphism of the tangent space $T_x(M)$. The characteristic multipliers of γ are the eigenvalues of this automorphism (one of them is always equal to 1).

of modulus 1; the set $W^s(x)$ (resp. $W^u(x)$) of points y such that $f^{np}(y) \to x$ as $n \to \infty$ (resp. $n \to -\infty$) is then a submanifold of M, and the generic property of Kupka–Smale is that every periodic point of f is hyperbolic, and that if x, y are two periodic points (distinct or not), the submanifolds $W^s(x)$ and $W^u(y)$ intersect transversally.

An important notion is that of a *structurally stable* diffeomorphism, i.e., a diffeomorphism f that has a neighborhood in Diff(M) consisting of diffeomorphisms conjugate to f. The structurally stable diffeomorphisms form a nonempty open set in Diff(M), and in fact every diffeomorphism is isotopic to a structurally stable diffeomorphism. When $M = S_1$, the set of structurally stable diffeomorphisms is dense in Diff(M), but this is no longer the case for manifolds of dimension >1.

Among the structurally stable diffeomorphisms are the *Morse–Smale* diffeomorphisms, which are those satisfying the Kupka–Smale property, having only finitely many periodic points and such that for each $x \in M$, the sequence $f^n(x)$ tends to a periodic point as n tends to $\pm\infty$. The set of these diffeomorphisms is always open and nonempty.

Another important class is the *Anosov diffeomorphisms*. To define these, assume that M carries a Riemannian structure, and hence a Euclidean norm on each tangent space $T_x(M)$. A diffeomorphism f is said to be an Anosov diffeomorphism if the tangent bundle T(M) decomposes into a direct sum $E^s \oplus E^u$ of vector bundles, each stable under the tangent mapping $T(f)$, and such that $T(f)$ (resp. $T(f^{-1})$) is a contraction on E^s (resp. E^u). The Anosov diffeomorphisms are structurally stable and form a (possibly empty) open set in Diff(M) [167].

There are analogous notions and results for differential equations on M (B 348, 374; BAMS 73 (Smale), 78 (J. Robbin), 80 (M. Shub); SAMS XIV; LN 206; [2], [48]).

5. Boundary-value problems

Many problems in analysis require solutions of a differential equation defined on an interval of **R** and satisfying various conditions at the endpoints (finite or not) of the interval. A typical example is to find solutions of a second-order equation that take given values at the (finite) endpoints of the interval. These problems have not been the subject of a general theory valid for equations of arbitrary order except in the case of *linear* equations, in the context of the spectral theory of operators (**C** III).

6. Connections with the natural sciences

Ever since the 17th century, almost all natural phenomena that involve certain quantities varying continuously as functions of a parameter (usually time) have led to problems about differential equations, and these problems have been a constant source of stimulation for the mathematical theory. We shall not attempt to list these innumerable applications. However, a special mention should be given to celestial mechanics, which historically was the first of these applications, and without doubt has instigated the largest quantity of important mathematical work ([168], [173]), the practical interest of which has been considerably augmented by its applications to the control of guided missiles and artificial satellites. As another example out of the common run, we may cite the research on the functioning of the heart and its representation by solutions of suitably chosen differential equations ([48] (Zeeman)).

7. The originators

We shall restrict ourselves to the *qualitative* theory, in which the most important ideas are due to the following mathematicians: H. Poincaré (1854–1912), A. Liapunov (1857–1918), G. D. Birkhoff (1884–1944), A. Denjoy (1884–1974), C. Siegel (1896–1981), A. Kolmogorov, S. Smale, A. Peixoto, V. Arnol'd, D. Anosov, J. Moser.

The following have also contributed substantially to the theory: R. Abraham, V. Alexyev, A. Andronov, R. Arenstorf, J. Auslander, A. Avez, J. Bendixson (1861–1936), N. Bhatia, R. Bowen (1947–1978), C. Camacho, L. Cesari, J. Chazy (1882–1955), R. Ellis, J. Franks, H. Furstenberg, W. Gottschalk, J. Guckenheimer, J. Hadamard (1865–1963), O. Hajek, J. Hale, P. Hartman, G. Hedlund, M. Herman, H. Hilmy, M. Hirsch, W. Kaplan, A. Kelley, H. Kneser (1898–1973), N. Kuiper, I. Kupka, J. La Salle, S. Lefschetz (1884–1972), A. Liénard, J. Littlewood (1885–1977), A. Manning, L. Markus, J. Massera, J. Mather, M. Morse (1892–1977), R. Moussu, V. Nemytskii (1900–1967), S. Newhouse, J. Palis, O. Perron (1880–1975), I. Petrowski (1901–1973), C. Pugh, L. Pontrjagin, G. Reeb, J. Robbin, R. Roussarie, A. Schwartz, H. Seifert, S. Shub, K. Sitnikov, J. Sotomayor, V. Stepanov (1889–1950), S. Sternberg, D. Sullivan, K. Sundman (1873–1949), G. P. Szegö, B. Van der Pol (1889–1959). T. Wazewski, H. Whitney, R. Williams, A. Wintner (1903–1958).

References

B: 17, 21, 217, 237, 264, 348, 374, 580.
LN: 35, 206, 468, 552, 583, 597, 668, 712, 738.
BAMS: 73 (Smale), 78 (Robbin), 80 (Shub).
SAMS: XIV.
Astérisque: 30, 31, 40, 49–51, 56.
[1], [2], [4], [48], [67], [74], [89], [101], [111], [133], [167], [168], [173], [220].

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Partial differential equations

The theory of partial differential equations has been studied incessantly for more than two centuries. By reason of its permanent symbiosis with almost all parts of physics, as well as its ever closer connections with many other branches of mathematics, it is one of the largest and most diverse regions of present-day mathematics, and the vastness of its bibliography defies the imagination.

For a long time, the theory of ordinary differential equations served more or less consciously as a model for partial differential equations, and it is only rather recently that it has come to be realized that the differences between the two theories are much more numerous and more profound than the analogies.

1. The local study of differential systems

Here the paradigm was the Cauchy–Lipschitz theorem of existence and uniqueness of solutions of ordinary differential equations: if I is an open neighborhood of a point $t_0 \in \mathbf{R}$, consider a differential equation

$$\frac{dx}{dt} = \mathrm{A}(x, t), \tag{1}$$

where x takes its values in an open set H in $\mathbf{R}^n$, and A is a function with values in H which is continuously differentiable on $\mathrm{H} \times \mathrm{I}$ (or at least satisfies a Lipschitz condition in x on a neighborhood of each point of $\mathrm{H} \times \mathrm{I}$); then for each $x_0 \in \mathrm{H}$, there exists a suitably small neighborhood $\mathrm{J} \subset \mathrm{I}$ of t_0 such that the equation (1) admits a unique solution $t \mapsto x(t)$ in J such that $x(t_0) = x_0$. [D, Chapter 10].

The only analogous general theorem we possess for partial differential equations is the Cauchy–Kowalewska theorem. Consider a system of r equations in r unknown real-valued functions $v_1, \ldots, v_r$ of $p + 1$ real variables $x_1, \ldots, x_{p+1}$, of the form manifold on which the Pfaffian system is defined (they are not, in general, locally closed submanifolds) [D, Chapter 18].

The study of properties of foliations from the point of view of differential topology was inaugurated by Ehresmann and Reeb in 1948, but only recently has it begun to attract the attention of many mathematicians: it is now a very active field of research. The emphasis is on the topological properties of the leaves, in particular on the existence of compact leaves (which generalizes the periodic trajectories in the theory of ordinary differential equations (**A** III)). The example that has served as a model is the Reeb foliation of the sphere $\mathbf{S}_3$, which is of codimension 1 and has only one compact leaf; only recently have generalizations to other compact manifolds been obtained (B 499). On the other hand, a completely integrable Pfaffian system on a manifold M determines a sub-bundle E of the tangent bundle T(M), satisfying the Frobenius condition at each point; but it has recently been realized that such a sub-bundle must also satisfy *global* conditions of a topological nature. The first of these was obtained by Bott, and expresses that the Pontrjagin classes of order k of the quotient bundle $\mathrm{Q} = \mathrm{T(M)/E}$ must vanish for $k > 2q$, where q is the rank of Q (LN 279). The present study of these problems depends on techniques from homotopy theory (in which homological algebra also plays a large part) introduced by Haefliger; it is necessary to enlarge somewhat the notion of foliation, by admitting "singular" leaves whose dimension is smaller than that of the "generic" leaves. The theory is also related to the cohomology of vector fields (**B** I) of Gelfand and Fuks (B 192, 339, 390, 393, 412, 434, 523, 524, 551, 574; LN 197, 206, 279, 392, 484, 493, 652, 712, 725; Nice I (Bott); BAMS 80 (Lawson); [34]).

3. Linear partial differential equations: general theory

A linear partial differential equation of order m on an open subset X of $\mathbf{R}^n$ is an equation of the form $P \cdot u = f$ where $P = \sum_{|\nu| \leq m} \mathrm{A}_\nu(x)\mathrm{D}^\nu$ is a linear combination of derivatives of order $\leq m$, the coefficients being functions of $x \in \mathrm{X}$, and f a given real-valued function on X. We shall limit our attention to the case in which the A_ν are real-valued functions of class C^∞; the notion of "solution" can then be extended to the case where f is a *distribution* [D, Chapter 17], and a solution is a distribution u satisfying $P \cdot u = f$.

In the same way we define systems of linear partial differential equations, or *vector* partial differential equations $P \cdot u = f$, where now $f = (f_1, \ldots, f_{\mathrm{N}''})$ is a given vector-valued function of $x \in \mathrm{X}$, $u = (u_1, \ldots, u_{\mathrm{N}'})$ is an unknown vector-valued function of x, and the $\mathrm{A}_\nu(x)$ are functions of x whose values are $\mathrm{N}'' \times \mathrm{N}'$ matrices. With the help of charts we pass from this situation to the case where f is a section of a vector bundle F over a differential manifold X, and u is a section of a vector bundle E over X (which may or may not be the

$2n - 1$) obtained by taking an $(n - 1)$-dimensional submanifold of (5) and forming the union of the integral curves of (6) that intersect this submanifold, and then we project these submanifolds into $\mathbf{R}^n$. The integral curves of (6) are called the *bicharacteristic bands* of the operator P, and their projections in $\mathbf{R}^n$ the *bicharacteristic curves* of P [D, Chapter 18].

Problems. As for ordinary differential equations, the essential problems of the theory of linear partial differential equations center on questions of *existence* and *uniqueness* of solutions of $P \cdot u = f$, but have many diverse aspects. They can be considered *globally*, where the domain X of the variables is fixed; but they can also (as for the Cauchy theorems) be studied from the *local* point of view, that is to say in an *unspecified* neighborhood of a point of X. In either case, the data for these problems will contain not only the given equation, but also various supplementary conditions on the given function f or the unknown function u; these may be either "boundary-value conditions" on the behavior of u and f in a neighborhood of the boundary of X (when X is embedded in a larger manifold), or they may be "regularity" conditions imposed on u or f. A general and suggestive way of presenting them is to envisage u and f as points of two topological vector spaces $\mathscr{A}$, $\mathscr{B}$ (distinct or not) defined by the given conditions; the problem then consists of studying the *image* and the *kernel* of P, regarded as a linear mapping of $\mathscr{A}$ into $\mathscr{B}$.

Once in possession of a theorem of existence and uniqueness, other problems present themselves: whether explicit formulas or, alternatively, methods of "approximation" (in various senses) can be given for the solution. Finally, it may be asked whether the solution varies "continuously" (in various senses) when the data (i.e., the function f, the coefficients of P, or the domain X) are subjected to variation. When this is the case, the problem is said, in Hadamard's phrase, to be "well posed" (cf. LN 316).

Techniques. A first point to consider is the judicious choice of the function spaces $\mathscr{A}$ and $\mathscr{B}$ in which to work. For maximum generality, one can use the general spaces of distributions such as $\mathscr{D}'(X)$, $\mathscr{E}'(X)$, or $\mathscr{S}'(\mathbf{R}^n)$ [D, Chapters 17 and 22] or, even more generally, spaces of "hyperfunctions" (B 214, LN 126, 287, 325, 449; Nice D 10 (Sato)). If on the other hand one seeks solutions that are as "regular" as possible (in view of applications to analysis or physics), it is natural to take spaces of C^∞ or analytic functions (or intermediates such as the "Gevrey classes"). A class of spaces that has turned out to be very useful is the *Sobolev spaces* and their many variants (LN 82). The simplest of these are the spaces $H^m(X)$, where m is a positive integer: $H^m(X)$ consists of the classes of functions in $L^2(X)$ whose partial derivatives (in the sense of distributions) belong to $L^2(X)$, up to and including order m, and is a Hilbert space

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