A Profile of Mathematical Logic

Howard DeLong

A PROFILE OF MATHEMATICAL LOGIC

HOWARD DELONG
Brownell Professor of Philosophy, Emeritus
Trinity College
Hartford, Connecticut

DOVER PUBLICATIONS, INC.
Mineola, New York

Copyright

Bibliographical Note

This Dover edition, first published in 2004, is an unabridged republication of the second (September 1971) printing of the work first published by Addison-Wesley Publishing Company, Reading, Massachusetts, in 1970.

Library of Congress Cataloging-in-Publication Data

DeLong, Howard, 1936–
A profile of mathematical logic / Howard DeLong.
p. cm.
Originally published: Reading, Mass. : Addison-Wesley, 1970.
Includes bibliographical references and index.
ISBN 0-486-43475-3 (pbk.)
1. Logic, Symbolic and mathematical. I. Title.

QA9D37 2004
511.3—dc22

2004043935

Manufactured in the United States of America
Dover Publications, Inc., 31 East 2nd Street, Mineola, N.Y. 11501

CONTENTS

CHAPTER 1

HISTORICAL BACKGROUND OF MATHEMATICAL LOGIC

§1 INTRODUCTION

Near the end of a work now called *Sophistical Refutations*, Aristotle apparently claims to have created the subject of logic [1928 183b 34ff].* The nearest analog of such a claim in our century is no doubt Freud's statement in 1914 that "... psychoanalysis is my creation; I was for ten years the only person who concerned himself with it..." [1953 7]. It seems probable that Aristotle's claim is as true as Freud's. Although Freud's claim is correct, it is nevertheless possible for the historian to find all kinds of hints and anticipations of psychoanalysis in the works of earlier thinkers; so if the works of Aristotle's predecessors were all intact, historians could no doubt perform a similar feat.

For example, Plato makes the following statement in the *Republic*: "The same thing cannot ever act or be acted upon in two opposite ways, or be two opposite things, at the same time, in respect of the same part of itself, and in relation to the same object" [1955 133 (436B)]. Aristotle claims that the most certain of all principles is that "the same attribute cannot at the same time belong and not belong to the same subject and in the same respect" [1928a 1005b 18ff]. This latter principle is Aristotle's formulation of the *Law of Non-Contradiction*, and it is tempting to say that Aristotle received not only this law, but many of his ideas on logic, from his predecessors. Nevertheless, one should resist this temptation because Plato makes this remark only in passing and there is no evidence that he, or anyone else before Aristotle, attempted to codify the rules of correct inference. Thus we may accept Aristotle's claim and ask what led him to create the subject of logic.

"All men by nature desire to know," Aristotle tells us in the famous opening sentence of the *Metaphysics*. Both he and Plato believed that philosophy begins in wonder, and there can be little doubt that this motive was strong in Aristotle's logical investigations. Yet it does not seem that this was the only or even the most pressing motive. Rather two other related but more practical aims were involved, one having to do with mathematics and the

* See the Bibliography for all references, as well as for explanation of reference system.

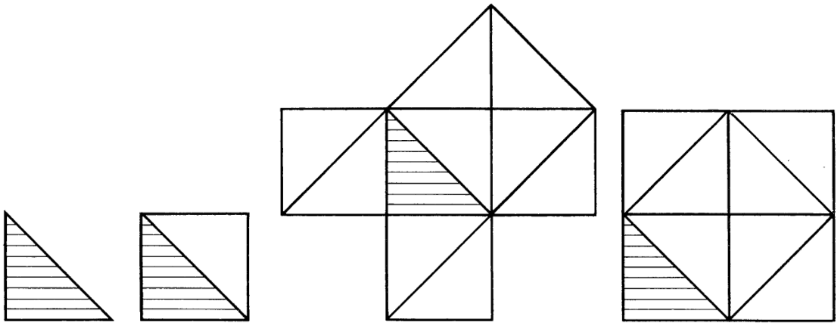
other with sophisms. If we wish to know what logic is all about we can do no better than to begin by asking about ancient Greek mathematics before Aristotle.

§2 MATHEMATICS BEFORE ARISTOTLE

The first Greek mathematician was Thales of Miletus (*c*.624–*c*.545 B.C.). Thales had visited Egypt and it is probable that he acquired some practical geometrical knowledge there. However, from what we now know of ancient Egyptian mathematics, it seems more likely that anything of value that the Greeks inherited in geometry they received ultimately from ancient Mesopotamia. The latter's geometric knowledge was vastly superior to Egypt's. If we can believe tradition, Thales must have been a very great mathematician indeed because he apparently was the first person both to conceive of general geometric propositions and to see the necessity of proving them. A number of geometric propositions are ascribed to him—for example, that the angles at the base of an isosceles triangle are equal—but unfortunately we do not have any idea how he proved them. The same must also be said for Pythagoras (*c*.566–*c*.497 B.C.), who according to tradition also visited Egypt and was a pupil of Thales. Pythagoras' most important contribution to Greek geometry is perhaps best summarized by Proclus (410–485 A.D.), who stated that after Thales, "Pythagoras transformed the study of geometry into a liberal education, examining the principles of the science from the beginning and proving the theorems in an immaterial and intellectual manner . . ." [quoted in Heath 1921 I 141]. If this be true, all the other mathematical achievements (real or alleged) of Thales and Pythagoras are insignificant by comparison, for it would mean that they were chiefly responsible for transforming geometry from an empirical and approximate science into a nonempirical and exact one. We do not, however, know enough to make this claim for them with any kind of assurance.

In any case Pythagoras, who was probably born in Samos, moved to the Greek city of Croton in southern Italy. There he formed a religious brotherhood based on numerous ascetic practices and beliefs. The members apparently believed in the transmigration of souls and in numerology. The specifics of the doctrine are obscure in part because the followers were pledged to secrecy. Although the achievements of Pythagoras are uncertain, it is likely that his order had followers for over a century, and its doctrines certainly influenced many important thinkers, including Plato and Aristotle.

One of the achievements of the school—perhaps even one of Pythagoras' achievements—was the first *proof* of the Pythagorean theorem, that is, the theorem which states that the sum of the squares of the sides of a right triangle is equal to the square of the hypotenuse. The *truth* of the theorem is

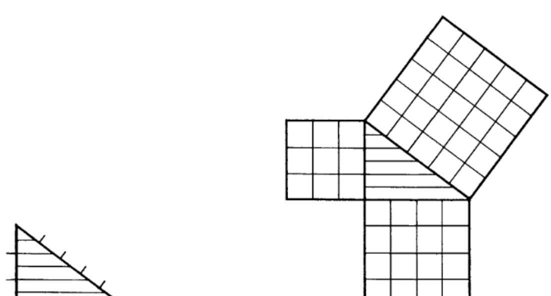


(a) (b) (c) (d)

Figure 1

not very difficult to see—especially in some of its special cases—and either it or some special case of it was discovered independently in a number of cultures, for example in Babylonia, India, and China. We do not know how the Pythagoreans proved the theorem. However, it seems probable that the person who was offering the proof would draw a diagram while speaking and would ask the person listening to the proof if he agreed as he went along. This is the procedure of Socrates in Plato's *Meno* (*c*.390 B.C.), where a special case of the theorem is in fact proved. It also seems likely, as often happens in mathematics, that special cases were proved first and later generalized. Finally, it is probable that the assumptions of the proof were not first stated but were appealed to in the course of the proof, and—sometimes, at least—were not clearly understood by either party of the proof. Supposing all this, we might consider the following a likely story.

Pythagoras started with an isosceles right triangle (as shown in Fig. 1a) and made a construction on that triangle (Fig. 1b and c). The proof of the



(a) (b)

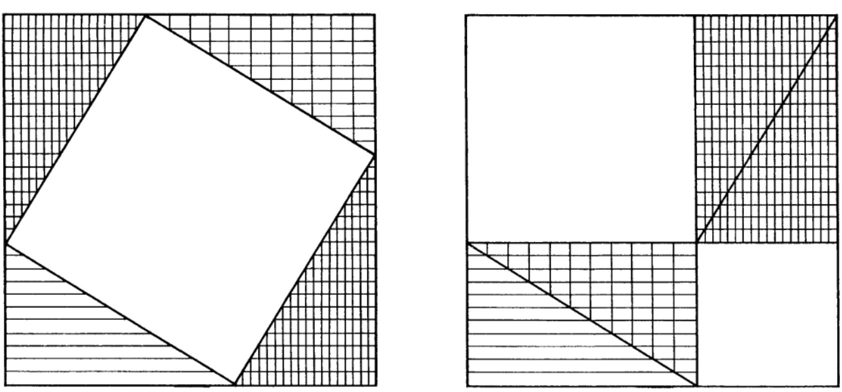
Figure 2

theorem can then be given by the process of counting the congruent triangles. Alternatively, he may have constructed one figure (Fig. 1d) and argued only after the construction. There is also evidence to suggest that he (like the ancient Babylonians who preceded him by 1200 years) knew that a 3, 4, 5 triangle is right. If so, he may have made constructions like those in Fig. 2, where it is possible to count unit squares.

Problem 1.* Construct a square seven units by seven units, analogous to Fig. 1(d), such that the theorem's truth for a 3, 4, 5 triangle can be seen without any further construction.

If he proved the theorem in its full generality, he may have used a construction such as that given in Fig. 3(a). It is obvious that any right-angled triangle can be duplicated four times, as indicated in the figure. Now the area of the square of the hypotenuse is equal to the total area of the square minus the area of the four congruent triangles. Rearrange the triangles as indicated in Fig. 3(b). Clearly the sum of the squares of the two legs of the right triangle is equal to the total area of the square minus the area of the four congruent triangles. This is the most intuitively clear proof of the general Pythagorean theorem that has yet been discovered. But just for this reason it is unlikely that Pythagoras discovered it, since it often happens that the first proof of a theorem is far from the easiest.

Rather, he probably used his theory of *proportion*. In modern form this proof could be described by saying that we take triangle ABC (Fig. 4), which is right-angled at B, and drop a perpendicular to the hypotenuse from B.



(a) (b)

Figure 3

* The answers to all problems are found beginning on page 237.

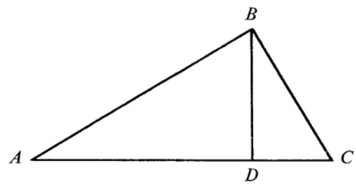


Figure 4

Now triangles *ABC*, *ADB*, *BDC* are all similar (that is, equiangular) and thus each have sides in the same ratio. Hence

$$AB : AD :: AC : AB,$$

$$BC : DC :: AC : BC.$$

Re-expressing these relations (by multiplying the means and extremes), we have

$$(AB)^2 = (AD) \cdot (AC),$$

$$(BC)^2 = (DC) \cdot (AC).$$

Adding, we get

$$(AB)^2 + (BC)^2 = (AD) \cdot (AC) + (DC) \cdot (AC)$$
$$= (AC) \cdot (AD + DC) = (AC)^2.$$

When we consider these possible proofs of Pythagoras, a number of features emerge. First, the assumptions are not clearly stated. The appeal at each point is merely to the intuitively obvious. Second, they are meant to be general, ideal, and exact. As Plato puts it,

[students of geometry] make use of visible figures and discourse about them, though what they really have in mind is the originals of which these figures are images: they are not reasoning, for instance, about this particular square or diagonal which they have drawn, but about *the* Square and *the* Diagonal; and so in all cases. The diagrams they draw and the models they make are actual things... while the student is seeking to behold those realities which only thought can apprehend [1955 225 (510 D, E)].

Third, no special notation was used. Except in Fig. 4, where the explanation would have become very long without it, only line shading was used, and Pythagoras may have used a similar device. However, he may not have known of the familiar device of labeling triangles with letters, as used in Fig. 4. This might seem utterly unimportant, but we know today that advances in science and mathematics have very often depended on advances in notation. Up to a point they seem merely a matter of convenience, but beyond this the notation itself serves the heuristic function of suggesting further developments of a substantive nature and of allowing a compactness of expression which makes understanding possible. We do not know who first thought of this simple device for naming points, lines, triangles, etc., but without it Euclid's *Elements* would not have been possible.

But the Pythagorean relation was not the most important mathematical theorem discovered by Pythagoras and his school; rather it was the discovery and proof of the existence of incommensurate lengths. *Commensurate* means having a common measure, and it is obvious that the unit is the common measure of all numbers; in fact, a number was understood as a multitude of units [cf. Euclid 1926 II 277]. The arithmetic unit was thought to be indivisible. Fractions such as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, etc., were not understood as representing a part of a unit, but always as 1 unit out of 2, 2 out of 3, 3 out of 4, etc. Given this arithmetic, it was probably obvious that there had to be an indivisible geometric entity so small that any length would be an even multiple of it. It would then follow that every two lengths would be in a definite fixed proportion to each other. What was meant by *definite fixed proportion* was that the relative size of any two lengths could be expressed as a ratio between numbers. That is,

$$\text{first length} : \text{second length} :: x : y,$$

where x is the number of indivisible entities in the first length and y the number in the second.

This theory of proportions makes understandable the creed of the Pythagoreans that the essence of things is numbers. For it was natural to identify the indivisible geometric entity with the numerical unit. But the Pythagoreans made a further identification: namely, that the unit (or indivisible entity) was also a physical atom. This belief prompted (or was prompted by) the Pythagoreans' discovery that all musical scales may be expressed as the ratio of the first four natural numbers: for example, octave 2 : 1, fifth 3 : 2, fourth 4 : 3. They saw special significance in the fact that $1 + 2 + 3 + 4 = 10$. As Aristotle tells us, Pythagoreans saw numbers everywhere:

In numbers they seemed to see many resemblances to the things that exist and come into being—more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity—and similarly almost all other things being numerically expressible); since, again, they saw that the modifications and the ratios of the musical scales were expressible in numbers;—since, then, all other things seemed in their whole nature to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number [1928a 985b 27ff].

Given this belief in number as the unifying principle of arithmetic, geometry, cosmology and philosophy—a belief which was instilled by all the artifice of religious practice—the discovery of incommensurate lengths must have been a real shock, the first of many clashes between science and religion in the West. It was said that the Pythagoreans were sworn never to reveal this discovery. Aristotle gives us a hint of how the existence of incommensurate lengths was first proved. The substance of the proof, in modern form, is as follows: Suppose we have unit square (see Fig. 1b) and consider the relation of the side (call it s) to the diagonal (call it d). According to the theory of proportions,

$$s : d :: x : y,$$

where x and y are natural numbers with no common divisor. Re-expressing this relation, we have

$$s/d = x/y,$$

from which, if we square both sides, we derive

$$s^2/d^2 = x^2/y^2.$$

By the Pythagorean theorem,

$$d^2 = s^2 + s^2 = 2s^2,$$

so we have

(1) $$s^2/d^2 = s^2/2s^2 = 1/2 = x^2/y^2.$$

That is, $y^2 = 2x^2$, from which it follows that y is even. Hence x must be odd, since x and y have no common divisor. If y is even, then $y = 2z$ and $y^2 = 4z^2 = 2x^2$, and so $x^2 = 2z^2$, from which it follows that x is even. Since it is impossible for a number to be both odd and even, s and d cannot be commensurate.

Problem 2. The proof was more general than required. In what way?

Since number was understood as a plurality of units, it followed that no number could correspond to the length of the diagonal of a unit square. Thus the harmony between numbers and lengths, or between arithmetic and geometry, was broken. The length of d ($= \sqrt{2}$) was neither a number, nor a ratio (cf. "rational") between two numbers. Rather, according to the Pythagoreans, d represented an *irrational* magnitude. Hence number cannot be the essence of geometry, much less of cosmology or philosophy.

The importance of this development for logic is that it represents the first scientific use of a *reductio ad impossibile* proof. In such a proof one derives a contradiction from a hypothesis and then concludes that the hypothesis is false. Its importance is that it enables one to refute a position held by either oneself or another. If what is derived is false, the argument is called a *reductio ad absurdum.* Thus the latter type of argument would include *reductio ad impossibile* arguments as well as arguments in which the derived conclusion is merely known to be false. This kind of distinction was no doubt not made until much, much later.

§3 ARGUMENTATION BEFORE ARISTOTLE

Mathematics developed in a number of significant ways between the time of the achievements of the early Pythagoreans and the time of Aristotle. However, from a logical point of view nothing really new was added to the proof and disproof procedure of the early Pythagoreans. But mathematics was not the only area which stimulated the development of logic; arguments in philosophy and the law courts did also.

The relevance of such arguments may be seen by considering the usefulness of developing a theory of logic in a situation in which there is both a lot of talk which is aimed at proving the truth of something or other and disagreement as to what the truth is. There would be no need to state logical principles if either there were no disagreements or only a small number of them (since in the latter case each could be considered individually). Conversely, the ability to elucidate logical principles presupposes agreement on some very simple arguments. Without this it is unlikely that communication would be possible at all.

Ancient Athens of Aristotle's time provided a great variety of viewpoints and thinkers. Some of the thinkers were from Athens, but many were from Greek colonies; either they would visit Athens or reports of their views would be brought by their disciples. Further, there were extant writings or oral traditions of a philosophical heritage that even then was well over 200 years old. Thales argued that the basic stuff of the world is water, Anaximander that it is not one thing but an indeterminate something or other; Heraclitus that all things are in motion, Parmenides that no things are; Protagoras that our ethical judgments are relative, Socrates that they are not, and so on. Thus, in order to refute the arguments of the sundry Sophists and philosophers whose conclusions Aristotle found either false or paradoxical, he tried to devise a set of principles by which one could determine whether any given argument is a good one.

Zeno of Elea was a typical example of a Pre-Socratic whose arguments Aristotle tried to refute. According to Plato, Zeno "has an art of speaking by which he makes the same things appear to his hearers like and unlike, one and many, at rest and in motion" [1937 I 265 (261D)]. We have no extant writings of Zeno, and it is even possible (although unlikely) that he wrote nothing at all. Nevertheless it is clear that Zeno devised a good number of puzzles which have philosophical interest. Commentary and criticisms of these riddles appeared early and the literature is still growing at a good pace. A typical example of one of his paradoxes is the so-called Achilles argument against motion. Aristotle tells us that

> ... it amounts to this, that in a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead [1941 335 (239b 14–17)].

One of the reasons that there has been so much commentary on the puzzles is the (for the most part) cryptic descriptions we have of them. The above reference is typical in this respect, but a probable reconstruction might be as follows:

Achilles, born of a goddess and the fastest of human runners, cannot catch even a tortoise, the slowest of moving creatures. For suppose we have a race in which the tortoise is given a lead. However fast Achilles runs to reach the point at which the tortoise was, he must take some time to do it. In that time the tortoise will move forward some (smaller) distance. But now we may repeat our argument again and again and again. It is clear that Achilles may get closer and closer to the tortoise but he cannot catch the tortoise.

There is no way to be certain, but it is possible that this *reductio ad absurdum* argument against the existence of motion was inspired by the *reductio ad impossibile* arguments of Pythagorean mathematics. At any rate, we know of no earlier use of this form of argument in philosophy. Its importance is that once the *reductio* form is learned, it tends to breed discussion and dispute rather than disciples who faithfully accept and promulgate the master's teaching. The creation of a heritage of discussion rather than one of truths laid down by authority is perhaps the most important contribution of the Pre-Socratic philosophers to our civilization, and is surely one of the greatest cultural achievements of all time.

Unfortunately, we know very little about the origins of this contribution. It is probable, however, that it originated in the playful element in human nature. Perhaps one of its first forms was the riddle such as that posed by the Sphinx in the Oedipus myth. "What creature," the Sphinx asked, "goes on four feet in the morning, on two at noonday, on three in the evening?" Oedipus' correct answer depended on an ambiguity: "Man, because in childhood he creeps on hands and feet; in manhood he walks erect; in old age he helps himself with a staff." The flash of insight saved Thebes, and it is likely that much of the teaching of the Pre-Socratic philosophers was similar: aphoristic wisdom which comes "out of the blue."

However, another play form developed in which there was a game whose object was to defeat one's opponent by words. Perhaps Zeno was an important influence in the development of it. We do know that there existed a class of teachers who came to be known as sophists. These sophists would travel, much like wandering minstrels, and for a fee would teach their students how to speak persuasively on many different kinds of topics. Sophists were also prepared to defeat any opponent in a public argument. The competitiveness of such a spectacle must have been very keen and the arguments often dramatic, so that we can understand why the arrival of an important sophist in town was the occasion of much excitement and why sophists were often able to command large fees.

Protagoras, often considered to be the greatest of the sophists, would no doubt be thought a great thinker if his works had survived. He is best known for his saying that "man is the measure of all things" and his humanism probably exhibited itself in ways we consider uniquely modern. The following ancient story about him, although probably apocryphal, indicates the kind of verbal pyrotechnics of which the sophists were capable. Protagoras had contracted to teach Euathlus rhetoric so that he could become a lawyer. Euathlus initially paid only half of the large fee, and they agreed that the second installment should be paid after Euathlus had won his first case in court. Euathlus, however, delayed going into practice for quite some time. Protagoras, worrying about his reputation as well as wanting the money, decided to sue. In court Protagoras argued to the jury:

Euathlus maintains he should not pay me but this is absurd. For suppose he wins this case. Since this is his maiden appearance in court he then ought to pay me because he won his first case. On the other hand, suppose he loses the case. Then he ought to pay me by the judgment of the court. Since he must either win or lose the case he must pay me.

Euathlus had been a good student and was able to answer Protagoras' argument with a similar one of his own:

Protagoras maintains that I should pay him but it is this which is absurd. For suppose he wins this case. Since I will not have won my first case I do not need to pay him according to our agreement. On the other hand, suppose he loses the case. Then I do not have to pay him by judgment of the court. Since he must either win or lose the case I do not have to pay him.*

Problem 3. Construct arguments for the defense and prosecution similar to those of Euathlus and Protagoras in the circumstances of the following story. It is taken from Cervantes' *Don Quixote.* Sancho Panza, the governor of the island of Baratavia, has the following case brought before him by a foreigner:

My Lord ... there was a large river that separated two districts of one and the same seignorial domain—and let your Grace pay attention, for the matter is an important one and somewhat difficult of solution. To continue then: Over this river there was a

* I have altered this story in inessential ways in order to bring out its logical form. For those who are interested in looking up the original story, see Gellius 1927 404 ff.

bridge, and at one end of it stood a gallows with what resembled a court of justice, where four judges commonly sat to see to the enforcement of a law decreed by the lord of the river, of the bridge, and of the seignory. That law was the following: "Anyone who crosses this river shall first take oath as to whither he is bound and why. If he swears to the truth, he shall be permitted to pass, but if he tells a falsehood, he shall die without hope of pardon on the gallows that has been set up there." Once this law and the rigorous conditions it laid down had been promulgated, there were many who told the truth and whom the judges permitted to pass freely enough. And then it happened that one day, when they came to administer the oath to a certain man, he swore and affirmed that his destination was to die upon the gallows which they had erected and that he had no other purpose in view . . . [1949 842].

It is clear that to straighten out such puzzles one has to inquire into *general* procedures of argument. The motive for such an inquiry might not be just to find out the truth but also to defeat one's opponent. Surely this latter motive influenced Plato (427–347 B.C.), who wrote a series of dialogs which leave him unsurpassed both as a thinker and as a writer. Plato was aware of the sportive element in his enterprise and at times—especially in later life—felt it, and artistic endeavor in general, to be unworthy of a true philosopher, who should seek truth without art or playfulness. The young Nietzsche captured Plato's élan:

What . . . is of special artistic significance in Plato's dialogues is for the most part the result of a contest with the art of orators, the sophists, and the dramatists of his time, invented for the purpose of enabling him to say in the end: "Look, I too can do what my great rivals can do; indeed, I can do it better than they. No Protagoras has invented myths as beautiful as mine; no dramatist such a vivid and captivating whole as my *Symposion*; no orator has written orations like those in my *Gorgias*—and now I repudiate all this entirely and condemn all imitative art. Only the contest made me a poet, a sophist, an orator" [1954 37–8].

In perhaps no dialog is the playfulness and competitive spirit of Plato better revealed than in the *Protagoras.* The heart of the dialog consists in a long conversation between Protagoras and Socrates before an audience. It concerns the question of whether or not virtue can be taught. Protagoras asserts that it can be; Socrates questions this. This leads to the question of what virtue essentially is. This dialog reveals more than just playfulness, however, when Plato has Socrates state at the end that

. . . the result of our discussion appears to me to be singular. For if the argument had a human voice, that voice would be heard laughing at us and saying: 'Protagoras and Socrates, you are strange beings; there are you, Socrates, who were saying that virtue cannot be taught, contradicting yourself now by your attempt to prove that all things are knowledge, including justice, and temperance, and courage, which tends to show that virtue can certainly be taught; for if virtue were other than knowledge, as Protagoras attempted to prove, then clearly virtue cannot be taught; but if virtue is entirely knowledge, as you are seeking to show, then I cannot but suppose that virtue is capable of being taught. Protagoras, on the other hand, who started by saying that it might be taught, is now eager to prove it to be anything rather than knowledge; and if this is true, it must be quite incapable of being taught.' Now I, Protagoras, perceiving this terrible confusion of our ideas, have a great desire that it should be cleared up [1937 I 129–30 (361A ff)].

This idea of an argument having a life of its own, the conclusions of which may be unexpected or unwanted by the formulator of the argument, was probably not new with Socrates. But it leads to the open quality of the Socratic dialogs, which in turn heightens interest because the outcome is unknown. It seems clear that Socrates was the first to give his supreme loyalty to inquiry itself rather than to some specific proposition which might be the result of inquiry. "... for you will come to no harm," he says on one occasion to his partner in philosophical conversation, "if you nobly resign yourself into the healing hand of the argument as to a physician without shrinking..." [1937 I 535 (475D)]. It is from Socrates that even today we get the ideal of honest inquiry, in which the inquirer follows the argument wheresoever it leads—even if this brings him embarrassment or disadvantage. The development of this attitude, although not a contribution to logic proper, is nevertheless of utmost importance for it. For without it a large part of the desire to argue correctly is dissipated.

This attitude leads to a reverence for philosophical conversation which keeps its integrity and does not degenerate into mere quibbling. Socrates believes that people

... often seem to fall unconsciously into mere disputes which they mistake for reasonable argument, through being unable to draw the distinctions proper to their subject; and so, instead of a philosophical exchange of ideas, they go off in chase of contradictions which are purely verbal [1955 151 (454A)].

Many examples of purely verbal contradictions and quibbling are given in Plato's dialogs, especially in the *Euthydemus*. In that dialog extravagant arguments are put in the mouths of speakers; arguments whose conclusions are, for example, that no one can tell a lie, that there is no such thing as a contradiction, that Socrates knows everything. An illustration is the famous exchange between Dionysodorus and Ctesippus:

... You say that you have a dog.

Yes, a villain of a one, said Ctesippus.

And he has puppies?

Yes, and they are very like himself.

And the dog is the father of them?

Yes, he said, I certainly saw him and the mother of the puppies come together.

And is he not yours?

To be sure he is.

Then he is a father, and he is yours; ergo, he is your father, and the puppies are your brothers [1937 I 161 (298D–E)].

Problem 4. Give two different definitions of one of the terms in the argument which would remove the ambiguity on which the argument's plausibility depends.

It would be foolish to think that Plato was not aware of the invalidity of arguments like those mentioned in the *Euthydemus.* Yet we are unaware of just how much difficulty he had in deciding about the validity of many of the arguments put in the mouths of his characters. Nevertheless Plato did preserve a large body of arguments in writing, and this provided a significant part of the corpus of arguments from which Aristotle developed his logic.

It is clear that this writing-down of arguments was one of Plato's main contributions to the prehistory of formal logic. In the course of the dialogs a few logical principles are enunciated, but the real impetus to logic of his writings is the portrayal of that peculiarly Socratic turn of mind which states that we should be

careful of allowing or of admitting into our souls the notion that there is no health or soundness in any arguments at all. Rather say that we have not yet attained to soundness in ourselves, and that we must struggle manfully and do our best to gain health of mind . . . [1937 I 475 (90E)].

Such an attitude requires that one not allow a philosophical argument to degenerate into *mere* quibbling, but it certainly does not require that kind of seriousness which excludes playfulness.

We may now summarize Aristotle's motivation in inventing logic. First, there is the desire to know the truth about the nature of argument, an intellectual curiosity which needs no further account or justification. Second, there is the desire to know the conditions under which something is proved. This question was perhaps most clearly focused in the case of geometry: How are we going to decide when a mathematical relationship really holds? But the problem was wider and was also, in metaphysical questions, acute. Zeno's arguments provide a good example of the latter. Third, there is the desire to refute opponents. Here there is perhaps an analogy with the invention of probability theory. The theory was initiated when Chevalier asked Pascal to solve certain problems having to do with odds in gambling. But probability theory is vastly more comprehensive, applying to many, many different areas, including all physical and social sciences. Similarly, logic is vastly more comprehensive and useful than merely a device which may be used to show that an opponent is wrong. Yet we should not overlook the egoism and spirit of competitiveness which marked its origin.

§4 ARISTOTLE'S LOGIC

A. Preliminaries

Aristotle (384–322 B.C.) must have been a man of almost boundless energy; the range of his intellectuality is astounding. He contributed in important ways to biology, physics, astronomy, political theory, and ethics. His achievement in logic was just one of many others, and even if it were all wrong we would nevertheless consider him an intellectual giant. But just because of the great diversity of his interests, as well as his conviction that logic is a *techné*, an art or a tool, it is unlikely that he would approve of our considering just this aspect of his thought. Indeed his logical theory is embedded in a number of other related metaphysical doctrines which, if that theory is to be considered comprehensively, would have to be discussed in detail. To avoid being diverted from our purpose of providing a background for mathematical logic, we shall forgo such a discussion.

Although Aristotle did not give a definition of *argument*, it is clear that he meant by it what we mean; namely, a set of propositions of which one is claimed to follow from the others. That is, the one is claimed to be true *if* the other proposition or propositions are also true. The proposition which follows from the other proposition or propositions is called a *conclusion*; that from which it follows is the *premiss* or *premisses.* Now a *valid* argument is one in which *if* the premisses are true the conclusion must necessarily also be true. An *invalid* argument is any argument which is not valid. The validity of an argument is in general independent of the truth or falsehood of the premisses. It is perfectly possible for a valid argument to have a false conclusion and for an invalid argument to have a true conclusion.

Problem 5. Construct an example of each kind of argument.

Since the requirements of validity and truth are in general independent, we shall follow standard logical practice in applying 'valid-invalid' to arguments only and 'true-false' to either the premisses or conclusion. This departs somewhat from ordinary speech, in which it is permissible to say 'valid premiss' or 'true argument'. But by avoiding such language we shall be avoiding some of the confusions of ordinary talk about arguments.

The distinction between valid and true and invalid and false was understood by Aristotle, but its first clear and accurate formulation was made by the Stoics. A similar statement must be made about the distinction between a sentence and a proposition, although on this distinction Aristotle at times seems confused. In English we normally make a distinction between 'sentence' and 'proposition' but it is not a hard and fast one. A good example would be the use of the word 'proposition' in Lincoln's Gettysburg address:

> Fourscore and seven years ago, our fathers brought forth upon this continent a new nation, conceived in liberty, and dedicated to the proposition that all men are created equal [cf. Church 1956 26].

One does not dedicate oneself to a sentence. Similarly, if one refers to the fifth postulate of Euclid, one does not generally mean something in the Greek language. Consider an English sentence. Translate it into a dozen languages. We would then have thirteen sentences, but only one proposition. A proposition is what is expressed by a sentence.

Aristotle does draw a distinction between a sentence and a proposition, but different from the one just described:

> Every sentence has meaning . . . by convention. Yet every sentence is not a proposition; only such are propositions as have in them either truth or falsity. Thus a prayer is a sentence, but is neither true nor false [1928 17a 1 ff].

In order to keep closer to contemporary usage, we shall not make the distinction in this way, but rather speak of declarative, interrogative, imperative, and exclamatory sentences or propositions.

Consider Socrates' prayer at the end of the *Phaedrus*:

> Beloved Pan, and all ye other gods who haunt this place, give me beauty in the inward soul; and may the outward and inward man be as one. May I reckon the wise to be the wealthy, and may I have such a quantity of gold as a temperate man and he only can bear and carry [1937 I 282 (297E)].

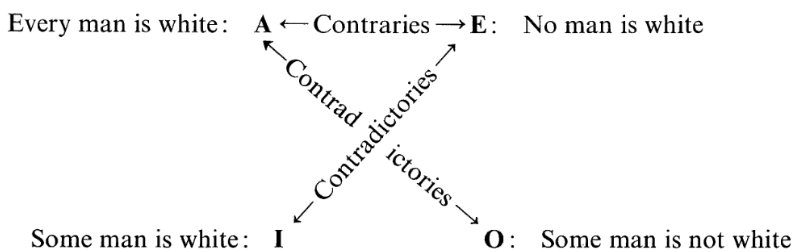
We shall say not that there are two sentences and no propositions, but rather that there are two Greek imperative sentences to which correspond two English (or German or French, etc.) sentences and which all express two imperative propositions. On the other hand, one sentence can be used to express two different propositions. Thus the sentence 'I am pleased with the results of the election' might be true if one person says it and false if another does. For this reason we shall say that the two utterances do not express the same proposition. Declarative propositions will be considered true or false in the primary sense, declarative sentences in the secondary sense (that is, they are true if they express true propositions); interrogative, imperative, and exclamatory sentences or propositions will not be understood as true or false. Logic, as we shall understand it, is primarily about propositions, and only secondarily about sentences. Further, we shall consider only declarative propositions. Relatively little work has been done on the logic of nondeclaratives.

B. Immediate Inferences

By a *simple proposition*, Aristotle means "a statement, with meaning, as to the presence of something in a subject or its absence" [1928 17a 23 f]; by a *composite proposition*, he means one that is made out of simple propositions. A proposition is *universal* if it affirms or denies the predicate to every instance of the subject; *particular* if it affirms or denies the predicate to some (that is, at least one) instance of the subject; *singular* if it affirms or denies the predicate of a subject which denotes exactly one thing. Aristotle considers four general types of propositions: the universal affirmative, the universal negative, the particular affirmative, and the particular negative. Medieval logicians later gave the names **A**, **E**, **I**, **O**, respectively, to these general types of propositions, apparently from the Latin *affirmo* (I affirm) and *nego* (I deny). We shall use these names for convenience. Examples, respectively, are 'Every man is white', 'No man is white', 'Some man is white', and 'Some man is not white'. The **A** proposition is often expressed by a variant using 'all' instead of 'every'. For example, 'All men are white'.

Singular propositions may be reduced to universal ones. For example, the singular affirmative proposition 'Socrates is white' may be rewritten as 'Every individual identical with Socrates is white'. The singular negative proposition 'Socrates is not white' may be rewritten as 'No individual identical with Socrates is white'. Hence singular propositions need not be considered separately.

Two propositions are *contradictories* when if either is true the other must be false, and if either is false the other must be true. **A** and **O** as well as **E** and **I** are thus contradictories when they have the same subject and predicate. *Contrary* propositions are such that, while both may be false, they cannot both be true. Thus **A** and **E** are contraries when they have the same subject and predicate. A diagram may be used to make Aristotle's doctrine clearer, although no diagram is found in his text.



Aristotle assumes that **A** implies **I**, and that **E** implies **O**. Thus, for example, 'every man is white' implies 'some man is white'. In later times the name *subalternation* was given to this relationship, the universal proposition being called the *superaltern*, its corresponding particular proposition being called the *subaltern*. Aristotle also indicates that he is aware of the relationship between the **I** and **O** propositions, although he does not explicitly define it nor give it a name. Later logicians have called the **I** and **O** propositions *subcontraries*.

Problem 6. Give a definition of the term *subcontraries* analogous to the one given for *contraries*. If you think about it you will see that the relationships already given allow only one possible definition for this term.

If you add the names *subalternation* and *subcontraries* to the above diagram in the appropriate places, you have the classical *Square of Opposition*; the word *opposition* being used as a generic name for all the possible logical relationships that the **A**, **E**, **I**, **O** propositions might have with one another. Alternatively, one could use a grid to make all these relationships explicit (Table 1).

Table 1

	A	E	I	O
1 **A** given as T				
2 **E** given as T				
3 **I** given as T				
4 **O** given as T				
5 **A** given as F				
6 **E** given as F				
7 **I** given as F				
8 **O** given as F				

In Table 1, 'T' is an abbreviation for 'true', 'F' for 'false'. Consider row (1). If **A** is T, then **A** will be T (given), **E** will be F (since **A** and **E** are contraries), **I** will be T (since **A** implies **I**), **O** will be F (since **A** and **O** are contradictories). Not all spaces can be filled in with a 'T' or 'F'. In such cases we shall use 'U' for 'undetermined'. Thus in row (5), where **A** is F, **E** is U. This can be seen by considering an example. If 'every man is white' is false, then 'no man is white' may be either true or false. It would be false under the condition that some men are white and some men are not white.

Problem 7. Fill in each space in Table 1 with a 'T' or 'F' or 'U'. Use an example if you are uncertain.

Aristotle was the first to use symbols in logic. It can be easily seen from what has been said that it is not the content of a proposition that is important but the form. In geometry, also, it was the triangle *qua* triangle that was important and the geometer had to abstract from the triangle he drew by ignoring any of its particular conditions. Apparently early in his career

Aristotle followed a similar procedure in logic. Thus he would use indiscriminately such statements as 'every man is white', 'every man is an animal', 'every pleasure is good', etc. At some time, however, he came to see that he could use a letter to stand for any term. This practice was perhaps suggested by the use of letters for points, lines, triangles, etc. in geometry. In any case this procedure not only reveals the structure of the proposition more clearly but it is also more convenient to use. Thus Aristotle began using what are now called *propositional forms*: 'Every *S* is *P*', 'No *S* is *P*', etc. The term *formal logic* arose to designate the study of the various kinds of relationships which these forms might have.

One of these relationships which Aristotle studied is called *conversion*, that is, the interchanging of the subject and predicate term. If 'no *S* is *P*', then it follows that 'no *P* is *S*'. Similarly, if 'some *S* is *P*', then 'some *P* is *S*'. With the universal affirmative proposition the situation is more complicated. If 'every *S* is *P*', it does not follow that 'every *P* is *S*'. But Aristotle asserted that 'some *S* is *P*' follows. Later logicians gave the name *limitation* to the procedure of changing an **A** (or **E**) proposition to an **I** (or **O**) proposition (with the subject and predicate remaining constant). Thus, by Aristotle's scheme, the converse of an **A** proposition does not follow, but the converse by limitation does. The inference from an **O** proposition to its converse is not a valid one. For if 'some *S* is not *P*', then it doesn't follow that 'some *P* is not *S*'. This can be seen by considering Aristotle's example of letting '*S*' stand for 'animal' and '*P*' for 'man'.

Aristotle also considered changing a term to what is now called its *complement*. If 'animal' is the term, its complement is 'that which is not an animal'. The latter would include the number seven, the Parthenon, and the sun—in fact, everything which is not an animal. By way of notation, if '*S*' is a term, we shall use 'non-*S*' to stand for its complement. Aristotle noted that if 'every *S* is *P*' it does not follow that 'every non-*S* is non-*P*', but that 'every non-*P* is non-*S*' does follow. This process is an example of what was later to be called *contraposition*; that is, the procedure of changing both terms to their complements and then converting.

Problem 8. Of the remaining categorical propositions—that is, **E**, **I**, **O**—it happens that for one contraposition is valid, for another only contraposition by limitation is valid, and the third has no valid contrapositive. Verify this last statement by determining which is which.

C. Syllogistic Theory

The kind of inference of Aristotle that we have been considering came to be called *immediate*, presumably because the inference depended on only one premiss, and thus the conclusion followed immediately without the introduction of any other premiss. Aristotle's main contribution to logic was not this theory of the immediate inference, but rather the theory of the *syllogism*, which involves *mediate inference*, that is, an inference depending on more than one premiss.

At first Aristotle understood a syllogism to be any argument, but later he took it to mean an argument with two premisses and a conclusion. He concentrated most of his attention on what later came to be known as *categorical syllogisms*, that is, those in which each of the three propositions involved is one of the four we have considered.

A syllogism must contain exactly three terms, each of which occurs twice (but not twice in the same proposition), and each term must be used in the same sense in each occurrence in the syllogism. An example would be the following: If 'all mammals are animals' and 'all men are mammals', then 'all men are animals'. Aristotle always called the predicate term of the conclusion the *major term*, the subject term of the conclusion the *minor term*, and the other term the *middle term*. Thus, in our example, 'animals' is the major term, 'man' the minor, and 'mammals' the middle. The *major premiss* is the one containing the major term and is always stated first; the *minor premiss* is analogously defined.

When we consider the possible arrangements of the middle term in the premisses, there are four. This can best be seen in a diagram such as Table 2, in which we use '*P*' for the major term, '*S*' for the minor, and '*M*' for the middle.

Table 2

	Figure 1	Figure 2	Figure 3	Figure 4
Major premiss	*M-P*	*P-M*	*M-P*	*P-M*
Minor premiss	*S-M*	*S-M*	*M-S*	*M-S*
Conclusion	*S-P*	*S-P*	*S-P*	*S-P*

Aristotle gave the name *figure* to these various arrangements, although (for reasons which are now of interest only to the historian of logic) he did not recognize the fourth figure. For ease of exposition we state each premiss or conclusion on a separate line, although Aristotle himself generally understood the syllogism as one 'if... then' proposition. Our example above about animals, mammals, and men is in the first figure. An example in the second figure is: If 'all animals are mammals' and 'all men are mammals', then 'all men are animals'. This syllogism is invalid, whereas the former was valid.

Now for each figure with a given conclusion there are 16 possible combinations of premisses:

AA, AE, AI, AO

EA, EE, EI, EO

IA, IE, II, IO

OA, OE, OI, OO

Since there are four possible conclusions there are 64 (4 × 16) possible syllogisms for each figure, and therefore 256 (64 × 4) possible syllogisms in all. Only 24 of these represent valid arguments according to the Aristotelian framework. (See Table 3, which indicates the 24 valid forms (V = valid).)

Armed with the theory of the syllogism, Aristotle felt that he could answer the question as to the conditions under which something is proved and thus refute the philosophical doctrine that scientific knowledge (that is, knowledge based on demonstration) is either impossible or circular. It was impossible if, in order to demonstrate a conclusion, one also had to demonstrate the premisses, and so on *ad infinitum*. It was circular if the conclusion was allowed in the premisses, which really comes to 'if A, then A'. "A simple way of proving anything," Aristotle remarks sarcastically.

To understand why anyone would hold either of these two views, we have to recall the situation in geometry when Aristotle was writing. It is likely that there were many treatises on the subject, and that what was assumed without proof in one was proved in another. As we have seen, the initial assumptions or axioms were probably not all made explicit. In such circumstances it is not unlikely that some thinkers would come to believe in the futility of demonstration because the demonstrator must always assume something he hasn't proved; whereas other thinkers would believe circular demonstration to be legitimate, since it was obvious that the geometers had knowledge.

"Our own doctrine," Aristotle tells us, "is that not all knowledge is demonstrative" [1928 72b 18]. Accordingly he provided a philosophical basis for demonstration. Demonstration must start with self-evident truths which are themselves not demonstrable. They must be clearly true and better known than anything that is subsequently proved from them. Aristotle's belief in such truths explains what seems to be an inversion of the relationship between demonstration and syllogism:

> Syllogism should be discussed before demonstration because syllogism is the more general: the demonstration is a sort of syllogism, but not every syllogism is a demonstration [1928 25b 28f].

Table 3

Figure	Conclusion	Premisses															
		AA	AE	AI	AO	EA	EE	EI	EO	IA	IE	II	IO	OA	OE	OI	OO
1st	A	V															
	E					V											
	I	V		V													
	O					V		V									
2nd	A																
	E		V			V											
	I																
	O		V		V	V		V									
3rd	A																
	E																
	I	V		V						V							
	O					V		V						V			
4th	A																
	E		V														
	I	V								V							
	O		V			V		V									

A syllogism—even a valid one—might have false premisses, whereas a demonstration is a valid argument based on true premisses. Today we would say that the syllogism can represent but one type of demonstration from premisses, whereas for Aristotle, who apparently believed that all correct reasoning could be made syllogistic (he wasn't always consistent in this belief), demonstration became a species of syllogistic reasoning. A demonstration, for Aristotle, was "a syllogism productive of scientific knowledge." A demonstration is productive of scientific knowledge because the premisses are true and the conclusion necessarily follows. Aristotle went further than this and asserted that the premisses in a demonstration must not only be true, but they must necessarily be true. This, as well as other considerations, makes it likely that he had in mind demonstration in geometry, although clearly he didn't mean to so limit it. In any case, he gave an answer to the question as to when something is proved in geometry, and his doctrine of the nature of demonstration influenced Euclid when he was organizing the *Elements.*

But this ideal of demonstrative knowledge did not just influence others; it brought about the theory of *reduction.* One syllogism can be reduced to another if it can be shown that the first will be valid if the second is. For example, consider an **EAE** syllogism in the second figure:

No P is M
Every S is M
Therefore no S is P

If we convert the major premiss we have:

No M is P
Every S is M
Therefore no S is P

This syllogism is in the first figure. Making use of more involved techniques, Aristotle is able to show that all valid syllogisms can be reduced to either **AAA** or **EAE**, both in the first figure. In other words, if we consider the latter two syllogisms as axioms, the other valid syllogisms can be derived from them. Aristotle's system of syllogisms represents, then, the first body of knowledge presented in an axiomatic way.

There are two other aspects of Aristotle's logic which should be mentioned, if only in passing. The first is that Aristotle developed a theory of modal logic in connection with his theory of the syllogism. *Modal logic* concerns inferences involving such notions as necessity, contingency, and possibility. An example would be the following:

It is possible that some white things are men.
It is necessary that all men are animals.
Therefore it is possible that some white things are animals.

Aristotle's theory of modal logic is quite complex and much of it is wrong. It represents, however, an attempt to deal with notions that any logic which aims at completeness must encompass.

The other aspect of Aristotle's theory that should be mentioned is his theory of definition. "A 'definition'," Aristotle tells us, "is a phrase signifying a thing's essence" [1928 101b 38]. Objects have properties that are both essential and accidental, but only the former enter into a definition of an object. Because Aristotle considered the difference between essential and accidental properties to be an objective fact, a definition, to him, was also objective and if correct, was necessarily true. In this he was thoroughly Socratic; indeed Aristotle gave Socrates the credit for being "the first to raise the problem of universal definition" and said that "it was natural that Socrates should be seeking the essence, for he was seeking to syllogize, and 'what a thing is' is the starting point of syllogisms..." [1928a 1078b 19 ff]. However, Aristotle did not consider a definition true until it had been shown that the word defined referred to something that existed and until the definition had indicated the thing's essential properties in terms which were better known than the word being defined. A definition, as Aristotle pointed out in a number of places, does not assert the existence of the thing defined. This must be either assumed or proved.

All in all, Aristotle's logic is a magnificent achievement; he started with virtually no predecessors and invented a theory which today is considered in many respects right and even complete. If from today's vantage point it also seems limited, it must be remembered that the discovery of its limitations is a rather recent achievement, and that it was 2000 years before anyone besides the Stoics made substantial progress in formal logic. Aristotle by no means claimed that his syllogistic theory covered all kinds of arguments. He was aware of others. For example, early in his career he formulated his famous argument on the necessity of philosophizing:

> Either we ought to philosophize or we ought not. If we ought, then we ought. If we ought not, then also we ought [i.e., in order to justify this view]. Hence in any case we ought to philosophize [cf. Kneale 1962 97].

Later in his career—indeed here and there throughout his logical writings—he considered or mentioned in passing a number of principles or examples or techniques which do not fit in with his main theory. There is even a hint that in some of his lost writings he dealt with asyllogistic inference in some detail. Thus when Aristotle's logical system was denounced in Renaissance times and later, the attack applied not so much to Aristotle as to those less-imaginative thinkers who followed him and who were not imbued with the Socratic spirit. This spirit was not absent even in his logical inquiries:

> ... all syllogism, and therefore *a fortiori* demonstration, is addressed not to the spoken word, but to the discourse within the soul... [1928 76b 23 ff].

When Aristotle made this statement he was merely betraying his Socratic allegiance, for Socrates says:

> ... the soul when thinking appears to me to be just talking—asking questions of herself and answering them, affirming and denying ... [To] form an opinion is to speak, and opinion is a word spoken—I mean, to oneself and in silence, not aloud or to another ... [1937 II 193 (190A)].

Dante, in a famous phrase, called Aristotle "the master of those who know." It would perhaps be more accurate to call him "a master of those who inquire."

§5 GREEK MATHEMATICS AND LOGIC AFTER ARISTOTLE

A. Mathematics

There are really only two further developments in Greek mathematics which turned out to be important for logic: The first is the systemization of geometrical knowledge, the second is the formulation of geometrical problems which the Greeks themselves could not solve. We shall consider each in turn.

The great system builder of ancient geometry is, of course, Euclid. Unfortunately nothing is known about him. Like the name *Bourbaki* in contemporary mathematics, it is possible that the name *Euclid* refers not to one man, but several. In any case, he (henceforth we shall ignore the possibility that the proper pronoun might be *they*) probably lived about 300 B.C. Tradition says he was a Platonist. If this is true, the following ancient story about him gains credibility, since Plato's theories did not emphasize practical applications:

> Some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, "But what shall I get by learning these things?" Euclid called his slave and said, "Give him threepence since he must make gain out of what he learns" [cf. 1926 I 3].

Euclid wrote other works beside his great *Elements*, but only the latter is important for the development of logic. The nature of Euclid's aim in the *Elements* is not altogether clear. The book contains no explanation of its purpose. Of course, Euclid wished to systematize geometrical knowledge, but he may have wished to do more than this. To see what this further aim may have been, let's return to Plato.

Plato established an Academy whose purpose was the advancement of knowledge. Above the entrance Plato is reported to have inscribed: "Let no one ignorant of geometry enter my door." Plutarch reports that Plato said that God continually geometrizes. These hints, together with several passages in the dialogues, confirm that Plato valued geometry highly. Indeed he emphasized the importance of geometry in education, and at one point [cf. 1937 II 573–574 (819E–820C)] extravagantly emphasized the importance of knowing about the incommensurables. There is evidence to suggest that, since the Pythagoreans had failed in their attempt to base their cosmology on arithmetic, Plato intended to base his cosmology on geometry. Apparently he thought his attempt was incomplete and only partly successful. In this light, Euclid's *Elements* may be looked on as an attempt to continue the work of Plato, that is, to base arithmetic and cosmology on geometry. Euclid dealt with arithmetic from a geometrical point of view, and the last section of the *Elements* concerns the five regular solids which played a part in Plato's cosmology. As we shall see [cf. §10A], Euclid showed that geometry could succeed, where arithmetic had failed, in dealing with the incommensurate. Whatever the lack of clarity of Euclid's motives, his achievement is clear: He produced the greatest textbook of all times. (On the topics of this paragraph, see Popper 1952 and Szabó 1967.)

The book starts out with a series of 23 definitions, 5 postulates and 5 common notions. From these Euclid deduced a large number of propositions or theorems. It is important to understand the difference between these four categories (definitions, postulates, common notions, theorems). They indicate the structure of the axiomatic method, which was destined to be closely tied to the development of mathematical logic. Only some of the definitions are omitted in the following extensive quotation from the opening passages of the *Elements* [1926 I 153–155, 241–242]:

Definitions

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.
15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.
16. And the point is called the *centre* of the circle.
19. *Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
20. Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

23. *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Propositions

Proposition 1. On a given finite straight line to construct an equilateral triangle.

Let AB be the given finite straight line (Fig. 5).

Thus it is required to construct an equilateral triangle on the straight line AB.

With centre A and distance AB let the circle BCD be described; [ ]
again, with centre B and distance BA let the circle ACE be described; [ ]

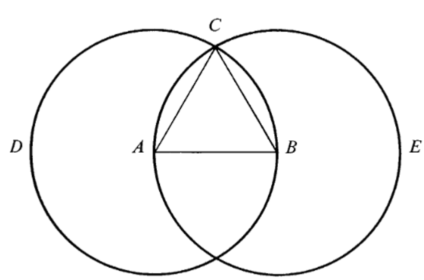


Figure 5

and from the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. [ ]

Now, since the point A is the centre of the circle CDB, AC is equal to AB. [ ]

Again, since the point B is the centre of the circle CAE, BC is equal to BA. [ ]

But CA was also proved equal to AB; therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; [ ]
therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB.

(Being) what it was required to do.

Problem 9. In each of the empty brackets at the right in the proof of Proposition I, place the number of the Definition, Postulate, or Common Notion which "justifies" the preceding statement.

Aristotle's view of the nature of demonstrative science apparently decisively influenced Euclid when he was organizing the *Elements*. In any case, what Aristotle has to say about definitions, postulates and common notions clarifies what Euclid was trying to do. We have seen that Aristotle considered definitions to be objective and true. They are objective in the sense that they may not violate established usage. Thus it would be wrong to define *circle* as that which is both equilateral and right-angled. They are true in a derivative sense. Strictly speaking, a definition refers only to a thing's essence, and says nothing about its existence. However, a definition becomes true if it is proved that the thing corresponding to the definition exists and that it has the essential properties which the definition requires. Thus, for example, Definition 20 gives a definition of *equilateral triangle*. It is objective in the sense that it fits in with established usage. It becomes true in conjunction with Proposition 1, which proves that equilateral triangles exist with the properties specified. Of course, one cannot *prove* that everything corresponding to definitions exists; according to Aristotle, in geometry one must *assume* the existence of some things: namely, points and lines.

One of the functions of the postulates is to make these assumptions explicit. Thus Postulate 1 asserts the existence of straight lines, Postulate 3 of circles. It is true that Euclid does not explicitly assume the existence of points, but we must either assume that he does this indirectly through the third definition (in which case it must be demonstrated that the first and third definitions coincide) or assume that this lack represents a failure of the *Elements* to meet Aristotle's requirements.

A second function of the postulates is to assert the possibility of certain constructions. For example, we are guaranteed by Postulate 1 the construction of not only a line but a straight line. Finally, the postulates also assert basic relationships which are needed to prove the theorems of the subject matter under investigation. Postulates 4 and 5 are examples of this sort.

Common notions—Aristotle uses the term *common opinions*—are basic assumptions that are common to a number of sciences, whereas postulates make use of terms that are specifically geometric. Thus, for example, Common Notion 1, "Things which are equal to the same thing are also equal to one another," is as applicable to arithmetic as it is to geometry. None of the postulates are applicable to arithmetic, at least not without changing the meaning of the terms used.

Finally, the propositions—or theorems—are meant to be derived from the definitions, postulates, and common notions without any additional assumptions. It is extremely difficult to do this in a systematic way. For example, even Euclid's Proposition 1 is deficient in that it makes an assumption that is not derivable from the definitions, postulates, and common notions: namely, that the constructed circles will meet at point *C*. However, this is surely a slip, and Euclid would no doubt have added a postulate to take care of this case, had he noticed it.

There is a final Aristotelian requirement for an axiomatic system: that the axioms be better-known than, simpler than, more certain than—in short, epistemologically prior to—the theorems. Otherwise the system commits the fallacy of *begging the question.* A person commits this fallacy either when he tries to prove what is already epistemologically prior (in which case the best that can be done is to covertly assume what one is trying to prove) or when he assumes as an axiom what is not prior (in which case simpler and more certain things than the axiom would follow from it). In the latter case the proof is fraudulent, since what is proved is less in doubt than the original premisses. Euclid's system seems to satisfy this last requirement in almost all respects: The postulates and common notions are not proved; indeed, there is no discussion whatever as to why we should accept them as true. The common notions are clearly self-evident. The first three postulates seemingly propose the simplest possible constructions; the fourth postulate, the determinateness of right angles. It is only with respect to the fifth postulate that he apparently failed, for two reasons: First, if it is true, it does not seem to be epistemologically prior; that is, it seems in need of proof from other statements that would be essentially simpler. Second, although it is plausible, it might not even be true, since other lines are known (for example, a hyperbola and its asymptote) which converge but do not meet.

These objections were raised but not answered in antiquity, and one of the legacies of Greek mathematics was to solve the problem raised by Euclid's fifth postulate; that is, either to prove it true or to get around its difficulties by, for example, giving another definition of *parallel* (for example, two straight lines are *parallel* if they are everywhere equidistant).

Problem 10. See if you can find where the mistake occurs in the following ancient argument, which apparently shows not only that it is possible for the lines in Postulate 5 not to meet but that they can't meet!

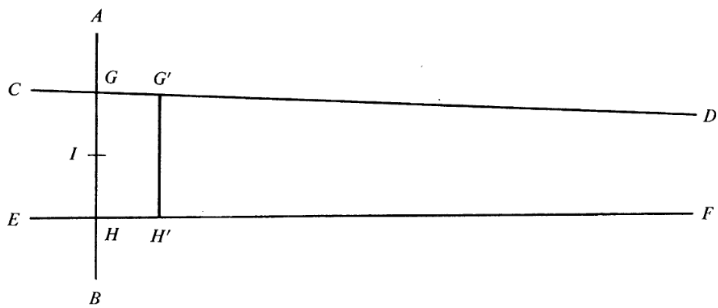


Figure 6

Let AB in Fig. 6 be the straight line which falls on two straight lines CD and EF such that the interior angles AHF and BGD are together less than two right angles. Bisect GH at I and make GG' and HH' each equal to GI. Form line $G'H'$. Lines CD and EF will not meet at a point both nearer to AB than G' on CD, and nearer to AB than H' on EF, for if they did meet at some nearer point L, then GL plus LH would be shorter than GH, which is impossible, since two sides of a triangle cannot be shorter than the third. Similarly, lines CD and EF cannot meet on $G'H'$, since two sides of a triangle cannot be equal to the third. Now repeat the same argument for $G'H'$ falling on CD and EF by forming (in an analogous way) G'' and H''. It is obvious that the argument can be repeated indefinitely, and thus lines CD and EF cannot meet.

Three other famous geometric problems were also proposed by the Greeks. First, the duplication of the cube, that is, constructing another cube with twice the volume of a given cube. Second, squaring the circle, that is, constructing a square with an area equal to that of a given circle. Third, trisecting an angle, that is, dividing an arbitrary given angle into three equal parts. In each case the problem was to make an exact construction with the aid of a straightedge and compass alone. Greek geometers failed to solve any of these problems under the condition that only a straightedge and compass be used; but by using more complex methods, they solved all of them. Some of the ancient Greeks seem to have been convinced that the problems were impossible to solve under the condition stated. However, no one, so far as we know, conceived of the idea that it might be open to proof that such a construction was impossible. So the Greeks left to posterity the task of deciding whether their failure to solve these problems under the condition stated was due to lack of ingenuity or to the intrinsic nature of the problems themselves. As we shall see, it wasn't until the nineteenth century that geometers made any real advance beyond the fundamentals set forth by the Greeks.

B. Logic

Ancient Greece developed another tradition in logic beside the Aristotelian. This was the Stoic logic, which evolved from the logic of the Megarians, which itself was developed from the philosophy of Parmenides and Zeno. Euclides (*c.* 430–*c.* 360 B.C.), who founded the Megarian school, had a pupil named Eubulides. This Eubulides formulated numerous paradoxes, only one of which is of interest today. However, this one is of great interest, and was destined to play an important part in the development of mathematical logic. It is called the *Liar paradox*, and is mentioned by St. Paul:

> One of themselves, a prophet of their own, said, "Cretans are always liars, wily beasts, lazy gluttons." This testimony is true. [*Titus* I : 12–13]

This prophet was probably Epimenides, who lived in the sixth century B.C. Sometimes the Liar is even called the Epimenides, but because he, like St. Paul, apparently did not see its logical interest, the credit for its formulation as a paradox should go to Eubulides. We do not have any ancient books on the subject of the Liar (though we know that many were written), and the references we do have are somewhat indirect and cryptic. Nevertheless it is probable that the ancient Greeks were familiar with such versions of the Liar paradox as the following:

1) Cretans always lie [uttered by a Cretan].
2) Whosoever says "I lie" lies and speaks the truth at the same time.
3) This proposition is not true.

The difficulty with the Liar paradox is that it leads to a contradiction, whether the proposition involved is considered true or false. Consider (3). Suppose that it is true. Then what (3) says is correct, and it says that it is not true. Hence, by *reductio ad impossibile*, (3) cannot be true. On the other hand, suppose that (3) is not true. Then it says of itself that it is not true, and this is of course true under the assumption. Hence, by *reductio ad impossibile*, (3) cannot be not true. In sum, if (3) is true, it is not true and if not true, it is true.

Problem 11. Show that (2) would require an additional (but plausible) hypothesis in order to generate a contradiction; that (1) would also require an additional but less plausible hypothesis in order to generate a contradiction.

In the formulation of this paradox the Megarians showed themselves to be the true heirs of Zeno. Of ancient resolutions of it we know little, but one of Chrysippus has come down to us. Chrysippus (280–207 B.C.) was the leading figure of the Stoic school of logic. If we possessed his works, it is probable that we would consider him, in logic, the equal of Aristotle. Perhaps we would even consider him Aristotle's superior, as some of the ancients did. In any case, he suggested that the Liar proposition—for example, (3)—has no meaning at all, and this anticipates one of the modern resolutions.

Problem 12. It must not be thought that the suggestion that (3) is meaningless easily resolves the antinomy. Determine at least an initial objection to this resolution by assuming the plausible principle that something meaningless is neither true nor false.

Stoic logic was surely very rich. For example, as we have seen, Aristotle was aware of the distinction between true and valid, but the first explicit and clear statement of it was made in Stoic writings. Nevertheless, since none of the modern innovators of mathematical logic knew anything of Stoic logic (Peirce is somewhat of an exception), it does not form part of the historical development of mathematical logic. Indeed it was not until 1927 that the Polish logician Łukasiewicz showed that in many ways Stoic logic was the unknown forerunner of contemporary logic. Thus much of what had been thought to be recent discoveries were really rediscoveries. We can get some idea of the extent of the Stoics' anticipation of modern logic by considering the fact that Stoic logicians defined inclusive and exclusive disjunction, discussed different kinds of implication and defined material and strict implication, devised an equivalent of our truth tables, discovered (in substance) the deduction theorem, developed a propositional calculus that may have been complete, made a distinction which is the virtual equivalent of the sense-denotation distinction of Frege, and indicated an awareness of the language-metalanguage distinction. We shall discuss most of these ideas in this book, when we consider mathematical logic.

Stoic logic, then, was a very great achievement. It is fundamentally simpler than Aristotelian logic, which cannot even be *systematically* presented without the equivalent of the Stoic logic of propositions. One is tempted to apply Santayana's scornful dictum, "Those who cannot remember the past are condemned to repeat it," to the innovators of mathematical logic. Before doing so, however, we should remember that although references to Stoic logic were scattered throughout many ancient writings, they are for the most part fragmentary; and that mathematical logicians not only rediscovered ancient ideas, but (because of a better mathematical heritage) went as far beyond the ancients in logic as contemporary physicists have outstripped their predecessors in physics.

§6 LOGIC FROM THE STOICS TO THE NINETEENTH CENTURY

Our ignorance of the past applies to medieval and Renaissance logic even more than to ancient logic. With respect to ancient logic, historians have at least checked all available material, and although our knowledge of it is inadequate, this is due to the destruction of our sources rather than to lack of interest or effort by historians. With respect to the history of post-ancient logic, however, many manuscripts are known to exist which even today have not been read, let alone translated and produced in critical editions. This is true even if we leave aside Arabian and Jewish logicians, about whom also relatively little is known, due to lack of competent investigators. In such a situation it is not surprising that post-ancient logic had practically no influence on the first formulators of mathematical logic, and it would perhaps be justifiable to omit here any reference to this logic at all. Most of the work that was done (of which we have knowledge) would belong to what is now known as the philosophy of logic. In purely formal logic, we know of little that we would today consider important. There were many reasons for this, the principal ones being the tradition-bound mentality of medieval writers, and the fact that many post-medieval thinkers scorned logic as a mere tedious game.

Nevertheless it might be worth while to briefly note a few of the developments that did take place in the medieval period. In a somewhat arbitrary manner we shall choose just one topic: the methods of deciding whether a syllogism is valid. This is mainly of historical interest, but it will serve as a useful contrast to what mathematical logicians later achieved when they tried similar things.

One of the ways was simply to introduce a mnemonic poem, which goes as follows:

Barbara celarent darii ferio baralipton
Celantes dabitis fapesmo frisesomorum;
Cesare camestres festino baroco; darapti
Felapton disamis datisi bocardo ferison.

This poem was apparently first published in the thirteenth century by William of Sherwood (*c.* 1205–*c.* 1268). To get some idea of the complications of this verse, consider part of its explanation: Each of the words represents a valid syllogism; the mood is represented by the first three vowels; the figure is indicated by the position of the punctuation mark, the first before the first semicolon, etc.; the first letter of each word (after the first four words) indicates which of the first four it should be reduced to; 's' indicates that one uses conversion (on the proposition that is indicated by the letter preceding 's') in the reduction, 'p' indicates that one similarly uses conversion by limitation, 'm' between the first two vowels of a word means that the order of the premisses is to be reversed, 'c' occurring after one of the first two vowels means that the respective premiss should be exchanged for the contradictory of the conclusion for a *reductio ad impossibile* type of reduction. Exactly why this poem seems to recognize 9 valid moods in the first figure, or why it recognizes only 19 valid syllogisms (when under the presuppositions made there should be 24) is unimportant from a logical point of view. However, the very complexity of this little poem gives some insight into the enormous labyrinth of medieval logic and indicates why, when the mood of the times became less tradition-bound, logic was treated with so much scorn by many thinkers.

Another less-artificial method was used to determine the validity of syllogisms. A series of rules was devised such that, if a syllogism satisfied all the rules, it was considered valid, and vice versa. To state the rules the technical term *distribution* is needed: A proposition *distributes* a term if it gives information (either positive or negative) about everything denoted by the term. Thus the **A** proposition (for example, 'all men are mortal') distributes the subject term but not the predicate term (since it says something about all men but not about all mortals); the **E** proposition (for example, 'no men are mortal') distributes both the subject and predicate terms (since it says that each and every man is not mortal and every mortal is not a man); the **I** proposition (for example, 'some men are mortal') distributes neither term (since it neither gives us information about all men or about all mortals), the **O** proposition (for example, 'some men are not mortal') distributes the predicate term (since it tells us that some men are excluded from the entire class of mortals). We can now state the rules for validity of a categorical syllogism. A categorical syllogism is valid when it satisfies the following conditions:

a) It does not contain two negative premisses.

b) It contains one negative premiss if and only if the conclusion is negative.

c) If the conclusion distributes a term it is also distributed in its premiss.

d) The middle term is distributed in at least one premiss.

Justification of these rules (which we shall here omit) was also given. What is interesting from the viewpoint of mathematical logic is that these rules represent an attempt to decide validity from a purely formal point of view. Moreover, it could be done in a few steps in a relatively short time.

Problem 13. Using Table 3 of §4C, verify that every invalid syllogism breaks at least one rule, and that the 24 valid ones break no rule. (Time: approximately 20–30 minutes.)

Problem 14. Suppose that condition (b) were changed to 'If it contains one negative premiss the conclusion is negative'. It is a curious fact that if the other conditions were kept the same, only one invalid syllogism would satisfy all the conditions. Find it. [*Hint*: Begin by ruling out an **E** conclusion by appealing to conditions (c) and (d).]

Nevertheless it might seem that these rules are somewhat arbitrary. This was the opinion of an Italian Jesuit priest named Gerolamo Saccheri (1667–1733), who published a book in 1697 called *Demonstrative Logic*. He wished to put logic on a basis as sound as geometry's:

When I speak of demonstrative logic I wish you to think of geometry—that rigorous method of demonstration which grudgingly admits first principles and allows nothing that is not clear, not evident, not indubitable [cf. Emch 1935 58].

Following Euclid's scheme, he founded his system of Aristotelian logic on a series of definitions, three common notions, and one postulate. However, after proving a large part of Aristotelian theory, he began to feel that the postulate was not really self-evident. He thus tried another method:

It is now my intention to follow another and, as I think, a very beautiful way of proving these same truths without the help of any assumption. I shall proceed as follows: I shall take the contradictory of the proposition to be proved and elicit the required result from this by a straight-forward demonstration [cf. Kneale 1962 346].

The essence of Saccheri's method is as follows: Suppose that a proposition A is true. Then, of course, A is true. Suppose that A is false. Now a demonstration is given which shows that the truth of A follows from that assumption. Since A must be either true or false, it follows in either case that it is true. Therefore it must be necessarily true. This argument was given the name *consequentia mirabilis*. An example which almost fits this form we have already seen from the young Aristotle:

Either we ought to philosophize or we ought not. If we ought, then we ought. If we ought not, then also we ought [i.e., in order to justify this view]. Hence in any case we ought to philosophize [cf. Kneale 1962 97].

This is not an exact example of the *consequentia mirabilis*, since the conclusion does not follow from its contradictory but only from the process of establishing the contradictory. The Stoics also used similar arguments to refute sceptics. Exact instances of the *consequentia mirabilis* can be found in a number of writers before Saccheri, including Euclid [cf. Problem 16 (§9)].

It appears, however, that Saccheri was the first to apply this form of argument to syllogistic theory. Here is an example:

Every syllogism with a universal major premiss and an affirmative minor premiss is an argument with a valid conclusion in the first figure.

No **AEE** syllogism is a syllogism with a universal major premiss and an affirmative minor premiss.

Therefore no **AEE** syllogism is an argument with a valid conclusion in the first figure.

This argument has true premisses and is itself an **AEE** syllogism in the first figure. Now if an **AEE** syllogism in the first figure is invalid, it is of course invalid. On the other hand, if an **AEE** syllogism in the first figure is valid, then the conclusion of the above argument is true (since the premisses are true) and thus an **AEE** syllogism in the first figure must be invalid. In sum, the assumption of the validity of such a syllogism implies its invalidity, and thus it is necessarily invalid.

Arguments of this form have a kind of self-reflective quality which at first makes one doubt that they are valid. However, not only are they valid but they provide an elegant method of proof in logical theory. And, as we shall see, another important use was made of this method which was significant for the development of mathematical logic [cf. §8].

Problem 15. What is *wrong* with the following argument?

All syllogisms which satisfy the four rules for the syllogism are valid.

All **AAA** syllogisms in the first figure are syllogisms which satisfy the four rules for the syllogism.

Therefore all **AAA** syllogisms in the first figure are valid.

§7 SUMMARY

Aristotle, with little help from his predecessors, formulated a rather extensive theory of logic, which consisted of the theory of syllogism, together with connected doctrines such as the theory of definition and modal logic. It was this theory which was to have an enormous influence on philosophy and theology for more than 2000 years. When a valid argument was found which could not be accounted for by that theory, it was either stated without developing any formal theory to account for it (as, for example, in the writings of Aristotle himself), or if some theory was provided it had little influence (as in the case of the works of the Stoics). Indeed, we even have the example of the physician Galen (*c.* 129–*c.* 199), who gave examples of arguments which could be accounted for by neither Aristotelian nor Stoic logic (for example: If 'Sophroniscus is father to Socrates', then 'Socrates is son to Sophroniscus'). Nevertheless the degree to which non-Aristotelian logic was known and was influential may perhaps be judged by the opinion of Immanuel Kant, often thought to be the greatest of modern philosophers. In 1787 he stated:

That *Logic*, from the earliest times, has followed this secure method [that is, of science], may be seen from the fact that since *Aristotle* it has not had to retrace a single step, unless we choose to consider as improvements the removal of some unnecessary subtleties, or the clearer definition of its matter, both of which refer to the elegance rather than to the solidity of the science. It is remarkable also, that to the present day, it has not been able to make one step in advance, so that, to all appearance, it may be considered as completed and perfect [1961 501].

In 1800, he added:

Aristotle has omitted no essential point of the understanding; we have only to become more accurate, methodical, and orderly [1885 11].

We can contrast this respectful and reverent attitude of Kant's with that of an important contemporary logician, Willard Van Orman Quine. The first sentence of one of Quine's books is: "Logic is an old subject, and since 1879 it has been a great one" [1949 vii].*

Now Kant's statement is incorrect and reveals an ignorance of the history of logic; and Quine's is surely an exaggeration. Logic was a "great" subject when Aristotle and Chrysippus were devising their respective theories. Nevertheless, as an indication of what many important thinkers believe both then and now, the statements are perhaps not misleading.

This radical change in attitude took place because of certain developments in the nineteenth and early twentieth centuries. This is the period of transition to which we now turn.

* 1879 was the date of publication of Frege's *Begriffsschrift*.

CHAPTER 2

PERIOD OF TRANSITION

§8 INTRODUCTION

The nineteenth and early twentieth centuries are called the *period of transition* because they were considered as such by the innovators of mathematical logic. Even today most logicians would probably consider this designation correct. Yet the transition was probably not as smooth and neat as this label might suggest. We know that a number of supposedly original discoveries of this period were only rediscoveries. For example, logicians of this period, without having any knowledge of Stoic logic, rediscovered much of the content of Stoic logic. If the gaps in our knowledge of logic from the Stoics to the nineteenth century were to be filled, no doubt we would find still more foreshadowings of modern logic. Thus the transition from that fusion of Aristotelian and some Stoic logic—often called *traditional logic*—to mathematical logic was rather complicated, and the various parts of the development did not take place simultaneously or at the same rate. Nevertheless, for our purposes, it is perhaps easier to organize this account around developments that took place in this period of transition, noting the known anticipations of these developments as they are described. And rather than sticking to a strict chronological account of them, we shall divide them into topics in order to discuss them more systematically.

If we wished to summarize the situation most simply, we could say with Quine that logic again became a "great" subject in the nineteenth century because of certain developments in mathematics. The progress of mathematics confronted mathematicians with new and profound logical problems, for which the traditional logic was of no help. These problems centered mainly around the nature of axiomatic systems and the elimination of contradiction from a given system. The attempts to solve these problems led to the application of mathematical methods to the problems. Thus it appears that logic became rejuvenated because of one of the same considerations that led to its initial development: the need to solve certain logical problems in mathematics. A new and more sophisticated logic was needed to equal the more sophisticated mathematics of the nineteenth century. Yet it must not be thought that the traditional logic was adequate for all the logical problems that arose before then. We have already cited an example of Galen's which fits no ancient framework. Here is a mathematical example of Galen's:

Theon has twice as much as Dio, and Philo twice as much as Theon; therefore Philo has four times as much as Dio [cf. Kneale 1962 185].

Nevertheless such arguments could be (and were) safely ignored, since the progress of mathematics did not depend on having a theory that would account for such an argument.

The case was otherwise in the nineteenth century. For then there arose in mathematics certain puzzling and surprising questions to which neither intuition nor the then current logical theory had any answer. Everyone can "see" that Galen's argument follows even if there is no logical theory to account for it. But the rise of a geometry different from Euclid's was a real surprise, and neither mathematicians nor philosophers were quite sure what to make of it. The same holds true both for the discovery of different sizes of the infinite and the logical inconsistency of some axiomatic systems with seemingly innocuous axioms. The latter occurred in a mathematical discipline called *set theory*. To understand what led to mathematical logic, we must first consider in turn each of these disciplines: non-Euclidean geometry and set theory.

§9 NON-EUCLIDEAN GEOMETRY

The reasons why a man continues to hold to a belief sometimes has little to do with what made him adopt the belief in the first place. Very often these reasons have to do with pride or self-esteem. This is especially true with regard to a belief with which a person identifies himself, as in the case of philosophical or religious beliefs. In such a case the person may develop a prejudice, that is, a refusal to consider impartially evidence on both sides of the question at hand. If the belief is widely held, the society may develop customs to make it more unlikely that the belief be questioned. We may then speak of a social prejudice.

In this sense there was in Europe at the end of the eighteenth century a social prejudice consisting of the belief that Euclidean geometry was true. There was good evidence for the truth of this system of mathematics. Was not Euclidean geometry more than 2000 years old? Yet the intellectual climate was such that a questioning of this tradition met not so much with refutation as with derision.

The security of those who laughed at the innovators of non-Euclidean geometry was greatly enhanced by the philosophy of Kant (1724–1804), which then prevailed. Further, it was the views that Kant presented in his *Critique of Pure Reason* (1781) that were influential, rather than some of his earlier views which, with the advantage of hindsight, we can see were more compatible with the development of geometrical knowledge than those that were laid down in the *Critique*. According to the *Critique*, all judgments can be divided into two kinds: The first kind is *analytic*, that is, the predicate belongs to the subject because it is overtly or covertly contained in it. The second kind is *synthetic*, that is, the predicate is connected with the subject but not contained in it. For example, the judgment that all bodies are extended is analytic since, according to Kant, only logical analysis of the concept body is needed in order to see its truth. However, the judgment that all bodies are heavy cannot be found to be true through analysis of the concept body, and thus is synthetic. Synthetic judgments extend our knowledge, whereas analytic judgments do not; they only clarify what we already know.

Kant tells us that all judgments based on experience (*a posteriori* judgments, as he calls them) are synthetic. However, just because they are based on experience they are not necessarily true, for at any time we might have a different sort of experience which would prove them to be false. Necessary judgments are all *a priori*, that is, independent of experience. In fact, since it is also true that all *a priori* judgments are necessary, we may use the criterion of necessity to decide whether or not a given judgment is *a priori*.

We have then seemingly four possible kinds of judgments: analytic *a priori*, analytic *a posteriori*, synthetic *a priori*, synthetic *a posteriori*. The second category, however, does not exist, since if a judgment is analytic no experience is needed to see that it is true. The problem as it appeared to Kant can be put thus: Analytic *a priori* judgments are necessary but they do not add to our knowledge. Synthetic *a posteriori* judgments add to our knowledge but are not necessary. How is it possible to have a judgment which adds to our knowledge and is necessary, that is, a synthetic *a priori* judgment? Kant never doubted that we had synthetic *a priori* knowledge; in particular, he believed most mathematical judgments to be of this sort. We may summarize the point reached so far with a diagram (Table 4) in which more examples are given:

Table 4

Judgments	Analytic	Synthetic
A priori	All bodies are extended. Every effect has a cause. A is A.	Seven plus five equals twelve. A straight line is the shortest distance between two points.
A posteriori		All bodies are heavy. Aristotle was a student of Plato.

Kant's view of geometrical knowledge was that he believed all judgments of geometry to be synthetic *a priori*, with the exception of "some few propositions." These latter are analytic. Although he did not explicitly say so, the apparent distinction he had in mind was the distinction between the common notions and postulates in Euclidean geometry. That is, the common notions are analytic, whereas the postulates—and those propositions which can be deduced from them—are all synthetic *a priori.* He said that "the whole is greater than its part" is analytic, whereas "that the straight line between two points is the shortest, is a synthetical proposition." His reasoning here was as follows:

> ... my concept of *straight* contains nothing of magnitude (quantity), but a quality only. The concept of the *shortest* is, therefore, purely adventitious, and cannot be deduced from the concept of the straight line by any analysis whatsoever. The aid of intuition, therefore, must be called in, by which alone the synthesis is possible [1961 529–30].

What this intuition is is difficult to characterize. The word is used in contrast to conception, which is discursive, whereas intuition is immediate. As Kant puts it:

> ... I construct a triangle by representing the object corresponding to that concept either by mere imagination, in the pure intuition, or, afterwards on paper also in the empirical intuition, and in both cases entirely *a priori* without having borrowed the original from any experience. The particular figure drawn on the paper is empirical, but serves nevertheless to express the concept without any detriment to its generality, because, in that empirical intuition, we consider always the act of the construction of the concept only, to which many determinations, as, for instance, the magnitude of the sides and the angles, are quite indifferent, these differences, which do not change the concept of a triangle, being entirely ignored [1961 421–2].

Kant argues that the conditions which determine this intuition are the conditions of having any experience whatsoever. An analogy might be useful in explaining this: One of the necessary conditions of sight in general is the existence of an eye. There is no sight without eyesight. So Kant is saying that one of the conditions of experience is that we experience things in space, in particular, in Euclidean space. Thus when we say 'a straight line is the shortest distance between two points', it is *a priori* because we know this to be one of the conditions of experience, and therefore it is not part of experience. It is the structure of our mind which makes space Euclidean. The world around us we must understand in terms of that structure and thus it too is Euclidean.

This then is Kant's answer to the problem of how synthetic *a priori* judgments are possible. To the question 'of what are Euclidean propositions true?' Kant now had an answer. They are true of our *a priori* intuitions.

It is easy to see how such a view might hinder the development of a geometry different from Euclid's. For if such a view were influential it would be unlikely that anyone would look for a non-Euclidean geometry. If someone did, it would take intellectual daring to assert the existence of a new geometry (remember that Euclidean geometry was thought necessarily true of the world), and finally it might take courage to publicly proclaim its existence. One can easily exaggerate the importance of Kantian philosophy in hindering the discovery of non-Euclidean geometry. The ancient authority of Euclidean geometry, as well as its seemingly unimpeachable presentation, were surely stronger factors. Nevertheless Kant's influence was not insignificant in this regard.

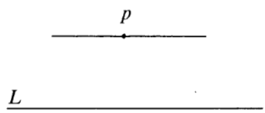
But what is non-Euclidean geometry? The story of its development goes back to the dissatisfaction with Euclid's fifth postulate. This postulate states 'that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles'. As we have seen, the doubts about this postulate were really two: First, if it is true, we should surely be able (so it was thought) either to prove it from the other common notions and postulates or to find an essentially simpler postulate from which it can be proved. Second, how can we be sure it is true, since other lines are known (for example, a hyperbola and its asymptote) which converge but do not meet?

So far as we know, no one who came between the time of Euclid and the nineteenth century doubted the truth of the postulate.* Thus efforts focused on proving the postulate or getting an essentially simpler substitute. However, all efforts at proving the postulate failed. We know today that they had to fail, since it cannot in fact be proved from the other assumptions. As for getting around the postulate by using simpler substitutes, two methods were tried: The first was to make a change in definition, the second to propose a simpler postulate. An example of the first method is the following: It was proposed that the definition of parallel straight lines be changed from 'straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet in either direction' to 'straight lines which are in the same plane and are everywhere equidistant'. The difficulty with this approach lay in proving that such lines exist and in showing that lines which are not parallel do meet somewhere. It was found that to prove these things one had to make an assumption logically equivalent to the fifth postulate. *Logically equivalent* here means that if we assumed the fifth postulate we could derive the new assumption with the help of the other common notions and postulates, whereas if we replaced the fifth postulate with the new assumption we could, using the latter (and, of course, the other common notions and postulates), deduce the fifth postulate. Thus from the Aristotelian point of view no proof would be given. All that would be shown would be a logical connection between the two statements; to claim a proof would be to commit the fallacy of begging the question.

---

* For some evidence that the postulate may have been doubted before Euclid, see Tóth 1969.

Of the many substitutes that have been made for the fifth postulate, the most famous and widely used is *Playfair's postulate*, so named because John Playfair (1748–1819), a Scottish mathematician, called attention to it, although it was known in ancient times. It states that 'through a given point, only one parallel can be drawn to a given straight line'. This may be illustrated by the following diagram:



That is, if we are given a line L and a point p not on L, there is only one line through that point parallel to L. (That at least one parallel line can be drawn can be proved from the other common notions and postulates; at least, it can be proved if the second postulate is understood as asserting the infinitude of straight lines.) Playfair's postulate is logically equivalent to Euclid's.*

Thus, again, if we are to avoid begging the question, we cannot say that we have advanced epistemologically from Euclid's geometry. If we had doubts about the fifth postulate we should have equivalent doubts about Playfair's postulate.

All attempts at either proving or finding a substitute for the fifth postulate failed until, in 1733, Saccheri finally tried a new method, in a book called *Euclid Freed of Every Flaw.* After studying what numerous commentators on Euclid had said before him (including even Arabic writers such as Omar Khayyam and Nasiraddin) and remembering the success of the *consequentia mirabilis* in his *Demonstrative Logic*, Saccheri tried to solve the problem by this indirect form of argument. Euclid himself had used this kind of argument, which, as we have seen, consisted essentially in proving a proposition by assuming the falsehood of that proposition. Since Saccheri believed that every "primal verity" could be thus established, it was natural for him to try it on the fifth postulate.

Problem 16. Saccheri said that Euclid used this form of argument to prove Proposition 12 in Book IX of the *Elements.* In modern form that proposition states the following: If in a finite geometrical progression†

* Euclid's fifth postulate, although it does not even contain the word 'parallel', is often called the *parallel postulate*; presumably because, on one hand, the whole Euclidean theory of parallels rests on it, and on the other, it has this equivalent which does involve the word *parallel.*

† A geometrical progression $(g_1, g_2, g_3, \ldots, g_m)$ is a sequence of numbers in continued proportion, that is, $g_i : g_{i+1} :: g_{i+1} : g_{i+2}$. Since the above sequence begins with 1, we have $1 : a_1 :: a_1 : a_2$, that is, $a_2 = a_1 \cdot a_1$. In general, $a_i = a_1 \cdot a_{i-1}$.

beginning with 1 $(1, a_1, a_2, \ldots, a_n)$, a prime number ($b$) evenly divides a_n, it (b) also evenly divides a_1. Euclid assumed that b evenly divided a_n but did not evenly divide a_1. He went on to show that nevertheless b did divide a_1. In his proof he also assumed (something proved earlier in the text) that if $c : d :: e : f$ and c and d had no common divisor, then c evenly divided e, and d evenly divided f. From these assumptions, see if you can prove Proposition 12. [*Hint:* Start with the assumption that b evenly divides a_n and show that this in conjunction with the other assumptions implies that b evenly divides a_{n-1}.]

In order to bring out the logical structure of the problem as he saw it, Saccheri considered a quadrilateral constructed as follows (see Fig. 7): On a line segment AB construct two equal perpendicular lines at the extremities of AB, that is, construct AC and BD. Join points C and D with a straight line.

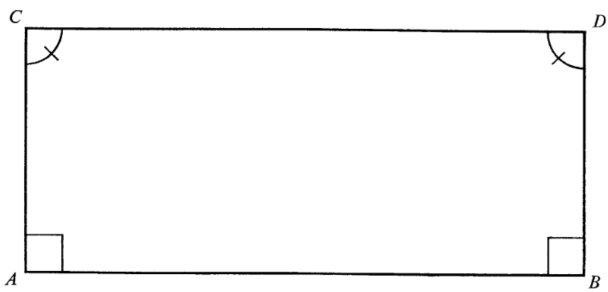


Figure 7

Without assuming Euclid's fifth postulate, Saccheri was able to prove that angles ACD and BDC were equal. Then Saccheri considered three hypotheses: First, that the angles were obtuse; second, that they were right; third, that they were acute. It can be shown that the hypothesis of the right angle is logically equivalent to the fifth postulate. Suppose that we deny this hypothesis and accept the hypothesis of the obtuse angle. Saccheri showed that the fifth postulate follows from this assumption. To do this he had to understand Euclid's second postulate—'to produce a finite straight line continuously in a straight line'—as asserting the infinitude of a line. Euclid also understood the postulate in this way, so that he was justified on a historical basis. Note, however, that the postulate does not explicitly state the infinitude of a line; it only suggests it in a more or less ambiguous way. In any case Saccheri was able to prove the fifth postulate from the hypothesis of the obtuse angle, which is contrary to the fifth postulate. That is, as Saccheri put it in his Proposition XIV:

The hypothesis of the obtuse angle is absolutely false, because it destroys itself [1733 59].

He then had only to dispose of the hypothesis of the acute angle in order to prove the fifth postulate. In this he failed, as today we know he had to, since there was no contradiction to be found. When he came to Proposition XXXIII he said:

The hypothesis of the acute angle is absolutely false; because repugnant to the nature of the straight line [1733 173].

What is "repugnant to the nature of the straight line" is the existence of two straight lines

AX, *BX*, existing in the same plane, which produced *in infinitum* toward the parts of the points *X* must run together at length into one and the same straight line, truly receiving, at one and the same infinitely distant point a common perpendicular in the same plane with them [1733 173].

He then went on to state and prove Proposition XXXVIII: "The hypothesis of the acute angle is absolutely false, because it destroys itself" [1733 225]. The proof, however, is erroneous. Saccheri himself apparently felt some uneasiness, for he states that there is

...a notable difference between the foregoing refutations of the two hypotheses. For in regard to the hypothesis of the obtuse angle the thing is clearer than midday light; since from it assumed as true is demonstrated ... the absolute falsity of this hypothesis.... But on the contrary I do not attain to proving the falsity of the other hypothesis, that of the acute angle, without previously proving that the line, all of whose points are equidistant from an assumed straight line lying in the same plane with it, is equal to this straight [line], which itself finally I do not appear to demonstrate from the viscera of the very hypothesis, as must be done for a perfect refutation [1733 233, 235].

To sum up, Saccheri thought he could prove the fifth postulate by means of the *consequentia mirabilis*. Some evidence that it is capable of being proved is given by the fact that Euclid himself proves the converse of the fifth postulate (Euclid's Proposition 17). Had Saccheri found the contradiction he was looking for, he would have proved the fifth postulate. But this would not have shown Euclid to be "freed of every flaw." On the contrary, it would have shown Euclid to have made an unnecessary assumption. When an initial assumption (or its negation) is derivable from other initial assumptions, it is called *dependent*; otherwise it is called *independent*. Saccheri would thus have showed the fifth postulate to be dependent on the other assumptions.

Saccheri (in his *Demonstrative Logic*) was perhaps the first person in history to study the idea of independence and to provide a criterion for it: Show that it is possible for all the other assumptions to be true and the one in question to be false. If it follows that the assumption in question must necessarily be false, the assumptions taken as a whole are inconsistent; that is, it is possible to derive a contradiction from them. The test for consistency of a set of assumptions then is: Is it possible for all the assumptions to be true? If so, the set is consistent; if not, not. Saccheri tried to prove that a contradiction followed from the hypothesis of either the acute or the obtuse angle. Thus, Saccheri tells us,

> . . . not without cause was that famous axiom assumed by Euclid as known *per se*. For chiefly this seems to be as it were the character of every primal verity, that precisely by a certain recondite argumentation based upon its very contradictory, assumed as true, it can be at length brought back to its own self [1733 237].

Ironically, it is only because Saccheri failed in the latter attempt that Euclid might be freed from the charge of an unnecessary or false assumption.

Problem 17. Consider two different axiom systems: the first Euclid's, in which the fifth postulate is replaced by its converse; the second Euclid's, in which the fifth postulate is replaced by the hypothesis of the obtuse angle. Determine, on the basis of information given in this section, whether the new postulate is independent and whether each system is or is not consistent.

In the course of trying to find the two contradictions he needed, Saccheri stated and proved a number of theorems which later became part of the body of non-Euclidean geometry. By taking the pose of being initially neutral about the fifth postulate, he was able to prove statements which no one else would be able to prove for nearly a century. Here are three examples:

1. According as the hypothesis of the obtuse angle, or right angle, or acute angle is proved true in a single case, it is respectively true in every other case.

2. According as the hypothesis of the obtuse angle, or right angle, or acute angle is true, the sum of the angles of a triangle is respectively greater than, equal to, less than, two right angles.

3. According as the hypothesis of the obtuse angle, or right angle, or acute angle is true, two straight lines always intersect; two straight lines intersect except in the single case in which they share a common perpendicular; two straight lines may or may not intersect, but (except in the case in which they share a common perpendicular) they either intersect or they mutually approach each other.

If we consider the last result and a few other things Saccheri either proved or easily knew, we can show that the hypotheses of the obtuse, right, and acute angles are respectively equivalent to the following statements:

Given a line L and a point p off that line, there are no lines through the point parallel to the given line; there is exactly one line through the point parallel to the given line; there are two lines through the point parallel to the given line (see Fig. 8 at the top of page 46).

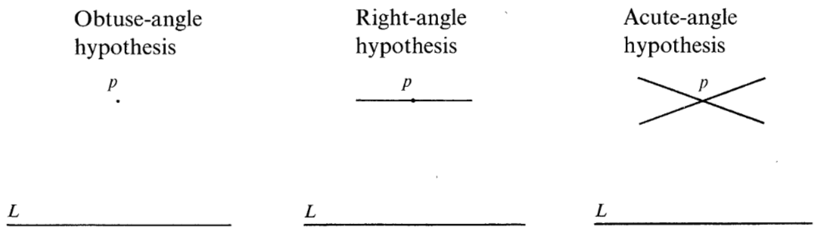


Figure 8

Saccheri showed that if there were two lines parallel, as in the case of the hypothesis of the acute angle, there were then an infinite number of such lines, namely, all those "between" the two given lines. He further showed there were always two determinate straight lines which separated those which are parallel from those which are not.

Nevertheless, because Saccheri didn't understand the nature of his discoveries, he—unlike Columbus— is not given credit for the discovery of a new-found land. The credit usually goes to three men: Gauss (1777–1855), who was German, Bolyai (1802–1860), who was Hungarian, and Lobachevski (1793–1856), who was Russian. Although this credit is not undeserved, the story of the development of non-Euclidean geometry from Saccheri to these men is complex. Saccheri's work was known in those days, but his influence is uncertain. The full story would involve more than a dozen names. To give just one example: Karl Schweikart (1780–1859), a lawyer, sent Gauss in 1818 a memorandum in which he asserted (among other things) that there were two kinds of geometry, Euclidean and another in which the sum of the angles of a triangle is less than two right angles. Schweikart did not publish any of his results. If he had, he might be given credit for the discovery of non-Euclidean geometry.

Gauss also did not publish his writings on the subject. However, his genius and versatility in mathematics were such that perhaps only Archimedes and Newton were in his class. This fact no doubt has led historians to credit mathematician Gauss with the discovery of non-Euclidean geometry, often without mentioning lawyer Schweikart. Gauss delayed publishing his results, probably due to the influence of both conservative mathematicians and Kantians. In a letter of 1824 Gauss states that

> ... the assumption that the sum of the three angles [of a plane triangle] is less than [two right angles] leads to a curious geometry, quite different from ours (the Euclidean), but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of the determination of a constant, which cannot be designated *a priori*. The greater one takes this constant, the nearer one comes

to Euclidean Geometry, and when it is chosen infinitely large the two coincide. [This fact had already been contained in Schweikart's memorandum which Gauss read in 1818.] The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. ... All my efforts to discover a contradiction, an inconsistency, in this non-Euclidean Geometry have been without success, and the one thing in it which is opposed to our conceptions is that, if it were true, there must exist in space a linear magnitude, *determined for itself* (but unknown to us). But it seems to me that we know, despite the say-nothing word-wisdom of the metaphysicians, too little, or too nearly nothing at all, about the true nature of space, to consider as *absolutely impossible* that which appears to us unnatural. If this non-Euclidean Geometry were true, and it were possible to compare that constant with such magnitudes as we encounter in our measurements on the earth and in the heavens, it could then be determined *a posteriori*. Consequently in jest I have sometimes expressed the wish that the Euclidean Geometry were not true, since then we would have *a priori* an absolute standard of measure [cf. Wolfe 1945 46–47].

Other evidence exists which shows that Gauss was delayed in announcing his results because of the anticipated reaction of the Kantians. But in February 1832 he received a letter from a friend of his, Wolfgang Bolyai (1775–1856). This contained an appendix which his son, Johann Bolyai (1802–1860), had written to a book of Wolfgang's. This appendix contained a description and many theorems on non-Euclidean geometry. Johann had been working on this approach on and off since 1823. He was a very excitable young man and correctly thought he had achieved something very important; "*out of nothing I have created a strange new universe*," he wrote his father in 1823. Because of Gauss' fame, his answer was eagerly awaited. It stated:

If I begin with the statement that I dare not praise such a work, you will of course be startled for a moment: but I cannot do otherwise; to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. On this account I find myself surprised to the extreme.

My intention was, in regard to my own work, of which very little up to the present has been published, not to allow it to become known during my lifetime. Most people have not the insight to understand our conclusions and I have encountered only a few who received with any particular interest what I have communicated to them. In order to understand these things, one must first have a keen perception of what is needed, and upon this point the majority are quite confused. On the other hand it was my plan to put all down on paper eventually, so that it would not finally perish with me.

So I am greatly surprised to be spared this effort, and am overjoyed that it happens to be the son of my old friend who outstrips me in such a remarkable way [cf. Wolfe 1945 52].

Johann was far from pleased with this reply, for it showed that either he had been preceded by Gauss or Gauss was lying. Sixteen years later he received another blow when he learned that Nikolai Lobachevski had published essentially the same results a few years before he wrote his own appendix. Johann wrote:

If Gauss was, as he says, surprised to the extreme, first by the *Appendix* and later by the striking agreement of the Hungarian and Russian mathematicians: truly, none the less so am I [cf. Wolfe 1945 56].

Again, it seems quite likely that Lobachevski discovered non-Euclidean geometry quite independently of Gauss and Bolyai. Since Lobachevski was the first to publish his work, this form of geometry is called *Lobachevskian.*

Lobachevskian geometry results when the fifth postulate of Euclid is replaced by the hypothesis of the acute angle or by the statement 'given a line L and a point p off that line, there are two lines through the point parallel to the given line'. We can examine the situation by dropping a perpendicular from p to L (see Fig. 9).

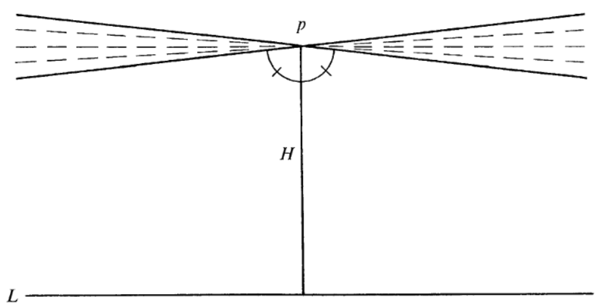


Figure 9

The lines which pass through p are divided into two classes: those which intersect L and those which do not. For convenience only those two lines which divide these two classes are called *parallel* (represented in the figure by solid lines); the others (represented by dashed lines) are called *non-intersecting*. That such parallel dividing lines exist was proved by Lobachevski (as it had been previously by Saccheri). Now the angle which one of these parallels makes with the perpendicular (Lobachevski shows them to be equal) is called the *angle of parallelism.* It would be interesting to know what this angle is. Lobachevski showed that it is acute and that it depends on the height H of the perpendicular. As the height of the perpendicular approaches infinity, the angle approaches 0; as the height of the perpendicular approaches 0, the angle approaches a right angle. Lobachevski shows that to every length H there corresponds exactly one angle of parallelism, and vice versa. Thus in Lobachevskian geometry there is an absolute measure of length. By *absolute* here is meant a standard implicit in the axioms themselves. For example, in Euclidean geometry angles have an absolute measure, whereas lengths do not. This follows from Euclid's definition of *right angle*, "when a straight line set up on another straight line makes the adjacent angles equal to one another, each of the equal angles is right," and his fourth postulate, "that all right angles are equal to one another." Thus in Euclidean geometry it does not matter how large or small a triangle is, the sum of the angles of a triangle is equal to two right angles. It is otherwise in Lobachevskian geometry, for there, although the sum of the angles of a triangle is less than two right angles, the sum approaches two right angles as the area of the triangle gets smaller, and approaches 0 as the triangle gets larger.

We can now understand what Gauss meant by saying that if this non-Euclidean geometry were true,

> there must exist in space a linear magnitude, *determined for itself* (but unknown to us). ... I have sometimes expressed the wish that the Euclidean Geometry were not true, since then we would have *a priori* an absolute standard of measure.

This standard would be the length H of the perpendicular which corresponds to the angle of parallelism for this world.

Lobachevskian geometry has as a limiting case Euclidean geometry. That is, the difference between Lobachevskian geometry and Euclidean becomes arbitrarily small depending on the choice of certain constants which occur in it. There is no *a priori* way of determining the constants. Thus the interesting question arises: Which of the two geometries is true? But the important question to notice here is: true of what? Since Euclidean and Lobachevskian geometry cannot both simultaneously be true, it follows that at most one of them can be true. Yet once Lobachevskian geometry is known it is hard to argue, as Kant did before it was known, that Euclidean propositions are true of our *a priori* intuitions. For what reasons can be *a priori*, given that Lobachevskian geometry is false? There don't seem to be any.

Thus it is very tempting to say that what geometrical statements are true of is the physical world. It becomes an empirical problem to find out whether Euclidean or Lobachevskian geometry is true. But one must be careful here. In a certain sense we are sure that neither Euclidean nor Lobachevskian geometry is true of the physical world. For consider the first two definitions of Euclid: "a *point* is that which has no part," and "a *line* is breadthless length." There seems to be nothing in our experience or the physical world which corresponds to these definitions. Of course, there are a lot of things which for one purpose or another might serve as a point or a line. But none will serve exactly, and in mathematics one is interested in absolute exactitude. This fact does not disturb the logical relationship between the axioms and the theorems, but it still leaves up in the air the question of truth.

To get at this question, let us say that a geometry is true of the physical world if it is presupposed by theoretical physics, that is, if the geometry serves the purpose of the theoretical physicist. The first hint of an attempt to consider geometry an empirical (that is, *a posteriori*) science appears in the work of Saccheri, who speaks of physico-geometric demonstrations; but Gauss was apparently the first person to attempt experiments. Gauss had been commissioned to make a survey of the earth around Hanover, and in the course of this work measured the triangle formed by three distant mountain peaks. He found that the difference between his measurement and two right angles was well within experimental error. Thus it seemed that Euclidean geometry was true of the world, but not true *a priori.* As he put it in one of his letters:

> I keep coming closer to the conviction that the necessary truth of our geometry cannot be proved, at least *by* the human intellect *for* the human intellect. Perhaps in another life we shall arrive at other insights into the nature of space which at present we cannot reach. Until then we must place geometry on an equal basis, not with arithmetic, which has a purely *a priori* foundation, but with mechanics [cf. Wolfe 1945 57].

Quite independently Lobachevski came to a similar view: In 1825 he said:

> The fruitlessness of the attempts made, since Euclid's time, for the space of 2000 years, aroused in me the suspicion that the truth, which it was desired to prove, was not contained in the data themselves; that to establish it the aid of experiment would be needed, for example, of astronomical observations, as in the case of other laws of nature [Bonola 1955 92].

This viewpoint led Lobachevski to work the calculations for a triangle one of whose sides was the radius of the earth's orbit and the opposite vertex a star. However, again the calculations showed that even for so large a triangle the deviation from the Euclidean would be within observational error.

To sum up the position reached by Gauss, Lobachevski and a few others by 1830: There are two mathematical theories of space, each consistent with itself but incompatible with the other. From a purely mathematical point of view, neither is superior to the other. On the other hand, there is only one theory of physical space; that is, that mathematical theory which must be presupposed by physics. Which of these two geometries is true of the world cannot be determined *a priori.* However, measurements seem to indicate that for all purposes—including theoretical physics—Euclidean geometry is sufficiently accurate. This did not preclude the possibility that further work in physics might, for some purposes, lead us to use Lobachevskian geometry, although this seemed at the time unlikely.

There the matter stood until Gauss was blessed with another pupil of superlative genius: Bernhard Riemann (1826–1866). Riemann had originally intended to become a theological student, but changed his mind and decided to go into mathematics and physics. Gauss (who was not easily impressed by the mathematical efforts of others) was very enthusiastic about Riemann's doctoral thesis, which he submitted when he was 25. Shortly thereafter Riemann sought a lectureship at the University of Göttingen, where it was the custom to have applicants present a trial lecture which the faculty attended. Riemann submitted three topics, for the first two of which he was well prepared. The third was on the foundations of geometry, and he did not expect it to be picked. But Gauss, who was on the faculty and at the very height of his fame and influence, saw to it that this was the topic selected. So on June 10, 1854, when he was 27, Riemann gave his lecture, "On the Hypotheses which Lie at the Foundations of Geometry." The lecture was very general and suggestive (the audience was not composed exclusively of mathematicians), but was incredibly profound. Gauss—who was to die within a year—was in the audience, and was quite excited by the lecture. Riemann's achievement of exciting this "prince of mathematicians" by lecturing on a topic on which he had spent 60 years was no mean feat.

What Riemann did was to propose a very general mathematical point of view from which physical space is merely a special case. As he put it:

> ... I have proposed to myself ... the problem of constructing the concept of a multiply extended magnitude out of general notions of quantity. From this it will result that a multiply extended magnitude is susceptible of various metric relations and that space accordingly constitutes only a particular case of a triply extended magnitude. A necessary sequel of this is that the propositions of geometry are not derivable from general concepts of quantity, but that those properties by which space is distinguished from other conceivable triply extended magnitudes can be gathered only from experience. There arises from this the problem of searching out the simplest facts by which the metric relations of space can be determined, a problem which in the nature of things is not quite definite; for several systems of simple facts can be stated which would suffice for determining the metric relations of space; the most important for present purposes is that laid down for foundations by Euclid. These facts are, like all facts, not necessary but of a merely empirical certainty; they are hypotheses; one may therefore inquire into their probability, which is truly very great within the bounds of observation, and thereafter decide concerning the admissibility of protracting them outside the limits of observation, not only toward the immeasurably large, but also toward the immeasurably small [Smith 1929 411–412].

On this last topic Riemann commented later in the lecture that

> ... the empirical notions on which spatial measurements are based appear to lose their validity when applied to the indefinitely small, namely the concept of a fixed body and that of a light ray; accordingly it is entirely conceivable that in the indefinitely small the spatial relations of size are not in accord with the postulates of geometry, and one

would indeed be forced to this assumption as soon as it would permit a simpler explanation of the phenomena [Smith 1929 424].

Problem 18. Can you see any connection between this latter quotation and the Achilles argument of Zeno?

It was from the general point of view suggested by the above quotations that Riemann was able to see the difference between the unbounded and the infinite. As Riemann expressed it:

> When constructions in space are extended into the immeasurably great, unlimitedness must be distinguished from infiniteness; the one belongs to relations of extension, the other to those of measure. That space is an unlimited, triply extended manifold is an assumption applied in every conception of the external world; by it at every moment the domain of real perceptions is supplemented and the possible locations of an object that is sought for are constructed, and in these applications the assumption is continually being verified. The unlimitedness of space has therefore a greater certainty, empirically, than any experience of the external. From this, however, follows in no wise its infiniteness, . . . [Smith 1929 423].

The geometry suggested by this lecture is one in which straight lines are unbounded, but not infinite. Precisely such a geometry results if we replace Euclid's fifth postulate by the hypothesis of the obtuse right angle and if we understand Euclid's second postulate—"To produce a finite straight line continuously in a straight line"—as asserting only the unboundedness of a straight line and not its infinity. However, in this geometry—now called *Riemannian geometry*—it is not universally true that given two points exactly one straight line may connect them. Thus, if a consistent geometry is to result, we must understand Euclid's first postulate—"To draw a straight line from any point to any point"—as meaning 'at least one straight line', not 'exactly one straight line'.

If these changes are made, Euclid's geometry is transformed into a consistent Riemannian geometry. The similarities and differences of the three plane geometries which so far have been distinguished may be summed up in a table (see Table 5). When we look at the table, one of the most interesting questions which arises is: How long are all the finite straight lines? In Riemannian geometry, as in Lobachevskian, the measurement of lengths as well as angles is absolute. It would be interesting to know the length of straight lines, as we would then have an absolute measure of size, that is, in terms of the length of a straight line. This length, however, cannot be stated *a priori*; it depends on the assignment of a constant. Well then, how about empirical observation? Is Riemannian geometry true of the world? Unfortunately very few people in the nineteenth century took this question seriously. Riemann did, William K. Clifford (1845–1879) in England did, and perhaps one or two others. But Riemann died at 39 and Clifford at 34. Had they

Table 5 Geometries

Euclidean	Lobachevskian	Riemannian
	Comparison of some postulates	
1. (Postulate 1) To draw a [that is, exactly one] straight line from any point to any point.	1. To draw a [that is, exactly one] straight line from any point to any point.	1. To draw at least one straight line from any point to any point.
2. (Postulate 2) To produce a finite straight line continuously in an [infinite] straight line.	2. To produce a finite straight line continuously in an [infinite] straight line.	2. To produce a finite straight line continuously in an unbounded straight line.
3. (Logical equivalent of Postulate 5) Given a line and a point off that line, there is exactly one line through that point parallel to the given line.	3. Given a line and a point off that line, there are at least two lines through that point parallel to the given line.	3. Given a line and a point off that line, there are no lines through that point parallel to the given line.
	Comparison of some theorems	
4. The sum of the angles of a triangle is equal to two right angles.	4. The sum of the angles of a triangle is less than two right angles, the sum approaches two right angles as the area of the triangle approaches 0, and the sum approaches 0 as the area of the triangle approaches the maximum area for a triangle.	4. The sum of the angles of a triangle is more than two right angles, the sum approaches two right angles as the area approaches 0, and the sum approaches six right angles as the area of the triangle approaches the maximum area for a triangle.
5. The ratio of the circumference of a circle to its diameter is π.	5. The ratio of the circumference of a circle to its diameter is greater than π; the ratio approaches π as the area of the circle approaches 0.	5. The ratio of the circumference of a circle to its diameter is less than π; the ratio approaches π as the area of the circle approaches 0.
6. Similar figures of different areas exist.	6. Similar figures of different areas do not exist.	6. Similar figures of different areas do not exist.
7. Straight lines are infinite.	7. Straight lines are infinite.	7. Straight lines are finite and they all have the same length.

both had long lives, perhaps they would have preceded Einstein in the Theory of Relativity, for that theory proposes that a generalized version of Riemannian geometry is true of physical space.

In any case, after Riemann's lecture it seemed clear that there were at least three different geometries, and the question became more insistent: Are they each consistent? The method which Saccheri used to test the consistency of a system—and few, if any, before him thought much about the problem—was to ask: Are the axioms true? Any collection of statements all of which are true is necessarily consistent. However, the converse is not correct, namely, that any consistent collection of statements is true. In particular it was believed that Lobachevskian and Riemannian geometries—although false of the physical world—were nevertheless consistent. But how could this be proved? There was no answer to the question on the basis of traditional logic. The fact that no one had found a contradiction—even though a number of very gifted men had worked on finding one—was not sufficient. Perhaps tomorrow someone would find an inconsistency in one or both of these geometries. Was there any way to settle the question once and for all?

In the nineteenth century great progress was made in doing just this. The method chosen was to reduce one system to another, that is, to show that the system in question had the same logical structure as a system believed to be consistent. This was really a *reductio ad absurdum* proof. If the system in question is inconsistent, then this known system is inconsistent. But this consequence is false. Therefore the system is consistent. Efforts focused, therefore, on reducing Lobachevskian and Riemannian geometry to known mathematical systems, in particular, to Euclidean geometry and to algebra. Unfortunately, the methods by which these reductions were made are quite complex. The following account of one possible reduction is a simplification, but perhaps it will not be misleading.

Consider the geometry of the surface of a sphere. We may define a *great circle* to be a circle with a radius equal to the radius of the sphere, and whose center is the same as the center of the sphere. Now one of the properties of a straight line in plane Euclidean geometry is that it is the shortest distance between two points. On the surface of a sphere, great circles have this property. This is why airlanes—especially those relatively close to the poles—are not straight. It is easy to see that one could extend Euclidean geometry to such a surface. We might even write down the postulates for such a geometry:

Postulates

Let the following be postulated:

1. To draw a great circle from any point to any point.

(that is, the way in which persons of normal vision understand their visual space) is Lobachevskian geometry. For example, railroad tracks (when one is between them and looks off into the distance) appear to converge in one direction and diverge in the other [cf. Luneburg 1947 and, for further discussion and references, Grünbaum 1963 152–157]. As soon as we try to mathematically represent our perceptions as a whole, the problems get even more complicated. As Bertrand Russell has noted, the space of sight and the space of touch are quite different [cf. 1929 120]. Even if we limit ourselves to physical space, complications can easily ensue. For example, Hans Reichenbach suggests that "it might happen that the geometry of light rays differs from that of solid bodies" [1962 137]. All this need not concern us, for we are now in a position to understand the influence of the growth of non-Euclidean geometries on the development of mathematical logic. There were at least four different ways in which the rise of non-Euclidean geometries was influential in that development.

First, the possibility of non-Euclidean geometries was a blow to the belief in intuition, in a twofold way. In the first place, the conviction of the Kantians that Euclidean geometry is *a priori* true of our intuitions was shown to be incorrect. This does not mean that Kantian philosophy as a whole, or even the general tenor of Kant's theory of space, is incorrect. It means only that this particular doctrine is. Even so, this was a powerful blow to the authority of metaphysicians who were claiming a necessary synthetic *a priori* truth, which didn't turn out to be *a priori* and might not even be true. In the second place, the reliance on intuition as a criterion for truth received a decisive blow. For centuries, any contrary to the fifth postulate seemed false to all who investigated the matter. Now it was found that some of these are individually consistent with the other axioms and thus possibly true. Henceforth mathematicians became sceptical of the claim of obviousness to prove truth.

Second, due in part to the rise of non-Euclidean geometries a new view of the nature of mathematical statements presented itself. To the question *Of what are geometrical propositions true?* a number of answers had been given. For example, they are true of patterns fixed in the nature of things (a Platonistic answer), they are approximately true of the physical world (an empiricist answer), they are true of our *a priori* intuitions (a Kantian answer). Now the possibility arose that they might not be true at all, not because they are false, but because they are not kinds of things which can be true. From this point of view the work of the pure mathematician is mainly or completely that of the logician; that is, the mathematician, like the logician, deduces conclusions from assumed premisses. Thus pure mathematics is not true of anything at all because it is not about anything. This point of view was (and is) far from universal among mathematical logicians; some of the very best logicians would deny it. Nevertheless it was (and is) influential.