Advanced Mathematical Thinking

Edited by

David Tall

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ADVANCED MATHEMATICAL THINKING

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PREFACE

Advanced Mathematical Thinking has played a central role in the development of human civilization for over two millennia. Yet in all that time the serious study of the nature of advanced mathematical thinking – what it is, how it functions in the minds of expert mathematicians, how it can be encouraged and improved in the developing minds of students – has been limited to the reflections of a few significant individuals scattered throughout the history of mathematics. In the twentieth century the theory of mathematical education during the compulsory years of schooling to age 16 has developed its own body of empirical research, theory and practice. But the extensions of such theories to more advanced levels have only occurred in the last few years.

In 1976 The International Group for the Psychology of Mathematics (known as PME) was formed and has met annually at different venues round the world to share research ideas. In 1985 a Working Group of PME was formed to focus on Advanced Mathematical Thinking with a major aim of producing this volume.

The text begins with an introductory chapter on the psychology of advanced mathematical thinking, with the remaining chapters grouped under three headings:

- the nature of advanced mathematical thinking,
- cognitive theory,

and

- reviews of the progress of cognitive research into different areas of advanced mathematics.

It is written in a style intended both for mathematicians and for mathematics educators, to encourage an interest in the cognitive difficulties experienced by students of the former and to extend the psychological theories of the latter through to later stages of development. We are cognizant of the fact that it is essential to understand the nature of the thinking of mathematical experts to see the full spectrum of mathematical growth. We therefore begin with an introductory chapter on the psychology of advanced mathematical thinking. This is followed by three chapters which focus on the nature of advanced mathematical thinking: a study of the mental processes involved, the essential qualities of mathematical creativity and the mathematician's view of proof.

The processes prove to be subtle and complex and, sadly, few of the more advanced processes are made available to the average student in an advanced mathematical course. Creativity is concerned with how the subtle ideas of research are built in the mind. Proof is how they are ordered in a logical development both to verify the nature of the relationships and also to present them for approval to the mathematical community.

However, there is a huge gulf between the way in which ideas are built cognitively and the way in which they are arranged and presented in a deductive order. This warns us that simply presenting a mathematical theory as a sequence of definitions, theorems and proofs (as happens in a typical university course) may show the logical structure of the mathematics, but it fails to allow for the psychological growth of the developing human mind.

We begin the part of the book on cognitive theory by considering the way in which formal mathematical definitions are conceived by students and how this can be at variance with the formal theory. As a result of mentally manipulating a (mathematical) concept an individual develops an idiosyncratic personal c*oncept imag*e which is the product of experience and mental activity. Empirical research shows how this can give rise to subtle conflicts that can cause *cognitive obstacles* in the mind of the developing student and act as a barrier to attaining the formal ideas in the theory. The next chapter looks at the mental objects that arc the material of mathematical thought – the *conceptual entities* that are manipulated in the mind during advanced mathematical thinking, and how these entities are represented by different kinds of symbolism. The final chapter in this part considers how these conceptual entities are formed – through the process of *reflective abstraction.* All advanced mathematical concepts arc "abstract". This chapter postulates a theory of how these concepts start as *processes* which are *encapsulated* as *mental objects* that are then available for higher level abstract thought. Such a theory can give insight into how mathematicians develop advanced mathematical ideas, yet may fail to pass these thinking processes on to students, and what might be done to improve the situation.

The remainder of the book is concerned with overviews of empirical research and theory in various specific topics. First the question of the nature of advanced mathematical thinking is addressed and how (if at all) it differs from more elementary thinking occurring in younger children. Then there follow chapters on functions, limits, analysis, infinity, proof, and the growing use of the computer in advanced mathematics. Each one of these reveals a wide variety of obstacles in students' mental imagery and often extremely limited conceptions of formal concepts which are the unforseen consequences of the manner in which the subject is presented to the student. A variety of more cognitively appropriate approaches are postulated, some with empirical evidence of success. These include:

- the participation of the student in the process of mathematical thinking through an active process of "scientific debate", rather than passive receipt of pre-organized theory,
- the direct confrontation of the student with conflict which occurs in developing new theoretical constructs, to help them reflect on the problem and build a new, more coherent, cognitive structure.
- the building up of appropriate intuitive foundations for the advanced mathematical concepts, through an approach which balances cognitive growth and an appreciation of logical development.
- the use of visualization, particularly utilizing a computer, to give the student an overall view of concepts and enabling more versatile methods of handling the information,

- the use of programming to cause the student to think through mathematical processes in a way which can be encapsulated by reflective abstraction.

In all these ways we believe that empirical research into advanced thinking processes related to complementary cognitive theory can have a significant effect in improving the education of students at an advanced level.

In every chapter the authors have been encouraged to impress their own personalities on their view of the phenomena, but this has been done within a framework of internal consultation. Each participant operates from personal constructs within a context of mutual support and constructive criticism from other authors and the final manuscript has been recast by the editor to enable it to be read throughout as a single text rather than as a collection of disconnected papers. This was made possible through the wonders of modem technology, using a Macintosh SE/30 computer to enable the editor to redraft the chapters and set the whole book as camera-ready copy.

The cognitive theory of advanced mathematical thinking is developing apace. This study is the first step in making the broad sweep of current ideas in the advanced mathematical education community available to a wider readership.

David Tall

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In preparing this book we acknowledge the assistance and support of a wide range of mathematicians and mathematics educators. First we thank the International Committee of PME for encouraging us to meet each year as a continuing Working Group of PME. Then we thank those members of the Working Group who do not appear as authors but who have helped in many ways – presenting papers, organizing and reporting sessions – including Janet Duffin, Ulla Kürstein-Jensen, Miriam Amit, Anna Sfard, Janine Rogalski. Within the group of authors, all have participated in criticizing and encouraging the writing of others. However, it is with grateful thanks that the editor wishes to acknowledge assistance beyond the call of duty from Tommy Dreyfus, Ted Eisenberg, Ed Dubinsky, and particularly from Gontran Ervynek, whose enthusiasm generated the original impetus for the creation of this book.

INTRODUCTION

CHAPTER 1

THE PSYCHOLOGY OF ADVANCED MATHEMATICAL THINKING

DAVID TALL

In the opening chapter of *The Psychology of Invention in the Mathematical Field*, the mathematician Jacques Hadamard highlighted the fundamental difficulty in discussing the nature of the psychology of advanced mathematical thinking:

> ... that the subject involves two disciplines, psychology and mathematics, and would require, in order to be treated adequately, that one be both a psychologist and a mathematician. Owing to the lack of this composite equipment, the subject has been investigated by mathematicians on the one side, by psychologists on the other ... (Hadamard, 1945, page 1.)

Exponents of the two disciplines are likely to view the subject in different ways – the psychologist to extend psychological theories to thinking processes in a more complex knowledge domain – the mathematician to seek insight into the creative thinking process, perhaps with the hope of improving the quality of teaching or research. Although we will consider the nature of advanced mathematical thinking from a psychological viewpoint, our main aim will be to seek insights of value to the mathematician in his professional work as researcher and teacher.

We begin by looking at pertinent psychological considerations which will lay the foundations for ideas introduced not only in the remainder of the chapter, but in the book as a whole. We then focus our attention on the full cycle of activity in advanced math–ematical thinking: from the creative act of considering a problem context in mathematical research that leads to the creative formulation of conjectures and on to the final stage of refinement and proof. We postulate that many of the activities that occur in this cycle also occur in elementary mathematical problem-solving, but the possibility of formal definition and deduction is one factor which distinguishes advanced mathematical thinking. We will also find that teaching undergraduate mathematics often presents the final form of the deduced theory rather than enabling the student to participate in the full creative cycle. In the words of Skemp (1971), current approaches to undergraduate teaching tend to give students the *product of mathematical thought* rather than the *process of mathematical thinking.*

Not only may current methods of presenting advanced mathematical knowledge fail to give the full power of mathematical thinking, it also has another, equally serious, deficiency: *a logical presentation may not be appropriate for the cognitive development of the learner.* Indeed, much of the empirical theory reported in the later chapters of the book reveals cognitive obstacles which arise as students struggle to come to terms with ideas which challenge and contradict their current knowledge structure. Fortunately, we are also able to report empirical evidence that appropriate sequences of learning and instruction designed to help the student actively construct the concepts can prove highly successful.

1. COGNITIVE CONSIDERATIONS

We begin by looking, not at the logic and order of the public evidence of mathematical thinking found in research articles and text-books, but at the way in which these coherent relationships are built in mathematical research and implications for how this might be implemented in teaching and learning.

1.1 DIFFERENT KINDS OF MATHEMATICAL MIND

Writing in the first decade of this century, the celebrated mathematician Henri Poincaré asserted:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban[1] who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard. (Poincaré, 1913, p. 210)

He supported his arguments by contrasting the work of various mathematicians, including the famous German analysts, Weierstrass and Riemann, relating this to the work of students:

> Weierstrass leads everything back to the consideration of series and their analytic transformations; to express it better, he reduces analysis to a sort of prolongation of arithmetic; you may turn through all his books without finding a figure. Riemann, on the contrary, at once calls geometry to his aid; each of his conceptions is an image that no one can forget, once he has caught its meaning.
>
> ... Among our students we notice the same differences; some prefer to treat their problems 'by analysis', others 'by geometry'. The first are incapable of 'seeing in space', the others are quickly tired of long calculations and become perplexed. (Poincaré, 1913, p. 212)

Of course, there are not just two different kinds of mathematical mind, but many. Kronecker agreed with Weierstrass that logical proof was of paramount importance and transcended intuitive visual arguments, but their fundamental beliefs in the nature of mathematical concepts were very different. Weierstrass declared that "an irrational number has as real an existence as anything else in the world of concepts", but Kronecker was unable to accept the actual infinity of real numbers, asserting that "God gave us the integers, the rest is the work of man". Based on the Weierstrassian notion of the actual infinity of real numbers, Cantor was able to produce an infinite counting argument to show that there are strictly "more" real numbers than algebraic numbers (solutions of polynomial equations with integer coefficients). He therefore claimed that there exists a real non-algebraic number, without giving an explicit method to construct one. This was anathema to Kronecker who caused Cantor's paper to be rejected from publication in Crelle's Journal in 1873.

---

[1] Sebastien de Vauban (1633-1707) was a French military engineer who revolutionized the art of siege craft and defensive fortifications.

Such arguments about the foundations of mathematics led to the development of several different strands of mathematical philosophy at the beginning of the twentieth century. The *intuitionist* view represented by Kronecker asserted that mathematical concepts only exist when their construction is demonstrated from the integers, the *formalist* view of Hilbert affirmed that mathematics is the meaningful manipulation of meaningless marks written on paper, whilst the *logicist* view of Russell, declared that mathematics consists of deductions using the laws of logic.

Practising mathematicians tend to distance themselves from esoteric arguments and simply get on with their work of stating and proving theorems. Thus the twentieth century has seen the demise of Kronecker's views and the triumph of a pragmatic mixture of formalism and logic. It has seen the creation of a large number of formal systems based on logical deduction from formal definitions and axioms – an approach that survived the apparently mortal blow struck by Gödel's incompleteness theorem, that any axiomatic system including the integers must contain true statements that cannot be proved by a finite sequence of steps within the system.

The textbook by Bishop (1967) on constructive analysis – which insists on algorithmic construction proofs and disallows proof by contradiction alone – seems but an isolated singularity in the dynamic flow of twentieth century mathematical creativity.

Nevertheless, the recent introduction of computer technology may yet see a new renaissance in constructibility because of the way that computers manipulate data:

> Computers have affected mathematics as inevitably as the development of railroads affected patterns of land development. With computers it is possible to test hypotheses and compile data with ease that formerly would have been accessible, if at all, only via the most sophisticated techniques. This has affected not only the sort of questions that mathematicians work on, but the very way that they think. One has to ask oneself which examples can be tested on a computer, a question which forces one to consider concrete algorithms and to try to make them efficient. Because of this and because algorithms have real-life applications of considerable importance, the development of algorithms has become a respectable topic in its own right. (Edwards, 1987)

The reason for raising these differences in mathematician's perceptions is to heighten the readers' awareness of their own part in life's rich tapestry, with a personal view of mathematics that will differ in many ways from the conceptions of others. It may come as a surprise when one first realizes that other people have radically different thinking processes. It happened to the author when using pictures to help students visualize ideas in mathematical analysis, at a time when he did not question the implicit belief that such an approach was universally valid. Whilst writing a text book on complex analysis, a colleague in the next room was engaged on a similar enterprise, yet the latter's book had almost no pictures at all. He only included a diagram illustrating the argument of a complex number after a great deal of heart searching. To him a real number was an element of a complete ordered field (satisfying specific axioms) and a complex number was an ordered pair of real numbers. The argument of a complex number (x,y) was defined as a real number a such that

$$\cos(\alpha) = \frac{x}{\sqrt{x^2+y^2}},\ \sin(\alpha) = \frac{y}{\sqrt{x^2+y^2}}$$

where sin and cos were defined by red power series. The theory did not require a geometrical meaning. He took this hard line to make sure that his arguments were the product of logical deduction and not dependent anywhere on geometric intuition. At the time the author was sympathetic to this philosophical viewpoint, but considered it too sophisticated for students. It was some considerable time later that the realization dawned that not all students shared the geometric point of view. No one view holds universal sway.

1.2 META-THEORETICAL CONSIDERATIONS

The discussion of the preceding session is a salutary reminder that any theory of the psychology of learning mathematics must take into account not only the growing conceptions of the students, but the conceptions of mature mathematicians. Mathematics is a shared culture and there are aspects which are context dependent. For example, an analyst's view of a differential may be very different ftom that of an applied mathematician, and a given individual may strike up different attitudes to this concept depending on whether it is considered in an analytic or applied context. We will see (chapter 11) that such attitudes can cause conflicts in students too.

At a far deeper psychological level we all have subtly different ways of viewing a given mathematical concept, depending on our previous experiences. For example, the "completeness axiom" for thereal numbers is viewed by some as "filling in all the gaps between the rational numbers to give all the points on the number line". Such a view may imply that there is "no room" to fit in any more numbers: the number line is now "complete". The "real" number line, in particular cannot contain "infinitesimals" which are smaller than any positive rational yet not zero. But, for others, "completion" is only a technical axiom to adjoin the limit points of cauchy sequences of rational numbers. In this case it is perfectly possible to embed the real numbers in a variety of larger number fields, which include infiitesimals and infinite numbers. It is this view which leads to the modern infinitesimal theory of "non-standard analysis". The latter idea, however, is anathema to many mathematicians, including Cantor, who denied the existence of infinitesimals on the grounds that it was not possible to calculate the reciprocal of an infiite number in his theory of cardinal infinities. Even today many mathematicians are troubled by the infinitesimal ideas of non-standard analysis; they may not deny its logic, but they sense a deep-seated psychological unease as to its validity.

Thus any theory of the psychology of mathematical thinking must be seen in the wider context of human mental and cultural activity. There is not one true, absolute way of thinking about mathematics, but diverse culturally developed ways of thinking in which various aspects are relative to the context.

1.3 CONCEPT IMAGE AND CONCEPT DEFINITION

In Tall & Vinner (1981). the distinction is made between the individual's way of thinking of a concept and its formal definition, thus distinguishing between mathematics as a mental activity and mathematics as a formal system. This theory applies to expert mathematicians as well as developing students:

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic that gives us insight, nor is it chance that makes us make mistakes ... We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. ... As the concept image develops it need not be coherent at all times.The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole. (Tall & Vinner 1981)

In this way it is possible for conflicting views to be held in the mind of a given individual and to be evoked at different times without the individual being aware of the conflict until they are evoked simultaneously.

The mature mathematician is not immune from internal conflicts, but he or she has been able to link together large portions of knowledge into sequences of deductive argument. To such a person it seems so much easier to categorize this knowledge in a logically structured way. Thus a mature mathematician may consider it helpful to present material to students in a way which highlights the logic of the subject. However, a student without the experience of the teacher may find a formal approach initially difficult, a phenomenon which may be viewed by the teacher as a lack of experience or intellect on the part of the student. This is a comforting viewpoint to take, especially when the teacher is part of a mathematical community who share the mathematical understanding. But it is not realistic in the wider context of the needs of the students. What is essential – for them – is an approach to mathematical knowledge that grows as they grow: a cognitive approach that takes account of the development of their knowledge structure and thinking processes. To become mature mathematicians at an advanced level, they must ultimately gain insight into the ways of advanced mathematicians but, en route, they may find a stony path that will require a fundamental transition in their thinking processes.

1.4 COGNITIVE DEVELOPMENT

There are many competing theories in psychology. Behaviourist theory, built on external observation of stimulus and response, refuses to speculate about the internal workings of the mind. It provides observable and repeatable evidence of the behaviour of animals, including humans, under repeated stimuli, but it has limited application to mathematical thinking beyond the mechanics of routine algorithms. Constructivist psychology, on the other hand, attempts to discuss how mental ideas are created in the mind of each individual. This may pose a dialectic problem for the mathematician with a Platonic ideal of mathematics existing independently of the human mind, but it proves to give significant insight into the creative processes of research mathematicians as well as the dfficulties experienced by mathematics students.

The great Swiss psychologist Piaget saw the individual's need to be in dynamic equilibrium with his environment as an underlying theme in his work. This equilibrium could be disturbed through the confrontation with new knowledge that conflicted with the old, and so a transition period might occur in which the knowledge structure is re-constructed to give a more mature level of equilibrium.

Piaget saw the child grow into the adult through a series of stages of equilibrium, each one richer than the one before. He identifed four main stages. The first is the *sensori-motor* stage prior to the development of meaningful speech, followed by a *pre-operational* stage when the young child realizes the permanence of objects, which continue to exist even if they are temporarily out of sight. The child then goes through a transition into the period of *concrete operations* where he or she can stably consider concepts which are linked to physical objects, thence passing into a period of *formal operations* in the early teens when the kind of hypothetical "if–then" becomes possible.

Piagetian stage theory has been extended to higher levels to encompass advanced mathematical thinking. For instance, Ellerton (1985) suggested that Piaget's cycle of sensori-motor, pre-operational and concrete is the first level of a spiral cognitive development in which the formal stage is the beginning of another cycle of the same type at a higher level of abstraction. Biggs & Collis (1982) suggested a repetition of formal operations at successively higher levels, each repeating the learning cycle: unistructural, multistructural, relational.

A difficulty of applying such theory to college mathematics teaching is that many – probably most – college students are not able to perform at the abstract level of formal operations, which Piaget reported occurring in children during their early teens. Ausubel criticized the stage theory:

> ... because such a high percentage of American high school and college students fail to reach this abstract level of cognitive logical operations. (Ausubel *et al* 1968, p. 230)
>
> Representative studies have indicated that only 15% of junior high school students ... 13.2% of high school students ... and 22% of college students were at this level. (*ibid,* p. 238)

The concrete/formal distinction has proved to be a useful starting point in developing local hierarchies of difficulty in extensive studies such as Hart (1981) in the 11 to 16 age range, and the development of early calculus concepts by Orton (1980). But a significant failure of Piaget's stage theory for the design of new teaching strategies is his own assertion that the movement from one stage to another cannot be greatly accelerated by the affects of teaching. Differences of cognitive demand have often been used in a *negative* sense to describe students' difficulties, but rarely to provide *positive* criteria for designing new approaches to the subject. Papert (1980) asserted:

> The Piaget of stage theory is essentially conservative, almost reactionary, in emphasizing what children cannot do. I strive to uncover a more revolutionary Piaget,one who see pistemological ideas might expand the known bounds of the human mind.

Advanced mathematics provides us with a useful metaphor which expands the vision of stage theory to a theory more valuable in the development of advanced mathematical thinking. Piaget used an analogy with group theory to underpin his sense of the dynamic equilibrium of cognitive growth. He saw the identity element as representing the stable state, and noted that stability couldbe maintained if any transformation from this state could be reversed, thus suggesting a group structure in which every element has an inverse. But the maintenance of a dynamic state of equilibrium has a more obvious mathematical metaphor in dynamical systems and catastrophe theory. Here a system controlled by continuously varying parameters can suddenly leap from one position of equilibrium to another when the first becomes untenable. Depending on the history of the varying parameters, the transition may be smooth, or it may be discontinuous. This analogy suggests that stage theory may just be a linear trivialization of a far more complex system of change, at least this may be so when the possible routes through a network of ideas become more numerous, as happens in advanced mathematical thinking.

1.5 TRANSITION AND MENTAL RECONSTRUCTION

A far more valuable aspect of Piaget's theory is the process of *transition* from one mental state to another. During such a transition, unstable behaviour is possible, with the experience of previous ideas conflicting with new elements. Piaget uses the terms *assimilation* to describe the process by which the individual takes in new data and *accommodation* the process by which the individual's cognitive structure must be modified. He sees assimilation and accommodation as complementary. During a transition much accommodation is required. Skemp (1979) puts similar ideas in a different way by distinguishing between the case where the learning process causes a simple *expansion* of the individual's cognitive structure and the case where there is cognitive conflict, requiring a mental *reconstruction.* It is this process of reconstruction which provokes the difficulties that occur during a transition phase.

Such transitions occur often in advanced mathematics as the individual struggles with new knowledge structure. Conflict is a phenomenon well-known to the mathematical mind.

1.6 OBSTACLES

The most serious problem occurs when the new ideas are not satisfactorily accommodated. In this case it may be possible for conflicting ideas to be present in an individual at one and the same time:

> New knowledge often contradicts the old, and effective learning requires strategies to deal with such conflict. Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be "kept around" if safely maintained in separate compartments. (Papert, 1980, p. 121)

The thesis of Comu (1983) studies the conceptual development of the limit process from school to university and underlines how the colloquial use of the term "limit" effects the mathematical usage. He discusses the notion of an "obstacle", introduced by Gaston Bachelard (1938):

> An obstacle is a piece of knowledge; it is part of the knowledge of the student. This knowledge was at one time generally satisfactory in solving certain problems. It is precisely this satisfactory aspect which has anchored the concept in the mind and made it an obstacle. The knowledge later proves to be inadequate when faced with new problems and this inadequacy may not be obvious. (Comu 1983, (original in French))

The obstacles found by Comu include the problems student face when they must calculate limits using techniques more subtle than simple numerical and algebraic operations. He discusses how the concept of infinity is introduced and is "surrounded in mystery", yet the new techniques "work" without the students understanding why. He demonstrates how students' experiences can lead to belief in the infinitely large and the infinitely small, with "nought point nine recurring" being a number "just less than one" and the symbol ε representing to many students a quantity that is smaller than any positive real number, but not zero. There are implicit assumptions that the limiting process "goes on forever", that the limit "can never be attained". (See chapter 10.)

Tall (1986a) suggests an explanation is given for these phenomena as the *generic extension principle:*

If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts.

For example, most convergent sequences described to beginning students are of a simple kind given by a formula such as $1/n$, which tends to the limit(in this case zero), but the terms never *equal* the limit. In the absence of any counter-examples students begin to believe that this is always so. The rich experience of colloquial language supports this belief (Schwarzenberger & Tall, 1978), with phrases like "gets close to" suggesting that the terms of a sequence can never be coincident with the limit. Thus the implicit belief is slowly formed that a sequence of terms converging to a limit gets closer and closer, *but never actually gets there.*

Furthermore, if all the terms of a sequence have a certain property, it is natural to believe that the limit has the same property. Thus the sequence 0.9, 0.99, ... has terms all less than 1, so the limit "nought point nine recurring" must also be less than one... This leads to the mental image of a limiting object termed a *generic limit* in Tall (1986a). Ageneric limit need not be a limit in the mathematical sense, but it is the concept of the limit that the individual holds in his or her mind as a result of extrapolating the common properties of the terms of the sequence.

This phenomenon happens not just with sequences of numbers, but sequences of functions and other mathematical objects hat share a common property. Historically this is enshrined in the "principle of continuity" of Leibniz:

In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included. (Leibniz in a letter to Bayle, January 1687.)

It arises even earlier in the work of Nicholas of Cusa (1401–1464) who regarded the circle as a polygon with an infinite number of sides, and inspired Kepler (1571–1630) to formulate a metaphysical "bridge of continuity" in which normal and limiting forms of a figure are characterized under a single definition. Thus Kepler (*Opera Omnia II* page 595) saw no essential difference between a polygon and a circle, between an ellipse and a circle, between the finite and the infinite, and between an infinitesimal area and a line.

The generic extension principle arises time and again in history. For example, Cauchy's assertion that the limit of continuous functions is continuous and Peacock's "Principle of Algebraic Permanence", in which the properties of extended number systems, such as the real and complex numbers, were based on the principle that the any algebraic law which held in the smaller system also held in the extension. The latter held sway for some time carries out what may be seen as a more general construct in particular cases and gives rise to a generic abstraction of the function concept. Given the theory just described, this suggests a further stage is necessary to pass from the generic example of programming, where the general is seen in the particular instances of functions programmed by the student, to the formal abstraction which requires a new level of abstract construction from the definition. Dubinsky formulates this transition within a Piagetian framework of *reflective abstraction*, in which processes are *encapsulated* as objects, so that the function *process* leads to the function *as a mental object.* This theory is further elaborated in chapters 7 and 15.

1.8 INTUITION AND RIGOUR

Mathematicians often regard the terms "intuition" and "rigour" as being mutually exclusive by suggesting that an "intuitive" explanation is one that necessarily lacks rigour. There is a grain of truth in this, for usually an intuition arrives whole in the mind and it may be difficult to separate its components into a logical deductive order. But the opposition between the two concepts is a false dichotomy as we shall soon see.

In a sense we have not one, but two brains. In attempting to assist patients who had serious epileptic fits, Sperry and his colleagues took the drastic action of partial or total severance of the corpus callosum that links the two hemispheres of the brain and found that each could essentially operate independently, though carrying out totally dfferent functions:

> Though predominantly mute and generally inferior in all performances involving language or linguistic or mathematical reasoning, the minor hemisphere is nevertheless clearly the superior cerebral member for certain types of tasks. If we remember that in the great majority of tests it is the disconnected left hemisphere that is superior and dominant, we can review quickly now some of the kinds of exceptional activities in which it is the minor hemisphere that excels. First, of course, as one would predict, these are all non-linguistic non-mathematical functions, largely as they involve the apprehension and processing of spatial patterns, relations and transformations. They seem to be holistic and unitary rather than analytic and fragmentary, and orientational more than focal, and to involve concrete perceptual insight rather than abstract, symbolic sequential reasoning.
>
> (Sperry, 1974)

This evidence resonates strongly with the observation of the two different kinds of mathematical mind suggest at the turn of the century by Poincaré. However, subsequent research suggests that the brains of different individuals need not follow such a simplistic division of functions. Gazzigna (1985) sees brain activity as a collection of different modules functioning independently in parallel, with a control unit (usually in the left brain) making decisions based on the information provided by the various modules. Thus it would be incorrect to divide human activity simplistically into two different modes, just as it is inappropriate to consider just two contrasting types of mathematical mind.In particular we may envisage that the human mind immersed in logical thought may eventually develop intuitions that are themselves logically based. Poincaré, speakmg of Hermite, said:

> His eyes seem to shun contact with the world; it is not without, it is within he seeks the vision of truth.
>
> ... When one talked to M. Hermite, he never evoked a sensuous image, and yet you soon perceived that the most abstract entities were for him like living beings. He did not see them, but

he perceived that they are not an artificial assemblage and that they have some principle of internal unity. (Poincaré, 1913, pp. 212, 220)

The conclusion is inescapable. Intuition is the product of the concept images of the individual. The more educated the individual in logical thinking, the more likely the individual's concept imagery will resonate with a logical response. This is evident in the growth of thinking of students, who pass from initial intuitions based on their pre-formal mathematics, to more refined formal intuitions as their experience grows:

We then have many kinds of intuition; first the appeal to the senses and the imagination; next, generalization by induction, copied, so to speak, from the procedures of the experimental sciences; finally we have the intuition of pure number... (Poincaré, 1913, p. 215.)

From a psychological viewpoint, Fischbein (1978) comes to similar conclusions, citing two different types of intuition:

Primary intuitions refer to those cognitive beliefs which develop themselves in human beings, in a natural way, before and independently of systematic instruction.

Secondary intuitions are those which are developed as a result of systematic intellectual training ... In the same meaning, Felix Klein (1898) used the term "refined intuition": and F. Severi wrote about "second degree intuition" (1951). (Fischbein, 1978, p. 161)

Thus aspects of logic too can be honed to become more "intuitive" to the mathematical mind. The development of this refined logical intuition should be one of the major aims of more advanced mathematical education.

2. THE GROWTH OF MATHEMATICAL KNOWLEDGE

As we have seen, the nature of mathematical thinking is inextricably interconnected with the cognitive processes that give rise to mathematical knowledge. We now focus on the full cycle of mathematical thinking to see mathematical proof as the final stage of this developmental process rather than just the formal framework of the completed knowledge structure.

2.1 THE FULL RANGE OF ADVANCED MATHEMATICAL THINKING

Mathematical proof, according to Hadamard (1945), is but the last, "precising" phase of mathematical thinking. Before a theorem can be conjectured, let alone proved, there is much work to be done in conceiving of what ideas will be fruitful and what relationships will be useful. Hadamard considers Poincaé's description of his own personal research activities and notes:

.. the very observations of Poincaré show us three kinds of inventive work essentially different if considered from our standpoint, *viz.*,

a. fully conscious work
b. illumination preceded by incubation
c. the quite peculiar process of the sleepless night. (Hadamard, 1945, p. 35)

Here Poincaré reports the necessity of working hard at a new problem, then relaxing to allow the ideas to incubate in his subconscious, during which time he had a sleepless night thinking vigorously about new ideas until suddenly, some time later, a sudden illumination bursts into his consciousness with a solution. After a further time had elapsed, at his leisure, he was able to analyse what had happened and build up a formal justification of his theory in the final "precising" phase when the results of the illuminative break-through are subjected to the cold analysis of the light of day, refining the assumptions so that the deductions will stand analytic scrutiny.

What becomes apparent is that the initial phases of the creative cycle may rely in part on logic and deduction, but they also need flexible mental activity to produce mental resonances between previously unconnected concepts. According to Gazzigna's model of brain activity, they may occur as juxtapositions from different modules in the brain processing simultaneously. Part of the success of this phase of mathematical thinking seems to be due to working sufficiently hard on the problem to stimulate mental activity, and then relaxing to allow the processing to carry on subconsciously.

2.2 BUILDING AND TESTING THEORIES: SYNTHESIS AND ANALYSIS

Poincaré was at pains to show the complementary roles of synthesis and analysis in mathematical thinking. Synthesis begins with the conscious act of the initial phase to begin to put ideas together, followed by a more intuitive activity, in which subconscious interplay between concept images takes place, until a powerful resonance forces the newly linked concepts to erupt once more into consciousness. Analysis, on the other hand, is a much more cool and logical conscious activity which organizes the new ideas into logical form and refines them to give precise statements and deductions.

Teaching of younger children emphasizes the *synthesis* of knowledge, starting from simple concepts, building up from experience and examples to more general concepts. The emphasis at this level is now changing to include more problem solving and open-ended investigations. Teaching at university often emphasizes the other side of the coin: *analysis* of knowledge, beginning with general abstractions and forming chains of deduction from them which may be applied in a wide variety of specific contexts.

Working with much younger children, Dienes (1960) proposed a theory for building concepts from concrete examples, yet Dienes & Jeeves (1965) formulates a far more general *deep-end principle* in which "there is a preference for extrapolation by leaps and interpolation, rather than always by step-by-step". They respond to their own question "When is it possible to generalize from a simple case to a more general case and when is it better for them to particularize from a more complex case to the simple case?" with the remark that "this is not likely to be answered by a simple positive or negative statement". They suggest that it is more a question of "the optimum degree of complexity required to start with" – a response which is just as valid for teaching and learning at more advanced levels. It is likely to require synthesis of knowledge to build up theories cognitively as well as analysis of knowledge to give the total structure a logical coherence.

2.3 MATHEMATICAL PROOF

Viewed as a problem-solving activity, we see that proof is actually the final stage of activity in which ideas are made precise. Yet so much of the teaching in university level mathematics *begins* with proof. In his preface to *The Psychology of Learning Mathematics*, Skemp succinctly refers to this as showing the students the product of mathematical thought, instead of teaching them the process of mathematical thinking. The splendid tomes of Bourbaki are a monument to the intellect of the mathematical mind, and may be used to help the learner appreciate the formal structure of mathematics. But once again, Poincaré has pertinent observations to make:

> To understand the demonstration of a theorem, is that to examine successively each of the syllogisms composing it and to ascertain its correctness, its conformity to the rules of the game? ... For some, yes; when they have done this, they will say: I understand. For the majority, no. Almost all are much more exacting they wish to know not merely whether all the syllogisms of a demonstrations are correct, but why they link together in this order rather than another. In so far as to them they seem engendered by caprice and not by an intelligence always conscious of the end to be attained, they do not believe that they understand. (Poincaré,1913,p.431)
>
> Perhaps you think I use too many comparisons; yet pardon still another. You have doubtless seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitiv enotions of our fathers, evenwhen we have abandoned them, still imprint their form upon the logical constructions we have put in their place. (*ibid,* p. 219)

Thus it is that so many mathematicians demand that a proof should not only be logical, but that there should be some over-riding principle that explains why the proof works. The proof of the four colour theorem, by exhaustion of all possible configurations using a computer search (Appel & Haken, 1976) *seems* logical, yet many professional mathematicians, though keen to see the theorem proved once and for all, are nevertheless sceptical that there may be some subtle flaw in the computer "proof", because there seems to be no rhyme or reason to illuminate why it works as it does.

Yet this principle is not always passed on to students. Sawyer (1987) reports how he tried to teach theorems in functional analysis by referring back to theorems in real variables that he expected his students to know, only to find that they had no recollection of them.

> The reason for this was that in their university lectures they had been given formal lectures that had not conveyed any intuitive meaning; they had passed their examinations by last-minute revision and by rote.

He tells how he was shocked to learn of a lecturer who became stuck in the middle of a proof, turned his back on the class to draw a picture to aid him, then erased it and carried on with the formal proof without enlightening the class how he had used his intuition to rebuild it. He observes:

> ... to teach calculus well is a very demandmg task. Three things have to be done: first to show by a drawing that some result is extremely plausible; second, to give counter-examples, which indicate

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