An Introduction to Category Theory

Category theory provides a general conceptual framework that has proved fruitful in subjects as diverse as geometry, topology, theoretical computer science and foundational mathematics. Here is a friendly, easy-to-read textbook that explains the fundamentals at a level suitable for newcomers to the subject.

Beginning postgraduate mathematicians will find this book an excellent introduction to all of the basics of category theory. It gives the basic definitions; goes through the various associated gadgetry, such as functors, natural transformations, limits and colimits; and then explains adjunctions. The material is slowly developed using many examples and illustrations to illuminate the concepts explained. Over 200 exercises, with solutions available online, help the reader to access the subject and make the book ideal for self-study. It can also be used as a recommended text for a taught introductory course.

An Introduction to Category Theory

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Preface

As it says on the front cover this book is an introduction to Category Theory. It gives the basic definitions; goes through the various associated gadgetry such as functors, natural transformations, limits and colimits; and then explains adjunctions. This material could be developed in 50 pages or so, but here it takes some 220 pages. That is because there are many examples illustrating the various notions, some rather straightforward, and others with more content. More importantly, there are also over 200 exercises. And perhaps even more importantly, solutions to these exercises are available online.

The book is aimed primarily at the beginning graduate student, but that does not mean that other students or professional mathematicians will not find it useful. I have designed the book so that it can be used by a single student or small group of students to learn the subject on their own. The book will make a suitable text for a reading group. The book does not assume the reader has a broad knowledge of mathematics. Most of the illustrations use rather simple ideas, but every now and then a more advanced topic is mentioned. The book can also be used as a recommended text for a taught introductory course.

Every mathematician should at least know of the existence of category theory, and many will need to use categorical notions every now and then. For those groups this is the book you should have. Other mathematicians will use category theory every day. That group has to learn the subject sometime, and this is the book to start that process. Of course, the more advanced topics are not dealt with here.

The book has been developed over quite a few years. Several short courses of about 10 hours have been taught (not always by me) using some of the material. In 2007, 2008, and 2009 I gave a course over the web to about a dozen universities. This was part of MAGIC, the

Mathematics Access Grid Instruction and Collaboration cooperative of quite a few University Departments of Mathematics in England and Wales. That was an interesting experience and helped me to split the material into small chunks each of the right length to fit into one hour. (The course is still being taught but someone else has taken over the wand.) Of course, the order in which material is taught need not be the same as the written account.

As someone once said, Mathematics is not a spectator sport. To learn and understand Mathematics you have to get stuck in and get your hands dirty. You have to do the calculations, manipulations, and proofs yourself, not just read the stuff and pretend you understand it. Thus I have included over 200 exercises to help with this process. I have also written a more or less complete set of solutions to these exercises. But these are not available in the book, for it is too easy simply to look up a solution. When you can't see how to do it you have to sweat a bit to find a solution. Someone else once said that horses sweat, gentlemen perspire, and ladies glow. However, I can't remember meeting many horses who could do mathematics all that well. In other words, although effort is important to learn mathematics you also need something else. You need help every now and then. That is why there are exercises *and* solutions. These solutions are available at

`www.cambridge.org/simmons`

The book is divided into six Chapters, each chapter is divided into several Sections, and a few of these are divided into Blocks (Subsections). Each chapter contains a list of Items, that is Definitions, Lemmas, Theorems, Examples, and so on. These are numbered by section. Thus item $X.Y.Z$ is in Chapter X, Section Y, and is the Zth item in that section. Where a section is divided into blocks the items are still numbered by the parent section.

Each section contains a selection of Exercises. These are numbered separately throughout the section. Thus Exercise $X.Y.Z$ is in Chapter X, Section Y, and is the Zth exercise of that section. Again, where a section is divided into blocks the exercises are still numbered by the parent section.

Occasionally you will see a word or two IN THIS FONT. This is a mention of a NOTION that is dealt with in more detail later. You should remember to come back to this place when you understand the notion.

There are several other books available on this subject. Some of these are introductory texts and some are more advanced. I have listed some of them in the bibliography. None of these are needed when reading this book, but some will certainly help broaden and advance your understanding of the subject. I have refrained from passing comment on these books, for I know that different people have different tastes. However, you should look around for different accounts. Some of these will help.

I first became aware of Category Theory in 1965 during a Summer Meeting in Leicester (England). Since then I have been trying to learn and understand the subject. It is patently obvious to me that Category Theory is a very useful tool. It helps to organize many parts of mathematics. It can sort out the 'routine' aspects of a proof and isolate the 'essential content' of the result. In some ways that is why Eilenberg and MacLane invented the subject. However, I am not one of those 42ers who think that Category Theory is the essential foundations for Mathematics, Life, and Everything. Of course Category Theory is something that every mathematician should know something about, but there are other things as well.

Many people have influenced this book. For several years Andrea Schalk has used the material to teach an introductory course. Hugh Steele, Roman Krenický, and Francisco Lobo have pointed out and sometimes corrected my eccentricities. And Wolfy has guided me through some of the deeper mysteries of LaTeX. Where would we be without the wonderful LaTeX?

There may still be mistakes, inaccuracies, or garbled bits in the book. I would be quite happy to pass on the blame, but I won't. I am not a politician. I am responsible for everything inside the cover. The outside cover is the work of others.

Any book of this kind must contain many diagrams, some of which must commute. I have used Paul Taylor's diagram package to do this job. If you don't know this package then I recommend you have a look at it. I have also used his lesser known tree drawing package at one place.

At Cambridge University Press my contact, Silvia Barbina, has been very helpful. I once taught her a little bit about football (and, as she reminded me, some Model Theory). Silvia has made writing this final version very enjoyable. She has kept me on the straight and narrow, so I didn't wander off to do something else. In her charming Italian style she asked me (instructed me) to cut out all the jokes. This was quite difficult since some of the official categorical terminology is a joke, but I have done my best.

Clare Dennison and Lucy Edwards oversaw the production period (getting my raw code converted into the material you have in front of you). Siriol Jones copy-edited the book and corrected many of my silly mistakes. I thank them all. Roger Astley was chief pie-man for the whole project.

On a more personal level I am very grateful to Bobby Manc and what he is achieving. I hope he continues for quite some time. The Lodge (Appleby Lodge) is at last getting back to what it should be. Ruth Maddocks kept me cheerful. She made me the odd cup of tea. A very odd cup of tea.

Enjoy yourself and learn something at the same time.

it is better to use 'arrow' for the abstract notion, and so distinguish between the general and the particular.

The word 'domain' already has other meanings in mathematics. Why bother with this and 'codomain' when there are two perfectly good words that capture the idea quite neatly. You will also see

$$f : A \longrightarrow B$$

used to name the arrow above. However, as we see later, you should not think of an arrow as a function.

All three of the notations

$$A \xrightarrow{\quad \boldsymbol{id}_A \quad id_A \quad 1_A \quad} A$$

are used for the identity arrow assigned to the object A. We will tend to use id_A. Notice that the source and the target of id_A are both the parent object A. Quite often when there is not much danger of confusion id is written for id_A. You will also find in the literature that some people write 'A' for the arrow id_A. This is a notation so ridiculous that it should be laughed at in the street.

Certain pairs of arrows are compatible for composition to form another arrow. Two arrows

$$A \xrightarrow{\quad f \quad} B_1 \qquad\qquad B_2 \xrightarrow{\quad g \quad} C$$

are composible, in that order, precisely when B_1 and B_2 are the same object, and then an arrow

$$A \longrightarrow C$$

is formed. For arrows

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

both of the notations

$$A \xrightarrow{\quad g \circ f \quad gf \quad} C$$

are used for the composite arrow. Read this as

$$g \text{ after } f$$

and be careful with the order of composition. Here we write $g \circ f$ for the composite.

We need to understand how to manipulate composition, sometimes involving many arrows.

Composition of arrows is associative as far as it can be. For arrows

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \xrightarrow{\;h\;} D$$

various composites are possible, as follows.

$$\begin{array}{l}
A \xrightarrow{\quad (h \circ g) \circ f \quad} D \\
A \xrightarrow{\;f\;} B \xrightarrow{\quad h \circ g \quad} D \\
A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \xrightarrow{\;h\;} D \\
A \xrightarrow{\quad g \circ f \quad} C \xrightarrow{\;h\;} D \\
A \xrightarrow[\quad h \circ (g \circ f) \quad]{} D
\end{array}$$

It is required that the two extreme arrows are equal

$$(h \circ g) \circ f = h \circ (g \circ f)$$

and we usually write

$$h \circ g \circ f$$

for this composite. This is the first of the axioms restricting the data.

The second axiom says that identity arrows are just that. Consider

$$A \xrightarrow{\;id_A\;} A \xrightarrow{\;f\;} B \xrightarrow{\;id_B\;} B$$

an arbitrary arrow and the two compatible identity arrows. Then

$$id_B \circ f = f = f \circ id_A$$

must hold.

Given two objects A and B in an arbitrary category $\boldsymbol{C}$, there may be no arrows from A to B, or there may be many. We write

$$\boldsymbol{C}[A, B] \quad \text{or} \quad \boldsymbol{C}(A, B)$$

for the collection of all such arrows. For historical reasons this is usually called the

hom-set

from A to B, although

arrow-class

would be better. Some people insist that $\boldsymbol{C}[A, B]$ should be a set, not a class. As usual, there are some variants of this notation. We often write

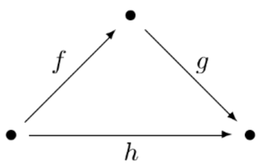
$$[A, B] \quad \text{for} \quad \boldsymbol{C}[A, B]$$

especially when it is clear which category $\boldsymbol{C}$ is intended. Sometimes

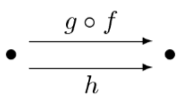
$$\mathrm{Hom}_{\boldsymbol{C}}[A, B]$$

is used for this hom-set.

We have seen above one very small diagram. Composition gives us a slightly larger one. Consider three arrows



arranged in a triangle, as shown. Here we haven't given each object a name, because we don't need to. However, the notation does *not* mean that the three objects are the same. For this small diagram, the triangle, the composite $g \circ f$ exists to give us a parallel pair



of arrows across the bottom of the triangle. These two arrows may or may not be the same. When they are

$$h = g \circ f$$

we say the triangle commutes. We look at some more commuting diagrams in Section 2.1, and other examples occur throughout the book.

Examples of categories

In the remaining sections of this chapter we look at a selection of examples of categories. Roughly speaking these are of four kinds.

The first collection is listed in Table 1.1 on page 6. These all have a similar nature and are examples of the most common kind of category we meet in practice. In each an object is a structured set, a set furnished, or equipped, with some extra gadgetry, the furnishings of the object. An arrow between two objects is a function between the carrying sets where the function 'respects' the carried structure. Arrow composition is then function composition. We look at some of these categories in Section 1.2.

Some categories listed in Table 1.1 are not defined in this chapter. Some are used later to illustrate various aspects of category theory, in which case each

Table 1.1 *Categories of structured sets and structure preserving functions*

Category	Objects	Arrows
Set	sets	total functions
Pfn	sets	partial functions
$\boldsymbol{Set}_\perp$	pointed sets	point preserving functions
RelH	sets with a relation	relation respecting functions
Sgp	semigroups	morphisms
Mon	monoids	morphisms
CMon	commutative monoids	morphisms
Grp	groups	morphisms
AGrp	abelian groups	morphisms
Rng	rings	morphisms
CRng	commutative rings	morphisms
Pre	pre-ordered sets	monotone maps
Pos	posets	monotone maps
Sup	complete posets	$\bigvee$-preserving monotone functions
Join	posets with all finitary joins	$\vee$-preserving monotone functions
Inf	complete posets	$\bigwedge$-preserving monotone functions
Meet	posets with all finitary meets	$\wedge$-preserving monotone functions
Top	topological spaces	continuous maps
$\boldsymbol{Top}_\star$	pointed topological spaces	point preserving continuous maps
$\boldsymbol{Top}^{\mathrm{open}}$	topological spaces	continuous open maps
$\boldsymbol{Vect}_K$	vector spaces over a given field K	linear transformations
$\boldsymbol{Set}$-R	sets with a right action from a given monoid R	action preserving functions
R-$\boldsymbol{Set}$	sets with a left action from a given monoid R	action preserving functions
$\boldsymbol{Mod}$-R	right R-modules over a ring R	morphisms
R-$\boldsymbol{Mod}$	left R-modules over a ring R	morphisms

Table 1.2 *More complicated categories*

Category	Objects	Arrows
$\boldsymbol{RelA}$	sets	binary relations
$\boldsymbol{Pos}^{\dashv}$	posets	poset adjunctions
$\boldsymbol{Pos}^{pp}$	posets	projection embedding pairs
$\widehat{S}$	presheaves on a given poset S	natural transformations
$\widehat{\boldsymbol{C}}$	presheaves on a given category $\boldsymbol{C}$	natural transformations
$\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$	chain complexes	

is defined when it first appears. Some categories are listed but not used in this book, but you should be able to fill in the details when you need to.

These simple examples tend to give the impression that in any category an object is a structured set and an arrow is a function of a certain kind. This is a false impression, and in Section 1.3 we look at some examples to illustrate this. In particular, these examples show that an arrow need not be a function (of the kind you first thought of).

An important message of category theory is that the more important part of a category is not its objects but the way these are compared, its arrows. Given this we might expect a category to be named after its arrows. For historical reasons this often doesn't happen.

Section 1.4 contains some examples to show that the objects of a category can have a rather complicated internal structure, and the arrows are just as complicated. These examples are important in various parts of mathematics, but you shouldn't worry if you cannot understand them immediately.

Table 1.2 lists some of these more complicated examples looked at in Sections 1.3 and 1.4.

Finally in Section 1.5 we look at two very simple kinds of categories. These examples could be given now, but in some ways it is better if we leave them for a while.

Exercises

1.1.1 Observe that sets and functions do form a category $\boldsymbol{Set}$.

1.1.2 Can you see that each poset is a category, and each monoid is a category? Read that again.

As suggested above, many categories fit into this 'algebraic' form. Each object is a structured set, and each arrow (usually called a morphism or a map) is a structure respecting function. Almost all of the categories in Table 1.1 fit into this kind, but one or two don't.

In a sense the study of monoids is the study of composition in the miniature. There is a corresponding study of comparison in the miniature. That is the topic of the next example.

1.2.2 Example A pre-order $\leq$ on a set S is a binary relation that is both reflexive and transitive. (Sometimes a pre-order is called a quasi-order.) A partial order is a pre-order that is also anti-symmetric.

A

preset poset

is a set S furnished with a

pre-order partial order

respectively. Thus each poset is a preset, but not conversely.

When comparing two such structures

$$(R, \leq_R) \qquad (S, \leq_S)$$

we use the carrying sets R and S to refer to the structures and write $\leq$ for both the carried comparisons. Rarely does this cause any confusion, but when it does we are a bit more careful with the notation.

Given a pair R, S of presets a monotone map

$$R \xrightarrow{\ f\ } S$$

is a function, as indicated, such that

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in R$. Note that this condition is an implication, not an equivalence. It is routine to check that for two monotone maps

$$R \xrightarrow{\ f\ } S \xrightarrow{\ g\ } T$$

between presets the function composition $g \circ f$ is also monotone.

This gives us two categories

$$\boldsymbol{Pre} \qquad \boldsymbol{Pos}$$

where the objects are

presets posets

respectively, and in both cases the arrows are the monotone maps. Each identity arrow is the corresponding identity function viewed as a monotone map. □

Consider a pair R and S of posets. Each is a preset, so we have the two collections of arrows

$$\boldsymbol{Pre}[R,S] \qquad \boldsymbol{Pos}[R,S]$$

in the categories. A moment's thought shows that, as sets of functions, these two sets are the same. Technically, this shows that $\boldsymbol{Pos}$ is a FULL SUBCATEGORY of $\boldsymbol{Pre}$.

The study of monoids is the study of composition in the miniature.

The study of presets is the study of comparison in the miniature.

What should we do to study these two notions together and in the large? Category theory! In a sense every category is an amalgam of certain monoids and presets, and that is a good enough reason why we should always keep these two simple notions in mind.

From the examples we have seen so far it is easy to get the impression that certain things always happen. The next example shows that some categories can be awkward (and sometimes cantankerous).

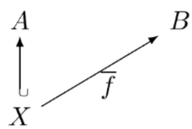
1.2.3 Example We enlarge the category $\boldsymbol{Set}$ of sets and total functions to the category $\boldsymbol{Pfn}$ of sets and partial functions. The objects of $\boldsymbol{Pfn}$ are just sets

$$A, B, C, \ldots$$

as in $\boldsymbol{Set}$. However, an arrow

$$A \xrightarrow{\ f\ } B$$

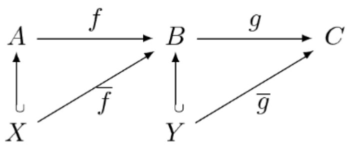
is a *partial* function from A to B. In other words, an arrow is a total function



from a subset X of the source A. (This is an example where the use of the word 'domain' for source can be confusing. The set X is the domain of definition of the partial function.) Notice that we need to distinguish between the total function $\overline{f}$ and the arrow f it determines. The notation has been chosen to emphasize that distinction.

We wish to show that these objects and arrows form a category $\boldsymbol{Pfn}$. To do that we must first produce a composition of arrows.

Consider a pair of partial functions.

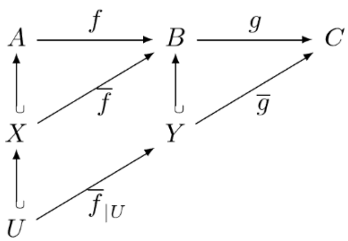


How might we compose these? We somehow want to stick $\overline{f}$ and $\overline{g}$ together, but these functions are not composition compatible.

We extract a subset $U \subseteq A$ by

$$a \in U \iff a \in X \text{ and } \overline{f}(a) \in Y$$

(for $a \in A$). Since $\overline{f}$ is defined on the whole of U we restrict $\overline{f}$ to U.



Now we do have composition compatible functions. Thus we take

$$g \circ f$$

to be that arrow (partial function) determined by

$$\overline{g \circ f} = \overline{g} \circ \overline{f}_{|U}$$

to produce a composition of arrows in ***Pfn***.

Notice here how the symbol '$\circ$' is overloaded. On the right it is the standard composition of total functions. On the left it is the defined operation on partial functions. If at first you find this confusing then write '$\bullet$' for the defined operation. Thus

$$\overline{g \bullet f} = \overline{g} \circ \overline{f}_{|U}$$

is its definition.

There is still some work to be done. For instance, we need to show that this composition of arrows is associative. That is left as an exercise. □

Once we see it the step from ***Set*** to ***Pfn*** is not so big. An arrow is still a function, but we have to take a little more care with composition. There is also a much neater way of handling ***Pfn***. Perhaps you can think about that.

We began this section by looking at the category $\boldsymbol{Mon}$ of monoids. We conclude by looking at two categories attached to each monoid.

1.2.4 Example Let R be a fixed, but arbitrary, monoid. A

left right

R-set is a set A together with an action

$$\begin{array}{ccc} R, A & \longrightarrow & A \\ r, a & \longmapsto & ra \end{array} \qquad \begin{array}{ccc} A, R & \longrightarrow & A \\ a, r & \longmapsto & ar \end{array}$$

where

$$\begin{array}{cc} s(ra) = (sr)a & (ar)s = a(rs) \\ 1a = a & a = a1 \end{array}$$

for each $a \in A$ and $r, s \in R$. Here the two definitions are given in parallel. These R-sets are the objects of two categories

$$R\text{-}\boldsymbol{Set} \qquad \boldsymbol{Set}\text{-}R$$

with left R-sets on the left and right R-sets on the right.

Given two R-sets A and B of the same handedness, a morphism

$$A \xrightarrow{\ f\ } B$$

is a function f such that

$$f(ra) = rf(a) \qquad f(ar) = f(a)r$$

for each $a \in A$ and $r \in R$. These are the arrows of the two categories. □

This may look a quite simple example but it is useful. Many aspects of category theory can be illustrated with these categories. We use them quite a lot in this book. They are also module categories in miniature. We can replace the monoid R by a ring and replace each set A by an abelian group. This gives the categories

$$R\text{-}\boldsymbol{Mod} \qquad \boldsymbol{Mod}\text{-}R$$

of left and right modules over R, respectively. These categories have quite a bit more structure, but we won't go into that too much here.

Exercises

1.2.1 The category $\boldsymbol{Pno}$ described in this exercise may look less than exciting, but it plays an important role in mathematics. (It was originally discovered by Dedekind without the category theory.)

The objects of $\boldsymbol{Pno}$ are the structures (A, α, a) where A is a set, and where $\alpha : A \longrightarrow A$ is a function, and $a \in A$ is a nominated element. Given two such structures a morphism

$$(A, \alpha, a) \xrightarrow{\ f\ } (B, \beta, b)$$

is a function $f : A \longrightarrow B$ which preserves the structure in the sense that

$$f \circ \alpha = \beta \circ f \qquad f(a) = b$$

hold.

(a) Verify that $\boldsymbol{Pno}$ is a category.

(b) Show that $(\mathbb{N}, \mathrm{succ}, 0)$ is a $\boldsymbol{Pno}$-object (where succ is the successor function).

(c) Show that for each $\boldsymbol{Pno}$-object (A, α, a) there is a unique arrow

$$(\mathbb{N}, \mathrm{succ}, 0) \longrightarrow (A, \alpha, a)$$

and describe the behaviour of the carrying function.

1.2.2 Consider pairs (A, X) where A is a set and $X \subseteq A$. For two such pairs a morphism

$$(A, X) \xrightarrow{\ f\ } (B, Y)$$

is a function $f : A \longrightarrow B$ that respects the selected subsets, that is

$$f(x) \in Y$$

for each $x \in X$. Show that such pairs and morphisms form a category $\boldsymbol{SetD}$, the category of sets with a distinguished subset.

1.2.3 Consider pairs (A, R) where A is a set and $R \subseteq A \times A$ is a binary relation on A. Show that these pairs are the objects of a category. You must find a sensible notion of morphism for such pairs.

1.2.4 A topological space $(S, \mathcal{O}S)$ is a set S furnished with a certain family $\mathcal{O}S$ of subsets of S (called the open sets of the space). This family is required to contain both $\emptyset$ and S, be closed under $\cap$ (binary intersection), and be closed under $\bigcup$ (arbitrary unions).

Consider an arrow F as above, so $F \subseteq B \times A$. For $a \in A$ and $b \in B$ we write bFa for $(b,a) \in F$. For two composible arrows

$$A \xrightarrow{\ F\ } B \xrightarrow{\ G\ } C$$

we define the composition $G \circ F$ by

$$c(G \circ F)a \iff (\exists b \in B)[cGbFa]$$

for $a \in A, b \in B$. We show that a is $G \circ F$ related to c by passing through a common element $b \in B$. It is easy to check that this composition is associative, and the equality relation on a set gives the identity arrow.

The two categories ***Set*** and ***RelA*** are connected in a certain way (which will be explained in more detail later). There is a canonical way

$$A \xrightarrow{\ f\ } B \qquad \longmapsto \qquad A \xrightarrow{\ \Gamma(f)\ } B$$

of converting a ***Set***-arrow into a ***RelA***-arrow with the same source and target. We simply take the graph of the function, that is we let

$$b\,\Gamma(f)\,a \iff b = f(a)$$

for $a \in A, b \in B$. □

The next example is important in itself, and also provides a miniature version of a central notion of category theory, that of an ADJUNCTION.

1.3.3 Example We modify the category ***Pos*** of posets, of Example 1.2.2, to produce a new category $\boldsymbol{Pos}^{\dashv}$. As with ***Pos***, the objects of $\boldsymbol{Pos}^{\dashv}$ are posets, but the arrows are different.

Given a pair S, T of posets, an adjunction from S to T is a pair of monotone maps as on the left such that the equivalence on the right

$$S \underset{g}{\overset{f}{\rightleftarrows}} T \qquad f(a) \leq b \iff a \leq g(b)$$

holds for all $a \in S$ and $b \in T$. We call

$$f \text{ the left adjoint} \qquad g \text{ the right adjoint}$$

of the pair, and sometimes write

$$f \dashv g$$

to indicate an adjunction.

Here we use the more common notation and write

$$S \underset{f_*}{\overset{f^*}{\rightleftarrows}} T$$

to indicate an adjunction $f^* \dashv f_*$. Sometimes a harpoon arrow

$$S \xrightharpoondown{f^* \dashv f_*} T$$

is used to indicate an adjunction. By convention, an adjunction points in the direction of its left component. Thus S is the source and T is the target. (You are warned that in some of the older literature this convention hadn't yet been established.)

Poset adjunctions are the arrows of $\boldsymbol{Pos}^{\dashv}$.

This gives us the object and arrows of $\boldsymbol{Pos}^{\dashv}$, but we still have some work to do before we know we have a category.

Consider a pair of adjunctions.

$$R \xrightharpoondown{f^* \dashv f_*} S \xrightharpoondown{g^* \dashv g_*} T$$

which ought to be composible. How should the composite

$$R \xrightharpoondown{(g^* \dashv g_*) \circ (f^* \dashv f_*)} T$$

be formed? The two left hand components are monotone maps that compose to give a monotone map. Similarly the two right hand components are monotone maps that compose to give a monotone map. Thus we have a pair of monotone maps

$$R \underset{f_* \circ g_*}{\overset{g^* \circ f^*}{\rightleftarrows}} T$$

going in opposite directions. We check that this is an adjunction and take that as the composite. Almost trivially, this composition is associative, and so we do obtain a category. □

It is not so surprising that any given monotone map may or may not have a left adjoint, and it may or may not have a right adjoint. It can have neither, and it can have one without the other. What is a little surprising is that it can have both adjoints where these are not the same. In fact, arbitrarily longs strings of adjoints can be produced. A simple example of this is given in Chapter 6.

Once we become familiar with categories we find that old categories can be used to produce new categories. Let's look at some examples.

1.3.4 Example Consider categories $\boldsymbol{C}$ and $\boldsymbol{D}$. To help us distinguish between these let us write

$$A, B, C \ldots \text{ for objects of } \boldsymbol{C} \qquad f, g, h \ldots \text{ for arrows of } \boldsymbol{C}$$

$$R, S, T \ldots \text{ for objects of } \boldsymbol{D} \qquad \theta, \phi, \psi \ldots \text{ for arrows of } \boldsymbol{D}$$

respectively. We form a new category, the product

$$\boldsymbol{C} \times \boldsymbol{D}$$

of $\boldsymbol{C}$ and $\boldsymbol{D}$ as follows. Each new object is an ordered pair of old objects

$$(A, R)$$

an object A from $\boldsymbol{C}$ and an object R from $\boldsymbol{D}$. A new arrow

$$(A, R) \longrightarrow (B, S)$$

is a pair of old arrows

$$A \xrightarrow{\quad f \quad} B \qquad\qquad R \xrightarrow{\quad \theta \quad} S$$

from the given categories. For composible new arrows

$$(A, R) \xrightarrow{(f, \theta)} (B, S \xrightarrow{(g, \phi)} (C, T)$$

the composite

$$(A, R) \xrightarrow{(g \circ f, \phi \circ \theta)} (C, T)$$

is formed using composition in the old categories. Almost trivially, this does give a category. □

That's not the most exciting example you have ever seen, is it? Here is a more interesting construction.

1.3.5 Example Given a category $\boldsymbol{C}$ we form a new category where the new objects are the arrows of $\boldsymbol{C}$. This is the arrow category of $\boldsymbol{C}$.

Consider the small graph

$$(\downarrow) \qquad\qquad \begin{array}{c} 0 \\ \big\downarrow \\ 1 \end{array}$$

with two nodes, here labelled 0 and 1, and with one edge. We use $(\downarrow)$ to convert $\boldsymbol{C}$ into a new category

$$\boldsymbol{C}^{\downarrow}$$

the category of $(\downarrow)$-diagrams in $\boldsymbol{C}$.

We think of $(\downarrow)$ as a TEMPLATE for diagrams in $\boldsymbol{C}$, and these diagrams are the objects of $\boldsymbol{C}^{\downarrow}$. Thus a new object is a pair of old objects

$$\begin{array}{c} A_0 \\ \big\downarrow{\scriptstyle \alpha} \\ A_1 \end{array}$$

and an old arrow between them. Given two new objects a new arrow

$$\begin{array}{c} A_0 \\ \big\downarrow{\scriptstyle \alpha} \\ A_1 \end{array} \xrightarrow{\quad f \quad} \begin{array}{c} B_0 \\ \big\downarrow{\scriptstyle \beta} \\ B_1 \end{array}$$

is a pair of old arrows

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ A_1 & \xrightarrow[f_1]{} & B_1 \end{array} \qquad f_1 \circ \alpha = \beta \circ f_0$$

such that the square commutes. Composition of new arrows is performed in the obvious way, we compose the two component old arrows. You should check that this does give a category. □

This is a simple example of a much more general construction, that of a FUNCTOR CATEGORY. We look at this once we know what a FUNCTOR is. Other simple examples of this construction are given in the exercises.

The idea of the previous example is to view *all* the arrows of the old category as the objects of the new category. Sometimes we want to do a similar thing but using only *some* old arrows.

1.3.6 Example Given a category $\boldsymbol{C}$ and an object S we form the two slice categories

$$(\boldsymbol{C} \downarrow S) \qquad\qquad (S \downarrow \boldsymbol{C})$$

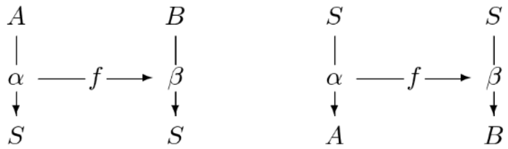
of objects

$$\text{over } S \qquad\qquad \text{under } S$$

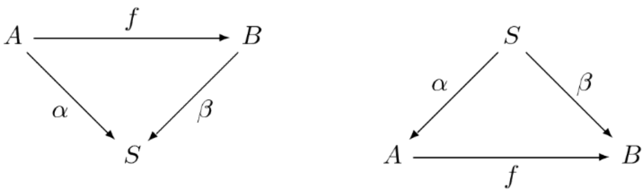
respectively. Each object of the new category is an arrow

to S from S

of $\boldsymbol{C}$. An arrow of the new category



is an arrow of $\boldsymbol{C}$



for which the indicated triangle commutes. Composition of the new arrows is obtained from composition of arrows in $\boldsymbol{C}$ □

As with Example 1.3.5 this construction is a particular case of a more general construction, that of a COMMA CATEGORY. Before we can explain that we need to understand the notion of a FUNCTOR.

Exercises

1.3.1 Consider the strictly positive integers $1, 2, 3, \ldots$ as objects. For two such integers m, n let an arrow

$$n \longrightarrow m$$

be an $m \times n$ matrix A (with real entries). Given two compatible matrices

$$n \xrightarrow{\quad B \quad} k \qquad\qquad k \xrightarrow{\quad A \quad} m$$

let the composite

$$n \xrightarrow{\quad A \circ B \quad} m$$

be the matrix product AB. Show that this gives a category.

Can you show that this example is a bit of a cheat?

1.3.7 Posets and certain adjoint pairs form another category $\boldsymbol{Pos}^{pp}$.

The objects of $\boldsymbol{Pos}^{pp}$ are again just posets. A $\boldsymbol{Pos}^{pp}$-arrow

$$A \xrightarrow[]{(f,g)} B \quad (\Rightarrow)$$

is a $\boldsymbol{Pos}^{\dashv}$-arrow

$$A \xrightharpoonup{f \dashv g} B$$

for which $g \circ f = id_A$. These arrows are sometimes called projection pairs.

Show that these projection pairs are closed under composition, and hence $\boldsymbol{Pos}^{pp}$ is a category.

You see here a useful little trick. It can be helpful to draw arrows in different, but related, categories in a different way. Thus here we have

$$\begin{array}{ll} \boldsymbol{Pos} & \longrightarrow \\ \boldsymbol{Pos}^{\dashv} & \rightharpoonup \\ \boldsymbol{Pos}^{pp} & \Longrightarrow \end{array}$$

for the three different kinds of arrows.

1.3.8 Consider the ordered sets $\mathbb{Z}$ and $\mathbb{R}$ as posets, and let

$$\mathbb{Z} \xrightarrow{\iota} \mathbb{R}$$

be the insertion.

(a) Show there are (unique) maps

$$\mathbb{R} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{Z}$$

such that

$$\mathbb{Z} \xrightharpoonup{\iota \dashv \rho} \mathbb{R} \xrightharpoonup{\lambda \dashv \iota} \mathbb{Z}$$

are adjunctions.

(b) Show also that this composite is $id_{\mathbb{Z}}$ in $\boldsymbol{Pos}^{\dashv}$ and the other composite, on $\mathbb{R}$, is idempotent.

(c) Show that $\iota \dashv \rho$ is a $\boldsymbol{Pos}^{pp}$-arrow, but $\lambda \dashv \iota$ is not.

1.3.9 For a poset S let $\mathcal{L}S$ be the poset of lower sections under inclusion. (A lower section of S is a subset $X \subseteq S$ such that

$$y \leq x \in X \Longrightarrow y \in X$$

for all $x, y \in S$.)

(a) For a monotone map

$$T \xrightarrow{\ \phi\ } S$$

between posets, show that setting $f = \phi^{\leftarrow}$ (the inverse image map) produces a monotone map

$$\mathcal{L}T \xleftarrow{\ f = \phi^{\leftarrow}\ } \mathcal{L}S$$

in the opposite direction.

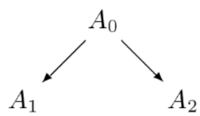
(b) Show that f has both a left adjoint and a right adjoint

$$f^{\sharp} \dashv f \dashv f_{\flat}$$

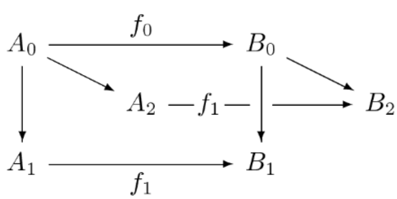
where, in general, these are different.

1.3.10 Let $\boldsymbol{C}$ be an arbitrary category. In Example 1.3.5 we used $(\downarrow)$ as a template to obtain a category $\boldsymbol{C}^{\downarrow}$ of certain diagrams from $\boldsymbol{C}$. The same idea can be used with other templates.

A wedge in a category $\boldsymbol{C}$ is a pair or arrows



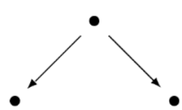
as shown. A wedge morphism



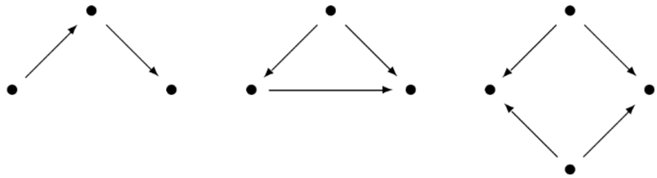
is a triple of arrows which make the two associated squares commute.

(a) Show that wedges and wedge morphisms form a category.

(b) This wedge example uses



as the template. Play around with other templates to produce other examples of categories. For example, consider each of



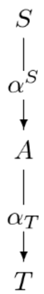
and worry about which cells are required to commute.

1.3.11 Let 1 and 2 be the 1-element set and the 2-element set, respectively. Describe the categories

$$(\boldsymbol{Set} \downarrow 1) \quad (1 \downarrow \boldsymbol{Set}) \quad (\boldsymbol{Set} \downarrow 2) \quad (2 \downarrow \boldsymbol{Set})$$

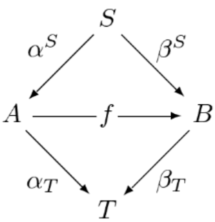
and show that you have met two of them already together with near relatives of the other two.

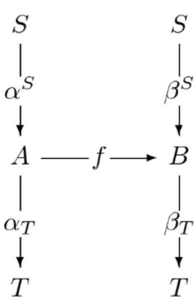
1.3.12 Given a category $\boldsymbol{C}$ and two objects S, T we form



$$(S \downarrow \boldsymbol{C} \downarrow T)$$

the butty category between S and T. Each object of the new category is an object A of $\boldsymbol{C}$ together with a pair of arrows from S and to T. An arrow of the new category is an arrow f of $\boldsymbol{C}$ to make the two triangles commute.





(a) Show that with the appropriate notion of composition this gives a category.

(b) Can you show that for an appropriate parent category $\boldsymbol{C}$ both the slice categories

$$(\boldsymbol{C} \downarrow T) \qquad\qquad (S \downarrow \boldsymbol{C})$$

are instance of the butty construction?

1.4 More complicated categories

From the examples we have seen so far you might conclude that category theory is making a bit of a fuss. It is true that objects are not just structured sets and arrows are not just functions, but the examples seem to suggest that we don't move too far from those ideas. Of course, as yet we have seen only comparatively simple examples of categories. One of the original aims of category theory was to organize and analyse what we now see as rather complicated categories. The simpler examples came along later. In this section we look at a couple of examples of the more complicated kind of category. You probably won't understand these at a first reading, but you should give it a go. You should come back to these examples as you learn more about category theory.

1.4.1 Example Let S be any partially ordered set. We describe the category $\widehat{S}$ of PRESHEAVES ON S. There is a more general notion where S is replaced by an arbitrary category, but we save that for later. We may think of $\widehat{S}$ as the category of 'sets developing over S'. At first sight the structure of $\widehat{S}$ looks quite complicated, but you will get used to it.

We think of S as a store of indexes $i, j, k, \ldots$ partially ordered

$$j \leq i$$

to form a poset.

A presheaf on S is an S-indexed family of sets

$$\mathsf{A} \qquad \big(A(i) \mid i \in S\big)$$

together with a family of connecting functions

$$\mathcal{A} \qquad A(i) \xrightarrow{A(j,i)} A(j)$$

one for each comparison $j \leq i$. Note these functions progress *down* the poset. These functions have to fit together in a coherent fashion. Thus

$$A(i,i) = id_{A(i)}$$

for each index $i \in S$, and the triangle

$$\begin{array}{ccc} A(i) & \xrightarrow{\quad A(k,i) \quad} & A(k) \\ & {\scriptstyle A(j,i)} \searrow \quad \nearrow {\scriptstyle A(k,j)} & \\ & A(j) & \end{array} \qquad\qquad A(k,j) \circ A(j,i) = A(k,i)$$

commutes for all $k \leq j \leq i$. These are the objects of $\widehat{S}$. Note the way the connecting functions are indexed.