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An Introduction to the Language of Category Theory

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Foundations

Before giving the definition of a category, we must briefly (and somewhat informally) discuss a notion from the foundations of mathematics. In category theory, one often wishes to speak of "the category of (all) sets" or "the category of (all) groups." However, it is well known that these descriptions cannot be made precise within the context of sets alone.

In particular, not all "collections" that one can define informally through the use of the English language, or even formally through the use of the language of set theory, can be considered sets without producing some well-known logical paradoxes, such as the Russell paradox of 1901 (discovered by Zermelo a year earlier). More specifically, if $\phi(x)$ is a well-formed formula of set theory, then the collection

$$X = \{\text{sets } x \mid \phi(x) \text{ is true}\}$$

cannot always be viewed as a set. For example, the family of all sets, or of all groups, cannot be considered a set. Nonetheless, it is desirable to be able to apply some of the operations of sets, such as union and cartesian product, to such families. One way to achieve this goal is through the notion of a **class**. Every set is a class and the classes that are not sets are called **proper classes**. Now we can safely speak of the *class* of all sets, or the *class* of all groups. Classes have many of the properties of sets. However, while every set is an element of another set, no class can be an element of another class. We can now state that the family X defined above is a class without apparent contradiction.

Another way to avoid the problems posed by the logical paradoxes is to use the concept of a set $\mathcal{U}$ called a **universe**. The elements of $\mathcal{U}$ are called **small sets**. Some authors refer to the *subsets* of $\mathcal{U}$ as *sets* and some use the term *classes*. In order to carry out "ordinary mathematics" within the universe $\mathcal{U}$, it is assumed to be closed under the basic operations of set theory, such as the taking of ordered pairs, power sets and unions.

These two approaches to the problem of avoiding the logical paradoxes result in essentially the same theory and so we will generally use the language of sets and classes, rather than universes.

The Definition

We can now give the definition of a category.

■ Definition

A **category** $\mathcal{C}$ *consists of the following:*

1) (**Objects**) *A class* **Obj**$(\mathcal{C})$ *whose elements are called the* **objects**. *It is customary to write* $A \in \mathcal{C}$ *in place of* $A \in \mathbf{Obj}(\mathcal{C})$.
2) (**Morphisms**) *For each (not necessarily distinct) pair of objects* $A, B \in \mathcal{C}$, *a set* $\hom_{\mathcal{C}}(A, B)$, *called the* **hom-set** *for the pair* (A, B). *The elements of* $\hom_{\mathcal{C}}(A, B)$ *are called* **morphisms**, **maps** *or* **arrows** *from* A *to* B. *If* $f \in \hom_{\mathcal{C}}(A, B)$, *we also write*

$$f\colon A \to B \quad \textit{or} \quad f_{AB}$$

The object $A = \mathrm{dom}(f)$ *is called the* **domain** *of* f *and the object* $B = \mathrm{codom}(f)$ *is called the* **codomain** *of* f.
3) *Distinct hom-sets are disjoint, that is,* $\hom_{\mathcal{C}}(A, B)$ *and* $\hom_{\mathcal{C}}(C, D)$ *are disjoint unless* $A = C$ *and* $B = D$.
4) (**Composition**) *For* $f \in \hom_{\mathcal{C}}(A, B)$ *and* $g \in \hom_{\mathcal{C}}(B, C)$ *there is a morphism* $g \circ f \in \hom_{\mathcal{C}}(A, C)$, *called the* **composition** *of* g *with* f. *Moreover, composition is associative:*

$$f \circ (g \circ h) = (f \circ g) \circ h$$

whenever the compositions are defined.
5) (**Identity morphisms**) *For each object* $A \in \mathcal{C}$ *there is a morphism* $1_A \in \hom_{\mathcal{C}}(A, A)$, *called the* **identity morphism** *for* A, *with the property that if* $f_{AB} \in \hom_{\mathcal{C}}(A, B)$ *then*

$$1_B \circ f_{AB} = f_{AB} \quad \textit{and} \quad f_{AB} \circ 1_A = f_{AB}$$

The class of all morphisms of $\mathcal{C}$ *is denoted by* **Mor**$(\mathcal{C})$. □

A variety of notations are used in the literature for hom-sets, including

$$(A, B), \quad [A, B], \quad \mathcal{C}(A, B) \quad \text{and} \quad \mathrm{Mor}(A, B)$$

(We will drop the subscript $\mathcal{C}$ in $\hom_{\mathcal{C}}$ when no confusion will arise.)

We should mention that not all authors require property 3) in the definition of a category. Also, some authors permit the hom-sets to be classes. In this case, the categories for which the hom-classes are sets is called a **locally small category**. Thus, all of our categories are locally small. A category $\mathcal{C}$ for which both the class **Obj**$(\mathcal{C})$ and the class **Mor**$(\mathcal{C})$ are sets is called a **small category**. Otherwise, $\mathcal{C}$ is called a **large category**.

Two arrows belonging to the same hom-set $\hom(A, B)$ are said to be **parallel**. We use the phrase "f is a morphism **leaving** A" to mean that the domain of f is A and "f is a morphism **entering** B" to mean that the codomain of f is B.

When we speak of a composition $g \circ f$, it is with the tacit understanding that the morphisms are **compatible**, that is, $\mathrm{dom}(g) = \mathrm{codom}(f)$.

The concept of a category is *very general*. Here are some examples of categories. In most cases, composition is the "obvious" one. We suggest that you just skim this list of examples at this point. If you are not familiar with some of the concepts in these example (such as smooth manifolds), not to worry. The purpose of this list is to give you a general idea of the wide range of categories in mathematics.

Example 1

The Category **Set** of Sets

Obj is the class of all sets.

hom(A, B) is the set of all functions from A to B.

The Category **Mon** of Monoids

Obj is the class of all monoids.

hom(A, B) is the set of all monoid homomorphisms from A to B.

The Category **Grp** of Groups

Obj is the class of all groups.

hom(A, B) is the set of all group homomorphisms from A to B.

The Category **AbGrp** of Abelian Groups

Obj is the class of all abelian groups.

hom(A, B) is the set of all group homomorphisms from A to B.

The Category $\mathbf{Mod}_R$ of R-modules, where R is a ring

Obj is the class of all R-modules.

hom(A, B) is the set of all R-maps from A to B.

The Category $\mathbf{Vect}_F$ of Vector Spaces over a Field F

Obj is the class of all vector spaces over F.

hom(A, B) is the set of all linear transformations from A to B.

The Category **Rng** of Rings

Obj is the class of all rings (with unit).

hom(A, B) is the set of all ring homomorphisms from A to B.

The Category **CRng** of Commutative Rings with identity

Obj is the class of all commutative rings with identity.

hom(A, B) is the set of all ring homomorphisms from A to B.

The Category **Field** of Fields

Obj is the class of all fields.

hom(A, B) is the set of all ring embeddings from A to B.

The Category **Poset** of Partially Ordered Sets

Obj is the class of all partially ordered sets.

hom(A, B) is the set of all **monotone** functions from A to B, that is, functions $f:P\to Q$ satisfying

$$p \le q \quad \Rightarrow \quad f(p) \le f(q)$$

The Category **Rel** of relations

Obj is the class of all sets.

hom(A, B) is the set of all binary relations from A to B, that is, subsets of the cartesian product $A \times B$.

The Category **Top** of Topological Spaces

Obj is the class of all topological spaces.

hom(A, B) is the set of all continuous functions from A to B.

The Category **SmoothMan** of Manifolds with Smooth Maps

Obj is the class of all manifolds.

hom(A, B) is the set of all smooth maps from A to B. □

▪ Example 2

The category of *all* categories does not exist, on foundational grounds. The well-known Russell paradox shows that the set of all sets does not exist and an analogous argument has been constructed to show that the category of all categories does not exist. However, the argument is a bit involved and is not really in the spirit of this introductory book, so we will not go into the details. On the other hand, the class $\mathcal{S}$ of all *small* categories does form the objects of another category, whose morphisms are called *functors*, to be defined a bit later in the chapter. □

Here are some slightly more unusual categories.

▪ Example 3

Let F be a field. The category $\mathbf{Matr}_F$ of matrices over F has objects equal to the set $\mathbb{Z}^+$ of positive integers. For $m, n \in \mathbb{Z}^+$, the hom-set hom(m, n) is the set of all $n \times m$ matrices over F, composition being matrix multiplication. Why do we reverse the roles of m and n? Well, if $M \in$ hom(m, n) and $N \in$ hom(n, k), then M has size $n \times m$ and N has size $k \times n$ and so the product NM makes sense and has size $k \times m$, that is, it belongs to hom(m, k), as required. Incidentally, this is a case in which the category is named after its morphisms, rather than its objects. □

▪ Example 4

A single monoid M defines a category with a single object M, where each element is a morphism. We define the composition $a \circ b$ to be the product ab. This example applies to other algebraic structures, such as groups. All that is required is that there be an identity element and that the operation be associative. □

▪ Example 5

Let $(P, \leq)$ be a partially ordered set. The objects of the category **Poset**$(P, \leq)$ are the elements of P. Also, hom(a, b) is empty unless $a \leq b$, in which case hom(a, b) contains a single element, denoted by ab. Note that the hom-sets specify the relation $\leq$ on P. As to composition, there is really only one choice: If ab: $a \to b$ and bc: $b \to c$ then it follows that $a \leq b \leq c$ and so $a \leq c$, which implies that hom$(a, c) \neq \emptyset$. Thus, we set $bc \circ ab = ac$. The hom-set hom(a, a) contains only the identity morphism for the object a.

As a specific example, you may recall that each positive natural number $n \in \mathbb{N}$ is defined to be the set of all natural numbers that preceed it:

$$n = \{0, 1, \ldots, n-1\}$$

and the natural number 0 is defined to be the empty set. Thus, natural numbers are ordered by membership, that is, $m < n$ if and only if $m \in n$ and so n is the set of all natural numbers *less than* n. Each natural number n defines a category whose objects are its elements and whose morphisms describe this order relation. The category n is sometimes denoted by bold face $\boldsymbol{n}$.□

▪ Example 6

A category for which there is *at most one* morphism between every pair of (not necessarily distinct) objects is called a **preordered category** (some authors use the term **thin category**). If $\mathcal{C}$ is a thin category, then we can use the *existence* of a morphism to define a binary relation on the objects of $\mathcal{C}$, namely, $A \preceq B$ if there exists a morphism from A to B. It is clear that this relation is reflexive and transitive. Such relations are called **preorders**. (The term *preorder* is used in a different sense in combinatorics.)

Conversely, any preordered class $(P, \preceq)$ is a category, where the objects are the elements of P and there is a morphism f_{AB} from A to B if and only if $A \preceq B$ (and there are no other morphisms). Reflexivity provides the identity morphisms and transitivity provides the composition.

More generally, if $\mathcal{C}$ is any category, then we can use the *existence* of a morphism to define a preorder on the objects of $\mathcal{C}$, namely, $A \preceq B$ if there is at least one morphism from A to B.□

▪ Example 7

Consider a deductive logic system, such as the propositional calculus. We can define two different categories as follows. In both cases, the well-formed formulas (wffs) of the system are the objects of the category. In one case, there is one morphism from the wff α to the wff β if and only if we can deduce β given α. In the other case, we define a morphism from α to β to be a *specific deduction* of β from α, that is, a specific ordered list of wffs starting with α and ending with β for which each wff in the list is either an axiom of the system or is deducible from the previous wffs in the list using the rules of deduction of the system. □

The Categorical Perspective

The notion of a category is extremely general. However, the definition is *precisely* what is needed to set the correct stage for the following two key tenets of mathematics:

1) Morphisms (e.g. linear transformations, group homomorphisms, monotone maps) play an essentially equal role alongside the mathematical structures that they morph (e.g. vector spaces, groups, partially ordered sets).
2) Many mathematical notions are best described in terms of morphisms between structures rather than in terms of the individual elements of these structures.

In order to implement the second tenet, one must grow accustomed to the idea of focusing on the appropriate *maps* between mathematical structures and not on the *elements* of these structures. For example, as we will see in due course, such important notions as a basis for a vector space, a direct product of vector spaces, the field of fractions of an integral domain and the quotient of a group by a normal subgroup can be described using maps rather than elements. In fact, many of the most important properties of these notions follow from their morphism-based descriptions.

Note also that one of the consequences of the second tenet is that important mathematical notions tend to be defined *only up to isomorphism*, rather than uniquely.

An immediate example seems in order, even though it may take some time (and further reading) to place in proper perspective.

▪ Example 8

Let V and W be vector spaces over a field F. The external direct product of V and W is usually defined in elementary linear algebra books as the set of ordered pairs

$$V \times W = \{(v, w) \,|\, v \in V, w \in W\}$$

with componentwise operations:

$$(v, w) + (v', w') = (v + v', w + w')$$

and

$$r(v, w) = (rv, rw)$$

for $r \in F$. One then defines the **projection maps**

$$\rho_1 \colon V \times W \to V \quad \text{and} \quad \rho_2 \colon V \times W \to W$$

by

$$\rho_1(v, w) = v \quad \text{and} \quad \rho_2(v, w) = w$$

However, the importance of these projection maps is not always made clear, so let us do this now.

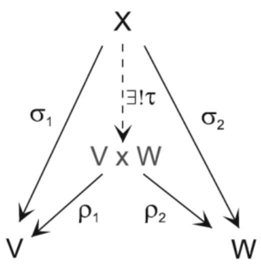

Figure 1

As shown in Figure 1, the ordered triple $(V \times W, \rho_V, \rho_W)$ has the following **universal property**: Given any vector space X over F and any "projection-like" pair of linear transformations

$$\sigma_1 \colon X \to V \quad \text{and} \quad \sigma_2 \colon X \to W$$

from X to V and W, respectively, there is a *unique* linear transformation $\tau\colon X \to V \times W$ for which

$$\rho_1 \circ \tau = \sigma_1 \quad \text{and} \quad \rho_2 \circ \tau = \sigma_2$$

Indeed, these two equations uniquely determine $\tau(x)$ for any $x \in X$ because

$$\tau(x) = (\rho_1(\tau(x)), \rho_2(\tau(x))) = (\sigma_1(x), \sigma_2(x))$$

It remains only to show that τ is linear, which follows easily from the fact that σ_1 and σ_2 are linear. Now, the categorical perspective is that this universal property is the essence of the direct product, at least up to isomorphism. In fact, it is not hard to show that if an ordered triple

$$(U, \lambda_1\colon U \to V, \lambda_2\colon U \to W)$$

has the universal property described above, that is, if for any vector space X over F and any pair of linear transformations

$$\sigma_1\colon X \to V \quad \text{and} \quad \sigma_2\colon X \to W$$

there is a *unique* linear transformation $\tau\colon X \to U$ for which

$$\lambda_1 \circ \tau = \sigma_1 \quad \text{and} \quad \lambda_2 \circ \tau = \sigma_2$$

then U and $V \times W$ are isomorphic as vector spaces. Indeed, in some more advanced treatments of linear algebra, the direct product of vector spaces is *defined* as *any* triple that satisfies this universal property. Note that, using this definition, *the direct product is defined only up to isomorphism*.

If this example seems to be a bit overwhelming now, don't be discouraged. It can take a while to get accustomed to the categorical way of thinking. It might help to redraw Figure 1 a few times without looking at the book. □

Functors

If we are going to live by the two main tenets of category theory described above, we should discuss morphisms between categories! Structure-preserving maps between categories are called *functors*. At this time, however, there is much to say about categories as individual entities, so we will briefly describe functors now and return to them in detail in a later chapter.

The unabridged dictionary defines the term *functor*, from the New Latin *functus* (past participle of *fungi*: to perform) as "something that performs a function or operation." The term *functor* was apparently first used by the German philosopher Rudolf Carnap (1891–1970) to represent a special type of function sign. In category theory, the term *functor* was introduced by Samuel Eilenberg and Saunders Mac Lane in their paper *Natural Isomorphisms in Group Theory* [8].

Since the structure of a category consists of *both* its objects and its morphisms, a functor should map objects to objects and morphisms to morphisms. This requires two different maps. Also, there are two versions of functors: *covariant* and *contravariant*.

▪ Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ *is a pair of functions (as is customary, we use the same symbol F for both functions):*

1) *The* **object part** *of the functor*

$$F\colon \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{D})$$

maps objects in $\mathcal{C}$ to objects in $\mathcal{D}$

2) *The* **arrow part**

$$F\colon \mathbf{Mor}(\mathcal{C}) \to \mathbf{Mor}(\mathcal{D})$$

maps morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$ as follows:

a) *For a* **covariant functor,**

$$F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$$

for all $A, B \in \mathcal{C}$, that is, F maps a morphism $f\colon A \to B$ in $\mathcal{C}$ to a morphism $Ff\colon FA \to FB$ in $\mathcal{D}$.

b) *For a* **contravariant functor,**

$$F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FB, FA)$$

for all $A, B \in \mathcal{C}$, that is, F maps a morphism $f\colon A \to B$ in $\mathcal{C}$ to a morphism $Ff\colon FB \to FA$ in $\mathcal{D}$. (Note the reversal of direction).

We will refer to the restriction of F to $\hom_{\mathcal{C}}(A, B)$ *as a* **local arrow part** *of F.*

3) *Identity and composition are preserved, that is,*

$$F1_A = 1_{FA}$$

and for a covariant functor,

$$F(g \circ f) = Fg \circ Ff$$

and for a contravariant functor,

$$F(g \circ f) = Ff \circ Fg$$

whenever all compositions are defined. □

As is customary, we use the same symbol F for both the object part and the arrow part of a functor. We will also use a double arrow notation for functors. Thus, the expression $F\colon \mathcal{C} \Rightarrow \mathcal{D}$

implies that $\mathcal{C}$ and $\mathcal{D}$ are categories and is read "F is a functor from $\mathcal{C}$ to $\mathcal{D}$." (For readability's sake in figures, we use a thick arrow to denote functors.)

A functor $F\colon \mathcal{C} \Rightarrow \mathcal{C}$ from $\mathcal{C}$ to itself is referred to as a **functor on** $\mathcal{C}$. A functor $F\colon \mathcal{C} \Rightarrow$ **Set** is called a **set valued functor**. We say that functors $F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$ with the same domain and the same codomain are **parallel** and functors of the form $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{C}$ are **antiparallel**.

The term *covariant* appears to have been first used in 1853 by James Joseph Sylvester (who was quite fond of coining new terms) as follows: "Covariant, a function which stands in the same relation to the primitive function from which it is derived as any of its linear transforms do to a similarly derived transform of its primitive." In plainer terms, an operation is covariant if it varies in a way that preserves some related structure or operation. In the present context, a covariant functor preserves the direction of arrows and a *contravariant* functor reverses the direction of arrows.

One way to view the concept of a functor is to think of a (covariant) functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ as a mapping of one-arrow diagrams in $\mathcal{C}$,

$$A \xrightarrow{f} B$$

to one-arrow diagrams in $\mathcal{D}$,

$$FA \xrightarrow{Ff} FB$$

with the property that "identity loops" and "triangles" are preserved, as shown in Figure 2.

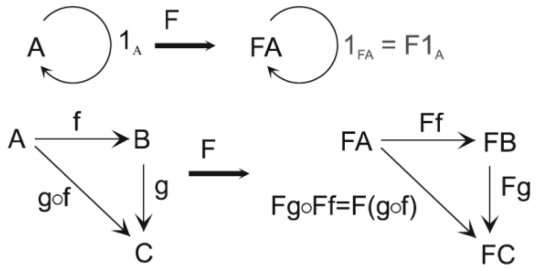

Figure 2

A similar statement holds for contravariant functors.

Composition of Functors

Functors can be composed in the "obvious" way. Specifically, if $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ and $G\colon \mathcal{D} \Rightarrow \mathcal{E}$ are functors, then $G \circ F\colon \mathcal{C} \Rightarrow \mathcal{E}$ is defined by

$$(G \circ F)(A) = G(FA)$$

for $A \in \mathcal{C}$ and

$$(G \circ F)(f) = G(Ff)$$

for $f \in \hom_{\mathcal{C}}(A, B)$. We will often write the composition $G \circ F$ as GF.

Special Types of Functors

Definition

Let $F: \mathcal{C} \Rightarrow \mathcal{D}$ be a functor.

1) *F is* **full** *if all of its local arrow parts are surjective.*
2) *F is* **faithful** *if all of its local arrow parts are injective.*
3) *F is* **fully faithful** *(i.e., full and faithful) if all of its local arrow parts are bijective.*
4) *F is an* **embedding** *of $\mathcal{C}$ in $\mathcal{D}$ if it is fully faithful and the object part of F is injective.* □

We should note that the term *embedding*, as applied to functors, is defined differently by different authors. Some authors define an embedding simply as a full and faithful functor. Other authors define an embedding to be a faithful functor whose object part is injective. We have adopted the strongest definition, since it applies directly to the important Yoneda lemma (coming later in the book).

Note that a faithful functor $F: \mathcal{C} \Rightarrow \mathcal{D}$ need not be an embedding, for it can send two morphisms from *different* hom sets to the same morphism in $\mathcal{D}$. For instance, if $FA = FA'$ and $FB = FB'$ then it may happen that

$$Ff_{AB} = Fg_{A'B'}$$

which does not violate the condition of faithfulness. Also, a full functor need not be surjective on $\mathbf{Mor}(\mathcal{C})$.

A Couple of Examples

Here are a couple of examples of functors. We will give more examples in the next chapter.

Example 9

The **power set functor** $\wp$: **Set** $\Rightarrow$ **Set** sends a set A to its power set $\wp(A)$ and sends each set function $f: A \to B$ to the induced function $f: \wp(A) \to \wp(B)$ that sends X to fX. (It is customary to use the same notation for the function and its induced version.) It is easy to see that this defines a faithful but not full covariant functor.

Similarly, the **contravariant power set functor** F: **Set** $\Rightarrow$ **Set** sends a set A to its power set $\wp(A)$ and a set function $f: A \to B$ to the induced *inverse* function $f^{-1}: \wp(B) \to \wp(A)$ that sends $X \subseteq B$ to $f^{-1}X \subseteq A$. The fact that F is contravariant follows from the well known fact that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$ □

Example 10

The following situation is quite common. Let $\mathcal{C}$ be a category. Suppose that $\mathcal{D}$ is another category with the property that every object in $\mathcal{C}$ is an object in $\mathcal{D}$ and every morphism $f: A \to B$ in $\mathcal{C}$ is a morphism $f: A \to B$ in $\mathcal{D}$.

For instance, every object in **Grp** is also an object in **Set**: we simply ignore the group operation. Also, every group homomorphism is a set function. Similarly, every ring can be thought of as an abelian group by ignoring the ring multiplication and every ring map can be thought of as a group homomorphism.

We can then define a functor $F: \mathcal{C} \Rightarrow \mathcal{D}$ by sending an object $A \in \mathcal{C}$ to itself, thought of as an object in $\mathcal{D}$ and a morphism $f: A \to B$ in $\mathcal{C}$ to itself, thought of as a morphism in $\mathcal{D}$.

Functors such as these that "forget" some structure are called **forgetful functors**. In general, these functors are faithful but not full. For example, distinct group homomorphisms $f, g: A \to B$ are also distinct as functions, but not every set function between groups is a group homomorphism.

For any category $\mathcal{C}$ whose objects are sets, perhaps with additional structure and whose morphisms are set functions, also perhaps with additional structure, the "most forgetful" functor is the one that forgets all structure and thinks of an object simply as a set and a morphism simply as a set function. This functor is called the **underlying-set functor** $U: \mathcal{C} \Rightarrow$ **Set** on $\mathcal{C}$. □

The Category of All Small Categories

As mentioned earlier, it is tempting to define the category of all categories, but this does not exist on foundational grounds. On the other hand, the category **SmCat** of all *small* categories does exist. Its objects are the small categories and its morphisms are the covariant functors between categories. Of course, **SmCat** is a *large* category.

Concrete Categories

Despite the two main tenets of category theory described earlier, most common categories do have the property that their objects are sets whose elements are "important" and whose morphisms are ordinary set functions on these elements, usually with some additional structure (such as being group homomorphisms or linear transformations). This leads to the following definition.

Definition

A category $\mathcal{C}$ is **concrete** *if there is a faithful functor $F: \mathcal{C} \Rightarrow$* **Set**. *Put more colloquially, $\mathcal{C}$ is concrete if the following hold:*

1) *Each object A of $\mathcal{C}$ can be thought of as a set FA (which is often A itself). Note that distinct objects may be thought of as the same set.*
2) *Each distinct morphism $f: A \to B$ in $\mathcal{C}$ can be thought of as a distinct set function $Ff: FA \to FB$ (which is often f itself).*
3) *The identity 1_A morphism can be thought of as the identity set function $F1: FA \to FA$ and the composition $f \circ g$ in $\mathcal{C}$ can be thought of as the composition $Ff \circ Fg$ of the corresponding set functions.* □

Categories that are not concrete are called **abstract categories**. Many concrete categories have the property that FA is A and Ff is f. This applies, for example, to most of the previously defined categories, such as **Grp**, **Rng**, **Vect** and **Poset**. The category **Rel** is an example of a category that is not concrete.

In fact, the subject of which categories are concrete and which are abstract can be rather involved and we will not go into it in this introductory book, except to remark that all small categories are concrete, a fact which follows from Yoneda's lemma, to be proved later in the book.

Subcategories

Subcategories are defined as follows.

■ Definition

Let $\mathcal{C}$ be a category. A **subcategory** *$\mathcal{D}$ of $\mathcal{C}$ is a category for which consists of a nonempty subclass* $\mathbf{Obj}(\mathcal{D})$ *of* $\mathbf{Obj}(\mathcal{C})$ *and a nonempty subclass* $\mathbf{Mor}(\mathcal{D})$ *of* $\mathbf{Mor}(\mathcal{C})$ *with the following properties:*

1) $\mathbf{Obj}(\mathcal{D}) \subseteq \mathbf{Obj}(\mathcal{C})$, *as classes.*
2) *For every $A, B \in \mathcal{D}$,*

$$\hom_{\mathcal{D}}(A, B) \subseteq \hom_{\mathcal{C}}(A, B)$$

and the identity map 1_A in $\mathcal{D}$ is the identity map 1_A in $\mathcal{C}$, that is,

$$(1_A)_{\mathcal{D}} = (1A)_{\mathcal{C}}$$

3) *Composition in $\mathcal{D}$ is the composition from $\mathcal{C}$, that is, if*

$$f: A \to B \quad \textit{and} \quad g: B \to C$$

are morphisms in $\mathcal{D}$, then the $\mathcal{C}$-composite $g \circ f$ is the $\mathcal{D}$-composite $g \circ f$.

If equality holds in part 2) for all $A, B \in \mathcal{D}$, then the subcategory $\mathcal{D}$ is **full**. □

■ Example 11

The category **AbGrp** of abelian groups is a full subcategory of the category **Grp**, since the definition of group morphism is independent of whether or not the groups involved are abelian. Put another way, a group homomorphism between abelian groups is just a group homomorphism.

However, the category **AbGrp** of abelian groups is a *nonfull* subcategory of the category **Rng** of rings, since not all additive group homomorphisms $f: R \to S$ between rings are ring maps. Similarly, the category of differential manifolds with smooth maps is a nonfull subcategory of the category **Top**, since not all continuous maps are smooth. □

The Image of a Functor

Note that if $F\colon \mathcal{C} \Rightarrow \mathcal{D}$, then the image $F\mathcal{C}$ of $\mathcal{C}$ under the functor F, that is, the set

$$\{FA \mid A \in \mathcal{C}\}$$

of objects and the set

$$\{Ff \mid f \in \hom_{\mathcal{C}}(A,B)\}$$

of morphisms need *not* form a subcategory of $\mathcal{D}$. The problem is illustrated in Figure 3.

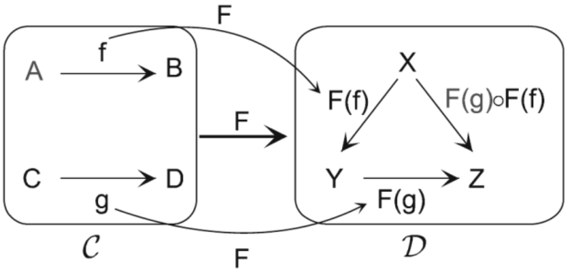

Figure 3

In this case, the composition $F(g) \circ F(f)$ is not in the image $F\mathcal{C}$. The only way that this can happen is if the composition $g \circ f$ does not exist because f and g are not compatible for composition. For if $g \circ f$ exists, then

$$F(g) \circ F(f) = F(g \circ f) \in F\mathcal{C}$$

Note that in this example, the object part of F is not injective, since $F(A) = F(C) = X$. This is no coincidence.

■ Theorem 12

If the object part of a functor $F\colon \mathcal{C} \Rightarrow \mathcal{D}$ is injective, then $F\mathcal{C}$ is a subcategory of $\mathcal{D}$, under the composition inherited from $\mathcal{D}$.

■ Proof

The only real issue is whether the $\mathcal{D}$-composite $Fg \circ Ff$ of two morphisms in $F\mathcal{C}$, when it exists, is also in $F\mathcal{C}$. But this composite exists if and only if

$$Ff\colon FA \to FB \quad \text{and} \quad Fg\colon FB \to FC$$

and so the injectivity of F on objects implies that

$$f\colon A \to B \quad \text{and} \quad g\colon B \to C$$

Hence, $g \circ f$ exists in $\mathcal{C}$ and so

$$F(g) \circ F(f) = F(g \circ f) \in F\mathcal{C}$$

Diagrams

The purpose of a *diagram* is to describe a portion of a category $\mathcal{C}$. By "portion" we mean one or more objects of $\mathcal{C}$ along with *some* of the arrows connecting these objects.

Informally, we can say that a diagram in $\mathcal{C}$ consists of a class of points (or nodes) in the plane, each labeled with an object of $\mathcal{C}$ and for each pair (A, B) of nodes a collection of arcs from the node labeled A to the node labeled B, each of which is labeled with a morphism from A to B.

The simplest way to form a diagram is with a functor—any functor.

Definition

*Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A $\mathcal{J}$-***diagram** *(or just* **diagram***) in $\mathcal{C}$ with* **index category** *$\mathcal{J}$ is a functor $J\colon \mathcal{J} \Rightarrow \mathcal{C}$.* □

Since the image $J(\mathcal{J})$ is indexed by the objects and morphisms of the index category $\mathcal{J}$, the objects in $\mathcal{J}$ are often denoted by lower case letters such as m, n, p, q. Figure 4 illustrates this definition.

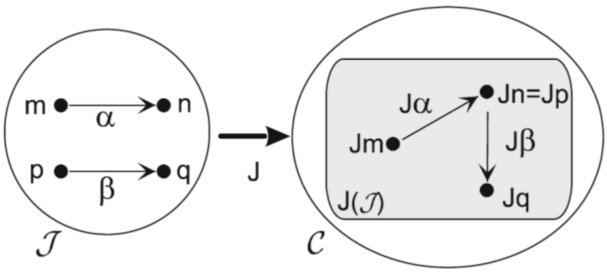

Figure 4

Observe that, as in this example, the image $J(\mathcal{J})$ need not be a subcategory of $\mathcal{C}$. In this example, J sends n and p to the same object in $\mathcal{C}$ but since α and β are not compatible for composition, the image of J need not contain the composition $J\beta \circ J\alpha$. Thus, the image of a functor simply contains *some* objects of $\mathcal{C}$ as well as *some* morphisms between these objects.

It is worth emphasizing that *any* functor $F\colon \mathcal{J} \Rightarrow \mathcal{C}$ is a diagram and so we have introduced nothing new other than a point of view and some concomitant terminology.

Definition

Let $\mathcal{C}$ be a category.

1) *A morphism $f: A \to B$ is* **right-invertible** *if there is a morphism $f_R: B \to A$, called a* **right inverse** *of f, for which*

$$f \circ f_R = 1_B$$

2) *A morphism $f: A \to B$ is* **left-invertible** *if there is a morphism $f_L: A \to B$, called a* **left inverse** *of f, for which*

$$f_L \circ f = 1_A$$

3) *A morphism $f: A \to B$ is* **invertible** *or an* **isomorphism** *if there is a morphism $f^{-1}: B \to A$, called the* (**two-sided**) **inverse** *of f, for which*

$$f^{-1} \circ f = 1_A \quad \textit{and} \quad f \circ f^{-1} = 1_B$$

In this case, the objects A and B are **isomorphic** *and we write $A \approx B$.* □

Note that the *categorical* term *isomorphism* says nothing about injectivity or surjectivity, for it must be defined in terms of morphisms only!

In fact, this leads to an interesting observation. For categories whose objects are sets and whose morphisms are set functions, we can define an isomorphism in two ways:

1) (Categorical definition) An isomorphism is a morphism with a two-sided inverse.
2) (Non categorical definition) An isomorphism is a bijective morphism.

In most cases of algebraic structures, such as groups, rings or vector spaces, these definitions are equivalent. However, there are cases where only the categorical definition is correct.

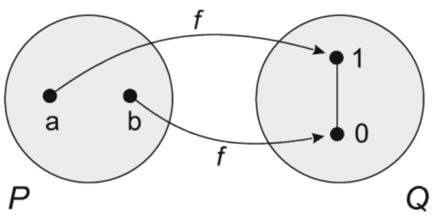

Figure 9

For example, as shown in Figure 9, let $P = \{a, b\}$ be a poset in which a and b are incomparable and let $Q = \{0, 1\}$ be the poset with $0 < 1$. Let $f: P \to Q$ be defined by $fa = 0$ and $fb = 1$. Then f is a bijective morphism of posets, that is, a bijective monotone map. However, it is not an isomorphism of posets!

Proof of the following familiar facts about inverses is left to the reader.

3) If a category $\mathcal{C}$ has Π^{op}, then it also has p^{op} (abbreviated $\Pi^{\text{op}} \Rightarrow p^{\text{op}}$).

The fact that

$$\Pi \Rightarrow p \quad \text{iff} \quad \Pi^{\text{op}} \Rightarrow p^{\text{op}}$$

is called the **principle of duality** for categories. Note that if Π is **self-dual**, that is, if $\Pi = \Pi^{\text{op}}$, then the principle of duality becomes

$$\Pi \Rightarrow p \quad \text{iff} \quad \Pi \Rightarrow p^{\text{op}}$$

Of course, the empty set of properties is self-dual. Moreover, the condition $\emptyset \Rightarrow p$ means that all categories possess property p. Hence, we deduce that

> if all categories possess a property p, then all categories also possess any dual property p^{op}

For example, all categories possess the property that initial objects (when they exist) are isomorphic. Hence, the principle of duality implies that all terminal objects (when they exist) are isomorphic.

New Categories From Old Categories

There are many ways to define new categories from old categories. One of the simplest ways is to take the Cartesian product of the objects in two categories. There are also several important ways to turn the morphisms of one category into the objects of another category.

The Product of Categories

If $\mathcal{B}$ and $\mathcal{C}$ are categories, we may form the **product category** $\mathcal{B} \times \mathcal{C}$, in the expected way. Namely, the objects of $\mathcal{B} \times \mathcal{C}$ are the ordered pairs (B, C), where B is an object of $\mathcal{B}$ and C is an object of $\mathcal{C}$. A morphism from $B \times C$ to $B' \times C'$ is a pair (f, g) of morphisms, where $f: B \to B'$ and $g: C \to C'$. Composition is done componentwise:

$$(f, g) \circ (h, k) = (f \circ h, g \circ k)$$

A functor $F: \mathcal{A} \times \mathcal{B} \Rightarrow \mathcal{C}$ from a product category $\mathcal{A} \times \mathcal{B}$ to another category is called a **bifunctor**.

The Category of Arrows

Given a category $\mathcal{C}$, we can form the **category of arrows** $\mathcal{C}^{\to}$ of $\mathcal{C}$ by taking the objects of $\mathcal{C}^{\to}$ to be the morphisms of $\mathcal{C}$.

Index of Symbols

S. Roman, *An Introduction to the Language of Category Theory,* Compact Textbooks in Mathematics, DOI 10.1007/978-3-319-41917-6,

$\Rightarrow$ Functor
$\leftrightarrow$ Bijection
$\dot{\rightarrow}$ Natural transformation
$\dot{\leftrightarrow}$ Natural bijection
$\approx$ Isomorphism
$\dot{\approx}$ Natural isomorphism
$\dashv$ Left adjoint
$\vdash$ Right adjoint
1_A Identity morphism
$(A \to \mathcal{C})$ Comma category of arrows leaving A
$(\mathcal{C} \to A)$ Comma category of arrows entering A
$(A \to G)$ Comma category of arrows leaving A entering G
$(G \to A)$ Comma category of arrows entering A leaving G
$\mathcal{A}(v, w)$ Set of arcs between v and w in a digraph
$\mathcal{B} \times \mathcal{C}$ Product category
$\mathcal{C}^{\mathrm{op}}$ Opposite category
$\mathcal{C}, \mathcal{D}, \mathcal{E}$ Categories
$\mathcal{C}^{\to}$ Category of arrows
$\mathbf{Cone}_{\mathcal{C}}(F)$ *or* $\mathbf{Cone}_{\mathcal{C}}(\mathbb{D})$ Category of cones
$\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, etc. Diagrams
$\mathbb{D}(F\colon \mathcal{J} \Rightarrow \mathcal{C})$ Diagram in $\mathcal{C}$ with functor F and index category $\mathcal{J}$
$\mathbf{dia}_{\mathcal{J}}(\mathcal{C})$ Category of diagrams
$\mathcal{D}^{\mathcal{C}}$ Functor category
$f^{\leftarrow}$ Follow by f
$f^{\to}$ Preceed by f
$\hom_{\mathcal{C}}(A, B)$ Hom-set
$\hom_{\mathcal{C}}(A, -)$ Hom-set category
$\hom_{\mathcal{C}}(A, \cdot\,)$ Hom-set functor
$\mathcal{K}, \mathcal{L}$ Cones and cocones
$\mathbf{Mor}(\mathcal{C})$ Morphisms of $\mathcal{C}$
$\mathbf{Obj}(\mathcal{C})$ Objects of $\mathcal{C}$
$\mathcal{V}(D)$ Vertex class of a digraph

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