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Applied Mathematics: Body and Soul [VOLUME 2]

Integrals and Geometry in $\mathbb{R}^n$

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Integrals and Geometry in $\mathbb{R}^n$

$$u(x_N) - u(x_0) = \int_{x_0}^{x_N} u'(x)\,dx$$

$$\approx \sum_{j=1}^{N} u'(x_{j-1})(x_j - x_{j-1})$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

A familiar example of this problem occurs when $f(x)$ is a velocity and x represents time so that the solution $u(x)$ of $u'(x) = f(x)$, represents the distance traveled by a body moving with instantaneous velocity $u'(x) = f(x)$. As the examples above show, we can solve this problem in simple cases, for example when the velocity $f(x)$ is equal to a constant v for all x and therefore the distance traveled during a time x is $u(x) = vx$. If we travel with constant velocity 4 miles/hour for two hours, then the distance traveled is 8 miles. We reach these 8 miles by accumulating distance foot-by-foot, which would be very apparent if we are walking!

An important observation is that the differential equation (27.1) alone is not sufficient information to determine the solution $u(x)$. Consider the interpretation when f represents velocity and u distance traveled by a body. If we want to know the position of the body, we need to know only the distance traveled but also the starting position. In general, a solution $u(x)$ to (27.1) is determined only up to a constant, because the derivative of a constant is zero. If $u'(x) = f(x)$, then also $(u(x) + c)' = f(x)$ for any constant c. For example, both $u(x) = x^2$ and $u(x) = x^2 + 1$ satisfy $u'(x) = 2x$. Graphically, we can see that there are many "parallel" functions that have the same slope at every point. The constant may be specified by specifying the value of the function $u(x)$ at some point. For example, the solution of $u'(x) = x$ is $u(x) = x^2 + c$ with c a constant, and specifying $u(0) = 1$ gives that $c = 1$.

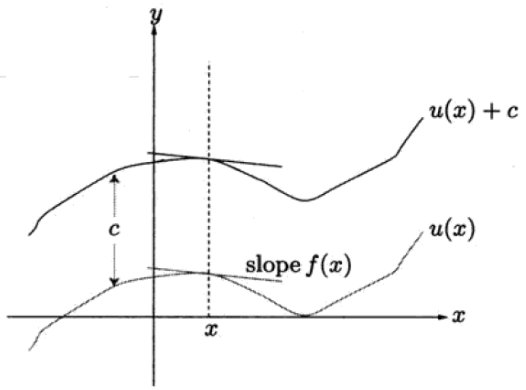

Fig. 27.2. Two functions that have the same slope at every point

More generally, we now formulate our basic problem as follows: Given $f : [a, b] \to \mathbb{R}$ and u_a, find $u : [a, b] \to \mathbb{R}$ such that

$$\begin{cases} u'(x) = f(x) & \text{for } a < x \leq b, \\ u(a) = u_a, \end{cases} \tag{27.2}$$

where u_a is a given *initial value*. The problem (27.2) is the simplest example of an *initial value problem* involving a differential equation and an initial value. The terminology naturally couples to situations in which x represents time and $u(a) = u_a$ amounts to specifying $u(x)$ at the initial time $x = a$. Note that we often keep the initial value terminology even if x represents a quantity different from time, and in case x represents a space coordinate we may alternatively refer to (27.2) as a *boundary value problem* with now $u(a) = u_a$ representing a given *boundary value.*

We shall now prove that the initial value problem (27.2) has a unique solution $u(x)$ if the given function $f(x)$ is Lipschitz continuous on $[a, b]$. This is the *Fundamental Theorem of Calculus*, which stated in words says that a Lipschitz continuous function has a (unique) primitive function. Leibniz referred to the Fundamental Theorem as the "inverse method of tangents" because he thought of the problem as trying to find a curve $y = u(x)$ given the slope $u'(x)$ of its tangent at every point x.

We shall give a constructive proof of the Fundamental Theorem, which not only proves that $u : I \to \mathbb{R}$ exists, but also gives a way to compute $u(x)$ for any given $x \in [a, b]$ to any desired accuracy by computing a sum involving values of $f(x)$. Thus the version of the Fundamental Theorem we prove contains two results: (i) the existence of a primitive function and (ii) a way to compute a primitive function. Of course, (i) is really a consequence of (ii) since if we know how to compute a primitive function, we also know that it exists. These results are analogous to defining $\sqrt{2}$ by constructing a Cauchy sequence of approximate solutions of the equation $x^2 = 2$ by the Bisection algorithm. In the proof of the Fundamental Theorem we shall also construct a Cauchy sequence of approximate solutions of the differential equation (27.2) and show that the limit of the sequence is an exact solution of (27.2).

We shall express the solution $u(x)$ of (27.2) given by the Fundamental Theorem in terms of the data $f(x)$ and u_a as follows:

$$u(x) = \int_a^x f(y)\, dy + u_a \quad \text{for } a \leq x \leq b, \tag{27.3}$$

where we refer to

$$\int_a^x f(y)\, dy$$

as the *integral* of f over the interval $[a, x]$, a and x as the *lower and upper limits of integration* respectively, $f(y)$ as the *integrand* and y the *integration variable.* This notation was introduced on October 29 1675 by Leibniz, who thought of the integral sign $\int$ as representing "summation" and dy as the "increment" in the variable y. The notation of Leibniz is part of the big success of Calculus in science and education, and (like a good cover of a record) it gives a direct visual expression of the mathematical content of the integral in very suggestive form that indicates both the construction of

27.5 A Quick Overview of the Progress So Far

Any function obtained by linear combinations, products, quotients and compositions of functions of the form x^r with rational power $r \neq 0$ and $x > 0$, can be differentiated analytically. If $u(x)$ is such a function, we thus obtain an analytical formula for $u'(x)$. If we now choose $f(x) = u'(x)$, then of course $u(x)$ satisfies the differential equation $u'(x) = f(x)$, so that we can write recalling Leibniz notation:

$$u(x) = \int_0^x f(y)\,dy + u(0) \quad \text{for } x \geq 0,$$

which apparently states that the function $u(x)$ is a primitive function of its derivative $f(x) = u'(x)$ (assuming that $u(x)$ is defined for all $x \geq 0$ so that no denominator vanishes for $x \geq 0$).

We give an example: Since $D(1+x^3)^{\frac{1}{3}} = (1+x^3)^{-\frac{2}{3}}x^2$ for $x \in \mathbb{R}$, we can write

$$(1+x^3)^{\frac{1}{3}} = \int_0^x \frac{y^2}{(1+y^3)^{\frac{2}{3}}}\,dy + 1 \quad \text{for } x \in \mathbb{R}.$$

In other words, we know primitive functions $u(x)$ satisfying the differential equation $u'(x) = f(x)$ for $x \in I$, for any function $f(x)$, which itself is a derivative of some function $v(x)$ so that $f(x) = v'(x)$ for $x \in I$. The relation between $u(x)$ and $v(x)$ is then

$$u(x) = v(x) + c \quad \text{for } x \in I,$$

for some constant c.

On the other hand, if $f(x)$ is an arbitrary function of another from, then we may not be able to produce an analytical formula for the corresponding primitive function $u(x)$ very easily or not at all. The Fundamental Theorem now tells us how to compute a primitive function of an arbitrary Lipschitz continuous function $f(x)$. We shall see that in particular, the function $f(x) = x^{-1}$ has a primitive function for $x > 0$ which is the famous *logarithm function* $\log(x)$. The Fundamental Theorem therefore gives in particular a constructive procedure for computing $\log(x)$ for $x > 0$.

27.6 A "Very Quick Proof" of the Fundamental Theorem

We shall now enter into the proof of the Fundamental Theorem. It is a good idea at this point to review the Chapter *A very short course in Calculus*. We shall give a sequence of successively more complete versions of the proof of the Fundamental Theorem with increasing precision and generality in each step.

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