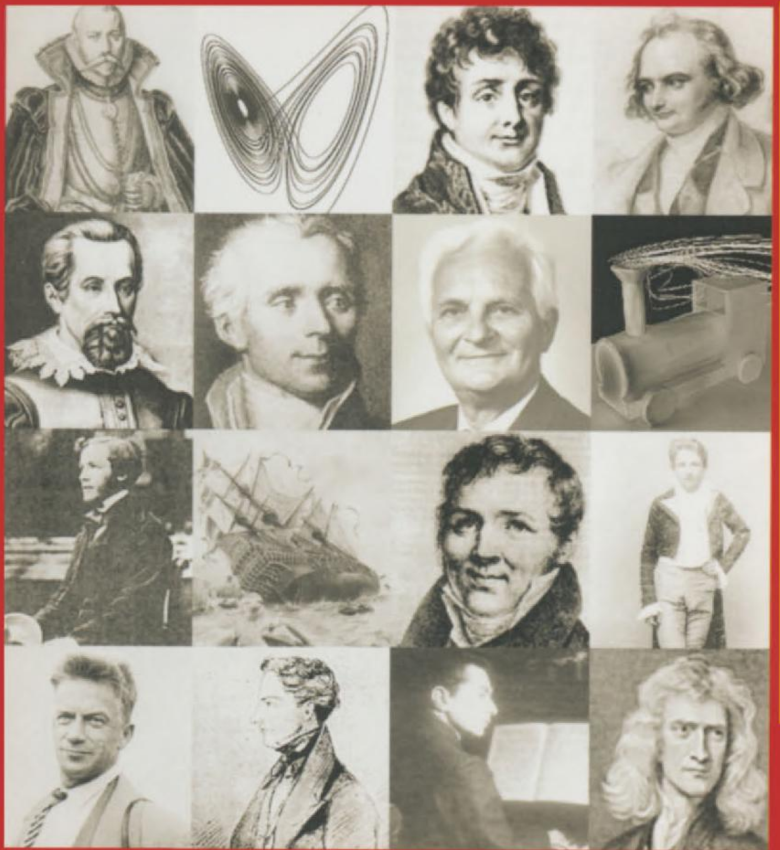

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Applied Mathematics: Body and Soul [VOLUME 3]

Calculus in Several Dimensions

Springer

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[VOLUME 3]

Calculus in Several Dimensions

Springer

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Calculus in Several Dimensions

$$\Delta u = \nabla \cdot \nabla u$$

$$\frac{\partial u}{\partial t} - \Delta u = f$$

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f$$

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + Vu$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u = f, \quad \nabla \cdot u = 0$$

$$\nabla \times H = J, \quad \nabla \cdot B = 0, \quad B = \mu H.$$

54
Vector-Valued Functions of Several Real Variables

> Auch die Chemiker müssen sich allmählich an den Gedanken gewöhnen, dass ihnen die theoretische Chemie ohne die Beherrschung der Elemente der höheren Analysis ein Buch mit sieben Siegeln blieben wird. Ein Differential- oder Integralzeichen muss aufhören, für den Chemiker eine unverständliche Hieroglyphe zu sein, ... wenn er sich nicht der Gefahr aussetzen will, für die Entwicklung der theoretischen Chemie jedes Verständnis zu verlieren. (H. Jahn, Grundriss der Elektrochemie, 1895)

54.1 Introduction

We now turn to the extension of the basic concepts of real-valued functions of one real variable, such as Lipschitz continuity and differentiability, to vector-valued functions of several variables. We have carefully prepared the material so that this extension will be as natural and smooth as possible. We shall see that the proofs of the basic theorems like the Chain rule, the Mean Value theorem, Taylor's theorem, the Contraction Mapping theorem and the Inverse Function theorem, extend almost word by word to the more complicated situation of vector valued functions of several real variables.

We consider functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that are vector valued in the sense that the value $f(x) = (f_1(x), \ldots, f_m(x))$ is a vector in $\mathbb{R}^m$ with components $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \ldots, m$, where with $f_i(x) = f_i(x_1, \ldots, x_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. As usual, we view $x = (x_1, \ldots, x_n)$ as a n-column vector and $f(x) = (f_1(x), \ldots, f_m(x))$ as a m-column vector.

As particular examples of vector-valued functions, we first consider *curves*, which are functions $g : \mathbb{R} \to \mathbb{R}^n$, and *surfaces*, which are functions $g : \mathbb{R}^2 \to \mathbb{R}^n$. We then discuss composite functions $f \circ g : \mathbb{R} \to \mathbb{R}^m$, where $g : \mathbb{R} \to \mathbb{R}^n$ is a curve and $f : \mathbb{R}^n \to \mathbb{R}^m$, with $f \circ g$ again being a curve. We recall that $f \circ g(t) = f(g(t))$.

The inputs to the functions reside in the n dimensional vector space $\mathbb{R}^n$ and it is worthwhile to consider the properties of $\mathbb{R}^n$. Of particular importance is the notion of Cauchy sequence and convergence for sequences $\{x^{(j)}\}_{j=1}^{\infty}$ of vectors $x^{(j)} = (x_1^{(j)}, \dots, x_n^{(j)}) \in \mathbb{R}^n$ with coordinates $x_k^{(j)}$, $k = 1, \dots, n$. We say that the sequence $\{x^{(j)}\}_{j=1}^{\infty}$ is a *Cauchy sequence* if for all $\epsilon > 0$, there is a natural number N so that

$$\|x^{(i)} - x^{(j)}\| \leq \epsilon \quad \text{for } i, j > N.$$

Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$, that is, $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$. Sometimes, it is convenient to work with the norms $\|x\|_1 = \sum_{i=1}^n |x_i|$ or $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$. We say that the sequence $\{x^{(j)}\}_{j=1}^{\infty}$ of vectors in $\mathbb{R}^n$ *converges* to $x \in \mathbb{R}^n$ if for all $\epsilon > 0$, there is a natural number N so that

$$\|x - x^{(i)}\| \leq \epsilon \quad \text{for } i > N.$$

It is easy to show that a convergent sequence is a Cauchy sequence and conversely that a Cauchy sequence converges. We obtain these results applying the corresponding results for sequences in $\mathbb{R}$ to each of the coordinates of the vectors in $\mathbb{R}^n$.

Example 54.1. The sequence $\{x^{(i)}\}_{i=1}^{\infty}$ in $\mathbb{R}^2$, $x^{(i)} = (1 - i^{-2}, \exp(-i))$, converges to $(1, 0)$.

54.2 Curves in $\mathbb{R}^n$

A function $g : I \to \mathbb{R}^n$, where $I = [a, b]$ is an interval of real numbers, is a *curve* in $\mathbb{R}^n$, see Fig. 54.1. If we use t as the independent variable ranging over I, then we say that the curve $g(t)$ is *parametrized* by the variable t. We also refer to the set of points $\Gamma = \{g(t) \in \mathbb{R}^n : t \in I\}$ as the curve Γ parameterized by the function $g : I \to \mathbb{R}^n$.

Example 54.2. The simplest example of a curve is a straight line. The function $g : \mathbb{R} \to \mathbb{R}^2$ given by

$$g(t) = \bar{x} + tz,$$

where $z \in \mathbb{R}^2$ and $\bar{x} \in \mathbb{R}^2$, is a straight line in $\mathbb{R}^2$ through the point $\bar{x}$ with direction z, see Fig. 54.2.

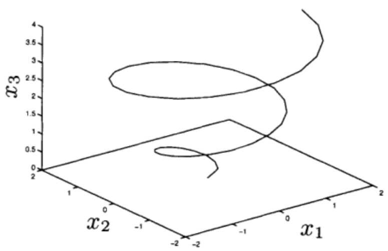

Fig. 54.1. The curve $g : [0, 4] \to \mathbb{R}^3$ with $g(t) = \big(t^{1/2}\cos(\pi t),\, t^{1/2}\sin(\pi t), t\big)$

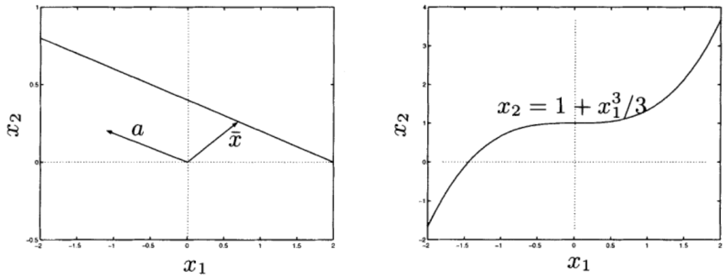

Fig. 54.2. On the *left*: the curve $g(t) = \bar{x} + ta$. On the *right*: a curve $g(t) = (t, f(t))$

Example 54.3. Let $f : [a, b] \to \mathbb{R}$ be given, and define $g : [a, b] \to \mathbb{R}^2$ by $g(t) = (g_1(t), g_2(t)) = (t, f(t))$. This curve is simply the graph of the function $f : [a, b] \to \mathbb{R}$, see Fig. 54.2.

54.3 Different Parameterizations of a Curve

It is possible to use different parametrizations for the set of points forming a curve. If $h : [c, d] \to [a, b]$ is a one-to-one mapping, then the composite function $f = g \circ h : [c, d] \to \mathbb{R}^2$ is a *reparameterization* of the curve $\{g(t) : t \in [a, b]\}$ given by $g : [a, b] \to \mathbb{R}^2$.

Example 54.4. The function $f : [0, \infty) \to \mathbb{R}^3$ given by

$$f(\tau) = (\tau\cos(\pi\tau^2), \tau\sin(\pi\tau^2), \tau^2),$$

is a reparameterization of the curve $g : [0, \infty) \to \mathbb{R}^3$ given by

$$g(t) = (\sqrt{t}\cos(\pi t), \sqrt{t}\sin(\pi t), t),$$

obtained setting $t = h(\tau) = \tau^2$. We have $f = g \circ h$.

54.4 Surfaces in $\mathbb{R}^n$, $n \geq 3$

A function $g : Q \to \mathbb{R}^n$, where $n \geq 3$ and Q is a subdomain of $\mathbb{R}^2$, may be viewed to be a *surface* S in $\mathbb{R}^n$, see Fig. 54.3. We write $g = g(y)$ with $y = (y_1, y_2) \in Q$ and say that S is parameterized by $y \in Q$. We may also identify the surface S with the set of points $S = \{g(y) \in \mathbb{R}^n : y \in Q\}$, and reparameterize S by $f = g \circ h : \tilde{Q} \to \mathbb{R}^n$ if $h : \tilde{Q} \to Q$ is a one-to-one mapping of a domain $\tilde{Q}$ in $\mathbb{R}^2$ onto Q.

Example 54.5. The simplest example of a surface $g : \mathbb{R}^2 \to \mathbb{R}^3$ is a plane in $\mathbb{R}^3$ given by

$$g(y) = g(y_1, y_2) = \bar{x} + y_1 b_1 + y_2 b_2, \quad y \in \mathbb{R}^2,$$

where $\bar{x}$, b_1, $b_2 \in \mathbb{R}^3$.

Example 54.6. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given, and define $g : [0,1] \times [0,1] \to \mathbb{R}^3$ by $g(y_1, y_2) = (y_1, y_2, f(y_1, y_2))$. This is a surface, which is the graph of $f : [0,1] \times [0,1] \to \mathbb{R}$. We also refer to this surface briefly as the surface given by the function $x_3 = f(x_1, x_2)$ with $(x_1, x_2) \in [0,1] \times [0,1]$.

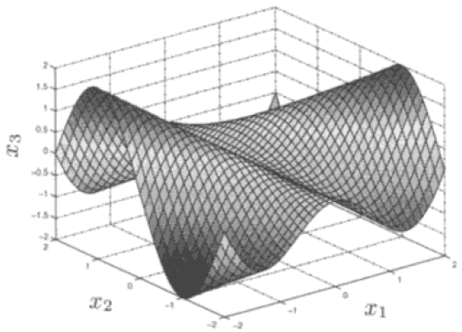

Fig. 54.3. The surface $s(y_1, y_2) = \big(y_1, y_2, y_1 \sin\big((y_1 + y_2)\pi/2\big)\big)$ with $-1 \leq y_1, y_2 \leq 1$, or briefly the surface $x_3 = x_1 \sin\big((x_1 + x_2)\pi/2\big)$ with $-1 \leq x_1, x_2 \leq 1$

54.5 Lipschitz Continuity

We say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on $\mathbb{R}^n$ if there is a constant L such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n. \tag{54.1}$$

We take $\mu = 1000$ and solve on the interval $[0, 10]$ with initial condition $u^0 = (2, 0)$. The time step sequence behaves as desired with only a small portion of the time spent on taking small damping steps. The cost is now $\alpha \approx 140$ and the cost reduction factor is $\alpha/\alpha_0 \approx 1/75$.

Chapter 57 Problems

57.1. Compute the stability factors $S_d(T)$ and $S_c(T)$ for the linear scalar IVP $\dot{u}(t) = -\lambda(t)u(t)$ for $t > 0$, $u(0) = u^0$, where $\lambda(t)$ depends on time t and (a) $\lambda(t) \geq 0$, (b) $\lambda(t) < 0$.

57.2. Compute $S_d(T)$ and $S_c(T)$ for the linear 2×2 system $\dot{u}_1 = u_2, \quad \dot{u}_2 = -u_1$ for $t > 0$, $u(0) = u^0$.

57.3. Implement adaptive IVP-solvers based on dG(0) and cG(1) and apply the solvers to different problems.

57.4. Show that the a posteriori error estimate for cG(1) may be written on the form $\|e(T)\| \leq S_c(T) \max_{0\leq t\leq T} \|k(t)(f(U(t)) - \bar{f}(U(t)))\| + S_d(T)\|e(0)\|$, where $\bar{f}(U(t))$ is the mean-value of $f(U(t))$ over each time interval.

57.5. Show that choosing in the dual problem $\varphi(T) = e_i$ gives control of error component $e_i(T)$.

57.6. Develop explicit versions of dG(0) and cG(1) based on fixed point iteration at each time step. Show that with diagonal scaling such an explicit method may work very well for some stiff problems.

58
Lorenz and the Essence of Chaos*

> I am convinced that chaos, along with its many associated concepts – strange attractors, basin boundaries, period-doubling bifurcations and the like – can readily be understood and relished by readers who have no special mathematical or other scientific background...
> (E. Lorenz, in Foreword to *The Essence of Chaos*)

58.1 Introduction

On December 29, 1972, the meteorologist Edward Lorenz presented in a session on the Global Atmospheric Research Program at the 139th meeting of the American Association for the Advancement of Science in Washington D.C., a talk with the title *Predictability: Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?* The talk by Lorenz with its "Butterfly effect" rocketed to fame a decade later during the development of "Chaos Theory" that became a fashion in mathematics and physics during the 80s, with the pretention of explaining a variety of phenomena from turbulent fluid flow to collapsing stock markets sharing qualities of *unpredictability.* A decade earlier, "Catastrophe Theory" played a similar role, while today very few remember this intriguing subject. Of course, unpredictability or "chaos" is a phenomenon that has long been familiar to mankind. The word "chaos" comes from early Greek cosmology and signifies the complete lack of order of the Universe before the creation of Gaea and Eros (Earth and Desire).

Lorenz' question is connected to the obvious difficulty of making reasonably reliable predictions of the daily weather over longer time than a week. A weather forecast is made by numerically solving an IVP modeling the evolution of the atmosphere, including variables such as temperature, wind speed and pressure. There are many sources of errors in a weather forecast made this way: errors in the initial value, modeling errors and numerical errors, and it seems that these errors are magnified at a rate that limits the predictions, depending on the scale from a few hours in very local models to weeks in global circulation models.

Lorenz' Butterfly analogy indicates that in certain dynamical systems, very small causes may have large effects after some time. We have already met such a problem in the form of a pendulum being released starting from the unstable top position: depending on the initial perturbation the position of the pendulum will be vastly different after some time (one side or the other). In meteorology, this corresponds to a situation where the weatherman can't say if a certain low pressure will take this way or that way, and thus can't be sure if it will rain in Göteborg tomorrow or not. In his book, Lorenz gives other examples of unstable systems such as a pinball machine, where very small changes in the action of the player can change the outcome of the game completely. Of course there are many other examples from real life of "small" causes having large effects, from soccer games to the assassination of Archduke Francis Ferdinand by the Serb nationalist Gavrilo Princip in Sarajevo on June 28, 1914, initiating the First World War.

58.2 The Lorenz System

Lorenz formulated an IVP of the form $\dot{u} = f(u)$ with $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$f(u) = \left(-10u_1 + 10u_2, 28u_1 - u_2 - u_1u_3, -\frac{8}{3}u_3 + u_1u_2\right),$$

which is the famous *Lorenz system.* Lorenz found that the solution of this system is very sensitive to perturbations. The system has some vague connection to a very simple model for fluid flow and has been given the role of explaining properties of fluid motion, such as turbulence. This was not Lorenz' original idea, who just wanted to make a connection to the apparent unpredictability and supposed sensitivity to perturbations of common meteorological models. If the seemingly very harmless and innocent Lorenz system could have unpredictable solutions, then there should be no surprise that also the weather could be unpredictable.

More precisely, Lorenz found that two solutions of the Lorenz system with very close initial data will stay close for some time but will eventually move apart completely. The Lorenz system is therefore very difficult to solve accurately using a numerical method over times longer than say 30 units. The numerical solution will stay close the the exact solution for some time, but will eventually move apart significantly. Of course there are many IVP:s sharing this property of instability. Even the simple pendulum has this property if the pendulum reaches the top position with small velocity. It is thus remarkable that the Lorenz system seemed to present some kind of surprise to the scientific world. But it did, and it has become quite popular to explain all sorts of phenomena, from turbulence to politics, by referring to the "strange attractor" supposedly being displayed in plots of solutions of the Lorenz system.

The Lorenz system in component form reads:

$$\begin{cases} \dot{u_1} = -10u_1 + 10u_2, \\ \dot{u_2} = 28u_1 - u_2 - u_1u_3, \\ \dot{u_3} = -\frac{8}{3}u_3 + u_1u_2, \\ u_1(0) = u_{01},\ u_2(0) = u_{02},\ u_3(0) = u_{03}, \end{cases} \tag{58.1}$$

and u_0 is a given initial condition. The system (58.1) has three equilibrium points $\bar{u}$ with $f(\bar{u}) = 0$: $\bar{u} = (0,0,0)$ and $\bar{u} = (\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$. The equilibrium point $\bar{u} = (0,0,0)$ is unstable with the corresponding Jacobian $f'(\bar{u})$ having one positive (unstable) eigenvalue and two negative (stable) eigenvalues. The equilibrium points $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$ are slightly unstable with the corresponding Jacobians having one negative (stable) eigenvalue and two eigenvalues with very small positive real part (slightly unstable) and also an imaginary part. More precisely, the eigenvalues at the two non-zero equilibrium points are $\lambda_1 \approx -13.9$ and $\lambda_{2,3} \approx .0939 \pm 10.1i$.

In Fig. 58.1, we present two views of a solution $u(t)$ that starts at $u(0) = (1,0,0)$ computed to time 30 with an error tolerance of $TOL = 0.5$ using an adaptive IVP-solver of the form presented in Chapter *Adaptive IVP-solvers*. We can think of $u(t) = (x(t), y(t), z(t))$ as the position at time t of a particle that moves according to the equation $\dot{u} = f(u)$. In Fig. 58.1 we thus plot the trajectory or path followed by the particle as the particle moves with increasing time. The plotted trajectory is typical: the particle is kicked away from the unstable point $(0,0,0)$ and moves towards one of the non-zero equilibrium points. It then slowly orbits away from that point and at some time decides to cross over towards the other non-zero equilibrium point, again slowly orbiting away from that point and coming back again, orbiting out, crossing over, and so on. This pattern of some orbits around one non-zero equilibrium point followed by a transition to the other non-zero equilibrium point is repeated with a seemingly random number of revolutions around each equilibrium point.

As noted by Lorenz, a close inspection of the trajectory in Fig. 58.1 reveals quite a bit of structure in the behavior of the solution. From the path of the trajectory, it seems that, roughly speaking, there are two flat "lobes" in which the orbits around the non-zero equilibrium points are

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