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APPLIED MATHEMATICS

A Very Short Introduction

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Preface

Before I speak, I have something important to say.

–Groucho (attributed)

When meeting new people at a party, it is customary, out of politeness, to ask what one does. When it happens, I usually mumble that I am a doctor, but not the real kind, just the academic kind. Then, if further questioned, I admit that I am mathematician, but not the real kind, just the applied kind. Then, the dreadful question comes: what is applied mathematics? In desperation, I use one of the usual witty platitudes ('It is like mathematics but useful and more fun', 'We are the social kind, we look at other people's shoes', 'Applied mathematics is to pure mathematics, what pop music is to classical music'). After the awkward pause that sums up most of my human interactions, I look for the closest exit, convinced that further contact would inevitably deepen my existentialist crisis. As I walk out I ask myself if I could really make an honest statement about my own field of study that has also become my way of life. Why is applied mathematics so different from scientific disciplines and so clearly unlike pure mathematics? How could I possibly explain the constant excitement and fun that it brings to my intellectual life?

The decision to write a *Very Short Introduction* for applied mathematics is an attempt to answer this single question: what is applied mathematics? Rather than giving an encyclopaedic description, my aim here is to give a feeling for the problems that applied mathematicians face every day and how they shape their views of the world. In most cases, I use historical perspectives to tell the story of how certain scientific or mathematical problems develop into modern mathematical theories and how these theories are still active fields of research with outstanding challenges.

Unavoidably, I introduce a few equations. It is always a danger but I do not apologize for it. You can hardly expect to open a book about French Literature and not expect to find a few French words. Equations are the language of mathematics. In the proper context, equations sum up concisely simple self-evident truths. The reader not familiar with such expressions should not unduly worry and can safely skip the technical parts. Equations are included as their absence would make them mythical quantities and their invocation without explicit mention would border on esotericism.

When I started writing this book, I still hoped that my sons could be convinced to study applied mathematics. How could they possibly resist the pressure of a perfect argument so nicely illustrated by so many examples? I certainly managed to convince myself that applied mathematics was indeed the queen of all sciences. However, it became clear that they are not likely to follow in my footsteps and that forcing them to read my writings isa form of cruel and unusual punishment. Yet, I have not given up hope that other mathematically inclined readers may learn about the topic and be charmed by its endless possibilities. There is great beauty in mathematics and there is great beauty in the world around us. Applied mathematics brings the two together in a way that it is not always beautiful but that is always interesting and exciting.

Reading playlist

Pure mathematics is often associated with classical music for its beauty and construction. When I work, I usually listen to classical music, but when I write I enjoy something with a little more energy and fun. Here are a few suggestions that are naturally associated with this book:

‘Should I stay or should I go?’ (The Clash)

‘What’s so funny ‘bout peace, love & understanding?’ (Elvis Costello)

‘Do you want to know a secret?’ (The Beatles)

‘Do you believe in magic?’ (The lovin’ spoonful)

‘Do you know the way to San Jose?’ (Dionne Warwick)

‘What’s the frequency, Kenneth?’ (R.E.M.)

‘Can you picture that?’ (Dr Teeth and the Electric Mayhem)

‘War, (what is it good for?)’ (Edwin Starr)

‘Where are we going?’ (Marvin Gaye)

‘Can you please crawl out your window?’ (Bob Dylan)

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Acknowledgements

I mostly blame Dyrol Lumbard for the existence of this book. It never crossed my mind to write it and I had no particular desire to do it until he twisted my arm and told me that I might actually enjoy doing it. I wish I could say he was wrong. Looking back, I am glad he pushed me out of my comfort zone.

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Chapter 1

What's so funny 'bout applied mathematics? Modelling, theory, and methods

Please accept my resignation. I don't want to belong to any club that will accept me as a member.

–Groucho

The modern world of mathematics is divided into different categories and if you are so lucky as to meet real-life mathematicians and engage them in a conversation, they will typically tell you that they are either mathematicians or applied mathematicians. You have probably heard of mathematics, but what is applied mathematics? A quick look on the Internet will give you conflicting definitions. It will also reveal that applied mathematics has found its place in modern academia. As such it is recognized by international scientific societies, journals, and the usual conferences. What is so special about applied mathematics? How is it different from mathematics, or any other scientific discipline?

Mathematics

Let us start with mathematics itself. Whereas philosophers still ponder the best definition, most scientists and mathematicians agree that modern mathematics is an intellectual discipline whose aim is to study idealized objects and their relationships, based on formal logic. Mathematics stands apart from scientific disciplines because it is not restricted by reality. It proceeds solely through logic and is only restricted by our imagination. Indeed, once structures and operations have been defined in a formal setting, the possibilities are endless. You can think of it as a game with very precise rules. Once the rules are laid out, the game of proving or disproving a statement proceeds.

For example, mathematicians have enjoyed numbers for millennia. Take, for instance, the natural numbers (0,1,2, …) and the familiar multiplication operation (×). If we take two numbers p and q together, we obtain a third one as $n = p \times q$. A simple question is then to do the reverse operation: given a number n can we find two numbers p and q such that $n = p \times q$? The simple answer is: of course! Take $p = 1$ and $q = n$. If this is the only possible way that a natural number n larger than 1 can be written as a product of two numbers, then n is called *a prime number*. Mathematicians love prime numbers and their wonderful, and oftentimes, surprising properties. We can now try to prove or disprove statements about these numbers. Let us start with simple ones. We can prove that *there exist prime numbers* by showing that the natural numbers 2, 3, and 5 have all the required properties to be prime numbers. We can disprove the naive statement that *all odd numbers are prime* by showing that $9 = 3 \times 3$. A more interesting statement is that *there are infinitely many prime numbers*. This was first investigated *c.*300 BC by Euclid who showed that new larger prime numbers can always be constructed from the list of all known prime numbers up to a certain value. As we construct new prime numbers the list of prime numbers increases indefinitely. Prime numbers have beautiful properties and play a central role in number theory and pure mathematics. Mathematicians are still trying to establish simple relationships between them. For instance, most mathematicians believe *there are infinitely many pairs of prime numbers that differ by 2*, the so-called *twin prime conjecture* (a *conjecture* is a statement believed to be true but still unconfirmed). For example, (5,7), (11,13), and (18369287,18369289) are all pairs of primes separated by 2, and many more such pairs are known. The burning question is: are there infinitely many such pairs? Mathematicians do believe that it is the case but demonstrating this seemingly simple property is so difficult that it has not yet been proved or disproved. However, at the time of writing, a recent breakthrough has taken place. It was established that there exist infinitely many pairs of prime numbers that differ by 246. This result shook the mathematical community and the subject is now a hot topic of modern mathematics.

Through centuries of formalization and generalization, mathematics has evolved into a unified field with clear rules. The rules have been systematically codified by the introduction of formal ideas such as the notions of definition, axiom, proposition, lemma, theorem, and conjecture. These, in turn, are the guidelines through which a stated truth may be systematically verified, step by step, given enough time and patience. Nowadays, mathematics presents itself as a well-organized discipline with well-defined sub-disciplines, some of which may even have familiar names. For instance, number theory, algebra, and geometry are all accepted branches of mathematics. Mathematics is sometimes referred to as pure mathematics to further reinforce its ethereal quality. Pure mathematics has reigned supreme in the pantheon of human intellectual constructs for more than 2,000 years. As the Greek philosopher Aristotle said: ‘The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.’

A Study in Contrast

I wish I could stand on the shoulders of 2,000 years of history and intellectual achievement, and affirm with pride that I am a mathematician in the tradition of the ancient Greeks. Unfortunately, the situation is not so pristine when it comes to mathematics’ oxymoronic sibling: applied mathematics. Applied mathematics is an intellectual bastard, born from an illicit relationship between the grand noble discipline of mathematics and its plebeian neighbour, the natural sciences; it is the illegitimate offspring of pure rationality and empirical wisdom. This misbegotten hybrid belongs in neither world. It lives in limbo, between the two, in a mathematical and scientific purgatory. Due to its unfortunate heritage, it is difficult to find a universal definition of applied mathematics, even among its practitioners and forebears. Richard Courant, who founded one of the first centres of applied mathematics, the Courant Institute at New York University, once said:

> Applied mathematics is not a definable scientific field but a human

attitude. The attitude of the applied scientist is directed towards finding clear cut answers which can stand the test of empirical observations.

The distinction between pure and applied mathematics is a relatively new concept. It will come as no surprise to the reader that mathematics' origins, not unlike society's nobility, can be traced to modest origins. Indeed, most branches of modern mathematics arose from earthly and human concerns. Arithmetic and numbers were developed for the purpose of trade and tax collection. Many ideas of geometry originated in problems related to measuring distances and making maps. The field of analysis is connected to physical or engineering problems related to mundane concerns such as the design of pendulum clocks, pipe flows, steam engines, or the construction of bridges. The great mathematicians of the past like Descartes, Newton, Euler, Gauss, Kovalevskaya, Maxwell, and Kelvin are all pure and applied mathematicians. Indeed, their work has been of equal importance to mathematics, physics, and engineering. The distinction between sciences and mathematics is a modern construct born from our eagerness to draw borders around disciplines and place ourselves into well-defined boxes. Before the 20th century, scientists or mathematicians were known simply as *natural philosophers*.

Natural philosophy is the general study of mathematics, nature, and the physical universe. It promotes a unified view of the sciences through mathematics, and the development of new mathematics through science that is still dear to many modern applied mathematicians.

Rather than engage in futile epistemological discussions, I will take a pragmatic approach. I will argue that applied mathematics includes the modelling of natural phenomena and human endeavours, the study of mathematical ideas originating from these models, and the systematic development of theoretical and computational tools to probe models, handle data, and gain insight into any problem that has been properly quantified (with special emphasis on *properly* and *quantified*, as there are many abuses of mathematics in areas where problems are neither quantified nor properly defined).

Applied mathematics is best characterized by three intertwined areas: modelling, theory, and methods.

Modelling refers to the intellectual steps that start with a question, phenomenon, set of data, or any process that attracts our curiosity; then, through suitable assumptions, identifies the key elements (the *variables*) that can be described mathematically, and derives the relations that these variables satisfy (the *equations*). In the best situations, modelling provides us with a well-defined mathematical problem from which one can hope to make predictions, extract information, or simply gain insight.

Theory is the conceptual framework that provides a systematic way to endow data and equations with a clear meaning based on fundamental or phenomenological principles. It includes mathematics itself as its central organizing language and logical framework as well as the theory of statistics that provides precise meaning to data. The formulation of mathematical models also relies on scientific and engineering theories that have been developed over past centuries.

Methods are the tools necessary to extract useful information from equations or data. It includes both the theoretical tools that help us solve equations, as well as the algorithms and computational techniques that are used to solve equations and manipulate data.

Any work in applied mathematics fits in one of these three categories or combines them judiciously. To illustrate these different aspects, let us briefly consider an example.

Burning Candles

In the 19th century the great English scientist Michael Faraday made fundamental contributions to electromagnetism and electrochemistry. He was also known as a gifted lecturer. In 1825, he initiated the Royal Institution Christmas Lectures, a series of public lectures that are still held every year (see Figure 1). In 'The Chemical History of a Candle', he begins:

1. Faraday and his lectures at the Royal Institution are celebrated on the Series E of the £20 British note, in circulation between 1991 and 2001.

> There is no better, there is no more open door by which you can enter into the study of natural philosophy than by considering the physical phenomena of a candle. There is not a law under which any part of this universe is governed which does not come into play, and is not touched upon, in these phenomena. I trust, therefore, I shall not disappoint you in choosing this for my subject rather than any newer topic, which could not be better, were it even so good.

Following in Faraday's footsteps and hoping that I shall not disappoint the reader, I shall enter the study of applied mathematics with the humble candle rather than any newer topic and ask a simple question: how quickly does a candle burn?

I know from experience that the candles we use on birthday cakes burn so quickly that it is hard to light eighty of them before the first one is completely consumed. I have also seen large pillar candles in churches burning for days. We could perform a number of experiments with candles of different sizes and obtain a fairly good understanding of the relationship between a candle's size (estimated from its diameter) and flame velocity (defined as the height of candle consumed per unit time). This empirical estimate will be useful but it would not give me insight into the underlying mechanism and I would remain forever curious about the beautiful phenomenon of flames.

Modelling the burning of a candle is surprisingly difficult. It requires a knowledge of fluid dynamics, combustion, and chemistry. However, simple mathematical reasoning can be applied to partially answer the question. To model a burning candle, we need to identify the key variables. But what are they? Many effects could play a role: the diameter of the candle, its composition, the width of the wick, the atmospheric pressure, the size of the room, its temperature, and so on. The crucial step in modelling is to retain important effects, and, as a first step, ignore minor contributions. We can therefore make the following assumptions:

- The candles are all placed in the same large chamber at the same atmospheric pressure and temperature. A candle is cylindrical with radius *R* and height *H*, measured in centimetres. We also assume that they all have the same wick's type and size.
- For comparison, candles are made of the same wax. So, the chemical energy density, *E*, stored in the wax is independent of the candle size and type. The variable *E* is defined as the energy per unit volume, let's say in joules per cubic centimetre. This means that a candle has volume $\pi R^2 H$ cubic centimetres and contains $E\pi R^2 H$ joules that can be eaten by the flame and transformed into heat and light.
- The rate of energy dissipation, that is how fast energy is lost, is independent of the candle height and radius. We call this quantity *P* and it denotes the rate at which energy is consumed through light and heat emission, say in joules per minute. We assume that all candles, big or small, have the same wick and release the same amount of energy per unit time. A lit candle dissipates *PT* joules during *T* minutes.

Calling on basic physics, we can model the flame dynamics as a process between the two flows of energy. Let *T* be the time taken for a candle of size *H* to be consumed. In that time, the candle consumes $EH\pi R^2$ joules and releases *PT* joules: the energy goes from wax to flame. The balance of these two processes leads to an *equation*:

$$PT = EH\pi R^2.$$

Since a candle of height *H* burns in a time *T*, the flame velocity is $\upsilon = H/T$ and using this model, we can express this velocity as

$$\upsilon = \frac{H}{T} = \frac{P}{\pi E} \times \frac{1}{R^2}.$$

This simple model does not tell us much about the velocity of the flame since we do not know P and E and, at first sight, it looks like we have replaced the unknown velocity by two unknown and mysterious quantities. Nevertheless, it provides interesting information when we realize that P and E are independent of the size of the candle. Therefore, we can extract information about the *scaling* of the process. We know that the velocity depends on the inverse of the radius squared, which implies that a candle of twice the diameter would burn four times slower; a candle ten times the diameter would burn a hundred times slower, and so on.

How good is this prediction? It is fairly easy to run a series of experiments with candles of different sizes and measure both the velocity and radius for each candle as shown in Figure 2. This figure is a log-log plot, a particular way to plot data that transforms any power relationship into a straight line as detailed in Chapter 2. The only important thing to know at this point is that if the data is close to a line of gradient −2, the candle follows our model. A look at the figure tells us that our prediction is quite good for large candles but not so good for smaller ones where other effects may be playing a role (the wick may be smaller, the wax may be melting away, …). Nevertheless, this model helps us understand candle flame propagation as a basic energy conversion problem. It also provides a simple law that we can test. This approach is often referred to as a *back of the envelope computation*: a simple computation that aims to understand the basic mechanisms involved in a physical process. Other examples will be studied in Chapter 2.