Architecture of Mathematics

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Preface

The proposed book is devoted to the description of the logical structure of Mathematics. This is prepared based on a special course lectured by the author for many years at the Mechanics and Mathematics Faculty of al-Farabi Kazakh National University (Almaty). Its form and content correspond to the subjective point of view of the author at the time of the last editing of this book. Its name goes back to N. Bourbaki's famous program article, and with the choice of floors, rooms, etc. as structural units, instead of the usual chapters and sections, it corresponds to Hardy's representation of the Mathematical World as a multi-story building.

Despite the undoubted influence of the ideas of Bourbaki, this significantly more modest work is qualitatively different from the famed treatise "*Elements of Mathematics*" in volume, form and content. In particular, the reader will not find complete and rigorous proofs here. As Lao Tzu said, the one who knows does not prove, the one who proves does not know.... Providing such material hardly brought us closer to the desired goal, but would greatly increase its volume, inevitably reducing the number of potential readers. We basically do not give a reference, since any list of sources will still be far from complete. Behind each of the many rooms of the Building under construction are thousands and thousands of items. In addition, this book does not pursue scientific, but educational goals. A serious reader, with a special desire, can push this frivolous book to a considerable extent and turn to more solid specialized literature with the necessary list of problems and theorems. Finding the necessary information at the current level of information technology is not difficult.

To participate in construction work and travel through numerous sections, blocks and floors, in principle, no knowledge or skills are required. However, the higher the level of mathematical culture of the reader, the more he will be able to learn from this not quite standard course. It is possible that, traveling through the majestic halls of the Building, in some distant room hopelessly forgotten by God and people, he will unexpectedly find for himself something interesting, some missing detail that had previously escaped his attention. Perhaps some incredulous reader will reproach the author for these or other sins or omissions, will not agree to the given constructions, and will propose his own architectural project. In this case, I will consider that it was not in vain that I spent time working on this book.

Our publication is the third attempt to present this subject. The previous editions of 1998 and 2006 were published in a small edition in Russian. Over the years, my views have undergone significant changes, which inevitably affected the contents of the book. There is a slight chance that after some time I will feel an urgent need to carry out a new reconstruction of the Building. In any case, I would really like to understand the subject. Therefore, I will be very grateful to the readers for their comments and suggestions, for any sharp criticism on individual issues and on the book as a whole. For those who want to respond, I inform my address: serovajskys@mail.ru.

I dedicate this book to the reviewer of its first edition, Professor ***Shaltay Smagulov***, whom I really miss. I would like to express my special gratitude to ***Yuri Kulakov***, ***Vladimir Shcherbak*** and ***Andrey Berlisev***, who expressed a number of valuable advice on certain provisions of this book, as well as to many employees and students of the Mechanics and Mathematics Faculty of Al-Farabi Kazakh National University who expressed interest in discussing questions given. I am also grateful to ***Yusif Gasimov***, ***Valery Romanovsky*** and ***Michael Ruzhansky*** for their support. I am very grateful to the author of the cover for the artist ***Israil Saitov***, with whom hundreds of kilometers of joint mountain hiking connect me. Invaluable help in preparing the computer version of the book was provided to me by ***Kumar Shashi***. I am extremely grateful to ***Alexander Teplov***, without whom this book would hardly have been written. Finally, I am grateful to my wife ***Larissa Ananyeva*** for understanding and support.

Introduction

The object of our attention is the delightful and inexhaustible Mathematics. This amazing science is so radically different from all the others that there is a serious doubt whether it can even be considered as a science in the classical sense of this word. Indeed, what do respectable representatives of the real sciences do? The brilliant physicists in the cool shadow of the synchrophasotron calmly split an atom and condescendingly observe the electromagnetic waves that go somewhere far into the distance. Brave chemists gaze intently into the intricate structure of the molecules and synthesize ethyl alcohol without much hassle. Curious biologists are able to count all bones of artiodactyls and know the innermost secret of the genetic code. Profound historians courageously find new reasons for the collapse of the Roman Empire and draw far-reaching conclusions from this. Self-confident economists skillfully exchange goods for money and get tangible profits.... All of them, to the best of their abilities and means, honestly examine real-life objects.

What do the sinful mathematicians spend their precious time on? They stubbornly study numbers and functions, operators and equations, algorithms and sets.... Have you ever had to meet any set or at least an ordinary number on your life path? You saw two apples or two cars, but not the number 2. This is not in nature! The mathematical objects are just products of our own intellection. They were invented by mathematicians themselves, in order to somehow justify their own existence. Therefore, it turns out that Mathematics is studying something that is not really there, that exists exclusively in our imagination. At the same time, mathematicians unceremoniously invade physics and chemistry, biology and sociology, psychology and economics.... And, most surprisingly, only after such unceremonious intervention do the latter receive the legal right to be called the serious sciences. Boring mathematical abstractions turn out to be amazingly effective in any subject area. This manifests the special role of Mathematics in universal human culture.

Unfortunately, the generally accepted system of mathematical education is far from perfect. It does not contribute to a holistic impression about Mathematics. The education is focused mainly on obtaining specific information, and not on the formation of a common mathematical culture. Over the years of study, schoolchildren, students, graduate students get acquainted with analysis, algebra, differential equations, but not with Mathematics, as a unified science. The result is a misleading impression that Mathematics is a set of disparate independent disciplines, each of which has its own subject, goals and methods. This not only complicates the process of mathematical education for scientists, engineers, economists, but also inevitably affects the quality of training of professional mathematicians.

Even professional researchers and university professors are, as a rule, extremely narrow specialists. Analysts are often closer and more accessible to the language of physicists or economists, but not algebraists. The geometer does not understand and is not interested in the vague propositions of probability theory. For a computer mathematician, the profound problems of number theory are similar to Chinese literacy. Even within a concrete historically established mathematical discipline, there are many small sections and subsections, in each of which established traditions and recognized authorities reign supreme. They confidently divided this discipline into feudal states and often do not even know what is happening in the neighboring sovereign mathematical power. All this greatly inhibits the development of Mathematics, a unified science, where between seemingly completely independent directions there are amazing, deep and sometimes unexpected connections, where everything is intertwined with thousands of threads invisible to the unprepared eye. It is clear that a successful advance, as a rule, requires the concentration of all forces in some narrow scientific direction. However, as Louis de Broglie noted, specialization narrows the horizons, complicates fruitful comparisons and analogies, and finally threatens the future of the human mind.

In the development of any science, two trends invariably manifest themselves. On the one hand, there is a constant intense search for the fundamental principles, elementary "bricks" that make up the magnificent Science Building. On the other hand, the general system of knowledge is steadily improving, i.e. universal principles for combining these "bricks" into a united whole. The combination of these two trends (analysis and synthesis), the gradual reduction in the number of fundamental principles and the refinement of ways to unify the information received determine the progress of science. For Mathematics, this process is highly typical. Each decisive break-through deep into the increasingly dense jungle of abstract mathematical constructions is invariably accompanied by an extension of the class of possible applications. Attempts to solve new problems that previously did not allow a comprehensive analysis, bring researchers to a qualitatively new level of abstraction, to improve the foundation of Mathematics. Clarification of the structure of Mathematics, penetration to its bottomless depths allows us to develop an increasingly powerful apparatus for solving specific applied problems, which, in turn, stimulate the development of fundamental theory.

A holistic view of Mathematics is possible from two perspectives. At first, this is a historical approach, which characterizes the evolution of the mathematical ideas and inextricably related with the development of civilization. This principle is implemented in the course "*History of Mathematics*", which is devoted to the author's book "*History of Mathematics. Evolution of Mathematical Ideas*", published in 2019 in Moscow, at the URSS publishing house. Another point of view is the analysis of the logical structure of Mathematics with its deductive description from the foundations to the applications. It is not very important here how the mathematicians obtained some notion or result. The general problem is the constructing of the most mathematical theories on the single foundation and determining clear logical links in the Building of Mathematics. In the process of such a construction, it suddenly turns out that the established historically habitual division of Mathematics into geometry, algebra, analysis, etc., which is based on the experience of millennia, should give way to something fundamentally different.

It is clear that historical and logical approaches are in no way opposed. On the contrary, they largely complement each other, forming a single spatiotemporal picture of the considered object. In this course, the subject of our attention is the "*Architecture of Mathematics*", and thus the logical principle was chosen as the basis for the presentation of the material under discussion.

A natural question may arise, is it worth studying such problems at all? Does knowledge of the Architecture of Mathematics bring us closer to solving a specific scientific and technical problem? Would it not be better to spend our precious time with undoubted benefit, in particular, for an in-depth study of various theoretical or applied mathematical disciplines that provide the key to directly solving the concrete problems. However, our world is so arranged that even when we start researching a specific applied problem, we, as a rule, are not able to fully evaluate what kind of apparatus will be needed for its final solving. We run the risk of drowning in the vast ocean of diverse mathematical information. We are literally overwhelmed by the rapid flow of published literature, both of a general theoretical and purely applied nature. In order to successfully navigate in an ever-changing environment, one must possess not only knowledge in a particular field, but also a certain mathematical culture. For its formation, the courses of *History* and *Architecture of Mathematics*, apparently, are the most effective.

Note that the applied mathematician and specialist of the non-mathematical sciences (physicist, economist, engineer, etc.), who feel the need for in-depth use of the mathematical apparatus, even more need to study these courses in comparison with the theoretical mathematician. A theoretical mathematician can still afford the luxury of plunging himself into, for example, the theory of finite abelian groups and completely ignoring the existence of other types of groups, not to mention other algebraic (and even more so not algebraic) objects. However, in the process of solving applied problems, many mathematical problems of a diverse nature arise, as a result of which a broad outlook and the ability to navigate in an unfamiliar environment are especially necessary.

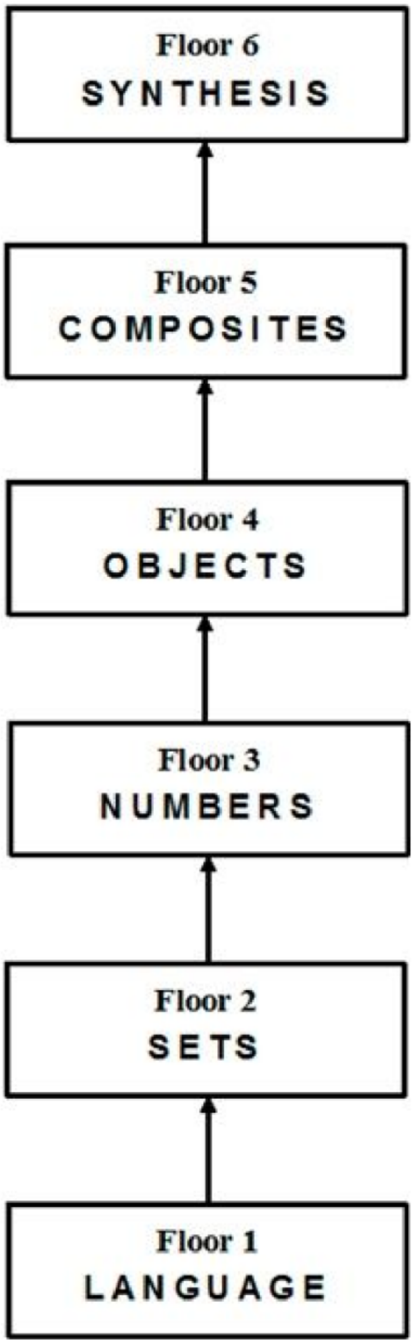
Turning to the Architecture of Mathematics, we must temporarily forget everything that we have been taught for a long time hard at the school and at university. Having resolutely embarked on this difficult path, we can no longer take anything on faith. So let us agree that we basically do not know anything, i.e. neither numbers, nor functions, nor algebraic operations, nor geometric figures... moreover, if we think carefully, such a categorical statement is not so far from the truth. Our grandiose construction begins on a completely empty place. For a better orientation in the labyrinth of mathematical constructions, each new floor, block or section will be accompanied by a more or less detailed plan of the structure.

The general scheme of the Building under construction (i.e. the location of its floors) is given below. The basis of any formalized theory is *language*, which is the only possible means of communication between different individuals. We put it on the first floor. The immediate foundations of all Mathematics are the *sets* located on the second floor. Next, on the third floor, using set theory, *numbers* are determined. There are the most important concepts that are models of the basic mathematical structures. On the fourth floor of the Building the various mathematical *objects* are considered. There are sets endowed with any specific properties. Considering different types of properties on the same object, we arrive at the fifth floor to *composites*. Finally, on the sixth floor, the *synthesis* of previously constructed structures into a single unit is carried out.

Of course, such a representation of Mathematics is extremely arbitrary and should in no way be taken as absolute truth. It should also be borne in mind that the Building of Mathematics is not something finally formed once and for all. The Building is continuously being rebuilt. Somewhere high in the clouds, new floors are being erected. At the very bottom there is a constant strengthening of the foundation. Obsolete blocks and sections are gradually being replaced. There is a persistent search for more advanced forms of building material and a more comfortable communications and life support system. New paths are opening up, connecting rooms that seem to be completely remote from each other. The new windows and doors that connect the building with the outside world are constantly opening. Mathematics is developing....

However, a lot has already been said. We need to get down to business. It is time to hit the road....

Constructing begins...


Floor 6
SYNTHESIS
Floor 5
COMPOSITES
Floor 4
OBJECTS
Floor 3
NUMBERS
Floor 2
SETS
Floor 1
LANGUAGE

Floor 1

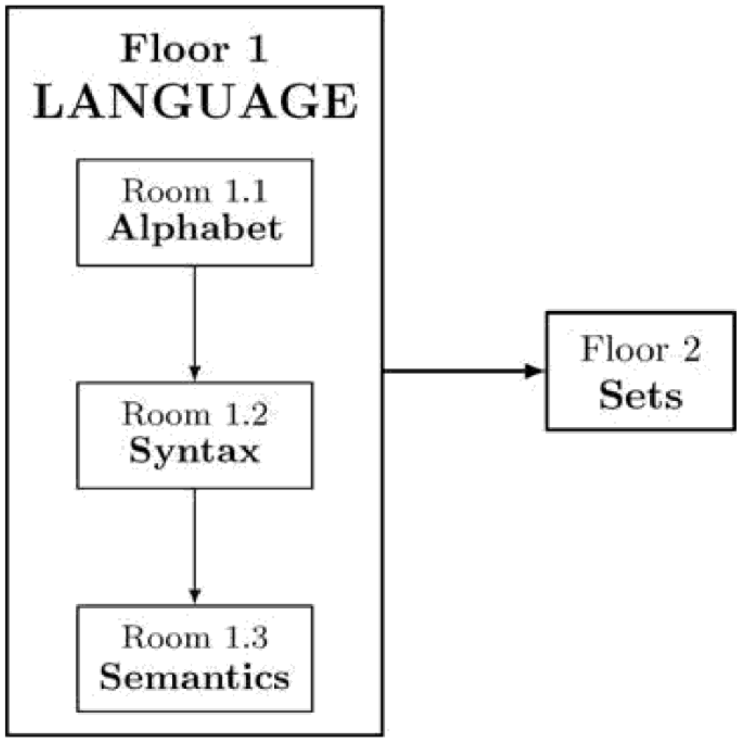
Language

Science and technology, literature and art, religion and philosophy are different forms of human interaction with the world around them. The existence of the human in this world is always associated with the collection, storage and processing of diverse information. Any information is necessarily presented in a language, understandable for a certain persons. Therefore, the construction of any arbitrary formalized theory should be preceded by a characteristic of the language that is the only possible means of communication between individuals. Only having mastered to the proper degree this language, can we safely go on a long and difficult journey in an unfamiliar country, not being afraid to meet with its mysterious inhabitants.

The description of any language, natural or artificial, begins with an alphabet. This is a list of admissible letters for the considered language. The letters are combined under the rules of syntax in independent blocks, called words. Empowering words and phrases with some sense by the rules of semantics, we get statements that can be true or false.

Alphabet, syntax and semantics form three rooms of Floor (see Figure 1). This is not yet Mathematics, as we usually imagine it. There will be no distinctly definitions, no clearly defined concepts, no strict formulas, no serious theorems.

When moving on the first floor, we are forced to rely not on flawless logic, but on vague intuition. We try not to clearly define the rules of the game, but only to try to understand what game we are all the same entering. We will confine ourselves to a brief description of the situation, with a hope of some further understanding of these far from obvious problems. Only having risen high enough and looking around, moving from floor to floor, getting acquainted with local inhabitants (sets, numbers, structures, categories, etc.), entering into a direct conversation with them, we may, to some extent, master the language of Mathematics. Is it possible to learn some other natural language somehow differently?



Room 1.1 Alphabet

The language is the necessary means of communication between individuals. Each statement in any language, regardless of its origin, is formulated using the elementary symbols that make up the alphabet.

Definition 1.1 *An* ***alphabet*** *is a collection of admissible symbols that are called* ***letters****.*

Remark 1.1 We understand that this definition is not fully correct. Particularly, the word "collection" is in fact the synonym with concept of the set that is fundamental for all subsequent constructions and is considered only on Floor 2. However, the presentation of set theory is necessarily carried out in a language that is characterized by a certain alphabet, and so the sets turn out to be secondary to the language. Besides, the word "symbol" is a synonym with concept of the letter and does not characterize it in any way. Another obvious drawback of this definition is associated with the mention in it of the language itself, which from the logical point of view is obtained only after the introduction of the alphabet and the rules of syntax and semantics. It remains for us to accept this sad situation and hope that we already have an intuitive idea of the alphabet.

Remark 1.2 By the position of Floor 2, we could say that the alphabet is a set, and its letters are the elements making up this set. In principle, any set can be interpreted as an alphabet, although this can hardly be of particular benefit. Particularly, the interpretation of the empty set defined on Floor 2 and the infinite sets (especially non-countable ones, see Floor 3) as an alphabet of a language would be strange enough. One usually understands an alphabet as a non-empty finite set of elements. However, the terms used here are meaningless so far. Therefore, we confine ourselves to what we have, namely, Definition 1.1.

To clarify the situation, let us consider an example related to natural language and, does not foretell seemingly no surprises.

Example 1.1 ***Latin alphabet.*** We have the easiest question. How many letters are there in the Latin alphabet? Any literate person will grin and confidently say that the Latin alphabet includes exactly 26 letters, namely, a, b,..., z. Such an answer does not seem to cause any objections. However, we are not interested in letters in a narrowly philological sense. We are trying to find out which characters are considered valid in the text based on the Latin alphabet, and therefore fall under the definition of the letter we have accepted. Can we assume that all the letters of the alphabet have already been listed? Are not other symbols used in Latin (English, French, etc.)? In particular, in any meaningful text, of course, there can also be capital letters A, B, ..., Z, which have their own peculiarities in language constructions, which means that they enter independently into our alphabet. Besides, we can also use in phrases the digits 0, 1, ..., 9. There are the letters too under Definition 1.1. However, our alphabet is not exhausted by this. All kinds of brackets, point, comma, dash, quotes, etc. also are the letters of this alphabet. These characters are also permissible and even necessary. Let us also recall different special symbols that have a completely definite meaning, for example, £ or ©. Remember the different mathematical symbols used sometimes in phrases. The text can sometimes contain letters of Greek, Russian, and other alphabets. Finally, the gap between the words in the sentence, i.e. the absence of any symbol, has some meaning, and therefore must be interpreted as a letter of the alphabet. Now we return to the initial question, how many letters are there in the Latin alphabet by Definition 1.1? It is hardly possible to give a clear answer to this quite innocuous question. However, there is no special need. It was important for us to emphasize that all symbols without exception, considered valid in the text in this language, should be recognized as letters of the alphabet. □

We will be interested, of course, not in the Latin (Greek, Russian) alphabet, but in the set of the symbols used in Mathematics. However, we do not presented their detailed list here. The necessary symbols will be introduced gradually in the course of the presentation of the material. As already noted, we are not concerned with the formal construction of Mathematics, but with the clarification of its logical structure. Therefore, now it was important for us only to note that the presentation of any formalized theory should begin with a description of the language, which is based on an alphabet. There is no need to begin a travel through the world of Mathematics with the declaration to the unfortunate traveler of a long and boring list of intricate symbols, the true meaning of which, often, will become clear only in the middle, if not at the end of the journey. Perhaps, it will not clear up at all...

By themselves, the letters do not yet form a language, just as bricks are just the building material from which buildings can be constructed. We still need to specify some formal rules that allow us to combine letters into words, and also be able to interpret the propositions. Thus, in order to get acquainted with the language, it is necessary to receive the an information about syntax and semantics.

Room 1.2 Syntax

The letters are elementary bricks that make up the language. Combining letters in blocks, we get a text that can be given a certain meaning. Thus, for the transfer of information, it is not the letters themselves that are important, but their combinations. As a result, we arrive at the following notion.

Definition 1.2 *A **word** is an ordered sequence of letters that is admissible by the rules of syntax.*

Remark 1.3 One uses also here the term ***formula***, which is more natural for the language of Mathematics.

Remark 1.4 The word is not usually understood as any, but only a finite sequence of letters. However, the meaning of the finiteness of the object will be revealed only at the beginning of the Floor 3.

Remark 1.5 Using the word "sequence", we mean that letters in a word are written in order one after another (from left to right or vice versa). In mathematical constructions, we can meet terms that seem completely inconsistent with this rule, for example, the following objects

a+b2,Σi=1n2-k,∫abf(x)dx.

However, we can transform it to, for example,

(a+b)/2, Σ (k=1,n,2 ↑ (-k)),∫ ↓ a ↑ bf(x)dx.

This saves the initial sense and accords with Definition 1.2. We do not usually adhere to the rules of the game in mathematical constructions, because this is due exclusively to the established tradition. However, when you enter any text into the computer, we set the necessary characters sequentially one by one in full accordance with the specified rules.

Example 1.2 ***Number language***. Let us consider the decimal representation of real numbers. The corresponding alphabet includes the standard symbols. There are ten digits, the decimal point for describing the fractional values (the comma is often used instead of the point), and the minus sign for describing the negative numbers. The real numbers are the words of this language. However, along with natural combinations of signs, for example, 235, 547.021, -0.023, we could form another terms like 3.00.5, 37-1.2, 0009, -000.5, 7.2300, which somehow would not like to recognize as acceptable mathematical constructs. We will use the following rules for the formation of words in a numerical language:

i. *the word can include no more than one letter ".", besides, this character can not be at beginning or end of the word, and after the "-" sign;*
ii. *the letter "-" can be at the beginning of the word only;*
iii. *the word can begin with the "0" letter only if it is followed by a decimal point;*
iv. *if one has the letter "0" after "-", then the next letter is ".";*
v. *if the word includes the letter ".", then it does not end with zero.*

These rules are the syntax of the considered language. □

Remark 1.6 The described syntax of the numerical language is not unique. Particularly, we could not require using of the last three rules. You could also enter additional letters in the alphabet, for example, "/" to indicate ordinary fractions; decimal order for writing floating-point numbers; separating symbol "," or a space, allowing to consider not one, but several numbers; signs of arithmetic operations and parentheses for writing not only numbers but also numeric terms. The introduction of any new symbol with preservation of its natural mathematical meaning is necessarily accompanied by a description of additional syntactic restrictions.

The considered numerical language is poor enough. This is characterized by the very simple syntax. If we consider a natural language (English, Russian, German, etc.) or a formalized language with a sufficiently meaningful mathematical theory, then we will inevitably have to use fairly complex grammatical rules.

Remark 1.7 Problems of syntax are studied (in addition to classical philology) by such disciplines as ***mathematical linguistics***, which consider the formal structure of languages (natural and artificial), and ***mathematical logic*** that explores the general principles of proving and the foundations of Mathematics. The problems of syntax include, in particular, the study of formal axiomatic theories as purely symbolic systems. There are, for example, many serious results on the consistency of concrete axiomatic theories, on the compatibility of different axioms, etc. We do not consider these results, as this is not part of our plans.

We did not determine the alphabet of mathematical language. Therefore, now we cannot to analyze the rules of syntax. We will note the limitations on the process of word formation as new symbols are determined along the way forward.

Room 1.3 Semantics

The syntax gives the formal rules, allowing combinations of letters of the alphabet into some blocks without taking into account their meaning. However, the interpretation of words and propositions, the study of information contained in the text, refers already to semantics. The concept of proposition is most important here.

Definition 1.3 *A word or a complex of words is called a **proposition** if it makes a sense by the rules of **semantics** and expresses a statement.*

We will use propositions to describe the considered objects, to characterize their properties, to formulate and prove assertions. There are made up of the words of the language used in accordance with the given rules and are given some meaning. Then the proposition can be ***true*** or ***false***. From the linguistic point of view, the proposition is characterized by a narrative sentence. This statement is usually contrasted with imperative, interrogative and any other sentences, an assessment of the truth or the falsity of which is impossible.

Remark 1.8 The verification of the truth or the falsity of the concrete propositions relating to a particular subject area is in fact the subject of any science. This is to the maximum extent applicable to any direction of Mathematics. We note here the ***Tarski theorem***, according to which, roughly speaking, the truth problem in every concrete enough substantial theory cannot be solved by the proper means of this theory.

Remark 1.9 The identification of the text goes through a triple system of filters. First, terms that contain symbols not provided by the alphabet are rejected. Then from all admissible sets of letters are chosen words, i.e. terms that satisfy the rules of syntax. After this, on the basis of the laws of semantics, exclusively meaningful parts of text are considered. There are propositions that can be true or false. This process is schematically depicted in Figure 1.1. Let us give the following natural interpretation of this procedure. Suppose that we need to write a computer program to solve a problem. First of all, the text of this program is compiled on the basis of the alphabet of a programming language. After this, the syntactic control is carried out, i.e. identify and eliminate the syntactic errors. A program without formal errors will not necessarily calculate (in particular, if some additional requirements that can be attributed to semantics get broken). Finally, if the result is obtained, then it can turn out to be either true or false, depending on whether the algorithm actually put into the program gives a solution to the problem posed.

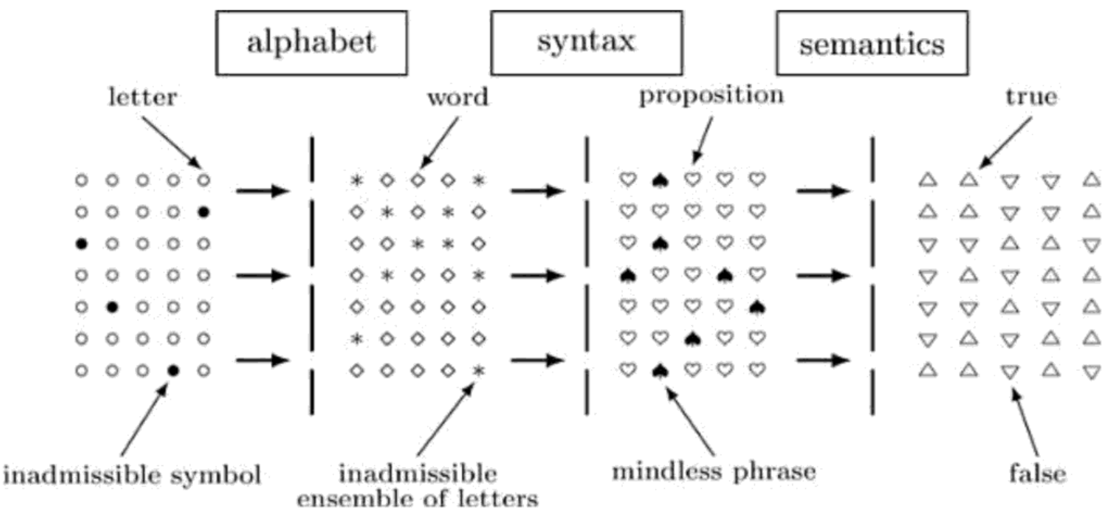


Figure 1.1: Identification of text.

Remark 1.10 The connection between syntax and semantics is studied by the ***model theory***, which is one of the main branch of mathematical logic. The ***model*** is a certain mathematical structure, which can be described with the help of the considered language. Particularly, the system of axioms of Euclidean geometry without the fifth postulate of Euclid admits a non-unified interpretation. In particular, as its model, along with ***Euclidean geometry***, ***Lobachevskian geometry*** also appears.

Example 1.3 ***Easiest arithmetic***. Consider the alphabet with letters a, b, c. Under the rules of syntax, two classes of words are admissible. There are the simple and complex words. The simple words contains finite ensembles of the symbol a, for example, aa, aaa, and $aaaaa$. The complex words have the form $xbycz$, where x, y, and z are simple words. Under semantics rules, a simple word is interpreted as a natural number that is equal to the quantity of the letter a in this word. The symbol b is the addition, and the symbol c is the equality. Then the complex word is a proposition that can be true or false. For example, the complex word $aabaaacaaaaa$ is interpreted as the true proposition $2+3=5$ and the complex word $aabaacaaa$ is the false proposition $2+2=3$. □

In the future, we will need some actions for the propositions. Let us have a proposition denoted by M. Denote by $\neg M$ the proposition that is true whenever M is false, and this is false if and only if the proposition M is true. The symbol $\neg$ is called the ***negation***.

Remark 1.11 We use here the language of Mathematics. Particularly, one considers two letters. The symbol M denotes the given proposition, and $\neg$ is the negation. By syntax, the term $\neg M$ is admissible, i.e. this is the word. However, the term $M\neg$ is not admissible without dependence of the sense of the proposition M. If M is a concrete proposition, then $\neg M$ has the clear sense. Therefore, this is the proposition. Now we consider the semantics.

Let us have propositions M and N. The term M⋀N denotes the proposition that is true whenever both propositions M and N are true. The term $M \vee N$ is the proposition that is true whenever at least one of the above propositions is true. The letters ⋀ and ⋁ are called the ***conjunction*** and the **disjunction**.

Remark 1.12 One use sometimes the symbol ∩ for the denotation of the conjunction and the letter ∪ for the disjunction. In Mathematics, despite its high degree of severity, there are sometimes discrepancies (see also Remark 1.14). From a linguistic point of view, this indicates the presence of several more or less common mathematical dialects.

Remark 1.13 Consider the electric interpretation of the last actions (see Figure 1.2). The considered propositions are interpreted as a switch in the electrical circuit, which can either be closed (the statement is true), or open (the statement is false). Then the disjunction corresponds to the parallel connection of two switches (there is the current in the circuit if at least one switch is closed), and the disjunction is their consistent connection (there is the current, if both switches are closed at the same time).

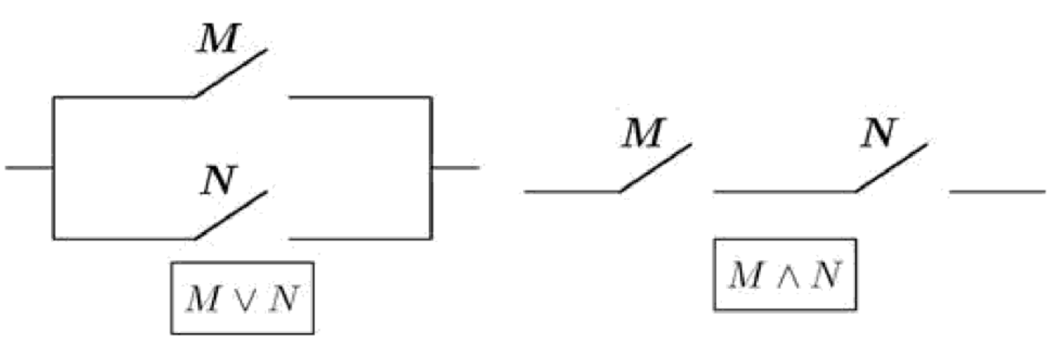


Figure 1.2: Electric interpretation of the logical operations.

If a proposition N is a corollary of a proposition M, then one uses the denotation $M \twoheadrightarrow N$, where the symbol $\twoheadrightarrow$ is called the ***implication***. More exact, the proposition $M \twoheadrightarrow N$ is false whenever M is true and N is false. Thus, the presence of an implication means that a false effect cannot be deduced from the true cause. If we have both propositions $M \twoheadrightarrow N$ and $N \twoheadrightarrow M$, then we write $M \equiv N$, where the letter $\equiv$ is called the ***equivalence***.

Remark 1.14 The implication can be denoted by symbols $\rightarrow$ and $\subset$. One denotes sometimes the equivalence by the symbols $\Leftrightarrow$, $\leftrightarrow$ or $=$.

Remark 1.15 The concept of equivalence will be used later also in another sense, in particular, for describing the relationship between two objects. Particularly, we determined the proposition $M \equiv N$ as an action with respect to the given propositions that is the result of a logical operation on them (see the following Remark). However, on the other hand, we can compare these two statements with each other and on the basis of such a comparison conclude that these propositions are equivalent in the sense that each of them is a consequence of the other. Comparing two objects (propositions), we interpret the equivalence as a relation on the set of propositions (see Floor 2).

Remark 1.16 By the terminology of Floor 4 (see Block B), negation, conjunction, disjunction, implication and equivalence are operations on a set of propositions. This means that the result of the operation is an element of the same nature as the original objects. In particular, the negation of a proposition is itself a proposition, the conjunction of two propositions will also be a proposition, etc. In this case, negation is an operation of the first order, because it transform one proposition to a concrete proposition that is its negation. The remaining operations are of the second order, since they connect the two initial propositions. The considered actions are called the ***logical operations***. We will consider operations over sets on the Floor 2, and operations over numbers on the Floor 3. The general theory of operations is the subject of ***algebra*** (see Block B). There is a natural relationship between logical operations and Boolean functions. A ***Boolean function*** of n arguments is a transformation that maps n elements of a set B consisting of two values 0 (false) and 1 (true), to a concrete element of the set B. From the point of view of algebra, such Boolean function is an operation of order n on the set B (see Block B).

Remark 1.17 The considered logical operations are not independent, i.e. it is possible to express some operations through others. Thus, our alphabet is redundant, i.e. there is no need to use a separate symbol for each of the logical operations. The inclusion of “extra” characters in the alphabet allows you to avoid unnecessarily cumbersome terms in mathematical constructions. We also note that we have determined only the most important of logical operations. In the general case, a logical operation is a way of constructing complex propositions from simple propositions, in which the truth of complex propositions is uniquely determined by the truth-values of the initial propositions. One of the directions of mathematical logic is the ***propositional calculus*** or ***propositional logic***, where complex propositions are analyzed, the principles of the transition from simple propositions to complex ones by means of logical operations without analysis of the structure of simple propositions.

Summarizing the brief visit to the Floor 1, we give the following definition of the language.

Definition 1.4 *A **language** is a set of letters, rules of word formation, and rules of statement interpretation.*

Remark 1.18 By the terminology of the Floor 2, the language is a triple of sets, called the alphabet, syntax and semantics and satisfying corresponding constraints.

This definition should be considered together with the preceding ones.

Remark 1.19 In principle, you can "simplify" the situation by using a separate letter to each object that is to be described by the given language (for example, for the numerical language each number will be denoted by an independent symbol). Then any word consists of unique letter. The need for an independent concept of the "word" does not exist at all. However, any rules of syntax, of course, will be superfluous, i.e. the second floor room will be completely empty. Rules of semantics will also be superfluous, since each letter will correspond to some object, i.e. meaningless text in such a language do not arise. Thus, the concepts "letter", "word" and "proposition" are identical. However, in this case, we have to keep in memory a huge number of symbols and be able to identify the objects under consideration by them. This task is extremely difficult in the presence of a sufficiently large quantity of objects and hopeless in principle, if one is to operate with an infinite (perhaps not even countable) set of concepts. The second special case of language is associated with the use of the alphabet, represented by a single letter. Any word will consist here of a set of identical symbols, which is the entire syntax of the language (see, in particular, the representation of the numbers in Example 1.3). By fixing any object for any such word, we again avoid meaningless text, as a result of which the necessity for the rules of semantics again disappears. However, in such a grandiose language, even with an small set of objects, it is necessary to operate with extremely long words. With the increase in the number of objects under consideration, such language finally loses all meaning. In practice, the composition of the language includes not too much, but not very few letters. With the increase in the size of the alphabet, it becomes possible to operate with words of sufficiently small length; this is good. However, serious difficulties arise in interpreting the corresponding texts. Naturally, the size of the computer keyboard increases with the complexity of the work of translators.

Now we already have some experience. Therefore, we can safely move up to Floor 2 and confidently move forward, gradually replenishing knowledge of the language. In fact, Floor 1 does not yet fully apply to Mathematics. This, if I may say so, is an office room. The real habitants of the Mathematical World live on the subsequent floors of our Building. This is not surprising, because the lower floor of a large building is not residential, as a rule.

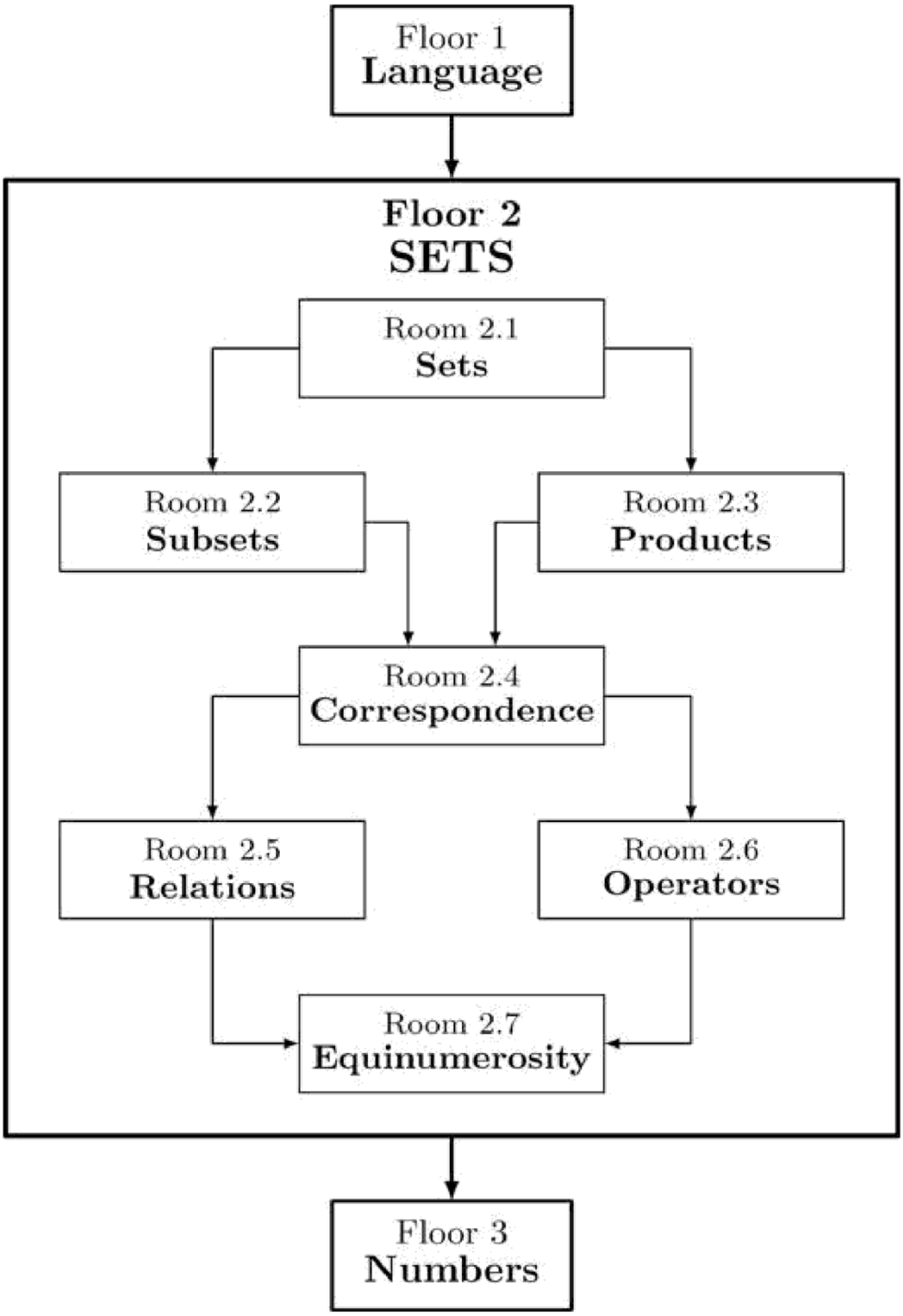
Floor 2

Sets

We discussed basic principles of building of mathematical constructions on the previous floor, trying to clarify the general rules of the game. Actually, mathematical notions appear only on Floor, where we have to get acquainted with the strange concept of the set. ***Set theory*** is the foundation of Mathematics and worthy of serious study. We confine ourselves only to a brief review of the most important set: theoretic concepts used directly in the future. We do not give a rigorous detailed description of the foundations of Mathematics, but we try to show how various mathematical constructions are built on these grounds. If somebody wishes to know more thoroughly the theory of sets or other directions of Mathematics, then he can turn to more serious literature.

Naturally, the initial positions of set theory related to the primary concepts of set, element, and property will be the least convincing. There are defined axiomatically by intuition and cannot in any way be logically derived from something simpler, because of its fundamental absence. One can only determine a connection between them (see Room 2.1). However, in the future the theory is gradually acquiring a certain severity, increasing with distance from very unreliable grounds.

All objects considered in the future turn out to be sets, elements or properties of some objects. In this regard, we would like to start to gain some experience with these concepts. The simplest way to learn it is to determine new objects from the existing building material. Here, on our way, we come across the concept of the subset, which is a collection of elements of the original set that satisfy an additional property (see Room 2.2). Another way of constructing sets is based on the concept of the product of sets, which is the set of all pairs of the given elements of these sets (see Room 2.3). Combining both techniques, we obtain new classes of sets. Thus, any subset of a product of sets is called a correspondence (see Room 2.4). Then we consider two independent classes of correspondences. There are relations (see Room 2.5) and operators (see Room 2.6). Together it lead to the concept of equinumerosity, which is the equivalence relation of sets and assumes the existence of an invertible operator acting from a set to another (see Room 2.7). A common characteristic of all equinumerable sets is called a cardinality. A cardinality is a coveted staircase that leads us to the numbers. This is Floor 3 of Mathematics Building. In particular, the natural numbers will be defined as cardinalities of non-empty finite sets.



Room 2.1 Sets

Our aim is to describe the logical structure of Mathematics. Any mathematical reasoning (geometric constructions, proofs of theorems, computational algorithms, etc.) is a logical chain of transformations. At each step of this process, some elementary procedure is performed, connected with the derivation of some result from already known facts. Thus, from a concrete cause, a certain consequence is deduced. However, a natural question arises, where did this reason come from? As a rule, this is itself a corollary of an earlier, deeper cause. However, what guarantees the truth of this new reason?

Turning the existing logical chain, we certainly come to some root cause, from which everything else is already output. However, what is the beginning of this original cause? It by definition cannot be a corollary of something more elementary. Consequently, we are dealing with an ***axiom***, its logical foundation is impossible in principle. So when laying the construction of the Building of Mathematics, the problem of the first brick inevitably arises.

Remark 2.1 The axioms are introduced based on the intuition, and not logically. As Hadamard said, strictness only illuminates what has been won by the intuition. The intuition is not flawless, and can fail. Niels Bohr said that there are small and big truths. The small truth is such that the rejection of it leads to a contradiction. The rejection of the big truth leads to another big truth. The small truth of Bohr in Mathematics is the theorem. This says that under certain conditions a concrete result takes place. It is worthwhile to admit that under these conditions another result is realized, as we inevitably arrive at a contradiction. Thus, a small truth fully conforms to the logic. The big truth is the axiom. By refusing it, we do not come to a contradiction, but we get another axiom, and hence another big truth. Let us recall, in particular, the dramatic story with the ***fifth postulate of Euclid***. Everything is determined in a world dominated by logic. Under the existing conditions, only this result can be obtained and no other result. Anyone who analyzes the situation under these conditions will make the same conclusion. There is practically no chance for the real creativity, for the freedom. The freedom exists if there is a choice. The freedom in Mathematics is the freedom to choose hypotheses, axioms, root causes. One can use this freedom by the intuition. However, the choice is limited. It is carried always from of objectively existing possible outcomes, which guarantee the consistency of the selected hypotheses.

Remark 2.2 Logic reduces many difficult statements to a few simpler assertions that are axioms. In Mathematics, there are often situations where a large class of complex objects is reduced to a few simpler objects called the ***basis***. Particularly, a word is a collection of letters (see Floor 1), a natural number is a product of prime numbers (see Floor 3), a vector is represented as a linear combination of unit vectors (see Floor 4, Block B), a function is represented as a Fourier series (see Floor 5), a neighborhood of a point is described by a neighborhood basis (see Floor 4, Block C), etc. Thus, the system of axioms is a kind of basis for a formalized mathematical theory.

The set is a most elementary notion of Mathematics. This is cannot deduced to easier objects. There is impossible to give a definition with classic form "*...is called the set*". Sometimes synonyms like "*collection*", "*class*", "ensemble", etc., are used to characterize the set, which does not clarify this very complicated situation.

Remark 2.3 Georg Cantor, trying to somehow characterize the notion of the set, said that he imagines it as an abyss.

We will use the following definition, if only it can be, in reality, considered as a definition.

Definition 2.1 *A* ***set*** *consists of* ***elements*** *with* ***properties*** *that allow to combine elements into a set.*

Here it is clearly noted that the mysterious something called the "*set*" has a certain internal structure, being composed of some strange objects called "*elements*" (one uses also the denomination "***members***"). There are united in a set, have certain distinctive features, named "*properties*". We are absolutely unable here to characterize none of these three elementary concepts separately. We can have only some very vague considerations about the relationship between them. Particularly, a set is something that results from combining some elements with common properties into a single whole. The elements are something, which are capable of possessing some properties and, being independent units, to unite into a whole set. Finally, the property is the unifying principle, through which separately existing elements can be grouped into a single set. Similar "explanations" do not have the desired degree of credibility, but in view of the apparent lack of something better, we still dwell on them. We note only that any of the introduced concepts can be reduced to two others, although we cannot consider any of them to be primary.

Remark 2.4 Definition 2.1 characterizes all relationships between each pair of three considered objects. Particularly, the sentence "*set consists of elements*" can be denoted by $X = \{x,...\}$, i.e. the set X includes the element x and maybe some others (the pair "set–element"). We can write also $x \in X$, i.e. the object x is the element of the set X (element–set). The part of proposition "*elements that have properties*" is denoted by $P(x)$ that is the property P is true for the element x (property–element). We could write also $x(P)$ that signifies that element x has the property P (element–property). Finally, the thought "*properties that allow to combine elements into a set*" is usually denoted by $X = \{x| \ P(x)\}$, i.e. the set the set consists precisely of those elements x that possess the property P. This proposition does not depend, in reality, from the element x. Therefore, we could use shorter denotation $X = \{|P()\}$ for it (set–property). We could use also the dual assertion $P = \{x|x \in X\}$, i.e. the property P is true for the elements x of the set X only or shorter $P = \{| \in X\}$, i.e. the property P is true for the elements of X (property–set). Note that not all of these denotations have practical use.

Remark 2.5 In the future, we have to consider exclusively with sets, elements and properties. However, this classification is very arbitrary. The concrete object, depending on the specific situation, can be identified as a set, as an element, or as a property. For example, in the sentence "*someone is a student of the biological faculty*", the faculty is a set, and the student is its element. On the other hand, in the statement "*the biological faculty is the subdivision of a university*", the faculty is an element of the set "university". A more meaningful example of this kind will be described on the next floor, when we already have a certain stock of mathematical objects. Note that the possibility of interpreting the same object as an element and as a set predetermines serious troubles (see below).

The objects have concrete names. Particularly, we can consider the set X, the element x, and the property P (see Figure 2.1), where the letters "X", "x", and "P" are used for the denotation of the considered objects. We use the formulas $x \in X$

and $P(x)$ for signification the relations between the element and other notions. The object x is an element of the set X by the first of them, and this element satisfies the property P by the second proposition.

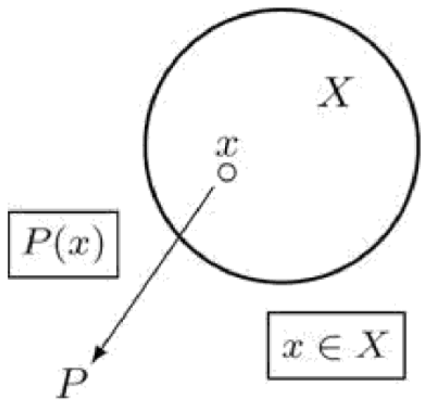


Figure 2.1: Set X, element x, and property P.

Remark 2.6 We have already some language structure here. Particularly, we determined six letters: "X", "x", "P", "∈", "(", ")" of alphabet with following rules of the syntax. The letter "∈" can be used after a symbol that denotes an element and before a symbol that denotes a set. Particularly, the letter sequence "$x \in X$" is admissible, i.e. this is a word, and "$\in xX$" is not admissible. Then after a letter that denote a property it is necessary to use the symbol "(". Then we have the name of element and the letter ")". Thus, the ensemble of letters "$P(x)$" is the word. Finally, the sense of the word "$x \in X$" is "*the element x belongs to the set X*", and "$P(x)$" has the sense "*the element x satisfies the property P*" by semantics. Now the considered words are transformed to propositions.

If the proposition $x \in X$ is false, then we use the denotation x∉X.

Remark 2.7 Note the redundancy of our alphabet. Particularly, the words $x \in X$ and ¬(x∉X) have the same sense, i.e. there are the synonyms. The value under the brackets is understood as a whole here. By mathematical logic, the proposition (x∈X)≡(¬(x∉X)) is true for all objects x and X here. Thus, it would be possible to exclude the symbol "∉" from the alphabet. As always, the expansion of the alphabet allows you to operate in shorter words.

Consider possible forms of determination of the sets that characterizes the relations between the set, elements, and properties. We know that "*the set consists of elements...*". Therefore, the most natural method of the set definition is the recitation of all its elements, for example (see Figure 2.2),

X={x,y,z}.

The symbol X here is the name of the considered set; the letter "=" (the equality symbol) declares that the term, which is present after it, determines this set. The letters "x", "y" and "z" that is located between the brackets "{" and "}" and separated by a symbol "," (comma) denote the concrete elements, and the letter "." (point) is used for the denotation the end of the proposition.

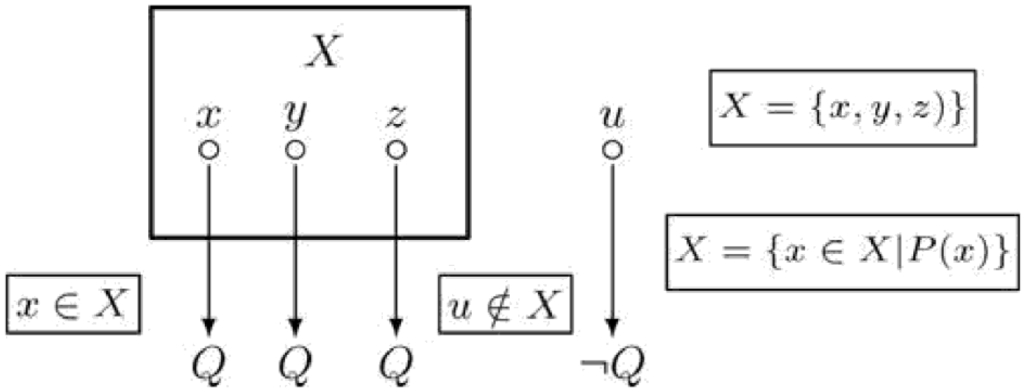


Figure 2.2: Determinations of the set.

Remark 2.8 One uses the symbol "=" not only for the definition of objects, but for the denotation of the objects equality. The use of the separator "," is not required. The word *xyz* can be interpreted as a list of elements too. If we would like to denote an element by many letters, we could use the brackets. For example, if we use the term (*xyz*), then the word *xyz* is the name of an element. In our case, "comma" is used solely for convenience of recording in accordance with accepted tradition and could be excluded from the alphabet. In the absence of brackets for combining letters into separate words, you can not do without commas, except in the case when it is stipulated in advance that all admissible words consist of the same number of letters. One realizes this situation, for example, in the universal genetic code, where each of the words consists in accuracy of three letters. Particularly, any amino acid is encoded by nucleotide triplets, called codons.

Remark 2.9 The order of elements in the definition of a set is not important in contrast to the concept of a pair, see Room 2.3. Particularly, the terms $\{x, y\}$ and $\{y, x\}$ denote the same set, because these contain the same elements.

Remark 2.10 One supposes occurring once inclusion of any element into the set. Sometimes the ***multisets*** are considered, where elements are included multiple times. Each element of the multiset is associated with its multiplicity. Particularly, the element a has the multiplicity 2, and the element b has the multiplicity 1 for the multiset $\{a, a, b\}$. Consider, for example, the multiset of divisors of a natural number (number 12 is characterized by the multiset of divisors $\{2, 2, 3\}$) and the multiset of roots of an algebraic equation (the roots of equation $x^3 - x^2 = 0$ is characterized by the multiset of roots $\{0, 0, 1\}$).

Unfortunately, it should be noted that only comparatively simple sets can be determined by the above method. In particular, we will consider infinite sets (see Floor 3), for which a complete enumeration of all elements is in principle impossible. However, there is another method of the denotation of sets, which is to use a property that is characteristic of the elements of the given set only. As a result, we have the following denotation (see Figure 2.2):

$$X=\{x|P(x)\}.$$

The above proposition means that the set X consists of those and only those elements that have the specified property, and the symbol "|" has a separate meaning. The term on the left of it denotes the name of the elements in question, and the word on the right characterizes their property.

Remark 2.11 We permanently meet in life with two methods of defining objects. For example, in order to explain to anyone what is meant by the word "sun", we can directly point to this object: "*the sun is that bright ball in the sky*". However, we can say differently: "*the sun is a star around which the Earth rotates*". In the first case, the identified object is presented directly, and in the second case, a property is specified that allows to select the given object among many others.

Remark 2.12 Two methods of set definition are considered differently from the positions of ***constructive mathematics***. In the framework of the latter, the existence of a mathematical object is rigidly linked to the fundamental possibility of its construction. As Andrei Markov Jr. said, "*Existence in mathematics is the potential feasibility of constructing*". An object is considered defined here only if there is an algorithm for its immediate location. The definition of the set by means of specifying the property, characteristic exclusively for its elements, cannot be considered satisfactory in constructive mathematics. This position is close to ***intuitionism*** that is one of the most original directions of constructing the foundations of Mathematics. It is clear that the element-wise definition of the set would be preferable, since it is usually much easier to restore the properties of already existing elements than to select an element that satisfies this property, let alone find all the elements with the specified set of properties. However, in the case when we do not have reliable means of explicitly determining the elements of a given set, it is better to give a clear description of their properties than not to have any information about the object under investigation. In modern Mathematics, it is customary to give preference to constructive methods of research, which corresponds to the explicit specification of the elements of the set. However, one should not ignore non-constructive methods, in particular, setting a set by specifying the properties of its elements, in those very common cases when nothing better can be proposed. In support of the meaningfulness of an non-constructive existence, the example of Hilbert is known. Among any group of people, for sure, there is a person with the most hair. The fact that we do not know who it is, can not serve as an obstacle in its description, as a real-world object.

Remark 2.13 The objects of modern Mathematics are characterized often accurate to isomorphism that is one-to-one correspondence (see Room 2.6) such that all the properties that are taken into account are preserved in the given subject area. Thus, we analyze not so much the specific elements themselves as their characteristic properties. As Werner Heisenberg said, the world is not divided into different groups of objects, but into different groups of relationships.

The problem of definition a set by specifying its defining property is far from trivial. Consider, for example, the concept of "*word*". Among the objects named by this term, i.e. among the numerous elements of the set under consideration, along with such expressions as "*man*", "*have*", "*red*", etc. we have the word "*word*". Thus, the element "*word*" belongs to a set with the same name. The fact that the same object appears in both a set and an element, as we have already noted, should not surprise us. Therefore, there are sets that contain themselves as an element. It is clear that not all sets have this property. In particular, the notion "*letter*" itself is not a letter. Now let the word $P(x)$ means that the object x is not an element of itself. Consider the set X with this property. Obviously, it contains the object "*letter*", no object "*word*". The question arises, is the property P true for the object X? If the set X is an element of itself? If $X \in X$ then the object X satisfies the property P, because this set contains the elements with property P only. However, the sense of this property is the absence of the considered inclusion, i.e. the condition X∉X. Suppose now the condition X∉X is true. Then the object X satisfies the property P, because the absence of this inclusion is the sense of the property P. Therefore, we get $X \in X$. Thus, the proposition $P(X)$ and its negation too lead to the contradiction (see Figure 2.3).

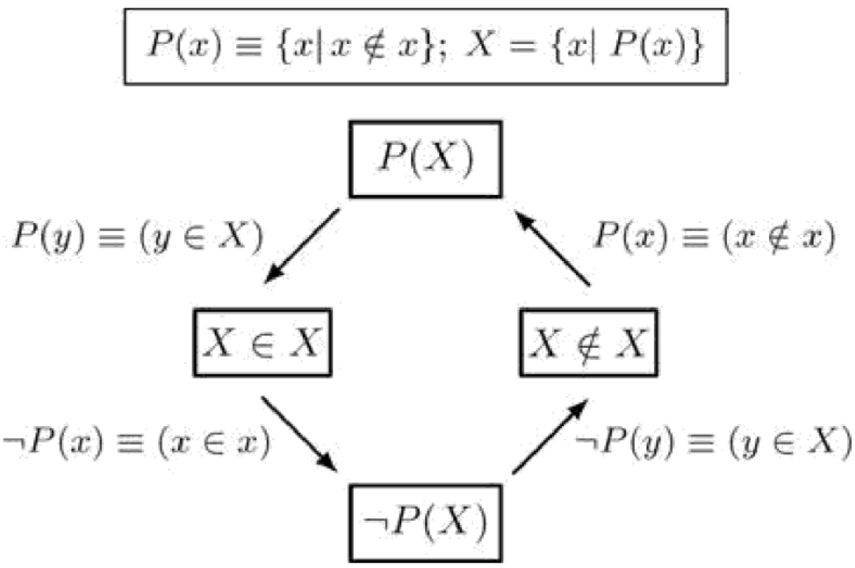


Figure 2.3: Russell paradox.

This ***Russell paradox*** and other similar effects, i.e. ***antinomies*** greatly undermine the credibility of the admissibility of the set definition by means of a defining property and are a serious argument in favor of intuitionism. However, the rejection of such a form of the definition of a set significantly reduces the capabilities of the mathematical apparatus and appears to be too expensive a payment for overcoming the paradoxes of set theory. In any case, the non-constructive task of the set should be treated with the utmost care.

Remark 2.14 Effects such as Russell paradox have long been known in logic. The same nature has the ***liar paradox***. Someone said: "I'm lying." Is it true? Let us also recall the ***barber paradox***. In a certain village, there lives a barber who shaves those and only those residents who do not shave themselves. Does he shave himself? Obviously, both a positive and a negative answers to these questions leads to a contradiction.

Remark 2.15 In order to lay the foundations of Mathematics and to circumvent Russell paradox and other similar troubles, the ***axiomatic theory of sets*** was developed. It involves the introduction of a certain system of axioms in such a way that it would be possible to describe the most important set-theoretic concepts, while eliminating the emergence of paradoxes. At present, there are several variants of such axiomatic, for example, ***von Neumann–Bernays–Gödel set theory*** (***NBG***) and ***Zermelo–Frenkel set theory*** (***ZF***). In particular, NBG uses the distinction between the concepts of set and class. The set can be considered as an element of a certain class here. At the same time, the class that is not a set is such a large collection that it can no longer be an element of something. We will meet these questions in category theory (see Floor 6). The ZF system works exclusively with sets. Particularly, it gives a description of the procedures that can be performed on sets in such a way that as a result sets will be obtained without fail. According to NBG, the property P considered in the Russell paradox sets a certain class of sets X, which, not being a set, cannot belong to something. In turn, in the ZF system, the object X simply cannot be defined, since its construction goes beyond the permissible set of procedures over sets.

Remark 2.16 As it is not strange, such questions can have practical sense. As you know, computer programs can be stuck, which hardly provokes positive emotions for programmers. In this regard, there is a natural desire to make a testing program that would check whether an arbitrary program will go in cycles or not without running it. If such a testing program is developed, it can easily be completed so that it loops if the program under test is working properly and stops if the program under test is looped. Since the program thus compiled is itself a class of computer programs, you can try to test it in the manner indicated above. If our program goes in cycles, then by construction it should stop. Conversely, if the program finishes normally, then it should go in cycles. As a result, we can conclude that the described program can not exist. These arguments are connected with the ***Turing halting problem***. Turing theorem is largely close to the famous ***Gödel incompleteness theorem***, which says, roughly speaking, that in any sufficiently substantial theory there is a saying that cannot be either proved or disproved. But its connection with the great ***Fermat theorem*** turns out to be quite unexpected. Suppose that the Turing theorem is not true. Then we have a universal program tester for their final stop. Then you can easily compile a program of simple enumeration of all possible values $n > 2$ and positive integers x, y, z. The program works until you find the four of the above mentioned numbers that satisfy the equation $x^n + y^n = z^n$. If this program ever completes its work, then the corresponding equation has a solution, and the Fermat theorem is not true. Our hypothetical tester should unequivocally specify, whether the given program will stop or not without starting the program. Thus, we certainly get an answer to the question whether the assertions of Fermat theorem are true or not. Unfortunately, the Turing theorem does not leave us the opportunity to obtain such an extravagant proof of the Fermat theorem.

Remark 2.17 Chapter 6 will discuss category theory, which allows one to describe many mathematical constructions from a unified method. An analogue of the concept of a set in category theory is ***topos***. Using the ***topos theory***, it is possible to describe almost all variants of the set theory.

Remark 2.18 One of the generalizations of a set is the concept of the ***fuzzy set***. Here, instead of the membership relation of an element to the set, the membership function with values from the interval [0,1] determines. In particular, if an element for sure does not belong to a given set, then its membership function is 0; if it belongs, then this is 1. In other cases, the membership relation of the element in question is considered fuzzy, and the corresponding value of the membership function lies between 0 and 1. The ***theory of fuzzy sets*** is used in computer science, cybernetics, and information theory.

Using the already existing set, one can construct other sets. The simplest methods of constructing new sets are described in subsequent rooms.

Room 2.2 Subsets

Consider general methods of determining new sets. The definition of a set is associated with specifying a property that is characteristic of elements of a given set and only for them. This circumstance suggests the simplest way of constructing new sets because of what is already available. It consists in determining an additional property on the set and selecting a class of elements of this set that satisfy this property.

Consider a set X and a property Q that can be true or false for the elements of X. Determine the set

$$Y=\{x|(x\in X)\wedge Q(x)\}$$

of all elements of X that satisfy the property Q (see Figure 2.4). We use the shorter denotation

$$Y=\{x\in X|Q(x)\}.$$

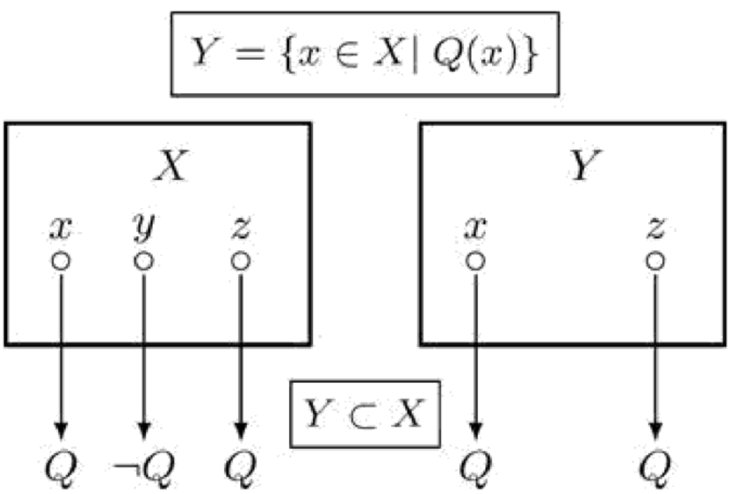


Figure 2.4: Y is the subset of X.

Definition 2.2 *The set Y is called the* ***subset*** *of the set X.*

Remark 2.19 This object Y is, in reality, a set, because it is determined by the concrete property of its elements.

Remark 2.20 Associated with the subset is the concept of a characteristic function defined in Room 2.6. At Floor 6, we will verify that the subset is a subobject of the category of sets. Numerous examples of subobjects of other categories (subgroup, topological subspace, etc.) are considered in Floor 4.

Remark 2.21 The transition from larger and poorer concepts to narrower and richer ones is accomplished by giving the preceding primary concepts additional properties. This is done by using the potentially existing internal reserves of the previous objects, rather than on the basis of some external intervention. Therefore, starting from the primary concepts of the set, element and property, we get the subsequent notion of a subset, carrying out actions exclusively on already existing primary objects. Subsequently, for example, the transition from a more general but also poorer concept of a groupoid to a less general but more meaningful concept of a semigroup is accomplished by introducing on the groupoid an associative property extra (see Floor 4, Block B). Definition of this property is carried out exclusively in terms of the theory of groupoids. Some groupoids are associative, forming a class of semigroups for which certain additional properties that are not characteristic of non-associative groupoids are satisfied. Objectivity of Mathematics is that something new is potentially already contained in the old regardless of the will and desire of man. The creative beginning of the researcher is manifested in the freedom of choice from a great number of potentially feasible mathematical concepts, among which a significant part is those that turn out to be the most important.

Obviously, if a set Y is a subset of a set X, all elements of Y belongs to X. The proposition "*a set Y is a subset of X*" is denoted by $Y \subset X$; besides this relation between considered sets is called the ***embedding***.

Remark 2.22 The condition $Y \subset$ can be denoted by equivalent form X⊃Y. Sometimes, one writes "⊆" and "⊇" instead "⊂" and "⊃".

Give the easiest classification of subsets of the given set (see Figure 2.5). Sometimes the additional property is true for all elements of the initial set. Then the subset Y can be deduced to X. Therefore, any set is the subset of itself. However, we can have the inverse case. The property Y can be false for all elements of X. Therefore, the subset does not have any elements. This strange object is called the ***empty set*** and denoted by ∅.

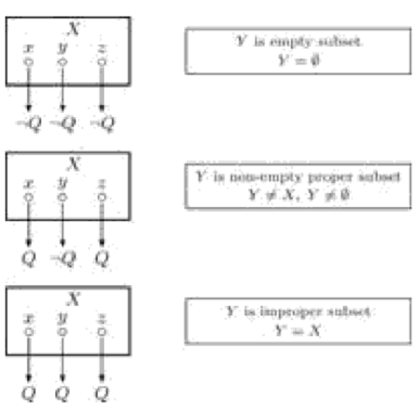


Figure 2.5: Classification of subsets.

Remark 2.23 In reality, the existence of the empty set is not obvious. It would seem there is an obvious contradiction with Definition 2.1, according to which the set must consist of something. However, we call the entered object a set, since it certainly corresponds to the definition of a subset, but it would not be entirely logical if any subset did not belong to the class of sets. Besides, we can denote it by a property, for example, $\varnothing=\{x|x\neq x\}$ Later, having entered Floor 3, we will see that the empty set (the undoubted and absolute "nothing") allows us to determine zero and other natural numbers. Thus, this concept turns out to be extremely meaningful and necessary for the erection of our Building. We declare (this is the axiom) that there is something that does not contain decisively any elements, but called for some reason a set.

The empty set is the subset of the each set.

Remark 2.24 We deduced from this property that the cardinality of the empty set is less than the cardinality of each non-empty set. It will be the basis for definition the number "zero" (see Floor 3). Note also that from this property it follows also that the empty set is an initial object of the set category (see Floor 6).

If a property Q is true for all elements of a set X, then we write $\forall x\in X{:}Q(x)$. The proposition $\exists x\in X{:}Q(x)$ means the existence of an element of X that satisfies this property. Sometimes one uses the shorter denotation $\forall x{:}Q(x)$ and $\exists x{:}Q(x)$, if it is clear, which set is considered. The symbols "$\forall$" and "$\exists$" are called ***quantifiers of universality*** and ***existence***.

Remark 2.25 In generally, a ***quantifier*** is a general name for logical operations that limit the truth of the propositions. Note also that propositions $\forall x{:}Q(x)$ and $\exists x{:}Q(x)$ does not depend from the symbol x, i.e. its sense does not change if we replace x by other symbol. One uses often the denotation $Q(x)\forall x$ or its more complete analogue $Q(x)\forall x\in X$ instead $\forall x{:}Q(x)$.

The sets X and Y are ***equal***, i.e. $X = Y$ if we have both conclusions $Y \subset X$ and $X \subset Y$. The elements x and y are equal, if $\{x\} = \{y\}$.

Remark 2.26 By the last definition, the following proposition holds

$$(X=Y)\equiv((Y\subset X)\wedge(X\subset Y)).$$

The term at the left-hand side here can be interpreted as a shorter denotation of its right-hand side. The replenishment of the alphabet with new letters pursues a single goal, that is, using too long words. By the way, it would be possible to reduce the equality of sets from the equality of elements, and not vice versa. Recall that the concepts of the set and the element we introduced simultaneously, but did not derive one of the other.

Remark 2.27 The symbol "=" is used here for the declaration of the relationship between two objects, not for its definition. These cases are quite easy to distinguish by the syntax rules. We write the term under the brackets for the definition of an object and the name of another object for the denotation an equality. The fact of using the same letter in words with different semantic loads should not cause any special surprise. We have the same situation for the natural languages too.

The equal sets have the same elements.

Remark 2.28 The fact of equality of two sets means that we have, in reality, the unique set with two different names. This is possible if specifying a set by enumerating its elements, a different order of location is used, for example, $\{x, y\} = \{y, x\}$. If we determine sets by properties of its elements, then the equality of the sets is realized if these properties are equivalent. For example, the set of even numbers coincides with the set of numbers obtained as a result of adding a unit to an arbitrary odd number.

If the proposition $X = Y$ is false, we use the denotation $X \neq Y$. If the set Y belongs to the set X without its equality, then Y is called the ***proper subset*** of the set X. The set X is ***larger*** than Y, and Y is ***narrower*** than X for this case, besides the embedding $Y \subset X$ is strict. Particularly, the embedding of the empty set to each non-empty one is strict.

Remark 2.29 The symbols $\subset$ and $=$ and its negations are the relations for the class of sets (see Room 2.5). Sometimes one uses the symbol $\subset$ for the denotation of the strict embedding only. Non-strict embedding is denoted by the symbol $\subseteq$ for this case. Remember that we used the letter $\subset$ for denoting the implication in Floor 1.

The set of all subsets of a given set is called its ***power set*** or ***Boolean***. The power set of a set X is denoted by P(X) that is $\{Y | \ Y \subset X\}$ (see Figure 2.6).

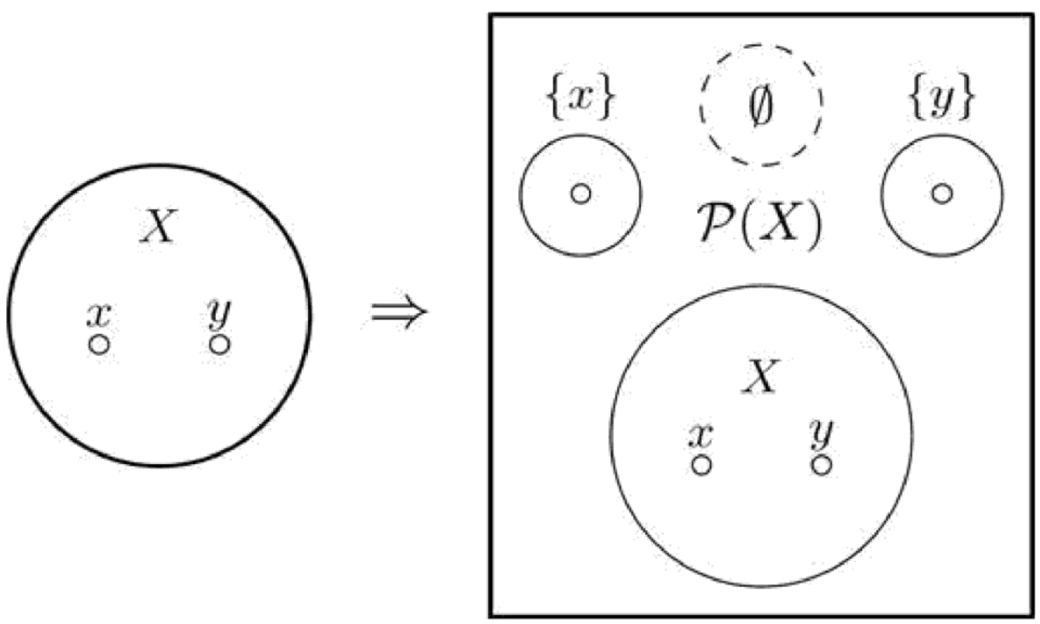


Figure 2.6: Power set of the two elements set.

Remark 2.30 The considered object is defined in accordance with the rules for defining the set. However, it is far from obvious that something made up of sets is itself a set. Earlier, it would seem that any set necessarily consists of elements, but not from other sets. Moreover, the use of the concept of a set of sets leads to the paradoxes of set theory (see, for example, the Russell paradox and Remark 2.72). However, in the framework of this course, we largely adhere to ***naive set theory***. The likelihood of experiencing major trouble will be relatively small here. A certain threat remains, for, as Hermann Weil remarked that it is surprising not that such contradictions arose, but that they appeared at such a late stage of the game.

Remark 2.31 Recall that the accepted separation of mathematical objects into sets, elements and properties is rather conditional. Particularly, the object X is a set with respect to an element x, if we have $x \in X$. However, the inclusion $X \in P(X)$ is true, because any set is the subset of itself and the element of its power set.

Remark 2.32 A categorical analogue of the power set will be given in Floor 6.

Consider a non-empty set X and its subsets M and N. Determine new subsets by the following equalities

$$M \cup N = \{x \in X | (x \in M) \vee (x \in N)\},$$
$$M \cap N = \{x \in X | (x \in M) \wedge (x \in N)\},$$
$$M \setminus N = \{x \in X | (x \in M) \wedge (x \notin N)\},$$
$$M \triangle N = (M \setminus N) \vee (N \setminus M).$$

Remark 2.33 This means that the result is subsets of X, i.e. objects of the same nature as the original sets. The general concept of the operation will be introduced on Floor 4. We could determine operations on arbitrary sets, and not on subsets of the same sets. However, the above Definition more accurately reflects the true property of the considered procedure. The geometric sense of the introduced constructions is illustrated by the ***Venn diagrams***, see Figure 2.7. Here we show operations on planar sets, i.e. subsets of the plane. Naturally, the very concept of a plane is not yet available. It can only be determined on Floor 3 after acquaintance with the real numbers.

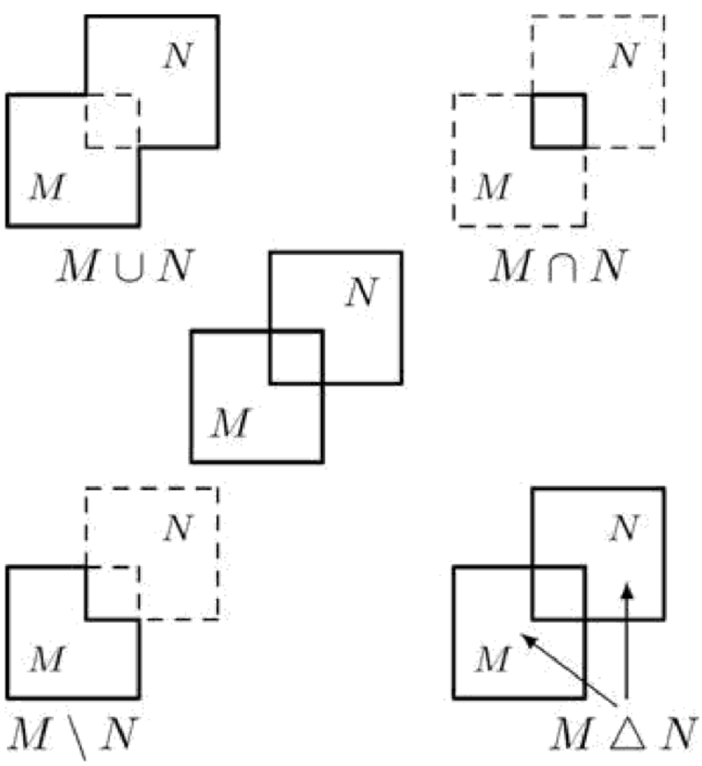


Figure 2.7: Operations on sets (Venn diagrams).

Remark 2.34 In Block B of Floor 4, we determine that the power set of a non-empty set with the operations $\cup$, $\cap$ and $\setminus$ forms a Boolean algebra.

If we have the embedding $N \subset M$, then the difference M\N is called the ***complement*** of the set N in M. The sets M and N are ***disjoint*** if its intersection is empty. The sets M and N determine the ***partition*** of a set X if there are non-intersect and its union is X. Particularly, if $N \subset M$, then the sets N and M\N determine the partition of the set M. These notions are illustrated by Figure 2.8.

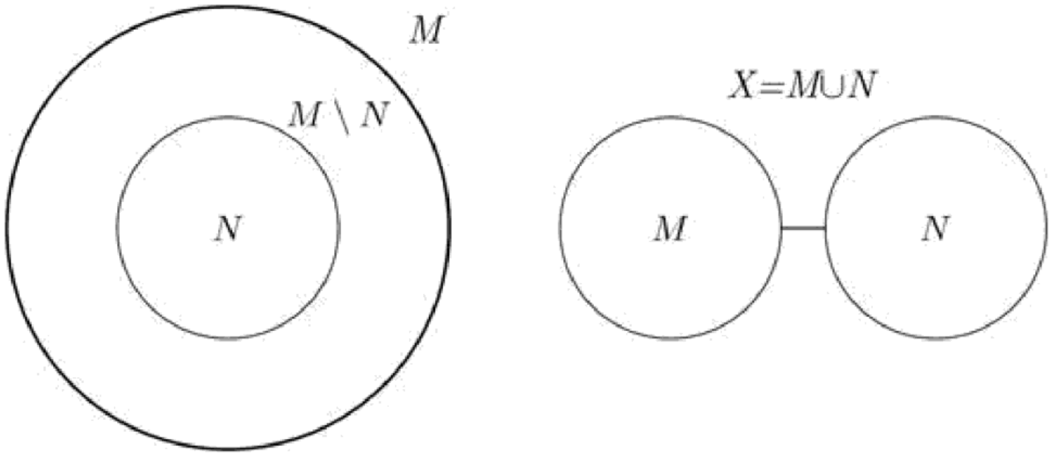


Figure 2.8: Partitions of sets.

We can determine new sets on the base of the set operations. However, there exists another method of solving resolution this problem.

Room 2.3 Set product

It is known that $\{x, y\}$ and $\{y, x\}$ are the same set or two equal sets. However, we can have an interest to constructions, where the order of elements is important. The ***pair*** of elements x and y is the set $(x, y) = \{\{x\}, \{x, y\}\}$.

Remark 2.35 The last formula determines, in reality, a set. The term (x, y) of its left-hand side (the word, see Floor 1) is the denomination (short form of the denotation) of the set that is definite by the concrete enumeration of the elements $\{x\}$ and $\{x, y\}$. In this way, we again call something consisting of sets a set. Moreover, the elements x and y have different properties here. Indeed, from the definition of the set equality it follows that the equality $(x, y) = (x' , y')$ is true whenever these sets consist of the same elements. The equality $\{x\} = \{x' , y' \}$ seems impossible (these sets have the different cardinalities, see Room 2.7). Then the considered pairs can be equal if $\{x\} = \{x' \}$, $\{x, y\} = \{x' , y' \}$. From the first equality it follows that $x = x'$. Then $y = y'$, because of the second previous equality. Thus, the objects (x, y) and (y, x) can by equal if $x = y$ only. Sometimes, one defines the pair by this properties, i.e. one declares that the pair (x, y) is a such object that the equality $(x, y) = (x' , y')$ can be true whenever $x = x'$, $y = y'$. This definition characterizes a pair, in reality. However, this is not constructive. By the way, the pair (x, y) can be characterized by a ***directed graph***, and the set $\{x, y\}$ describes an ***undirected graph*** (see Figure 2.9). For the general case the ***graph*** is a pair (X, Y), where X is a set of elements that are called the ***vertices***, and Y is a set of elements $\{x, y\}$ (the ***arrow*** connecting vertices x and y) or (x, y) (an arrow from a vertex x to a vertex y), where $x, y \in X$. The properties of graphs are considered by ***graph theory***. We will use graphs for analysis of the categories in Floor 6.

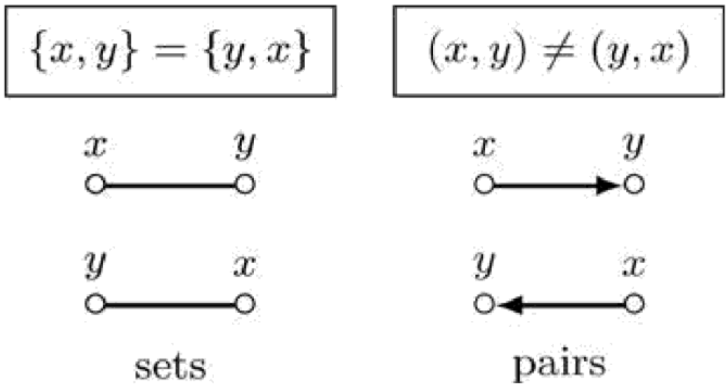


Figure 2.9: Sets and pairs.

Remark 2.36 One can also introduce the concept of a triple of elements (x, y, z) as a term $\{\{x\}, \{x, y\}, \{x, y, z\}\}$. The finite ordered ensemble $(x_1,..., x_n)$ can be determined analogically. This is called the ***tuple***, the ***word*** or the ***vector*** of n degree.

Using the pairs, we can determine an additional method of the definition of new sets.

Definition 2.3 *The **product** of sets X and Y or more complete its **direct product** or **Cartesian product** is the* set

$$X\times Y=\{(x,y)|(x\in X)\wedge(y\in Y)\}.$$

Remark 2.37 The definition of product based on the categories theory is given in Floor 6.

Remark 2.38 The transition from individual elements to their ordered ensemble with the subsequent transition from the sets of such elements to their product is carried out with the aim of uniting individual objects into a single whole. Particularly, the sets of individual numbers, determining the coordinates and velocities of a moving body as a result of this procedure, turn into a point of a ***phase space***, which is a product of sets of the corresponding coordinates and velocities.

Remark 2.39 The product of sets looks like a usual operation, like union or intersection. However, there is a fundamental difference here. The result of the operations considered earlier are objects of the same nature as the original sets, in particular, subsets of a set X. In the last case, we get an object of a completely different structure. Thus, the product of linear sets is a flat set (see Figure 2.10). Elements of the product are pairs, i.e. sets of sets, while the original sets consist of ordinary elements. Thus, the product is not an algebraic operation on a power set of a non-empty set (see Floor 4).

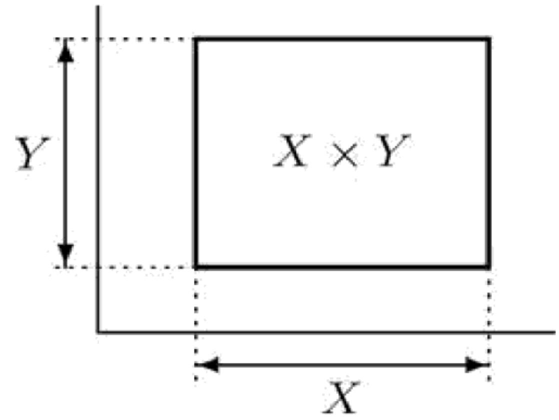


Figure 2.10: Product of sets.

Remark 2.40 One of the approaches to axiomatization of the set theory and overcoming paradoxes arising in it is ***type theory***. It is based on the special classification of all mathematical objects by types. In particular, ordinary elements have the first type, natural sets consisting of elements have the second, sets of sets have the third, etc. In this case, the inclusion of $x \in y$ is admissible (the syntax rule!) only if the type of the object x is one less than the type of the object y. According to this classification, the power set and the pair of elements belong to the third type, since they are composed of sets. Then the product of sets must already be assigned to the fourth type, because it consists of pairs, i.e. objects of the third type. Thus, the product of sets differs significantly from their intersection, union, etc. of ordinary sets having the second type.

Remark 2.41 The set product is not commutative and associative, i.e. we have the inequalities

$$X \times Y \neq Y \times X; (X \times Y) \times Z \neq X \times (Y \times Z).$$

for the general case. The first relation follows from the definition of the pair. The object $(X \times Y) \times Z$ contains of pairs, where the first component is a pair, the object $X \times (Y \times Z)$ contains of pairs, where the second component is a pair, and $X \times Y \times Z$ contains of triples.The product $X \times \varnothing$ is empty for all X, because the absence of any pair. The product $X \times X$ is denoted by X^2 (the high degrees of set are definite analogically). Particularly, R2 is the Euclidean plan that is the product of the set R (the numerical line) by R.

Remark 2.42 Both methods of determination of new sets (subsets and products) will be using for the definition of set scale and the general notion of mathematical structure (see Floor 6).

Using the set product, one determines correspondences that is the way to Floor 3.

Room 2.4 Correspondences

We considered before two methods of definition of new sets on the base of given sets. Combining these methods, determine new objects.

Definition 2.4 *Any subset of a set product is called a **correspondence** of* *considered sets.*

Remark 2.43 Sometimes, this is called the relation. However, we will use this name for the special class of the correspondences (see next room). One uses also the name multivalued function (see Room 2.6).

Let ϱ be a correspondence between sets X and Y. If $(x, y) \in \varrho$, then the element x is in the correspondence ϱ with the element y and wrote $x\varrho y$. The sets

$$D(\rho)=\{x\in X|\exists y\in Y:x\rho y\};R(\rho)=\{y\in Y|\exists x\in X:x\rho y\}$$

are called the ***domain*** and the ***range*** or the ***codomain*** of the correspondence ϱ of the sets X and Y (see Figure 2.11). These sets are called also the ***projections*** of the correspondence ϱ to the sets X and Y.

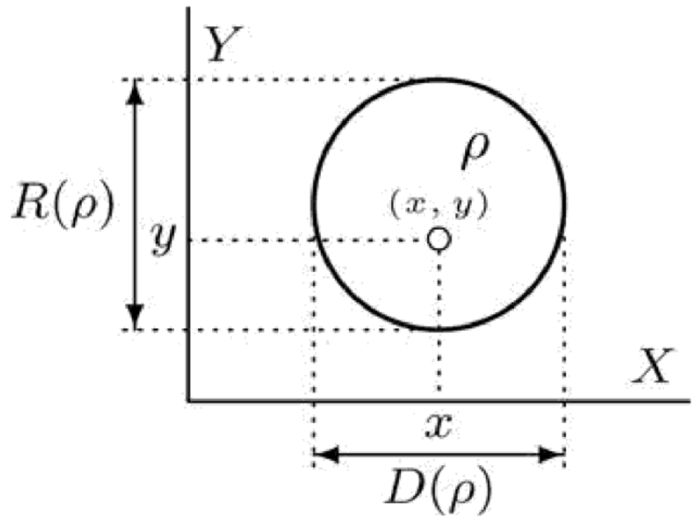


Figure 2.11: Correspondence ϱ of the sets X and Y.

For all elements $x \in X$ and $y \in Y$ the sets

$$\rho(x)=\{y\in Y|\exists x\in X:x\rho y\};\rho^{-1}(y)=\{x\in X|\exists y\in Y:x\rho y\}$$

are called the ***image*** and the ***preimage*** of the considered elements by the correspondence ϱ (see Figure 2.12). Note also the following obvious equalities

$$\rho(x)=\varnothing\ \forall x\notin D(\rho);\rho^{-1}(y)=\varnothing\ \forall y\notin R(\rho).$$

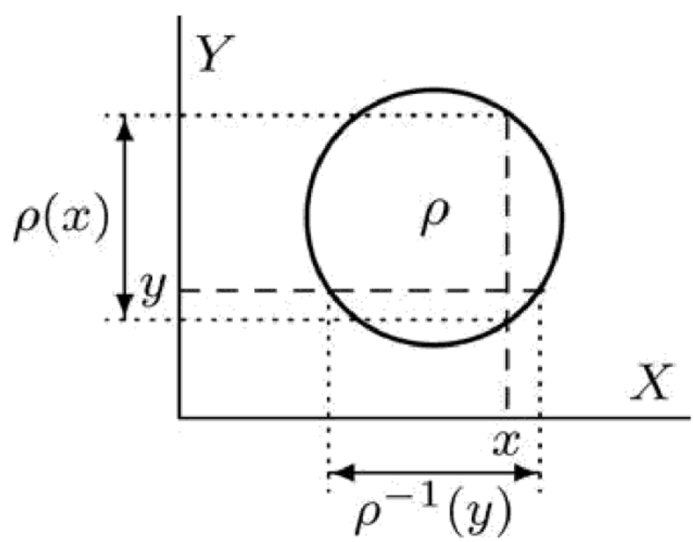


Figure 2.12: Image and pre-image of elements by a correspondence.

Remark 2.44 In principle, the inclusion $\in$ could be interpreted as a correspondence between elements and sets. Indeed, the condition $x \in X$ means that the element x belongs to the set X. Besides, the corresponding image of an element x is the collection of sets that include x, and the pre-image of a set X is the collection of all its elements, i.e. the set X itself.

Example 2.1 Let us have the sets $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and its correspondence ϱ determined by the equality

ρ ={(x1,y2),(x1,y4),(x3,y1),(x3,y2),(x3,y3)}.

We get $D(\varrho) = \{x_1, x_3\}$, $R(\varrho) = Y$. Determine the images and the pre-images of the elements

ρ (x1)={y2,y4}, ρ (x2)=∅, ρ (x3)={y1,y2,y3};

ρ -1(y1)={x1}, ρ -1(y2)={x1,x3}, ρ -1(y3)={x2}, ρ -1(y4)={x1}.

The considered correspondence can be presented by the matrix A=(aij), where $a_{ij} = 1$ if $x_i \varrho y_j$ and $a_{ij} = 0$ otherwise, $i = 1, 2, 3$; $j = 1, 2, 3, 4$. Thus, we find the ***correspondence matrix***

A=(010100000111).

Another form of the determination of correspondences is the ***correspondence graph*** (see Figure 2.13). □

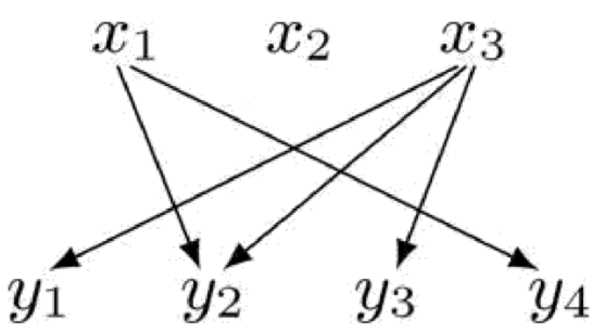


Figure 2.13: Correspondence graph of Example 2.1.

Remark 2.45 We will use the description of the correspondences by graphs and matrixes for the analysis of the ordered objects (see Floor 4, Block A), which are determined by special correspondences.

For any correspondence ϱ of sets X and Y one determines a ***dual correspondence*** ϱ^* of the sets Y and X such that the condition $y\varrho^* x$ is true whenever $x\varrho y$ (see Figure 2.14).

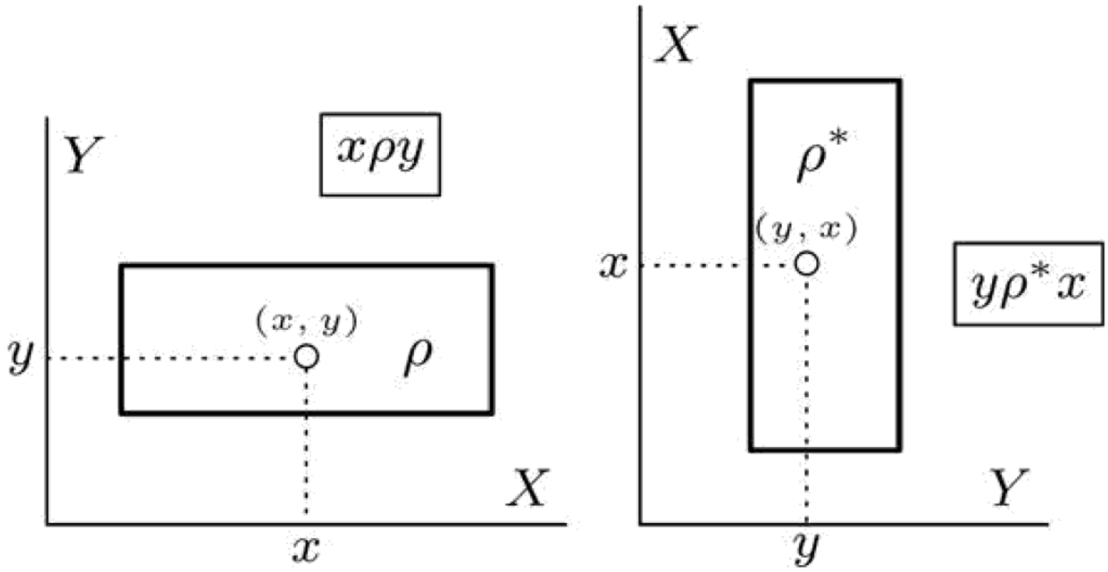


Figure 2.14: Determination of the dual correspondence.

Remark 2.46 The correspondence matrix and the correspondence graph for the dual correspondence are the ***transpose matrix*** (it is obtained after changing of the order of indexes for all elements of the initial matrix) and the ***dual graph*** (it is obtained after inverting of all arrows of the initial graph). Different forms of the duality will be considered in future (see Floor 6).

Now we consider two special classes of correspondences. The order of their consideration is not important. We will use both classes in Room 2.7, which is placed near the stair to Floor 3.

Room 2.5 Relations

One of the most important class of correspondences is the relations.

Definition 2.5 *A correspondence between a set X and itself is called a* ***relation*** *on X.*

Remark 2.47 The relation on a set X is a subset of X^2. This is ***binary relation***, because it connects two elements of the given set. One considers also relations of n order that are subsets of the set X^n. An example of a third order relation is the condition of ***collinearity*** of points on a plane, i.e. the location of three points on a same line. The ***coplanarity*** condition of the location of four points of three-dimensional space on the same plane is an example of a fourth order relation. A first order relation is a subset of the set X.

Consider most important classes of relations. A relation ϱ on the set X is called ***reflexive*** if $x\varrho x$ for all $x \in X$; this is ***symmetric*** if from $x\varrho y$ it follows that $y\varrho x$ for all x, $y \in X$; this is ***transitive*** if from $x\varrho y$ and $y\varrho z$ it follows always $x\varrho z$ (see Figure 2.15).

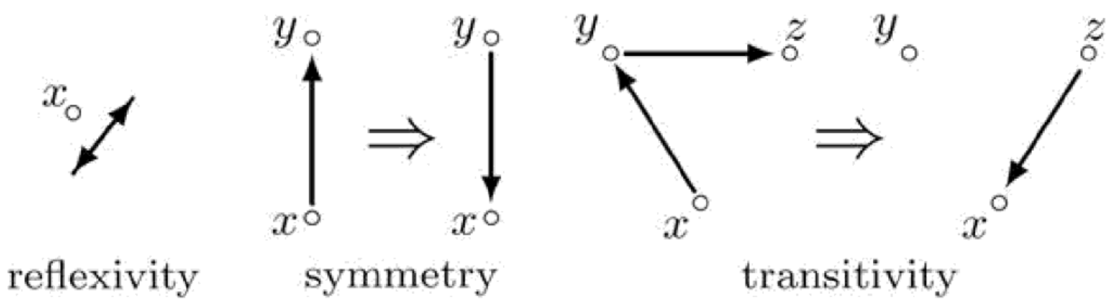


Figure 2.15: Classes of relations.

Remark 2.48 We consider also other relations. For example, antisymmetric relations will be used for the definition of the ordered objects (see Floor 4, Block A). We can determine the dual relation. Partially, if the relation ϱ is symmetric, then $\varrho^* = \varrho$.

The multiplicity of natural numbers, the parallelism of lines, the embedding of sets, the relation of "being countrymen" for people are reflexive. The relation "the sum is even" for natural numbers, the perpendicularity of lines, the intersection of sets, the relation of "to be spouses" for people are symmetric. The relations "more" for the real numbers, "location below" for the function graphs, are transitive; embedding of sets, causality of events are transitive too. The relations possessing all the above properties simultaneously are extremely important.

Definition 2.6 *The reflexive symmetric transitive relation is called the* ***equivalence****.*

Remark 2.49 We determined the equivalence as the logic operation in Floor 1. Now this name makes another sense; this is a relation.

Remark 2.50 It seems that the definition of the equivalence is not correct. Indeed, from $x\varrho y$ it follows always the condition $y\varrho x$, because of the symmetry. If we have $x\varrho y$ and $y\varrho x$ too, then we get $x\varrho x$ by transitivity. We do not have any constraints with respect to the element x. Therefore, the previous condition is true for all x that is the reflexivity. Thus, we follow this property from the symmetry and the transitivity. It seems we do not have any necessity to use the reflexive property for the definition of the equivalence. However, we supposed the element x is in the relation with an element of the given set. Therefore, we cannot guarantee the truth of the condition $x\varrho x$ for the arbitrary x.

The examples of equivalence are the parallelism of lines, the similarity of triangles, the coincidence of any questionnaire feature in people (citizenship, age, height, weight, specialty, etc.). The propositions x and y are equivalent, if the condition $x \equiv y$ is true. The set of all elements of X that is equivalent to x with respect to the relation ϱ is called the ***equivalence class*** of the element x. This is denoted by $[x]$ or more complete [x]ρ.

Definition 2.7 *The set X/ϱ of all equivalence classes of the set X with respect to the relation ϱ is called the* ***quotient set*** *or the* ***quotient space****.*