Arithmetic

PAUL LOCKHART

ARITHMETIC

Paul Lockhart

The Belknap Press of
Harvard University Press
Cambridge, Massachusetts
London, England

Printed in the United States of America
First Harvard University Press paperback edition, 2019
First printing

Library of Congress Cataloging-in-Publication Data

Names: Lockhart, Paul, author.
Title: Arithmetic / Paul Lockhart.
Description: Cambridge, Massachusetts : Harvard University Press, 2017. |
Includes index.
Identifiers: LCCN 2016055136 | ISBN 9780674972230 (cloth : alk. paper) |
ISBN 9780674237513 (pbk.)
Subjects: LCSH: Arithmetic. | Arithmetic--History.
Classification: LCC QA115 .L713 2017 | DDC 513--dc23 LC record available
at https://lccn.loc.gov/2016055136

CONTENTS

Dear Reader,

It's fun to count and arrange things. We like doing it, and we have even developed it into something of a folk art. This art is called *arithmetic*. Arithmetic is the skillful arrangement of numerical information for ease of communication and comparison. It is a fun and enjoyable activity of the mind and a relaxing and amusing pastime—a kind of "symbol knitting," if you will. But please understand that's all it is. Being good at it doesn't make you particularly smart or mathematically inclined or anything like that. Similarly, being unskilled at arithmetic does not mean that you are stupid or that you do not have a mathematical mind. Arithmetic is just like any other craft; you can get good at it if you want to, but it is no big deal either way. My hope is that by reading this book you will be inspired to try it out for yourself and to experience firsthand the simple pleasure and satisfaction that comes with numerical fluency. Have fun!

THINGS

We find ourselves in a world full of things: plants and animals, rocks and trees, the stars in the sky and the sand on the beach. We are surrounded by multitudes. And we *count* them. Why exactly do we bother counting things? The truth is, we usually don't. Most situations do not really call for any careful counting, just a sense that "we have plenty enough" or "we're way short." The main reason why we count is to *compare*. Usually these comparisons are easy to make: "So these together are about sixteen bucks . . . Oh, I have a twenty. No problem." In fact, most transactions these days sound less like "sixteen seventy-one out of twenty, that's three twenty-nine. Have a nice day," and more like "Beep. Boop. *Swipe*." Most of the time we don't need to do much actual counting. Still, there are times when we do want an accurate idea of exactly how many what-have-yous we happen to have. What sorts of things do we keep track of, and why?

One of the oldest is *time*. We can imagine meetings between prehistoric tribes being scheduled for so-many moons hence, or scratches carved in a tree counting the days the hunting party has been away. We certainly do keep track of time—too much, if you ask me. Money, of course, is another one. People definitely keep careful track of that stuff (don't get me started). Property also: "Are all of my marbles here in my marble bag?" "I hope we have enough silverware." Land, grain, cattle, the inventories of the ages. Seven years of famine, seven of plenty. Boy, do we like to count!

Then there is curiosity counting—counting for the sake of pure wonder. "How many stars are in the Milky Way?" "For how many seconds can I hold my breath?" Or the more mathematical: "How many ways can I arrange the books on my bookshelf?"

Whatever the purpose or rationale, we occasionally find ourselves wanting to know how many of something there are. This is where arithmetic begins, with *desire*. There's no point counting something you don't care about. Don't ever do that. It's boring, and it will make you hate counting. There are actually a lot of people who hate arithmetic (far too many to count!), and it makes me sad. Usually it's because they were made to do something they weren't interested in doing. Let's not have that be you.

The point is that there are things, and we sometimes want to count them. Actually, there are some quantities that we don't so much count as *measure*. What we do when we measure something—and this could be an amount of milk, or someone's height, or a piece of land—is we take some agreed-upon amount, known as a *unit* (e.g., a cup, an inch, or an acre), and we count the number of those. This has the effect of turning something smooth and continuous into something discrete and lumpy. So let's confine ourselves to thinking about counting distinct individual items.

As I said before, counting is really all about comparison. Even when you count something out of curiosity—say you happen to have thirty-two pennies—the only sense in which you "know" how many you have is by comparison with certain other amounts (e.g., thirty), which have special names in your language and serve as familiar benchmarks of size. Just saying the word *thirty-two* means you have unconsciously performed a comparison: you have a little more than three double handfuls.

Comparison is always the issue. "Do we have enough?" "Is that all going to fit?" "There were more of them before. Where'd they all go?" We count so that we can compare amounts. Now, it would be great if we could just look at two collections and know which one was bigger, as though we had a "number sense" akin to smell or taste. Maybe there are creatures like that who can just *tell*. Occasionally, some humans come along (usually severely neurologically impaired ones) who can do essentially that, but most of us are pretty bad at it. I think pretty much everyone can perceive "three-ness"—knowing that there are three things without having to literally count them—but past about six or seven it starts to get a little dicey.

LANGUAGE

One way to think about counting is that we are measuring the size of a collection. Implicit in this is the recognition of the fact that the number of things is independent of how they are arranged. If I put seven marbles in my pocket, it doesn't matter how I jiggle them around; when I take them out, there will still be seven (assuming I don't jiggle them so hard they shatter, and that I don't have a hole in my pocket). At some point in our mental development, we become aware of this *number permanence*. What this means is that we get to arrange and rearrange our collections in various ways without disturbing the information we are interested in—namely, how many things the collection has in it.

This is all very well and good when the things are relatively easy to hold, carry, and manipulate (e.g., marbles, coins, or jelly beans). Often, however, the things we want to count are inconveniently large, far away, or, even worse, moving. And sometimes, as in the case of time, what we wish to count is ephemeral and fleeting.

What we do in these cases, of course, is to substitute. We replace the things we are actually interested in counting by something that is easier to work with. We make a *representation* of the situation and work with that instead.

It is hard to say when this idea first originated, but it is clear from cave paintings and carved figurines that we have been representing things by other things for quite some time.

Although there are instances of this behavior in other species (primate gestures, bumblebee dances, etc.), nothing comes close to the extent and frequency with which humans perform these substitutions, often unconsciously. Language itself is nothing other than a vast interconnected set of sense-memory substitutions, with words used to represent instances of similar sensory experiences. It is safe to say that representation is one of the fundamental ingredients of consciousness.

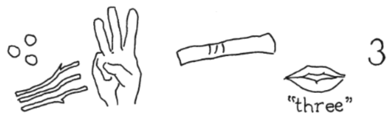
In fact, we are so good at making representations that we often lose track of the distinction between the representation and the thing itself. The word *cat*, for instance, is a string of three letters. It does not have a tail and it does not purr. The string of letters is a *code*—it stands for the thing with a tail that purrs. This is what Juliet is referring to when she muses to herself,

> What's in a name? That which we call a rose
> By any other name would smell as sweet.

In particular, when we start designing arithmetic systems, it will be crucial to keep clear the distinction between a number and its representation. A lot of people are walking around in a muddle about this, and I don't want you to be one of them.

Anyway, at some point—let's say two hundred thousand years ago—people started representing things by other things. One big advantage of doing this is *portability*. If I use pebbles to represent my sheep, I can then easily carry them around in a bag and use them to buy and sell and keep track of my flock, without having to carry around any actual sheep. (This is not only much easier on my back but is also vastly preferred by the sheep.) This is undoubtedly the way money originated. Similarly, a football coach can use a diagram of Xs and Os to represent players and can then move the symbols around on paper instead of having to position actual large and cumbersome football players.

A caveman wanting to convey some vital numerical information, such as how many caribou he spotted down by the water hole or (perhaps more important) how many lions, could quickly gather some rocks or berries, matching each animal with its more portable representative, and return with the information literally in the palm of his hand. That's what numbers are, really: *information*. And the point is that the replacement of caribou by rocks doesn't change the information. It does, however, change the ease with which the information is held and transmitted. So that's why we do it. The history of language (and arithmetic in particular) is a history of increasingly abstract and portable substitutions: from actual antelope to drawings of antelope, to rocks, sticks, or fingers, to scratches on bones, to verbal utterances (i.e., words), to abstract symbols.



My guess is that each of these developments in the evolution of language arose from the need to explain and communicate information. Mathematics, in particular, is an explanatory art form, and ultimately all of its structures arise as "information carriers" for the purpose of explaining a pattern or idea. Numbers are simply the most basic kind of mathematical information.

So in this book I'm going to tell you the story of arithmetic, and in a large part this is a history of representation—different choices and strategies for organizing and communicating numerical information. Throughout this story, it will be important to keep in mind the distinction between a number in and of itself, as opposed to the way in which it is being represented. For concrete things like cats this isn't usually too difficult—I don't think too many people are going around petting the letters C, A, T. But numbers are a bit more elusive. I can easily understand how a conflation of the number three and the symbol 3 can occur. If 3 is merely a symbol representing the number three, then what exactly is three itself, independent of its representation? I've held cats and petted them, but how can I feel or touch *three*?

Of course, it's not as bad as some abstract ideas, such as hope or love—at least I can hold three oranges or three feathers in my hand. Nevertheless, *threeness*, that property of collections, is a bit intangible and wispy. Be that as it may, three has properties and truths (such as being odd) that are independent of its representation.

It is especially vital that we be clear about this distinction if we are going to examine representation systems other than those with which we are familiar. For example, the number thirteen does not "start with a one and end with a three." Numbers don't start or end at all; they just *are*. The Hindu-Arabic decimal place-value representation is the thing that starts and ends and has a look to it. Thirteen does a lot of things—it is prime, it is a sum of two square numbers (namely four and nine), and it is odd—but it doesn't "look like" anything. I myself like to think of numbers as creatures of some kind that exhibit various behaviors, and this helps me play and work with them, but I don't want to get stuck into thinking of them as having any particular visual form—or rather, I understand that the form they take is up to me, and I can represent them well or badly depending on my choices.

Among my favorite ways to represent quantities is to imagine them as piles of rocks. As we've seen, depending on the size of the collection there will be a myriad of possible visual arrangements, some more organized (and thus easier to remember and recognize) and some less so.

For each number of rocks up to twelve,
what is your favorite arrangement?

For example, the arrangements of pips on a die are particularly simple and easy to remember:

(By the way, the five shape ⚄ is called a *quincunx*, a word one does not often get the opportunity to use. Naturally, I jumped at the chance.)

If the only numbers we ever wanted to communicate were the numbers one through six—as in a game of Chutes and Ladders, for instance—then these dice arrangements would constitute a perfectly valid and convenient number language. It's the same with fingers. When someone (such as a referee) holds up two fingers of one hand and all five fingers of the other, everyone can immediately recognize the finger pattern of seven. The real problem in designing an intelligent numerical representation system is what to do when the quantities get big. And as we will see, the problem is not just that we run out of fingers. The problem is that we run out of *memory*.

Can you invent a way to represent the numbers
one through twenty using only the fingers of one hand?

among the many options being the waltz pattern of three beats per measure: TA-ta-ta / TA-ta-ta. Other standard choices include two-, four-, six-, eight-, and twelve-beat patterns.

Similarly, an arithmetician will make a choice of grouping size convenient for the counting problem at hand. Entire number languages—words, symbols, and calculating devices—are designed around such choices and become part of the tribal culture. The five-barred gate system, for instance, places emphasis on the number five and makes that particular quantity feel more "solid" or "clean" than other amounts. Nobody really likes leftovers; they feel like unpleasant remnants, or at any rate, details that will have to be remembered and kept track of. How nice to be able to say "four handfuls" instead of "three handfuls and (grumble) two leftovers."

Of course, one's grouping size is a completely arbitrary and personal choice. There is nothing particularly special about the number five, other than the fact that we happen to have that many fingers on each hand. Over the centuries people have experimented with many different grouping sizes in many different contexts. We can still see traces of these choices in the number words of various languages. The words *dozen* in English and *zwolf* in German reveal a history of grouping by twelves (which we still do with inches and eggs), and an ancient choice of twenty as a grouping size still resonates in the words *vingt* (French) and *score* (English).

Aside from the fact that we are curious and easily bored (which has indirectly led to art, science, and technology), another great hallmark of humanity is that we are *lazy*. We get tired of making busloads of tally marks, and we want to find a way out of doing it. The solution is *abbreviation*. This is the way we always use language. Any time we find ourselves saying the same thing over and over, we simply come up with a new, shorter word for it. This is true for both written and spoken language.

Imagine we are shepherds in nearby pastures (say in ancient Greece), and I want to tell you how many of my sheep were eaten by wolves during the night. Let's say we have worked out a system (a language, really) whereby I bang two sticks together for each lost sheep. Just as we tire of looking at endless rows of tally marks, we also aren't very good at keeping track of long sequences of identical sounds. So one idea would be to use a different sound—maybe a somewhat lower pitch—to indicate a group of five. Just as we write 𝍸𝍸𝍸||, we could tap out *clop-clop-clop-click-click* and save time and energy that way.

Similarly, if we get tired of making five-barred gates all day long, we could invent a new shorthand symbol, such as a horizontal stroke –, in place of the more time-consuming 𝍸. Then our number above could be written simply as –––||, or even ≡||. That's the kind of thing we humans do. We get lazy and bored and then we use our intelligence and creativity to come up with clever new ways to get out of doing work. We are all Tom Sawyers on some level. Necessity may be the mother of invention, but boredom is surely the father.

However we wish to communicate numbers—by words, gestures, or in written form—repetition, grouping, and abbreviation are the natural linguistic means. It's up to the users of such languages to decide on a convenient grouping size and how they wish to indicate both groups and leftovers.

So how do we decide on a good grouping size? What makes one size better than another? Of course it depends on what you are counting, who you will be communicating with, and, most important, how large the numbers are going to get.

If you are only counting a small handful of things (e.g., your marble collection), then it doesn't really matter whether or not you organize it into groups or what size groups you use. On the other hand, if the numbers you are using are very large, you may want to put a little thought into choosing an appropriate grouping size.

If your choice of grouping size is too small, say two or three, then it kind of defeats the whole purpose of grouping. At some point, even with the five-barred gate, the numbers will get so large that you will need a ton of groups. So many, in fact, that you will no longer be able to grasp at a glance how many groups you have. You will experience a "higher level" perception problem. As an illustration, can you tell which collection is larger?

𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸   𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸
𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 || 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |||

Clearly the thing to do is to organize the gates themselves into groups. And here we have an interesting choice. Each gate represents a group of five individuals; that is, we chose five as our grouping size. But now it is the gates themselves that become the individuals, and for the same reasons as before (perception and comparison), we want to organize *them* into groups. How should we bundle them? Should we group them by fives again or by some different number?

Suppose we decide to group the gates into fours. (This would be natural if we were thinking of the gates as handfuls, so that all our fingers and toes together make four hands' worth.) A nice abbreviation for four gates might be a square shape, □ (the four corners remind me that it stands for four gates). This allows us to write rather large numbers quite easily: □□□𝍸𝍸||.

The point is that with any repetition system, as the numbers get larger we will continually need new words and symbols. Each time we hit the next "perception wall" where we start losing track, we will need to decide on a grouping size and a representation for it. Throughout the centuries there have been hundreds of ingenious systems for doing this, some very simple and elegant, others somewhat awkward and annoying. I will be showing you some of my favorites to give you an idea of the range of possibilities.

Can you design your own system of repetition and grouping?

THREE TRIBES

I want to imagine a period before recorded history—let's say thirty thousand years ago, by the banks of the Nile. (This way I can make up stuff and not worry about being contradicted by actual facts). Let's imagine three groups of early humans living and interacting with each other, each tribe using its own unique number language with its own particular choice of grouping size.

We'll start with the Hand People. In this tribe, the handful—five fingers' worth—is the conventional grouping size, just as in the five-barred gate system. Being a preverbal society, the Hand People communicate using a system of hand gestures. The number one is represented by a single clap, two by two claps, and so on. Five claps are considered difficult to distinguish from four, so a group of five is instead represented by a thump on the chest with a closed fist. Thus, the number that we call seven would be communicated by the gesture: *thump-clap-clap*. The Hand People possess no spoken or written forms of communication.

Can you count to twenty in the language of the Hand People?

Our second tribe is the Banana People. This group expresses its numbers verbally. The word for one is *na*, two is *na-na*, and so on. The Banana People group in fours, and their word for a group of four (a bunch) is *ba*. So the number seven would be rendered in Banana language as *ba-na-na-na*.

Naturally, those Banana tribesmen who wish to converse with the neighboring Hand People will have to translate between the two systems. A Banana merchant would become quite fluent in both systems—either by intentional study or (more likely) simply from constant daily use—and immediately upon seeing a *thump* will think to herself *ba-na*, perhaps even unconsciously.

Try counting to twelve in Banana language.

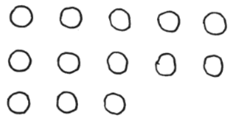
Our third group is the Tree People. In this tribe, numbers are communicated not by words or gestures but in written form—scratches on bark, let's say. Their grouping size is seven, and their first six numbers signs are:

Now, I myself can imagine getting a little confused by the last two. That is, I don't really trust my own sixness perception. But let's suppose that the Tree People can handle it.

Such a notation system clearly cannot continue indefinitely, so naturally they reserve a special symbol for a group of seven, namely a tree: . For the Tree People the number seven is quite special and sacred, being a solid and reliable number with no obnoxious leftovers. Of course, any Tree People who wish to exchange information (or bananas) with the other tribes would need to be able to translate fluently among the three systems.

Just to be clear, I am choosing to mention this hypothetical (and admittedly rather far-fetched) period of human history in order to illustrate a point. My point is that the artful rearrangement of numerical information—in particular, *the translation among different grouping sizes*—is the soul and essence of arithmetic.

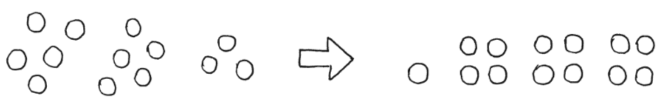
As you would expect, each of these three tribes has its own cultural norms and expectations, and grouping size is no exception. Growing up in a world where people group things in fours endows that number with a certain mystical and emotional significance. When, from your earliest memories, you hear people counting: *na*, *na-na*, *na-na-na*, *ba*, *ba-na*, *ba-na-na*, *ba-na-na-na*, *ba-ba*, the rhythm of it gets into your bloodstream and consequently *ba-ba* starts to feel like a very solid, safe, and understandable quantity, from which a number like *ba-ba-na-na* is an unpleasant departure.



In this scheme, each row represents a handful, or *thump*, with the last row being an incomplete row of leftovers.

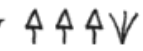
Either way, you now have a representation of the number that you can actually see and touch. And being primates with primate brains, we sure do like seeing and touching! That's why we make those "busy box" toys for infants—so they can turn the crank, open the door, and look in the mirror. (There are plenty of adult versions as well.)

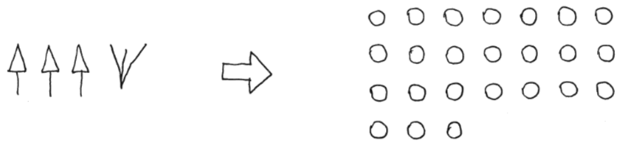
As humble as it may appear, the Piles of Rocks system is actually a very powerful calculating device. To use it, you simply rearrange the rocks into whatever different grouping sizes you wish. For example, our Hand-Banana translator (that is, *you*) can simply slide the rocks around to make a new pattern:



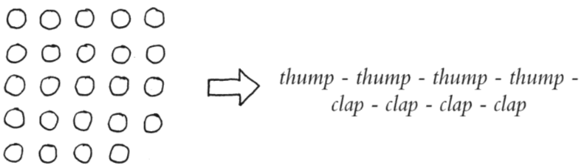
Now it is easy to see the new groups of size *ba*, and we see that we have a leftover as well. (Perhaps even more artfully, you could simply move one rock in the earlier diagram from the second row to the third.) So we can confidently tell our fellow Banana tribesmen that the offer is *ba-ba-ba-na*.

Piles of Rocks is the world's first calculator. Notice that the user does not need to have any special knowledge or skill, just the ability to put rocks in piles (and not lose any, I guess). The downside is having to find (or carry around) a bunch of rocks or other convenient objects.

Let's try another example. Suppose now that the Tree People also wish to trade for some Hand People tools. They have carved wooden beads to trade and will pay  beads for each tool. You, as a somewhat inexperienced Tree-Hand translator, decide to get out your Piles of Rocks:



Rearranging this into handfuls (crudely or cleverly, depending on your taste and skill) yields the arrangement:



This is what is known as "doing arithmetic." You have taken numerical information presented in one form, and you have reorganized it into another. Congratulations. Arithmetic doesn't actually get much more complicated than this, so if rearranging piles of rocks into different-sized groups makes sense to you, then it's probably going to be pretty smooth sailing from here on out. Of course, there's the question of getting *really* good at it—but that's entirely up to you and how much you want to practice and play around with it.

Now, you may have noticed that the number in the preceding example was rather large as numbers go. In fact, it is the largest number conveniently sayable in the language of the Hand People. Of course, they could say (and try to hear and remember) a number like *thump-thump-thump-thump-thump-thump-thump-thump*, but this brings back the whole perception problem again, which was the motivation for grouping and naming in the first place. The natural thing to do when numbers get big would be to start grouping the groups.

We'll focus on the Banana People. I want to imagine that a few centuries have gone by and that they have picked up the idea of using tools from the Hand People and also the idea of writing from the Tree People.

While we're at it, let's say that an entire Banana Civilization has come into full flower and with it the need to communicate larger numbers and more complicated transactions. The idea of four as a special and satisfying quantity remains, however.

The new written system might look like this. For each leftover we make a curved stroke, \ (this being, as we all know, the universal symbol for a single banana). For a bunch of four bananas, we use a four-sided figure, □. Thus, the number ten (*ba-ba-na-na*) would be written □□\\.

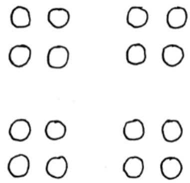
Naturally, after a certain point, the number of *ba*'s would become so large that we would want to start grouping *them* as well. How many should constitute a group of groups? This is one of the more important decisions one has to make when designing a numerical representation system.

You might think that there's no question about it; we chose four (*ba*) as our grouping size, so that's that. Of *course* we will group our groups into fours—as opposed to what? Grouping things into fours and then grouping our groups into sixes? That would be insane!

But it happens all the time. Inches are grouped into twelves to make feet, and then three feet make a yard. And the old British monetary system had twelve pence to the shilling and twenty shillings to the pound. Such mixed-base number systems have always existed and continue to exist. The Babylonians, for instance, grouped their tens into "big groups" of sixty; that is, six groups of ten was their grouping of groups. Still, if one had one's way, it seems simplest to pick a grouping size and stick with it.

So let's suppose that for the same reasons that the Banana

People group their bananas into bunches, they also want to group their bunches into bunches. This quantity (*ba-ba-ba-ba*) is called *la* and is written ⊞. This is a nice symbolic representation of a bunch of bunches:



Now larger numbers can be more easily represented and compared. For example, the number that the Tree People write as ⍋⍋⍋ would be understood by the Banana People as ⊞□\ (*la-ba-na*).

How would the Banana People say and write ⍋⍋⍋⍋?

These three tribal number languages are examples of what are called *marked-value systems*, meaning that each word or symbol has a fixed, definite meaning. No matter where the symbol ⍋ or ⊞ occurs, it always represents the same quantity. In particular, for representation systems like these, the order of the symbols doesn't matter. Just as the change in your pocket can jiggle around without affecting its value, the representations

⊞□□\\\   \⊞\□\□   \□⊞\\□

all refer to the exact same number. This is a very convenient feature and makes systems like these particularly easy to work with. Marked-value systems (such as money) are very sturdy; you can scramble up all the symbols on the page (or all the coins in the jar) and it won't affect any of the information. Of course, it will affect the *form* that the information is in, so if you wanted it organized in a particular way then you might want to be a little more careful.

In particular, most people like to arrange their numbers (and often their money as well) into a form that makes for easy comparison. Usually this is done by collecting together the largest denominations first, and then working your way down to the smallest amounts. In other words, even though we can write our symbols in any order, it pays to put them down *in order of importance*. This makes it easy to see what the big picture is: "Oh. I've got five hundred and something," as opposed to giving a penny and a twenty-dollar bill equal status.

Any pile of rocks on the table represents that number of things no matter how it is arranged, but some arrangements are more communicative than others. Even though the representations ⊞□□\\\ and \\⊞□\□ refer to the exact same number, the first one screams out the headline "*la* and then some," whereas the second one hides its largest quantity under a bushel of bananas.

So we will usually want to express our numbers this way, from largest to smallest (whether that is written from left to right or not is a cultural choice). Again, the point of doing arithmetic is for comparison, and expressing our quantities in this way makes comparison particularly convenient.

For the same reason, we usually prefer our number representations to be "packed up" into nice groups, rather than spilled out all over the place like so many socks and underwear. It can be confusing to compare packed and unpacked representations:

⊞⊞□\\   □□□□□
□□□□□

Here the second number is actually larger, even though it has no *la*'s explicitly written. So although the second representation is perfectly meaningful and unambiguous, it is somewhat inconvenient for comparison purposes.

On the other hand, nothing could be simpler than to compare two numbers that are both nicely packed into groups and organized by size.

Rather than allow this, the Egyptians had a different plan: nine is three rows of three.

nine

So the stacking patterns allow us to tell at a glance what we've got. As with dice, the designs are simple enough to be easily learned and recognized. We even have some flexibility in our choice of stacking patterns. The Egyptians often wrote four as two on top of two, for instance.

Since our grouping size is ten, we will need a special symbol for it. The Egyptians chose ∩, which is thought to represent a heel mark (presumably coming from the measurement of plots of land by digging in one's heels every ten paces).

Naturally, we use the same stacking patterns for groups as we did with leftovers. Thus, the number that we call forty-five would be written ∩∩∩∩ |||||. It is also perfectly fine to write it the other way, as ||||| ∩∩∩∩, depending on whether you are writing right to left or left to right. The Egyptians did both.

Of course, as the numbers we are dealing with get larger— and civilization tends to make that happen, what with storing grain and building armies and raising taxes—we will need symbols of greater value to denote groups of groups and so on.

The Egyptian symbol for a group of groups (that is, one hundred) is 𓍢, supposedly a coil of rope. For a group of 𓍢's (what we call one thousand) they used the lotus flower symbol, 𓆼.

The Banana People have been conquered by the Egyptians
and now they must learn to use the Egyptian system.
How would these numbers be translated:
⊞□□\\, ⊞⊞□\\\\\\, ⊞⊞⊞□□\\\\ *?*

Another wonderful innovation, which replaces Piles of Rocks as a calculation tool, was the introduction of *counting coins*. The idea is simply to make objects (usually wooden or ceramic chips of some kind) that are marked so as to indicate their value. These coins do not need to have any actual worth whatsoever, just as long as they *stand* for specific values. In a way, this is another level of abstraction: a coin marked ∩ represents a pile of ten rocks, which in turn represents whatever it is that you are actually counting.

Let's imagine that we have a large bag full of these counters, marked with the various symbols | , ∩ , 𓍢 , and 𓆼 . We can then spill them out onto the table (also known as the counter) and sort them and arrange them into piles or rows however we wish. This then becomes a simple and convenient calculating device, or *abacus*. An abacus is simply a manual representation system; that is, a way of representing numbers by things that can be held and manipulated. Just as with Piles of Rocks, counting coins can be grouped and rearranged easily, and calculating can be done relatively quickly. Then the results can be written down at the end, if desired.

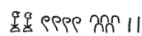
Make your own set of Egyptian counting coins.

Of course, for very simple calculations, an abacus system is usually unnecessary; you can often do such computations in your head, rearranging symbols and keeping track of information as you go. This requires a decent memory, however, and many people throughout history have found this sort of thing annoying and mentally painful. On the other hand, lots of people really enjoy mental arithmetic and even consider the use of an abacus system to be cheating or in some way beneath their dignity. No matter. If you enjoy storing and moving around detailed information in your head, then please do. If you would prefer to use some sort of calculating device, then be my guest. Both approaches to arithmetic are fun and full of surprises and amusement.

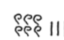
Anyway, to use the Egyptian counting coins is simplicity itself. We merely translate the numbers directly, each written symbol being replaced by the corresponding marked coin, like so:

And now our number can be manipulated (literally!), set aside, or combined with other quantities easily.

Imagine you are an Egyptian scribe, circa 1850 BC. The Pharaoh has asked for an accurate accounting of the grain supplies. The three granaries contain the following amounts, measured in bushels:

Thebes Granary: 

Giza Granary: 

Aswan Granary: 

What is the total amount of grain?

Assuming you are not comfortable calculating this total in your head, or you don't want to risk making an error and incurring the wrath of the Pharaoh, you make your way to the counting house and get behind the counter. Taking out your sack of counting coins, you proceed to lay out the numbers carefully:

To calculate the total, you simply push the piles of coins together:

The great thing about marked-value abacus systems is their wonderful flexibility—it doesn't matter how you scramble them up or rearrange them, the coins will always represent the same quantity. (Later, we will see some more modern abacus systems that are far more delicate but have their own advantages.)

So that's it, you've done the computation! The total is exactly the amount sitting there on the table. There are a couple of problems with this representation, however. First, it is inconvenient to have to carry a bunch of coins around (and you could accidentally drop a few, which, if they happened to be lotus coins, would be disastrous). We would prefer a more portable written representation. Secondly, for the purposes of comparison (which will undoubtedly occur at some point), we would want the number packed up into tidy groups, not eighteen coils of rope in a row.

So the next step is to "cash in" or *exchange* the coins until they are as compact as possible. In particular, if we have a group of ten coins of the same kind, say heel marks ∩, we can trade them in for one coil of rope 𓍢. This clearly doesn't change the actual number we are talking about, because 𓍢 literally means ten ∩'s. This exchange maneuver does however affect the *form* our representation takes, and that is in fact the entire point. Every number can be represented in a variety of ways, and we want to choose a form that is as useful and convenient as possible. Sometimes, happily, this means doing absolutely nothing. Other times, we may want to "clean up" the representation a bit. It all depends on the circumstances and what we want in the particular situation. In this case,