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Artificial Mathematical Intelligence

Cognitive, (Meta)mathematical, Physical, and Philosophical Foundations

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Contents

Acronyms

m.-s. Morpho-syntactic: It denotes the purely symbolic part of a conceptual entity together with the domain-specific syntactic rules that configure its atomic parts. For instance, the morpho-syntactic part of a mathematical entity is understood as the specific notation used for describing it symbolically.

Chapter 1
Global Introduction to the Artificial Mathematical Intelligence General Program

1.1 A Quite Revolutionary "Artificial Mathematical" Vision

More than eight decades ago, a brilliant scientist astonished the mathematical community with his simple and, at the same time, powerful formal notion of what an (autonomous) machine should be. With his new precise concept, he was able, on the one hand, to set initial bounds to the deductive scope that such machines possess regarding decision paradigms in formal mathematical thinking. On the other hand, he was able implicitly to show how powerful, useful, and universal his new devices could be with regards to enlightening and manipulating not only numerical, but also conceptual issues in mathematics.

Fourteen years, he once again surprised a broader community with an even simpler, more suggestive idea: could it be possible to verify in later, a pragmatic way if a physical realization of his formal machines can imitate "human intelligence" in an indistinguishable manner? Perhaps without knowing it, he indirectly also established a global interest in determining more precisely the real bounds for the general pragmatic scope that (his conceptual) machines have regarding the wide spectrum of intellectual activities that human beings can do.

More specifically, the former question is a generalization of the following deep request: Can we construct a specific instance of the former kind of machines with the ability to imitate an ideal universal mathematician,[1] when we restrict the conversation to the search of a (human-style understandable) solution of a specific

[1]By "an ideal universal mathematician," we mean a human being with extraordinary mathematical abilities, who has a universal mathematical ability and a basic knowledge of all mathematics done until the present time. In particular, (s)he should be able to provide solutions to (meta-)mathematical problems, (resp. problems with a mathematical nature) in the style of Arquimedes, Diophantus, Gauss, Euler, Riemann, Cauchy, Hilbert, Klein, Gödel, or/and von Neumann, among others.

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mathematical inquiry?[2] If one takes a closer look into the intellectual and 'secret' work of our inspiring thinker,[3] one can perceive that he was inspired by questions similar to the former one for inventing most of his astonishing concepts and devices. So, as some readers may already suspect, our mysterious figure is one of the leading founders of modern computer science and artificial intelligence—Alan Mathison Turing.[4]

If we compare the legacy of Alan Turing, as a mathematician and logician, with the work of his (contemporary and predecessor) colleagues, we can affirm that he was essentially the first one who quite seriously and in a pragmatic way offered structural insights to the question concerning the possibility that the intellectual human activity of doing (abstract) mathematics could be simulated concretely by an artificial engine.

Inspired by the most outstanding scientific and practical achievements of Alan Turing, we would suggest that the construction of a real artificial device able to perform mathematical intelligence at a global level, with a human-style manner and simulating, and even improving the (mathematical) minds of the most outstanding mathematicians (as individuals as well as a group) could be considered as one of the most important scientific programs and implicit visions that Alan Turing indirectly gave us. This vision is heuristically supported by the fact that mathematical generation, in general, obeys quite clear deductive and methodological rules, which are closely related to mechanical and systematic (artificial) processes.

Let us call the former global vision *Artificial Mathematical Intelligence* (AMI), in honor of one of the seminal founders of Artificial Intelligence: Alan Turing.

We will now systematically explain the central foundational principles of this vision.

First, let us note that this (AMI) vision possesses a broad multi- and interdisciplinary methodological nature. In other words, the fact that this kind of updated Turing's vision involves concepts like mathematics, intelligence, simulation and the (human) mind implies that it requires the integration and subsequent combination of results belonging not only to mathematics, metamathematics, logic, computer science and artificial intelligence, but also (cognitive) linguistics, (cognitive) psychology, neuroscience, cognitive science, physics and philosophy (of mind), among others. This methodological requirement is highly desirable and virtually necessary due to two facts: First, on the one hand, we want to get a "surgical" knowledge of how the human mind proceeds by solving a mathematical inquiry. On the other hand, the scientific knowledge that we have about the mind (together with its physical

[2]By mathematical inquiry, we mean any kind of mathematical formal computation, exercise, and conjecture at an arbitrary degree of sophistication.

[3]It is worth mentioning that during the time between the initial invention of his (conceptual) machines and the publication of his seminal ideas about intelligence and machines, he was applying all the pragmatic power of his technological instruments in a special team that was able to shorten the second world war by at least 2 years and save millions of lives worldwide.

[4]For the references supporting the former paragraphs see [28, 48, 49] and www.turingarchive.org.

"mirrors" like the brain and more generally the body) is not centralized and, in fact, it is spread among several disciplines like the ones mentioned above.

Second, due to the fact that some aspects of the AMI vision have already been intensively studied since the beginning of AI, most of the partial solutions offered have a mono-disciplinary nature (e.g., each of them possesses a purely metamathematical, linguistic, or philosophical character). For instance, Turing himself, as well as Kurt Gödel and Alonso Church, offered local negative answers, mainly from a logical point of view, to a purely metamathematical reading of the AMI plan [9, 17, 48]. More specifically, by doing a closer and detailed reading of the former classic results, one observes the following:

On the one hand, the concepts and arguments used have a structural mathematical character when the central demonstrations are presented. Nonetheless, the initial and final conclusions are extended at a meta-level and therefore lastly they have a metamathematical scope (e.g. the notion of Turing machine can be completely formalized in mathematical terms and simultaneously be used to infer metamathematical results like the insolubility of the Entscheidungsproblem). Contrastingly, a purely logical approach to the AMI vision turns out to be essentially blind regarding concept formation in pure mathematics (see, for instance, [31]), which is one of the most fundamental aspects underlining mathematical generation.

Now, if we wear a "methodological glasses" coming from cognitive science, then we can affirm based on the former fact that those results were essentially a cognitive product of three (quite brilliant) minds described with an intrinsic mathematical style. So, inspired by the most outstanding results in cognitive science regarding formal reasoning (e.g., assuming the veracity of the computational nature of the mind, at least in relation with mathematical creation [29]), there is no methodological obstacle for the construction of a kind of universal mathematical and logic artificial agent that is able to give answers similar to those given by Turing (himself), Gödel and Church, to the same kind of questions.[5,6]

Therefore, the former results are valuable more from a mono-disciplinary point of view. However, if we assume a more pragmatic perspective regarding the way in which mathematics is concretely produced at a global scale, we see (after enough time of search) that roughly speaking more than 90% of the mathematical results

[5]It is worth mentioning at this point that although the former three classic results possess a high level of brilliance in their central ideas, none of them develop an explicit explanation of how human-made mathematics are done from a cognitive perspective. Perhaps the only one who was able to achieving that was Turing with the development of his seminal concept of Turing machine.

[6]It is also important to mention that a lot of methods used by the former authors for obtaining their limiting results were based on some kind of meta-physical assumptions about the nature of space, time, and spacetime, like the existence of infinite collections of numbers and temporal processes. Now, these assumptions are, strictly speaking, not based on standard physical laws, therefore can be classified from a physical and cognitive perspective, as subjective mental assumptions from the corresponding authors, with a wide level of (subjective) acceptance by the general community. For a deeper discussion about these quantitative issues please see Chap. 5.

produced each year involve solvable and decidable mathematical inquiries.[7] So, our AMI program has a fundamental pragmatic importance in spite of the former limiting results.

A further ontological revision of the former three classic meta-theorems reveals that each of them use as implicit foundational principle—the existence of the natural numbers at a classic indefinite basis, i.e., the existence of infinite collection of objects (e.g., numbers) generated sequentially; or an equivalent version of this fact. Enhancing our cognitive glasses with additional “formal lenses” coming from seminal results in modern physics, one can see that the former hypothesis possesses more a mental nature than a practical and authentic physical substratum (for more details see Chap. 5). Hence, this shows that since those meta-results were based implicitly on a mental construct (e.g., a classic understanding of the natural number as a sequential structure without final quantity), the deductive scope should be preponderantly mental and less pragmatic as argued previously.

Third, after the groundbreaking work of Alan Turing regarding the foundations of AI at the beginning of the twentieth century, we have seen a tremendous number of theoretical and practical advances in our global understanding of how the mind works and how we can simulate, and sometimes improve, specific aspects and abilities of it through artificial devices (see, for instance, [4, 12, 35] and Sect. 3.1, Chap. 3). This gives the necessary inspiration to think about the AMI vision in terms of the creation of a *universal mathematical artificial agent* (UMAA) whose percepts and actions (in a classic AI sense [43, Ch.2]) consist exactly of mathematical structures (e.g., mathematical concepts, theorems, facts, conjectures) described in a clear, syntactic, and human-style way; its environment is delimited by the mathematical information that a user would like to provide it; and finally its sensors, actuators, and internal engine are formed in terms of specific and computationally-feasible formalizations of the most fundamental cognitive abilities used by the mind during mathematical research, among others (see, for instance, part II).

Finally, its performance measure is structured in direct relation to the conceptual solutions that it provides to the problems asked, for example, in the form of (mathematical) total or partial proofs or counterexamples.

So, arguing from a more philosophical perspective, an UMAA would be capable of reminding us an ideal universal mathematician-logician whose formal thinking is rapid, effective and not so constrained by physical, sensorial, or cultural influences. An abstract thinker who is able to create sophisticated mathematical concepts and (counter-)examples based on the questions[8] and (formal) evidence[9] provided by the interlocutor (e.g., user). In particular, this universal mathematician-logician should

[7] Here, the reader can conduct a self-tour through the articles and books published at renowned mathematical journals and by publishers, to obtain a stronger conviction of this global qualitative claim. For example, one can verify that more than 90% of the last 100 papers (until the first semester of 2019) published in renowned journals like Annals of Mathematics involve solvable problems dealing with the usage of classic concepts and with the constructions of new mathematical concepts, theories, and some physical applications.

[8] For example, a mathematical conjecture.

[9] For instance, successful cases where the conjecture was verified.

be able to solve formal mathematical inquiries and generate mathematical structures at a new level of sophistication.[10]

In other words, we conceive the fulfillment of the AMI program more from an interactive point of view than from an automatic one. Explicitly, let us imagine that the user provides not only the formal version of a mathematical conjecture, but also additional information describing special cases where the conjecture turns out to be truth (e.g., “formal specific evidence”). Such “special” interactive conditions and extra information can cut back several additional challenges that a purely automatic approach could potentially possess.

At this point, it is worth mentioning that the notions of algorithmic complexity and efficiency should be conceived with slightly different “eyes” within the AMI program. Effectively, one of the main goals of our vision will be to produce detailed and gradually explainable solutions to formal (domain-specific) mathematical problems, which would take less time to be found than the time required by a professional researcher (in mathematics or related areas). For example, our vision goes in the direction of imaging that a robust version of a UMAA which should require some months of (interactive) work (e.g., 6 months) for solving a Ph.D.-level mathematical problem, which turns out to be fine in comparison with the standard time that a Ph.D. thesis takes to be done (e.g., 3–4 years). Additionally, suppose that the rate of success of it is around 80%, this would represent a huge step towards an universal and cognitively-inspired (interactive) mechanization of mathematics.[11]

In summary, we want to “resurrect” through the AMI program a new form of what we could call, in modern terms, one of the biggest dreams of Alan Turing: the fulfillment of artificial intelligence within the special domain of (pragmatic) mathematical creation/invention.

> “Those who can imagine anything, can create the impossible.”
>
> Alan Turing

1.2 Towards Conceptual Computation

Let us make a simple comparison between the brain (and in an extended manner the (embodied) mind) and the computer: First, regarding working speed, the human mind processes information (roughly speaking) around six million eight hundred thousand times slower than an (average) computer and it is around one billion (10^9)

[10] See, for example, Chap. 11 for initial formal evidence regarding the universal way in which such an (artificial) researcher should be able to generate several mathematical structures from different mathematical domains.

[11] A quite simpler version of the AMI vision was described initially by the author in [20] at a very elementary and condense way.

times less accurate (regarding the error's rate per number of operations performed) [1]. On the other hand, the human mind is able to do very simple (and at the same time very powerful) conceptual inferences like if y =cymbal and Y =tambour, then $yYyYyYy$ =drum set; or, if A =house and B =boat, then AB =houseboat;[12] while this ability is essentially non-existence in (modern) computers. Furthermore, more than 95% of the scientific results generated in history are product of the (research of the) human mind, with all the former limitations and strengthens. Thus, why not to simulate the deductive-pragmatic functioning of the mind (taking into account the former spectrum of features) with all the strengthens of modern computation?

This kind of simple methodological approach is not so common in automatic deduction or computational logic. In fact, one of the main methodological approaches is to reduce the conceptual complexity of the problem to solve until it can be fully verified or refused computationally, instead of modeling computationally the manner in which the mind approaches the problem (without performing necessarily an ontological reduction on the way). The latter form of obtaining computationally-feasible solutions should be explored more deeply, because we have a sufficiently robust constellation of results in cognitive sciences (and related fields) describing a wide spectrum of (deductive) features of the mind. This motivates the quest for a new form of *conceptual computation* paradigm in computer science and artificial intelligence. In fact, the initial motivation of Alan Turing to create its famous Turing machines was to find a concrete and pragmatic formalization of the way in which a mathematician's mind perform quantitative tasks (involving, for example, the calculation (by hand) of a function on the natural numbers) [48]. Therefore, we can think in extending the classic (Church-)Turing Thesis (or Turing Theorem (TT)) to a general metamathematical conceptual framework:

Thesis 1.1 (Towards a Conceptual Extension of the Church–Turing Thesis) *A Mathematical structure (e.g., a concept, a proof, a counterexample, a theory) is effectively calculable (i.e., generated) by a human being('s mind) if and only if it can be computed by a "conceptual" (Turing) Machine (e.g., UMAA (Universal Mathematical Artificial Agent)).*

One of the main purposes of this book is to offer a general idea of how such a conceptual machine should look likes in terms of initial formalizations of a global taxonomy of fundamental cognitive mechanisms (see Chap. 10). In fact, we will prove in Chap. 9 that just with the cognitive ability of conceptual substratum it is possible to recover and to reinterpret cognitively the classic Church–Turing Thesis. Thus, this represents a starting evidence of the fact that conceptual machines should be at least as powerful as classic Turing machines. On the other hand, in Chaps. 7 and 11 it will be extensively shown that conceptual machines are a more appropriate formal device for producing artificial conceptual generations of

[12] Here, we assume the standard and relatively sophisticated meaning of the word "houseboat" in English.

dozens of mathematical structures from the most simple until the most sophisticate ones, which currently is not the case for the contemporary literature in automated deduction (and related fields).

1.3 Former and Current (Local) Advances Towards the AMI Vision

A lot of (mechanical and computational) aspects of our AMI vision have captured the attention of a considerable number of researchers during the last decades. Most of them have done amazing, valuable work which can be seen in our context as local evidence and support in favor of its ("near") fulfillment. In this section, we will mention some of the most outstanding results together with further remarks concerning their main original goals.[13]

At its very beginning, the research field of automated deduction had, as part of its central motivations, the construction of software able to generate (and implicitly solve) concrete mathematical work (e.g., outstanding mathematical theories/books). For example, Whitehead and Russell's Principia Mathematica [38, 51]; elementary plane Euclidean geometry [16], (some parts of) Newton's Principia [14], and, of course, many instances of propositional calculus [3], among (a few) others. On the other hand, some specific mathematical challenges as the Robbins problem, the four color's problem, the Kepler's theorem, and the Feit–Thompson theorem have given (significant) additional inspiration for developing more sophisticated (automated) theorem provers [22, 23, 25, 36].

Furthermore, nowadays there are many kinds of (free and paid) computer programs which can assist the researcher in mathematics (and related areas) on different tasks. Now, these have always involved a relatively small collection of mathematical areas, for instance, numerical and symbolic computation, the drawing of technical graphics, solving particular classes of systems of equations, inequalities, Diophantine and differential equations and quantifier elimination, among others [7, 34, 46, 52] (for a more general list see "The Guide to Available Mathematical Software"[14]).

Other kinds of outstanding software are used for finding proofs in several classes of propositional calculi and for proof verification and proof generation in some specific logics which, in principle, do not cover completely the scope of the mathematics done every day, not only by professional mathematicians, but also for researchers working in related fields [3, 42]. Furthermore, some instances of the later kind of software mentioned possess, in general, such a highly technical syntax that for the non-specialized mathematician (or related researcher) it is not

[13]It is not the purpose of the present section to give an exhaustive list of all of them, due to the fact that the literature in this direction is considerably vast.

[14]https://gams.nist.gov.

straightforward to begin to use it in his/her daily work, mainly because it would require several weeks (or even months) of regular and quite technical study to understand and manipulate practically their main semantic and syntactic features.

There are also a third kind of valuable programs aiming to produce human-style proofs by integrating the more robust account of the linguistic dimension involved in mathematical generation. However, its scope involves only very particular kinds of problems within quite specific theories, e.g., metric space theory (see [15] and the references there).

Furthermore, there are new proposals for setting general foundational frameworks for mathematics that aim to facilitate the implementations of mathematical proofs in computers, for instance, the univalent foundations project [50].

Most of the former works belong to what can be roughly called "The (classic) Mechanization of Mathematics" (see, for instance, [2]). They are "classic" in the sense that their methodologies and goals possess essentially a more purely logical, metamathematical, and algorithmic nature, and, on the other hand, they provide a reduced (and often nonexistent) formal account of the cognitive causes that underlines the origin and structure of the corresponding explanatory frameworks. In other words, an (implicit) external ontological point of view of mathematics is perceived in those works, i.e., the corresponding mathematical phenomena is analyzed as external entities that may or may not have a cognitive origin.

For instance, one (classic) trend in this direction, and one not so intimately related with a cognitively-inspired model of mathematical invention, is formed by the main techniques coming from resolution theorem proving, whose slightly different motivation and orientation emerges more from the need of finding efficient methods in proof verification and proof generation [42, Ch. 2].

So, keeping in mind the great value and brilliance some of those results can have, most of them possess the limitation that they cannot be generalized in a straightforward way to other mathematical domains, because their explanatory ontology depends structurally on the particular mathematical entities in consideration.

However, there is a complementary trend of formal and computational frameworks based more on a cognitive understanding of mathematical generation and starting with the identification and formalization of fundamental cognitive abilities used by the mind during creative thinking (see, for instance, [5, 8, 13, 18, 19, 21, 32, 33, 37, 44, 45, 47]).

Other quite outstanding works explore and exploit domain-specific mathematical heuristic at a computational basis for (automated) concept and conjecture generation and verification [10, 11, 39].

Moreover, there are more traditional treatises with a more philosophical touch, and simultaneously with deep insights regarding the identification of fundamental (mathematical) heuristics and cognitive strategies used in mathematical creation/invention like the classic works of G. Polya and I. Lakatos [31, 40, 41]. In fact, [39] presents a creative computational account of Lakatos' work where an initial formalization of the social dimension of mathematical generation plays a central role.

The former second type of results represent strong formal evidence in favor not only of the thesis that specific mathematical thinking can be gradually understood and subsequently simulated in a computational way, but also bringing an additional and enlightening new perspective into the AMI vision that classic purely logical approaches have only been barely able to suggest. Nonetheless, these more cognitively- and heuristically-based works present a clear constrain regarding the level of sophistication of the mathematical structures meta-analyzed and simulated. These works deal essentially with elementary problems belonging to, for example, geometry, real and complex analysis, algebra and number theory, among others. So, more abstract and general mathematical sub-disciplines are virtually not meta-studied, for instance, modern algebraic geometry [24, 27, 30], which represents an integrative, illuminating and fascinating case of study for the fulfillment of the AMI program. This is due to the high level of technical sophistication and elegance of its concepts and methods.

1.4 A New Foundational and Integrative Program

After exploring in the former section the strengths and weaknesses of some of the existent (local) results towards the fulfillment of the AMI vision, we will now describe the precise way in which we aim to used and integrate classic and modern techniques and perspectives and to create new ones for filling some important foundational and pragmatic gaps existent in the literature, as well as for setting a stronger inter- and multidisciplinary basis.

Virtually all the former (local) results towards a positive solution of the AMI program, with exception of the univalent foundations project and the classic works of Polya and Lakatos, propose almost immediate algorithmic formalizations of the particular classes of mathematical inquiries to be solved. This is usually done without a previous solid and deeper exploration and search into the foundational properties of the mathematical structures to be computationally modeled.

Methodologically speaking, this can be done without problem, however, such straightforward approaches have the limitation that they should create highly technical representations for the semantic content of the corresponding mathematical structures involved, which has the cost of sacrificing the "cognitive naturalness" of the whole framework. Furthermore, an ontological gap used to remain implicit "in the air" between the intrinsic nature and (cognitive) meaning of the mathematical entities involved, and the corresponding "artificial" juxtapositions of symbols used for representing them syntactically as well as semantically in artificial devices.

Contrastingly, aiming to go directly to the development of algorithmic frameworks after having analyzing (only) local mathematical data has the clear limitation of implicitly ignoring further heuristic, syntactic, morphological, and semantic principles that ground other mathematical areas and that should be mandatory for any kind of global explanatory (cognitive) metamathematical framework. In fact, we will see in further chapters that the meta-analysis of several kinds of conceptual substrata (Chap. 9) is necessary for the subsequent fulfillment of the AMI vision.

In particular, it involves an integrative symbolic, semantic, and cognitive meta-study of prototypical substrata belonging to a not-small collection of mathematical sub-disciplines, which remains (as far as we know) a non-accomplished task in (classic) metamathematics. Therefore, the present work focuses essentially on the theoretical foundations of the AMI program, together with some relevant remarks for the algorithmic aspects.

Explicitly, we propose in the first part of this book the establishment of a research program aim to set new cognitive foundations for mathematics, which includes implicitly a computational component.

We describe in Chap. 3 the main reasons supporting the necessity of a new foundational program for mathematics and its most fundamental future challenges. In addition, we discuss seminal issues involving the cognitive substratum of a mathematical proof in a wide generality. Further, we enlighten a central fact of the most successful natural machine producing mathematics, i.e., we describe the methodological implications for the AMI program from the fact that the whole mathematical product that we know today is basically the systematic accumulation of billions of conscious outputs of the human mind, considered also collectively.[15]

Inspired by "cosmological" and "synthetic" considerations, we do a deeper philosophical exploration (in Chap. 4) into the (cognitive) reality of mathematics and into the mathematics of the (cognitive) reality. Moreover, in our methodological framework we update the notions of observer's perspective at the macro, mecro, [16] and micro level. We argue in favor of the thesis that nature at any level of observation possess a kind of mathematical precision, and that, in fact, entities in nature possess a real mathematical substratum which structure them. So, mathematics understood in the widest sense of the word constitutes an existing dimension of the universe that structures any part of it. Subsequently, we state and support the existence of a kind of unpredictability principle at the mecro level. In other words, we argue that natural human will represent an example of a concrete entity in nature that qualitatively bounds the predictive scope of the kind of phenomena that any form of UMAA could model. Further, taking inspiration from some thought experiments and the development of a "continuous notation" for real numbers, we show the incompatibility of the following: the fulfillment of the "singularity" as an extreme form of artificial intelligence [6], and the fact that space and time can be modeled with a continuous framework, e.g. using the set of real numbers.[17] In summary, we conduct a concise intellectual exploration using philosophical, physical, mathematical, and logic tools for estimating more precisely the ontological status of the explanatory scope that a UMAA can have, not only concerning mathematical questions, but also (mathematical reformulations of) questions belonging to other scientific disciplines.

[15]This simple fact has important consequences on the way in which some cognitive abilities required in mathematical research are identified and formalized (see part II).

[16]For a more concrete description of this notion see Chap. 4, Sect. 4.3.

[17]Strictly speaking, we use a third fact for doing that, namely the unsolvability of the halting problem [48].

Chapter 2 serves as a compact and quite concise (meta)mathematical preparation for the non-specialist (mathematician/logician) reader. It briefly revises the notions of propositional and predicative logic, the most outstanding logical frameworks for modern mathematics (e.g., ZFC and NBG set theory, Peano arithmetic), and the notion of category and some of its derived notions. Moreover, a short description of fundamental algebraic, topological, and geometric notions are presented that are mostly required in Chaps. 7 and 11.

In Chap. 5, we start with the development of a seminal (cornerstone) topic within the new cognitive foundations' program. In other words, inspired by a formal and multifaceted analysis of our basic understanding of (mental) counting processes, we propose a concrete cognitive refinement of one of the most well-known structures in mathematics, namely the natural numbers. In fact, we present the *physical numbers* as a more precise quantitative notion which includes, and at the same time, refines classic perceptions that we use on a daily basis when we estimate the "number" of elements of collections of objects.

More explicitly, we enhance the standard notion of "counting" by the new notion of partitioning, and we show that the former can be considered a particular form of the latter where our minds can potentially gain a more global and precise perception of numerical (and subsequently mathematical) entities.

Additionally, we state that the physical numbers have an initial as well as a final entity, which is bound by the number of physical quanta in our universe. This allows us to make a finer (cognitive) taxonomy of the natural numbers, (i.e., natural number n "smaller" or equal that such a bound, which we denote by ω, are considered "physical natural numbers" since they count on a physical support represented by collections of (external) entities (e.g., elementary particles) having exactly n elements). On the other hand, a natural number m strictly surpassing ω will be considered simply as a "mental natural number," because it can be (cognitively) produced recursively as a concrete conceptual blend of physical natural numbers. It is a well-known fact that conceptual blending (see Chap. 7) can produce purely mental objects by combining two input concepts which have (or have not) physical realizations in the external realm [13].

Moreover, we establish a quite significant distinction between the (mental) notion of "infinity" (in its several variants) and the (more physical) notion of "immensity." In particular, we propose notions of small and immense numbers based on the specific conscious and unconscious patterns required for the mind in order to understand them.

In addition, we explore in a global way the fact that the explanatory range of mathematical frameworks based on numerical structures being finite as a whole, is mature enough to allow us to develop a lot of our most fundamental mathematical and physical theories.

Finally, we offer an initial formal framework for the physical numbers with (physical) division as main operation. We also propose a new kind of research heuristic in (classic) number theory which can be informally called "physical number theory." This essentially consists of doing an initial verification of the (non-)validity of an arithmetic conjecture for the physical (natural) numbers in order to test firstly its "physical (or external) veracity," and after that using (eventually) additional methods for the "proof" of a more mental component of it.

It is important to note here that the development of coherent refinements on the way in which we understand and manipulate pragmatically as well as theoretically (the notion of the) natural numbers and the particular "counting" (cognitive) processes underlying them would possess a huge influence not only on the foundations of mathematics, but also on the foundations of (theoretical) computer science and physics, among many others. So, this topic represents a central pillar of the AMI program with consequences beyond the AMI vision.

Delving deeper into the quantitative dimension of the new foundations' program, we show explicitly in Chap. 6 a "singular" phenomenon happening into "foundational bricks" of mathematics, (i.e., Zermelo–Fraenkel set theory with Choice (ZFC)). In other words, we prove formally that we can construct an identical (i.e., meta-isomorphic) version of (standard) mathematics (i.e., mathematics classically constructed from ZFC), called "Dathematics" (or Dual Mathematics), where instead of sets, one uses a special kind of proper classes as foundational bricks. This fact turns out to be surprising not only from a purely metamathematical perspective, but even more from a cognitive point of view. Effectively, proper classes are, strictly speaking, mental constructions without any kind of physical realization at any quantitative level. So, the fact that we can support in a semantic way a meta-isomorphic copy of our (in some sense daily life's) mathematics strictly based on objects that do not have any kind of physical counterpart in nature (by definition) implies that our current basic logic-deductive frameworks are grounding the semantic content of mathematical structures more in a purely formal and syntactic way and much less in a physical and (more) "tangible" manner. This implies, among other things, that deeper intuitions about mathematical structures could be highly limited by the mono-thematic formalizations that have been developed for them classically.[18]

In the second part, we focus our attention on the specific cognitive mechanisms used by the mind during mathematical creation/invention. Here, we take inspiration from a wide spectrum of classic and new results in cognitive science, cognitive linguistics, psychology, and from the classic works on the philosophy of mathematics of G. Polya and I. Lakatos.

First, we dedicate an entire chapter to the study of one of the most fundamental of these processes–conceptual blending (Chap. 7), or, informally, the ability of the mind to create genuine conceptual fusions of two (or more) input concepts. Explicitly, we use a classic formalization of conceptual blending in terms of colimits embedded in a categorical many-sorted framework for mathematical concepts. For such a formalization one can generate implementations of concrete blends in the Heterogeneous Tool Set (HETS). Moreover, we show how to generate fundamental notions of Fields and Galois theory recursively only in terms of conceptual blends starting from five elementary concepts coming from different mathematical subdisciplines like group theory, fixed point's theory, and abstract algebra. This is

[18]This issue will be illuminated in Sect. 3.1, Chap. 3.

an initial case study regarding concept generation from a cognitive as well as a logic perspective that aims to fill the gap existing in the automated deduction's literature.[19]

In Chap. 8, we present an initial cognitively-based formalization of (atomic and best) analogy and analogical space of two formulas, starting with a classic Hilbert's style calculus for propositional logic. Additionally, we illustrate the explanatory power of the former notions for offering meta-descriptions of the generation of classic (elementary but non-trivial) proofs of some tautologies. Moreover, a new formalization of conceptual blending is described in terms of the former notions. Finally, some notions are extended to a first-order setting.

In Chap. 9, the new cognitive (metamathematical) mechanism of conceptual substratum is introduced (i.e., the ability of producing specific morpho-syntactic configuration of symbols with intrinsic meaning, which allow our minds to manipulate essential formal features of (mathematical) concepts in sophisticated deductive tasks). We present two formalizations of this notion in different deductive contexts and their relations with classic tools in automated deduction and logic like Skolemization and Diophantiveness. Furthermore, based on a first-order formalization of (functional) conceptual substratum, we state an explicit cognitive characterization of the Church–Turing Thesis, which can be seen as a modern (and more cognitively-supported) description of this classic and foundational principle. Moreover, we show how to construct equivalent versions of the sequent calculus for first-order logic with equality over a language L, including deductive rules codifying (functional) conceptual substratum inside. Such deductive systems can be seen as slightly improved versions of the classic (Gentzel) sequent calculus from the point of view of their (increased) cognitive soundness. Lastly, we introduce conceptual lining as the dual cognitive ability of conceptual substratum.

In Chap. 10, we present an initial global taxonomy of the most fundamental cognitive mechanisms used in mathematical research, together with the corresponding formalizations in terms of a more global notion of mathematical concept (and mathematical structure) than the one initially presented in Chap. 7. In that chapter, one can see in a more concrete way the multi- and interdisciplinary nature of the whole AMI program from a methodological point of view. The formalizations are presented assuming a minimal robustness of the logical frameworks underlining the (local) mathematical theories that can be used as the object of meta-study. In particular, the meta-notions are presented at a level of generality that includes the possibility of meta-analyzing (local) concepts described over a wide spectrum of logics. In addition, more general versions of the three initial cognitive abilities of conceptual blending, analogical reasoning, and conceptual substratum are presented to match the level of abstractness needed subsequently in further chapters.

This chapter has an additional central relevance from the point of view of cognitive science because it offers (as far as we know, for the first time) a global and formal classification of all the essential mechanisms that the mind uses during

[19] We will tackle this issue in quite more detail in Chap. 11.

mathematical research. In particular, the intellectual activity of producing abstract mathematics is broad enough to represent an outstanding case study towards the development of more general formal frameworks explaining the general functioning of the mind which is a central research goal by cognitive scientists.

Furthermore, Chap. 10 has also a seminal relevance for the foundations of computer science because the mechanisms described there are more plausible to be modeled symbolically as well as algorithmically. And, in some sense, the concrete formalizations developed there, which possess a more finite nature, begin to "knock down" methodological "walls" that could emerge from "over-extrapolations" of classic (unsolvability) results.

In the last part, we present in an explicit and extended manner the concrete evidence for the universality of all the formal meta-tools developed so far.

Explicitly, in Chap. 11 we offer global formal support to fill the gap that the majority of the classic methods used in standard automated deduction have regarding the development of meta-explanations of conceptual generation [2, 26, 42]. In other words, we show explicit cognitive meta-generations (i.e., meta-explanations) not only of the proofs of two classic theorems in elementary geometry and number theory, but also of dozens of fundamental mathematical concepts belonging to several mathematical disciplines like topology, set theory, abstract algebra, category theory, sheaf theory, commutative algebra, and (classic and modern) algebraic geometry. These cognitive meta-constructions can be seen as explicit evidence of the creative power of an ideal (non-necessarily embodied) mathematical artificial agent (which is one of the main goals of the AMI program).

In particular, we exhibit a recursive and explicit cognitive meta-generation of one of the conceptual cornerstones of modern algebraic geometry (i.e., the notion of (mathematical) scheme). This notion was chosen in advance due to its technical sophistication to show that the multifaceted tools developed in the previous chapters are strong enough to generate higher abstract mathematics. This fact starts to fill simultaneously another gap existing in the literature involving the elementary scope that more cognitively-inspired accounts of conceptual creation possess.[20] It is worth mentioning at this point that we could have potentially chosen any other sophisticated mathematical notion instead of the one of scheme, however, we choose it due to the central role that it plays in modern mathematics and even beyond algebraic geometry.

In addition, all these cognitive meta-generations are presented more from the point of view of a global version of a UMII. So, some of them have more qualitative commonalities with the original historical reconstructions of the corresponding concepts (e.g., (mathematical) categories), some possess less and can be seen as new ways of generating those concepts (following the integrative guidelines of the AMI vision) (e.g., sheaves).

In Chap. 12, we present the most outstanding (future) challenges of the AMI program not only from a theoretical and foundational perspective, but also from a

[20] See, for example, the references presented at the end of Sect. 1.3.

more pragmatic and algorithmic point of view. In addition, we describe plausible extensions of the AMI vision to others scientific disciplines close to mathematics in some foundational aspect.

Along the lines of such an extension is exactly where one perceives the importance that a mature version of the AMI vision can have regarding the way in which we currently do scientific research at essentially a purely human level.

Finally, due to the multifaceted methodological dimension and the cognitive nature of our new metamathematical (AMI) program, we can also use the more classic name of *Cognitive Metamathematics* for it. In fact, this more neutral name has the advantage that, on the one hand, it stresses deeply the integrative scientific discipline grounding the AMI vision, and, on the other hand, it emphasizes with a new clarity the theoretical aspect of the AMI vision in a concise way, extricating along the way the AMI program from being understood only in terms of the computational challenge behind it.

1.5 Ethical Considerations

From the very beginning, this new inter- and multidisciplinary AMI program was conceived for improving and enhancing our theoretical, constructive, and practical understanding of mathematics in the widest sense of the word, and, subsequently our understanding of nature at several levels of observation. So, from a middle- and long-term perspective, any new product, technology, invention, and community emerging and largely (in-)directly based on (applications coming from) the AMI program should pursue respectful, deserving, peaceful, and integrative purposes regarding a pacific living with our fellows and with nature. So, the AMI program and all its future applications are strictly envisioned to increase and to protect (our quality of) life inside and outside earth at any stage of development. More generally, the Alisomar principles of the Future of Life Institute represent a valuable source for the global ethical principles that should be observed on any materialization of Artificial Mathematical Intelligence.[21]

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[21] For more details, please consult the website https://futureoflife.org/ai-principles/.

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Chapter 2
Some Basic Technical (Meta-)Mathematical Preliminaries for Cognitive Metamathematics

2.1 Introduction

In this chapter, we will introduce some classic logic and (meta-)mathematical terminology needed in several chapters of this book. So, the present chapter is mainly devoted to non-mathematicians (e.g., cognitive scientists, AI specialists) who want to acquire a minimal technical knowledge of some of the fundamental theories used implicitly along the AMI meta-program.[1] In this presentation, we will describe essentially foundational notions and results without proofs. It is worth to clarify that we offer in this chapter the minimal syntactic descriptions of most of the notions needed to get a better technical understanding of the initial applications of the AMI formal framework to the meta-generation of a wide spectrum of concepts in pure mathematics.[2]

2.2 Propositional and First-Order Logic

The main reference for this section is the classic treatise of E. Mendelson [11]. Propositional logic deals with one of the most simple ways of articulate deductive (semantic and syntactical) procedures among propositions. In other words, one generates recursively more complex propositions starting with atomic ones and using a suitable collection of logic connectives with some resemblance with natural language (e.g., "and" ($\wedge$), "if $\cdots$ then $\cdots$" ($\rightarrow$)). The possible truth value of

[1] Therefore, the working mathematician or logician can easily skip most of this chapter without any substantial lack of preparation for understanding the rest of this book.

[2] Hence, we recommend that the (non-specialist) reader consult also the references (better simultaneously) for a more complete and detailed presentation of the corresponding topics.

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a proposition is either "true" or "false" and the truth value of a compounded proposition depends completely on the truth value of these atomic components and of the (truth tables of the) corresponding logical connectives. Additionally, one possesses fixed deductive rules and (logic and proper) axioms. We will use the following concrete system for propositional logic in this book.

2.2.1 A Formal System for Propositional Logic

Explicitly, the basic symbols of our language are $\neg, \to$, (,), (primitive connectives and parenthesis) and upper-case letters $A, B, \ldots$ (statement letters). As usual, a well-formed formula (wf) is defined recursively as follows: all statement letters are wfs and, if $\mathscr{A}$ and $\mathscr{B}$ are wfs, then $(\neg\mathscr{A})$ and $(\mathscr{A} \to \mathscr{B})$ are wfs.

Let us use the special symbols $\#_1, \#_2$, and $\#_3$ to denote propositional variables ranging over all wfs. So, for any assignation of particular wfs on the former variables the following wfs are (logic) axioms of our system:

(A1) $(\#_1 \to (\#_2 \to \#_1))$
(A2) $((\#_1 \to (\#_2 \to \#_3)) \to ((\#_1 \to \#_2) \to (\#_1 \to \#_3)))$
(A3) $(((\neg\#_2 \to \neg\#_1) \to (((\neg\#_2) \to \#_1) \to \#_2))$

We have three axiom schemes in our formal system. The only inference rule that we use is modus ponens (MP), namely, we can deduce directly a wf $\#_2$ from the wfs $\#_1$ and $\#_1 \to \#_2$.

By a (formal) proof (in propositional logic) we will understand a constructive proof in the following sense: $\mathscr{W}$ is a syntactic consequence of $\Gamma = \{\mathscr{H}_1, \cdots, \mathscr{H}_n\}$ (i.e., $\Gamma \vdash \mathscr{W}$) if and only if there exists a collection of wfs $\mathscr{A}_1, \cdots, \mathscr{A}_m$; such that $\mathscr{A}_m = \mathscr{W}$, and for any j, $\mathscr{A}_j$ is either an exemplification of an axiom scheme, or one of the $\mathscr{H}_i$, or it is a direct consequence (by MP) of two of the formers $\mathscr{A}_k$, $(k < j)$.

We can speak also of an (indirect) "proof" of the fact that $\mathscr{T}$ is a consequence of the axioms of our system (i.e., $\mathscr{T}$ is a theorem, $\vdash \mathscr{T}$), in the sense of being able to verify that the wf $\mathscr{T}$ is a tautology (i.e., for any assignation of truth values of its atomic components, the resulting truth value is always true). Clearly, this is equivalent to the fact that there exists a formal proof of $\mathscr{T}$ in the former sense due to the completeness theorem for propositional logic, in other words, the notions of theorem (syntactic consequence) and tautology (semantic consequence) coincide [11, Ch. 1].

2.2.2 First-Order Logic

We want to extend the former (zeroth-order) logic system by including the possibility of expressing universal and existential quantifications of formal variables. For example, we wish to formalize sentences like "For all natural numbers x and y, model M of Γ is also a model of ψ) if and only if ψ is a syntactic consequence of Γ, $\Gamma \vdash \psi$ (i.e., there exists a formal proof of ψ starting from wf formulas of Γ (and possibly instantiations of axioms)). In other words, in the former kind of theories the notions of semantic and syntactic consequence coincide [11, Ch. 2]. A theory T is *consistent* if one cannot derive syntactically a formal contradiction, there is no wf formula ϕ such that $\vdash_T \phi$ and $\vdash_T \neg\phi$. It is equivalent to the existence of a model for the theory T [11, Ch. 4].

A *many-sorted first-order theory* is an important variant of a first-order theory, where one has an additional collection of *sorts* in the language which is meant to be used in order to create a taxonomy on the particular range of each of the variables (and indirectly on the domain of definition of the relation and function symbols). This technical trick can be reconstruct by a classic first-order theory by defining basically an (explicit) unary relation symbol in the language for any sort. Thus, both notions are meta-equivalent. We will see an enlightening example of this kind of theories in Chap. 7.

2.3 Foundational Instantiations of First-Order Theories in Mathematics

2.3.1 Zermelo–Fraenkel Set Theory with the Axiom of Choice (ZFC)

One of the most used and famous foundational (first-order) theory for (a large part of) modern mathematics is the theory of sets, originally developed in a primitive form by Georg Cantor, and based on the proper axioms of Ernst Zermelo and Abraham Fraenkel, including the axiom of choice [8]. From a cognitive perspective, one of the biggest reasons why ZFC is so widely used is its wide easy-going appealing use of mental and intuitive images as part of the phenomenological way of understanding its main objects and the relations between them.

ZFC set theory is a first-order theory with a canonical membership (binary) relation ($\in$), so $a \in b$ is expressed as "a is an element of b" or "a belongs to b."[7] Let us describe the proper axioms of ZFC in a compact and intuitive way. In later chapters one can gain a deeper idea of the way in which these axioms can be written more formally. However, it is more enlightening for an initial presentation if we mainly appeal to intuition.

[7] One can also add the equality relation as a primitive relation (assuming the fulfillment of the standard properties as proper axioms) or one can define it in terms of the membership relation. Here, we assume the first variant to stress prominently the main intuition behind it and to avoid excessive technicalities.

1. **Axiom of Extensionality.** Two sets are equal when (and only when) both have exactly the same elements.
2. **Axiom of Pairing.** Given two sets a and b there exists a (unique) set $\{a, b\}$ having exactly a and b as its elements.
3. **Axiom of Union** For any set a there exists a set b containing exactly all the elements of a. This set is called the union of a and is denoted as $\cup a$.
4. **Axiom of Power Set** For any set a there exists a set containing as elements all the subsets of a. This set receives the name of power set of a and is denoted as $P(a)$.
5. **Axiom of Infinity** There exists a set with infinitely many elements.
6. **Axiom of Regularity** Any nonempty set a has an element e, such that a and e has no common elements.
7. **Axiom Schema of Separation** Let $Q(a, b)$ be a formula (describing an (unary) property where the free variable b is fixed, i.e., b is a parameter). Then, for any (fixed) set w and (fixed) parameter b, there exists a set v containing precisely the elements of w fulfilling Q, in other words, $v = \{a \in w : Q(a, b)\}$.
8. **Axiom Schema of Replacement** Let $\psi(x, y, p)$ be a formula that describes a function F with parameter p,[8] (i.e., for any sets x, y, and z, if $\psi(x, y, p)$ and $\psi(x, z, p)$ hold, then $y = z$. This unique element y is also denoted as $F(x)$). Thus, the image of any set u under ψ is a set. In other words, for any u there exists a set v such that $v = \{F(d) : d \in u\}$.
9. **Axiom of Choice** Any family of nonempty sets (which can be expressed as $\cup a$ for some set a) possesses a choice function f, i.e., $f : a \mapsto \cup a$ and for any $x \in a$, $f(x) \in x$.

2.3.2 Von Newmann–Bernays–Gödel (Class and) Set Theory (NBG)

In some mathematical (modern) theories (like category theory) one usually needs to construct notions involving the collection of all sets and (very large) sub-collections of it. Now, due to the fact that such collections do not represent sets anymore,[9] one needs a kind of suitable extension of ZFC that do not increase the potential for the existence of inconsistencies that ZFC already possesses.

In this sense, Von Newmann–Bernay–Gödel set theory (NBG) is a coherent candidate for a broader theory maintaining the "working feeling" very similar to the one in ZFC and allowing to talk about classes as natural (semantic) extensions of sets.

[8]In this axiom p can be replaced by several parameters $p_1, \cdots, p_n$. However, for simplicity we describe the version with only one.

[9]This is based on the existence of Paradoxes emerging from the assumption that such collections are sets, e.g., Russell's paradox [13].

Here we adopt essentially the approach presented in [11, Ch. 4]. NBG set theory is a first-order theory with a binary (enlarged) membership relation ($\in$) and a primitive notion of class. We define equality between classes in terms of extensionality: $A = B$ stands for $(\forall C)(C \in A \leftrightarrow C \in B)$.

A *set* is defined as a class that belongs to some other class. A class that is not a set is called a *proper class*. As a matter of terminology, we denote classes by upper-case letters (e.g., X, Y, and Z) and sets by lower-case letters. Let us describe the proper axioms of NBG set theory in a more formal way:

1. **Axiom T**

$$A = B \rightarrow (\forall C)(A \in C \leftrightarrow B \in C)$$

2. **Axiom P (Axiom of Pairing)**

$$(\forall a)(\forall b)(\exists c)(\forall d)(d \in c \leftrightarrow d = a \vee d = b)$$

3. **Axiom N (Empty Set)**

$$(\exists a)(\forall b)(b \notin a)$$

4. **Axiom F (Axiom of Regularity)**

$$(\forall a)(a \neq \emptyset \rightarrow (\exists b)(b \in a \wedge a \cap b = \emptyset)$$

5. **Axiom E1 (Set-theoretical membership Relation-Class)**

$$(\exists A)(\forall b)(\forall c)(\langle b, c\rangle \in A \leftrightarrow b \in c)$$

6. **Axiom E2 (Intersection of Classes (Conjunction))**

$$(\forall A)(\forall B)(\exists N)(\forall c)(c \in N \leftrightarrow c \in A \wedge c \in B)$$

7. **Axiom E3 (Complement of a Class (Negation))**

$$(\forall A)(\exists C)(\forall b)(b \in C \leftrightarrow b \notin A)$$

8. **Axiom E4 (Domain (Existential Quantifier))**

$$(\forall A)(\exists B)(\forall c)(c \in B \leftrightarrow (\exists d)(\langle c, d\rangle \in A))$$

9. **Axiom E5 (Product by the Universal Class)**

$$(\forall A)(\exists B)(\forall c)(\forall d)(\langle c, d\rangle \in B \leftrightarrow c \in A)$$

10. **Axiom E6 (Circular Permutation)**

$$(\forall A)(\exists B)(\forall c)(\forall d)(\forall e)(\langle c, d, e\rangle \in A \leftrightarrow \langle d, e, c\rangle \in B)$$

11. **Axiom E7 (Transposition)**

$$(\forall A)(\exists B)(\forall c)(\forall d)(\forall e)(\langle c, d, e\rangle \in A \leftrightarrow \langle c, e, d\rangle \in B)$$

12. **Axiom U (Union (Sum) Set)**

$$(\forall a)(\exists b)(\forall c)(c \in b \leftrightarrow (\exists d)(c \in d \wedge d \in a)$$

13. **Axiom W (Power Set)**

$$(\forall a)(\exists b)(\forall c)(c \in b \leftrightarrow (\forall e)(e \in c \rightarrow e \in a))$$

14. **Axiom S (Subsets)**

$$(\forall a)(\forall B)(\exists c)(\forall d)(d \in c \leftrightarrow (d \in a \wedge d \in B))$$

15. **Axiom R (Replacement)**
    Let V be the universal class containing all the sets. If A is a class, let $Fnc(A)$ be the formal statement saying that A is a function, i.e.,

$$A \subseteq V^2 \wedge (\forall a)(\forall b)(\forall c)(\langle a, b\rangle \in A \wedge \langle a, c\rangle \in A \rightarrow b = c).$$

    Then

$$(\forall A)(Fnc(A) \rightarrow (\forall b)(\exists c)(\forall d)(d \in c \leftrightarrow (\exists e)(\langle e, d\rangle \in A \wedge e \in b))$$

16. **Axiom I (Axiom of Infinity)**

$$(\exists a)(\emptyset \in a \wedge (\forall b)(b \in a \rightarrow b \cup \{b\} \in a))$$

17. **Axiom G (Axiom of Global Choice)**

$$(\exists G)(Fnc(G) \wedge (\forall a)(a \neq \emptyset \rightarrow (\exists b)(b \in a \wedge \langle a, b\rangle \in G)))$$

We describe explicitly each of the former axioms for its particular foundational importance. However, the former list is, strictly speaking, non-minimal, i.e., some of the axioms can be deduced from some of the remaining ones. Nonetheless, we would not discuss this kind of technical issues in this very short presentation, mainly for pragmatic reasons.[10]

NBG set theory is a finitely axiomatizable theory, due to the fact that one can codify the membership and the equality relation for sets, together with the notions of intersection, complement, domain, and product (at the level of sets) as particular (binary) classes.

NBG set theory turns out to be an extension of ZFC set theory, without strictly bigger chances of generating formal contradictions. In other words, NBG is a conservative extension of ZFC, i.e., NBG extends formally a ZFC and one of them is consistent if and only if the other one so is [11, Ch,. 4].

A second fundamental working meta-principle assumed by a large portion of working mathematicians and logicians regarding ZFC is that one can simulate and ground almost any mathematical structure (e.g., concept and notion) used for example in mathematical analysis, abstract algebra, and (differential and algebraic) geometry (among many others) with set-theoretical structures. Notwithstanding, this theoretical meta-fact possesses more a platonic importance, because in the concrete mathematics done at a daily basis in research centers, it is pragmatically impossible to do the concrete and real grounding explicitly.

2.3.3 Peano Arithmetic

In this section, we will define the first-order theory needed for obtaining a modern syntactic formalization of the natural numbers in order to be able to establish the grounding framework for formal number theory.[11]

The *language of arithmetic* (in this formal context) includes a single predicate letter for equality (=), and individual constant (0) and three function letters f_1^1, f_1^2, and f_2^2, with the following conventions on notation $f_1^1(a) = a'$, $f_1^2(a, b) = a + b$, and $f_2^2(a, b) = a \cdot b$. Finally, the proper axioms are the following:

1. $(\forall y_1, y_2, y_3)(y_1 = y_2 \rightarrow (y_1 = y_3 \rightarrow y_2 = y_3))$
2. $(\forall y_1, y_2)(y_1 = y_2 \rightarrow y_1' = y_2')$
3. $(\forall y_1)(y_1 \neq 0)$
4. $(\forall y_1, y_2)(y_1' = y_2' \rightarrow y_1 = y_2)$
5. $(\forall y_1)(y_1 + 0 = y_1)$
6. $(\forall y_1, y_2)(y_1 + y_2' = (y_1 + y_2)')$
7. $(\forall y_1)(y_1 \cdot 0 = 0)$
8. $(\forall y_1, y_2)(y_1 \cdot y_2' = (y_1 \cdot y_2) + y_1)$
9. For any wf formula $\mathscr{D}$, $\mathscr{D}(0) \rightarrow ((\forall y)(\mathscr{D}(y) \rightarrow \mathscr{D}(y')) \rightarrow (\forall y)\mathscr{D}(y))$

[10]Our main goal here is to give a global view of the most foundational axiomatic aspects for mathematics, reviewing only some of the most seminal notions, axioms, and results.

[11]Here we adopt the terminology given, for instance, in [11, Ch. 3].

2.4 Further Seminal (Categorical and Set-Theoretical) Mathematical Notions

In this section, we will present additional mathematical concepts that will be used as test samples of the validity of our meta-formal taxonomy of fundamental cognitive mechanisms employed during abstract mathematical creation (see Chap. 10). Most of the following concepts can be seen in two ways: firstly, as instances of first-order theories, where the explicit conditions defining the structures corresponds to the proper axioms and, simultaneously, they give the essential elements of the (first-order) language in consideration; and, secondly, as categories, where the (proper) class corresponds to the collection of all models of the corresponding first-order theory, and the morphisms are given by the corresponding class of functions preserving the fundamental algebraic properties characterizing the corresponding mathematical structures (e.g., (group, ring) homomorphisms). So, the next conceptual examples have a double purpose, enlightening the former logic and (meta)mathematical notions, and preparing the way for a deeper understanding of the whole AMI program. For a more detailed description of the notions described in this section the reader can consult [4–6, 12] and [9, Ch5–6].[14] Note that the following notions are only required in Chaps. 7 and 11, the rest of the book can be read without an explicit knowledge of them.

Definition 2.1 A *relation* r (between a set a and a set b) is simply a subset of the Cartesian product between them, i.e., $r \subseteq a \times b$. A *(mathematical) function* f with domain a and codomain b is defined by a relation $f \subseteq a \times b$ such that for any $x \in A$ and $y, z \in b$, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$. If g is another function from b to c, then one defines the *composition function* $g \circ f$ from a to c, by the rule $g \circ f(x) := g(f(x))$, for any $x \in a$. A function $f : a \mapsto b$ is *injective* if for any $x, y \in a$, if $a \neq b$ then $f(a) \neq f(b)$; it is *surjective* if for any $z \in b$, there exists a $x \in a$ such that $f(x) = z$; it is *bijective* if is both injective and surjective.

Let a be a set. A relation $r \subseteq a \times a$ is an *equivalence relation* if it is *reflexive* (i.e., for all $x \in a$, $(x, x) \in r$), *symmetric* (i.e., for all $x, y \in a$, $(x, y) \in r$ if and only if $(y, x) \in r$, and *transitive* (i.e., for all $x, y, z \in a$, if $(x, y) \in r$ and $(y, z) \in r$, then $(x, z) \in r$). An equivalent relation r generates a partition of a into *equivalent classes* which are the subsets $c_w \subseteq a$ (where $w \in a$) such that for all $x \in a$, $a \in c_w$ if and only if $(w, a) \in r$.[15] The collection of equivalence classes is denoted as a/r and, sometimes, it is referred as the *quotient set* (where the relation should be clear from the context).

[14]Most of the notions are well-known structures in contemporary mathematics, others are less-known robust concepts which have an intermediate usage within the book, and, therefore, possess a local naming.

[15]Note that the element w is not canonical. In fact each element in c_w can also represent its equivalent class.

Definition 2.2 An abelian group is a set A with a binary operation $+$ and a special (neutral) element $0 \in A$ such that the following axioms hold:

1. $(\forall a \in A)(a + 0 = 0 + a = a)$.
2. $(\forall a \in A)(\exists b \in A)(a + b = b + a = 0)$.
3. $(\forall a, b, c \in A)((a + b) + c = a + (b + c)))$.
4. $(\forall a, b \in A)(a + b = b + a)$.

A is a group if it fulfills conditions 1–3.

The most elementary example of an (abelian) group are the integers $\mathbb{Z}$ with the addition operation and the zero element.

Definition 2.3 A pointed (abelian) group is a set B with a binary operation $*$ and a distinguished element $b \in B$ such that $(B \setminus \{b\}, *_{|B\setminus\{b\}\times B\setminus\{b\}})$ is an (abelian) group and, $b * c = c * b = b$ for all $c \in B$.

Well-known examples of pointed abelian groups are the rational, real, and complex numbers with the zero element and the product operation, respectively. Moreover, for any nonempty set with a distinguished element there exists at least one structure of pointed group for it. In fact, it could be shown that this statement is equivalent to the axiom of choice [7].

Definition 2.4 A distributive space consists of two sets D y K with two operations $\oplus : D \times D \to D$ and $\otimes : K \times D \to D$ such that

$$(\forall x \in K)(\forall y, z \in D)(x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)).$$

Instances of distributive spaces are Boolean algebras, the space of square matrices with entries over a field (e.g., the real or complex numbers) with the standard sum and product operations, and clearly the natural, integer, rational, real, and complex numbers with addition and multiplication, respectively. In all these cases, $D = K$.

Contrastingly, we also obtain an example of a distributive space if $(D, \oplus)$ is a vector space over a field K and $\otimes$ denotes the corresponding scalar product. If $\dim D > 1$, then clearly $D \neq K$.

Definition 2.5 An action of a group $(G, +, 0)$ on a set X is simply a function $* : G \times X \to X$ such that the following two conditions hold:

1. $(\forall a, b \in G)(\forall x \in X)((a + b) * x = a * (b * x))$.
2. $(\forall x \in X)(0 * x = x)$.

Definition 2.6 An algebraic substructure $\mathbb{S} = ((A, +_A, 0_A), (B, +_B, 0_B), i : A \to B)$, consists with two sets, two binary operations defined over each of them, two special constants, and an (structural) embedding i fulfilling the following properties:

1. i is an homomorphism: $i(0_A) = 0_B$ and $\forall x, y \in A(i(x +_A y) = i(x) +_B i(y))$.
2. i is injective: $(\forall x, y \in A)(i(x) = i(y) \Rightarrow x = y)$.

3. $(\forall x \in B)(\forall y \in A)((x +_B i(y) = 0_B) \Rightarrow (\exists z \in A)(i(z) = x))$.

The last condition can be rephrased as follows: the "potential inverses" of elements of A, considered as elements in B, belong as well to A.

Usual examples of algebraic substructures are given by the natural injections $i_1 : \mathbb{Z} \to \mathbb{Q}$, $i_2 : \mathbb{Q} \to \mathbb{R}$, and $i_3 : \mathbb{R} \to \mathbb{C}$ (as well as the remaining meaningful combinations) with the addition operation and the zero element, respectively.

The main intuition of this definition is that when $(B, +_B, 0_B)$ has additionally an algebraic structure, as the one of a monoid, a semi-group, or a group, then $(A, +_A, 0_A)$ would automatically inherit the same structure.

This definition is a stronger notion than the one of embedding (i.e., an injective morphism) commonly used in the mathematical literature, since, in principle, sets A and B have a basic algebraic structure; e.g., we do not even require associativity for the corresponding operations. However, we impose the typical conditions for an embedding in (1) and (2) and additionally, we request condition (3) for including potential inverses of the smaller structure into itself. If we restrict ourselves to the category of monoids, semi-group, and groups, these two notions coincide, because we can prove that under these hypothesis, (3) would follow from (1) and (2).

Definition 2.7 If X denotes a set and F is a collection of functions from X to X, then a subset Y of X is called the space of fixed points of F, if

$$(\forall x \in X)((\forall f \in F)(f(x) = x) \leftrightarrow x \in Y).$$

Typical examples of spaces of fixed points appear in topology and in the setting of retractions between topological spaces [12].

Definition 2.8 A field is a set $(F, +, 0, *, 1)$, such that $(F, +, 0)$ and $(F \setminus \{0\}, *, 1)$ are abelian groups and the operation $*$ distributes with respect to $+$.

Canonical examples of fields are the rational, the real, and the complex numbers with the corresponding operations of addition and multiplication and the distinguished constants zero and one.

Definition 2.9 A bigroup is a set Q with two binary operations $+$ and $*$ such that $(Q, +, 0)$ is an abelian group and $(Q, *)$ is a pointed abelian group with distinguished element 0.

Examples of bigroups are the rational, real, and complex numbers with the standard operations and the zero element as distinguished constant in any case. In fact, let us show a concrete example of a bigroup which is not a field. Let us define in the group $(R = \mathbb{Z}/4\mathbb{Z}, +)$, the following second binary operation $*$: $a * b = 0$, if either $a = 0$ or $b = 0$. For the subset $R' = R \setminus \{0\}$, we define $*$, in terms of the following bijection $\phi : \mathbb{Z}/3\mathbb{Z} \to R'$, defined by $\phi(0) = 3, \phi(1) = 1$, and $\phi(2) = 2$. Here, we translate the addition in $\mathbb{Z}/3\mathbb{Z}$ to R' by means of ϕ . It is straightforward to show that $(R, +, *)$ is a bigroup. However, $1*(1+1) = 1*2 = 3$ and $1 * 1 + 1 * 1 = 2 + 2 = 4 = 0$, thus $1 * (1 + 1) \neq 1 * 1 + 1 * 1$. So, R is not