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Andrei Rodin
Institute of Philosophy
Russian Academy of Sciences

Department of Liberal Arts
State University of Saint-Petersburg
Saint-Petersburg
Russia

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Chapter 1
Introduction

Logical and mathematical concepts must no longer produce instruments for building a metaphysical "world of thought": their proper function and their proper application is only within the empirical science.

Ernest Cassirer

Mathematics is a part of physics. It is a part of physics where experiments are cheap. [..] In the middle of the 20th century there were attempts to separate mathematics from physics. The results turned to be catastrophic.

Vladimir Arnold

The main motivation of writing this book is to develop the view on mathematics described in the above epigraphs. Some 200 years ago this view used to be by far more common and easier to justify than today. It is sufficient to say that it made part of Kant's view on mathematics, and that Kant's view on mathematics remained extremely influential until the very end of the nineteenth century. When Cassirer defended this Kantian view in the beginning of the twentieth century it was already polemical. When Arnold defended it in the end of the twentieth century and in the beginning of this current century it already sounded as an intellectual provocation, and so his words sound today. Kant, Cassirer and Arnold do not speak about the same mathematics: each speaks about mathematics of his own time. So the growing polemical attitude to their shared view reflects not only a change of the common opinion about the subject but a change of this subject itself. It is a common place that the modern mathematics is more abstract and more detached from physical experience than it used to be in Euclid's times and in Kant's times. When I say that I nevertheless want to defend the view on mathematics as a part of physics this means that I also want to contribute to changing the character of current mathematics, but not only to changing the common views about it.

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The above is a motivation behind this book but not its purpose. The purpose is much more limited. In order to justify the view on mathematics as part of physics I would need to write at least as much about physics as about mathematics. But this book is mainly about mathematics and about logic; physics is mentioned in it only occasionally. Yet more specifically I shall focus on the Axiomatic Method and Category Theory (including the categorical logic, which is a part of modern logic using category-theoretic methods). Let me explain why.

When Arnold talks about recent attempts to separate mathematics from physics he has in mind *Elements of Mathematics* by Nicolas Bourbaki (1939–1988) that aims at developing the whole of mathematics systematically from the first principles, i.e., on an axiomatic basis. Bourbaki's *Elements* continue the long tradition of presenting renewed foundations of mathematics in the form of *Elements*: this tradition begins with Euclid's *Elements* (and earlier versions of Greek *Elements* that have been lost) and continues through the whole history of mathematics until today. (I say a bit more about this tradition in the introductory part of Part **I**.) Arnold sees the key to the problem in Bourbaki's Axiomatic Method, and takes a notoriously hostile attitude towards the Axiomatic Method in general. I observe on my part that the problem of separating mathematics from physics concerns the specific form of the Axiomatic Method used by Bourbaki rather the Axiomatic Method in general. It is clear, in particular, that Euclid's method does not produce the same effect. And I further observe that Bourbaki's Axiomatic Method is a version of Hilbert's Axiomatic Method presented in Hilbert's *Foundations of Geometry* of 1899, which is another example of renewed mathematical *Elements* playing a more special but perhaps even more important role in the twentieth century mathematics than Bourbaki's *Elements*. So I conclude that the origin of Arnold's problem should be traced back at least to the beginning rather than only to the middle of the twentieth century. This explains my focus on Axiomatic Method and its history.

Why Category Theory? The mathematical notion of category (which has no immediate relation to the philosophical notion widely known under this name) was invented in 1945 by Eilenberg and MacLane for general purposes, some of which I explain in Chap. **9**, see also Kromer (2007) for details. In his thesis defended in 1963 (Lawvere 1963) and a series of papers based on this thesis (Lawvere 1964, 1966a,b, 1967). Lawvere put forward a program of categorical (i.e., category-theoretic) foundations of mathematics and opened a new research field known today under the name of categorical logic, see Marquis and Reyes (2012) for the most recent historical account. Although Lawvere and other people who pursued the program of categorical foundations have never explicitly challenged Hilbert's Axiomatic Method (albeit they did and do challenge some special applications of this method, most importantly its applications in the standard axiomatic set theories) I shall try to show in this book that some recent works in categorical logic and new foundations of mathematics effectively modify Hilbert's Axiomatic Method and develop it in a wholly new direction. As it always happens in the intellectual history this new development continue some earlier developments, which I shall also take into account. In the last Chapter of this book I generalize upon these tendencies and describe a hypothetical New Axiomatic Method, which admittedly does not yet exist in the form of precise logical and mathematical procedure. I hope that my proposed general philosophical vision of this new method will contribute to its future technical development and also help to use it outside the pure mathematics and its philosophy.

As the reader shall see the New Axiomatic Method establishes closer relationships between mathematics and physics and so suggests a solution of Arnold's problem. Although I cannot fully justify this claim in this book (because I am not going to discuss physics systematically) I do prepare a philosophical background for such a justification. The issue of relationships between mathematics and physics is a hardcore philosophical issue, and I believe that Arnold's problem cannot be solved without taking this philosophical issue seriously. Another hardcore philosophical issue that comes into the play as soon as one discusses the use of Axiomatic Method in mathematics is the relationships between mathematics and logic. This latter philosophical issue unlike the former is in the focus of this book. The main philosophical dilemma that I consider is, roughly, this: either (i) logic is fundamental in the sense that it gives us an independent access to an ideal space of logical possibilities where the actual world exists side-by-side with plenty of other possible worlds, which can be explored only mathematically, or as Cassirer insists in the above epigraph, (ii) logic and mathematics must stick to the actual world as we know it through empirical sciences, and by all means must avoid producing possible "metaphysical worlds of thought" even if these appear more logically coherent and more mathematical beautiful than our actual world. With many important reservations that this rough formulation requires I shall defend the latter view. The former view (which also obviously needs a more precise formulation) I call *logicism*, and when it is applied to mathematics I call it *mathematical logicism*. Beware that this meaning of "mathematical logicism" is broader than Russell's radical version of mathematical logicism according to which mathematics *is* logic (Russell 1903). So a central purpose of this book is to refute mathematical logicism and defend an alternative way of thinking about logic and mathematics.

Talking about these philosophical issues I would like to stress that I study primarily their implementation in mathematics. When in the beginning of the twentieth century Cassirer, Russell and other people discussed hot philosophical issues concerning mathematics and logic they not only made general philosophical arguments but also referred to the actual state of affairs in their contemporary science and to the history of these subjects. They also often contributed themselves to the ongoing research in mathematics and logic. In this book I follow the same pattern of philosophical discussion paying a lot of attention to some recent mathematical works and to the history of the subject but without trying to make any mathematical contribution.

Before I summarize the content of this book chapter by chapter let me say a few more words about its style and its methodology. I stick to the traditional idea according to which philosophy and its history naturally combine together. When this view is applied to the philosophy of science and mathematics the result is sometimes called the *historical epistemology* (Rheinberger 2010). So what I am doing in this book can be described as a historical epistemology of logic and mathematics.

However one important reservation is here in order. In my understanding the past history, the present state of affairs and the anticipated future of a given discipline are parts of the same whole. This whole can be described as the current state of affairs in a broader sense of the word, which includes both the historical reflection upon the past and the projection towards the future of the given discipline. When I talk in this book about mathematics and its philosophy I think about these subjects in this way. When such a view is called historical this should mean the attention to development of the given discipline but not the exclusive attention to its past.

Although I write about logic and mathematics I don't use myself any formal logical or other mathematical means for expressing and justifying my arguments. A century ago this point would be hardly worth mentioning but since using formal methods in philosophy in general and in philosophy of mathematics in particular is nowadays popular (particularly in the philosophical school that calls itself *Analytic philosophy*) this point requires some explanations. Without going into a long discussion on this sensitive issue let me boldly express my believe that the natural language and the philosophical prose remain so far the best instruments for historical and philosophical work, or at least for the kind of such work that I want to do. The clarity and the exactness that formal methods bring to philosophy come with a price, which for my purposes is unacceptable. This price amounts to certain philosophical assumptions, without which these formal methods cannot work. I am not prepared to pay this price until I can see clearly these assumptions and thus know the price exactly. A philosophical and historical analysis of the notion of logical formalization is a part of my present project (see particularly Chaps. **3** and **10**). Even if a formal theory of formalization is possible I cannot see that it can be useful for this purpose. I shall not return to the question of using formal methods in philosophy in what follows but the reader will see that my analysis of the idea of logical formalization hardly supports the idea of using it as an universal instrument for philosophizing.

Although I am not going to use formal methods for philosophical purposes the reader will find below a lot of rudimentary mathematics. Since this book is about mathematics, and a part of this book is about very recent mathematics, which still remains a work in progress (see Sects. **7.9–7.10**), this is not surprising. So let me explain my strategy of presenting the relevant mathematical content and mention some mathematical prerequisites for reading this book. My intention is to make this book readable both for a working mathematician interested in philosophy and history of this discipline and for a philosopher like myself, who studies (or wants to study) mathematics and its history, and finds a broad philosophical inspiration in this discipline. To present a fragment of modern mathematics to a wider audience is a very challenging task, which normally should not be combined with any philosophical agenda. I certainly do have a philosophical agenda, which I have already outlined earlier in this Introduction. This is why writing this book I have tried to reduce the burden of explaining mathematics to minimum. At the same time I tried to avoid any *metaphoric* talk about mathematical concepts – even if some people would argue that any talk about mathematics outside the pure mathematics is doomed to be metaphoric. So I could not avoid the burden of explaining some mathematics completely but tried to use the most elementary examples and also tried to use some existing introductory expositions when such were available. In each particular case I refer to the existing mathematical literature and chose this literature accordingly to my specific purpose.

For the first superficial reading the given book is self-sustained and, as I hope, it gives a right idea of what I am after. A more attentive critical reading is by far more demanding. The ideal judge of this book is a working mathematician who is also a working philosopher and working historian of mathematics having some broader philosophical and scientific interests, which include some interest in physics, its history and its philosophy. I know several people who at some degree of approximation fit this description but I rather imagine an average reader of this book as a person like myself who during these recent years has learnt some philosophy, some mathematics and some history of both subjects, and who tries to make these ends meet. I shall say more about the mathematical prerequisites and give some suggestions for reading (in addition to references found in the main text) in the following summary of the Chapters.

Part **I** of this book treats the history of Axiomatic Method. As I have already explained this history is not only about the past. Only Chap. **2** on Euclid concerns what is indeed in the past (albeit in Sect. **2.5** I show that even in this case the past continues to live in the present); Chap. **3** on Hilbert treats (in the original historical context) what remains today the standard notion of Axiomatic Method; Chap. **5** on Lawvere treats what I suggest as a conceptual basis of the New Axiomatic Method. So these three Chapters of this book present, roughy, the past, the present and the anticipated future of the Axiomatic Method. Chapter **4** is reserved for studying the fate of Hilbert's Axiomatic Method in the twentieth century mathematics.

Instead of trying to reconstruct a general history of Axiomatic Method, I decided to choose these three key figures and look at the relevant parts of their work more attentively. Although a historical discussion on Euclid found in Chap. **2** may appear out of place in a book about today's mathematics it is important for me for several reasons. According to a common view (supported by Hilbert himself at some occasions), Hilbert's Axiomatic Method improves upon Euclid's method in terms of logical rigor and logical clarity. Of course, in such a general formulation this view can hardly be challenged. However in order to see how exactly this improvement on rigor and clarity has been achieved in the twentieth century we need first to study Euclid's method on its own rights. This requires some special hermeneutical techniques, which are well-known to historian of mathematics but are less familiar to logicians, mathematicians and philosophers who also write about this subject. We shall see that in some respects Euclid's and Hilbert's method are different in principle, so that the difference between these methods does not reduce to differences in degrees of continuous magnitudes like rigor and clarity. In addition to my attempt to reconstruct Euclid's mathematical reasoning in its proper terms (and in some terms borrowed from Greek philosophy) I explain in this Chapter the relevance of Euclid's geometry to Kant's philosophy of mathematics. In the end of this Chapter I point to some Euclidean patterns of reasoning in the recent mathematics. The main textual reference in this Chapter is obviously Euclid's *Elements*, which is now available in a new English translation (Euclid 2011). An interested reader who would like to study the history of Greek mathematics more broadly and would like to better understand Euclid's special place in this history (this is an important subject that I wholly skip in this book) is advised to begin with (Heath 1981, 2003) and then study more recent secondary literature.

Chapter **3** on Hilbert is also written in a historical style and contains extended quotes from Hilbert's writings. Although I leave outside the scope of my discussion most of the contemporary context of Hilbert's work I follow the development of Hilbert's own ideas rather closely and distinguish in it several stages. In its narrow historical aspect my treatment of Hilbert's work contains nothing original. However I also make an attempt to reconstruct the history of some relevant notions (or at least to keep track of their changing meaning) including the notion of being formal. This historical discussion is combined with an explanation of Hilbert's Formal Axiomatic Method, which can be used by a non-mathematical reader for the first acquaintance with this basic method of modern mathematical reasoning. Someone well acquainted with this method will find here an analysis of certain assumptions required by this method, which remain tacit when this method becomes an intellectual habit and is used automatically. I shall pay a lot of attention to philosophical remarks made by Hilbert in his presentations of Axiomatic Method trying to reconstruct Hilbert's thinking and its philosophical motivation. I also discuss in this Chapter some related subjects including the notion of logicality, diagrammatic and symbolic thinking and some others. This Chapter presents (in its historical original form) the core notion of modern Formal Axiomatic Method, which I contrast in what follows to more traditional Euclid's method, on the one hand, and to some later versions of Axiomatic Method including the anticipated New Axiomatic Method, on the other hand.

The main suggested reading for Chap. **3** is Hilbert's *Foundations of Geometry*, which exist in multiple editions including the English edition (Hilbert 1950) and some later English editions. I highly recommend this reading also to a non-mathematical reader of this book because the real subject-matter of this short masterpiece is the Axiomatic Method itself rather than geometry, and so this short book can be used as a shortcut to the modern style of mathematical thinking. For a later more developed systematic presentation of Formal Axiomatic Method and its underlying philosophy I refer the reader to Tarski's textbook (1941). This textbook presents in a very clear form a philosophical view on logic and mathematics that I discuss in my present book.

In Chap. **4** I talk about applications of Hilbert's Axiomatic Method in the twentieth century mathematics and stress the fact that it has hardly ever been used in its original form and for its originally intended purpose. I discuss from this point view some formal studies of axiomatic set theories, Bourbaki's *Elements of Mathematics Bourbaki:1939–1988* and more specifically an unpublished Bourbaki's draft (Bourbaki 1935–1939). My main observation amounts to saying that both the modern set theory and Bourbaki's structural mathematics can be described in Hilbert's terms as a *metatheory* or in Tarski's terms as a *model theory* of certain Hilbert-style axiomatic theory or, more typically, of a number of such theories. Since this metatheory or model theory itself is developed by some other means (i.e., *not* axiomatically in Hilbert's sense) one can say that the mainstream mathematics widely applies Hilbert's Formal Axiomatic Method only with a pinch of salt. In the mainstream structural mathematics of the twentieth century this method serves as a method of definition and constructing new concepts rather than method of building deductive theories. On the basis of this observations I criticize Hilbert's Axiomatic Method arguing that it is not apt to support mathematical theories useful in the modern physics. Finally I consider in this Chapter Tarski's topological model of intuitionistic propositional logic (Tarski 1956) and stress its unusual character: although, technically speaking, there is no big difference between modeling a given formal theory and modeling a given logical calculus, philosophically it makes a huge difference and requires a rethinking of the whole idea of Axiomatic Method. Although Tarski himself does not draw from this work such far-reaching conclusions I use this example in the following Chapter as a historical prototype of the New Axiomatic Method.

In addition to the literature referred to in Chap. **4** I suggest reading the classical introduction (Bar-Hillel et al. 1973) to the modern axiomatic set theory including its last philosophical chapter, and Galileo's *Two New Sciences* (Galilei 1974) where the author stresses the constructive experimental character of the New Science against the background of the earlier Scholastic patterns of doing science.

Chapter **5** plays a central role in this book because here I first introduce the notion of category and discuss a new notion of Axiomatic Method, which emerges in category theory and, more specifically, in categorical logic. Although categorical logic is already a well established subject (see Marquis and Reyes 2012 for a historical introduction) I decided to follow here the pattern of the first two Chapters and focus my attention on the work of one particular person, namely Lawvere, who founded this discipline in 1960s; as before I combine here a historical and a systematic orders of presentation and pay a minute attention to Lawvere's philosophical comments found throughout his writings. After presenting Lawvere's categorical axiomatization of (the category of) sets (Lawvere 1964) and of the category of categories (Lawvere 1966a), which gives the first idea of using the category theory for axiomatization, I turn to Lawvere's critique of the standard Formal Axiomatic Method as "subjective" and explain his idea of *objective* conceptual logic realized by category-theoretic means. I begin this latter discussion by considering two Lawvere's papers (Lawvere 1966b, 1967) that mark the birth of the categorical logic, and in the same context explain Lawvere's notion of quantifiers as adjoint functors to the substitution functor. Then I make a digression on Curry's *combinatorial logic*, type theory and the so-called *Curry-Howard correspondence*, and show how these conceptual developments combine in Lawvere's notion of Cartesian closed category. Then after a brief discussion on Lawvere's notions of hyperdoctrine (that conceptually connects to the discussion on homotopy type theory found in Sect. **7.9**) and functorial semantics (further discussed in Sect. **10.2**) I turn to philosophical issues and discuss the role of Hegel's dialectical logic in Lawvere's thinking, which Lawvere stresses himself at many instances. Here I provide a philosophical reconstruction of Hegel's distinction between the *objective* and the *subjective* logic and then describe how this philosophical distinction is realized by Lawvere with the technical means of categorical logic. This discussion helps me then for interpreting the groundbreaking paper (Lawvere 1970b) where Lawvere suggests his axiomatization of topos theory and demonstrates the strength of his notion of internal logic of a given category. In the last Chap. **10** I use Lawvere's axiomatization of topos theory as a basic example of the new axiomatic approach, which I try to describe in general terms under the title of New Axiomatic Method.

For a better understanding of Chap. **5** it would be useful if the reader get some knowledge of basic category theory beforehand (albeit this is not an absolutely necessary requirement and the reader can also follow references during the reading). For a non-mathematical reader or a reader with a modest mathematical background I recommend (Lawvere and Schanuel 1997; Lawvere and Rosebrugh 2003) co-authored by Lawvere as a very accessible introduction into the subject. For a mathematical reader not familiar with categorical logic I recommend (MacLane and Moerdijk 1992) that covers most of the mathematical material that I discuss in this Chapter (but unfortunately skips hyperdoctrines). There is a huge gap in terms of required mathematical skills between these two suggested readings and by the present day this gap has not been yet filled in spite of many very valuable attempts such as Reyes et al. (2004). I believe that there is a principle and not only technical and pedagogical difficulty involved with the project of writing a fairly elementary introduction to category, topos theory and categorical logic. The problem is that the elementary introductions like Lawvere and Schanuel (1997), Lawvere and Rosebrugh (2003), and Reyes et al. (2004) begin with considering the category of finite sets, which are first introduced naively as bags of dots and then are treated in terms of their maps. Although such an introduction is geometrical in its character the basic geometry reduces here to the geometry of bags of dots, which is a geometry of a very special sort. A genuine continuous geometry appears then only at the much more advanced level and in a much more abstract form of Grothendieck topology and Grothendieck topos, which are systematically treated in MacLane and Moerdijk (1992) and other books of the same advanced level. So it still remains, in my view, a challenging task to follow Hilbert's example and rewrite Euclidean or other simple intuitive geometry in new categorical terms. Voevodsky Univalent Foundations discussed in Sect. **7.10** appear to be a step in this direction.

Talking about elementary introductions to category theory and topos theory I would like also to mention (1992) by McLarty. The expression "elementary' theory" in the title does not stand for being easy to grasp by a beginner but is used in the technical sense of being a first-order theory in the sense of modern logic and the standard Formal Axiomatic Method. This book is a systematic presentation of category and topos theory which fully complies with the requirement of Formal Axiomatic Method and at the same time treats the internal logic of a given topos and the idea of internal description of a given topos with its internal language. So for a logically-minded philosopher habituated to formal methods this book may also serve as an introduction into the subject. I would like to stress however that since in the present book I discuss specific features of Lawvere's axiomatic thinking, which fall apart from the standard Formal Axiomatic Method, studying McLarty's book does not replace studying Lawvere's original works even if, formally speaking, McLarty's book fully covers the same subject.

Part **II** is devoted to the notion of identity (in mathematics). This may appear as a side subject with respect to the general theme of this book but it is actually not. A mathematical logicist argues like this: in order to build a mathematical theory in an axiomatic form one needs first to fix some basic logical notions like that of being the *same* (or being equal). Unless this is done beforehand and quite independently from the content of any particular mathematical theory, so the argument goes, no axiomatic construction of mathematical theories is possible. A similar point can be made, of course, about other logical notions including logical connectives "and", "or", the notion of logical inference, of truth-value, etc. This standard logicist argument does not go through in the case of categorical logical, or at least it does not go through immediately, because the categorical logic *internalizes* the logical notions, i.e., reconstructs them in terms of a given mathematical theory (see Sects. **5.9** and **10.3**). This applies to logical connectives, the relation of inference, quantifiers, truth-values and to some other logical notions. It also applies to the logical identity relation but this case turns to be both more difficult and more mathematically and philosophically interesting than other cases. So I treat it systematically in the two consequent Chapters making the Part **II**.

In Chap. **6** I consider the question of identity/equality in mathematics in general beginning with some naive observations and historical examples. In particular, I briefly consider Plato's view according to which the mathematical equality is a weak form of strict identity: while the latter applies only the ideal world of Forms the former applies in the world of mathematics, which takes an intermediate position between the world of immutable Forms and the world of changing material beings. Plato's theory is an echo of the modern mathematical structuralism discussed later in Chap. **9**. In Chap. **6** I also show the significance of discussions about identity in mathematics in Frege's and Russell's works for establishing the logicist view on mathematics in the end of the nineteenth and the beginning of the twentieth century. Then I turn to more theoretical subjects including a discussion on classes and individuals, and a discussion of the distinction between logical extension and logical intension. This Chapter resumes with a discussion on Martin-Löf's intuitionistic type theory (Martin-Löf 1984) that provides a theory of identity types, which is very non-trivial in the intensional case. I compare Martin-Löf's approach to identity with Frege's approach and reconsider Frege's famous *Venus* example through the optics of Martin-Löf's type theory.

Chapter **7** continues to treat the issue of identity but this time with new approaches coming from category theory and some related fields. In the beginning of this Chapter I stress the conceptual similarity and the conceptual difference between the logical notion of relation and geometrical notion of transformation aka mapping or simply map. On this basis I re-introduce the notion of category with a naive geometrical example, stress the geometrical origin of categorical thinking and the relationships between category theory and Klein's *Erlangen Program*. (I come back to this topic in Sect. **9.6**). Then I turn to more advanced geometrically motivated categories and show how they realize the idea of identity as a map (rather than a relation). In particular, I consider Bénabou's *fibered categories* (Bénabou 1985) and higher categories (aka n-categories) – first in an abstract form and then in the geometrical form of homotopy categories. So I approach the hot subject of *homotopy type theory*, which brings together identity types of Martin-Löf's type theory and the geometrical approaches to identity and the homotopical higher category theory. When I began to study these two subjects about 10 years ago the precise mathematical connection between them was not yet established and the mathematical discipline of homotopy type theory did not yet exist. So it was for me a great relief to learn that these ideas combine not only at the level of speculative philosophy but also in precise mathematical terms. I conclude this Chapter with a presentation of Voevodsky's new foundations of mathematics that he calls Univalent Foundations (Voevodsky 2010, 2011; Voevodsky et al. 2013). In Chap. **10** I refer to the Univalent Foundations as an example of a new form of axiomatic presentation along with the example of Lawvere's axiomatic topos theory.

As a general mathematical reading for Part **II** I recommend Leinster's book (2004) on higher category-theory, which has great pedagogical advantages, Granstrom's book (2011) on type theory, which also provides a philosophical perspective on this theory, Jacob's book (1999) that stresses the link between categorical logic and type theory. The homotopy type theory has been not yet exposed in textbooks but there are very clear expository papers and the collective monograph (Awodey and Warren 2009; Awodey 2010; Voevodsky et al. 2013).

Last Part **III** of the book treats two different subjects, which fall under the scope of Hegel-Lawvere's distinction between objective and subjective features of logic and mathematics. In Chap. **8** I discuss the issue of mathematical intuition from a historical perspective and argue using some historical examples that mathematical intuitions change through the historical time at least as rapidly as do mathematical concepts. The main purpose of this Chapter is to refute the popular opinion according to which mathematics always develops by increasing its degree of abstractness and according to which the highly abstract character of modern mathematical concepts does not allow for a faithful intuitive representation in principle. I suggest an alternative picture of the historical development of mathematics where concepts and intuitions develop side-by-side but sometimes the conceptual development takes over the intuitive development and sometimes, on the contrary, the intuitive development takes over the conceptual one.

I expect that a phenomenologically-minded philosophical reader may object that what I discuss is not the strict philosophical notion of intuition but rather a commonsensical meaning of the word "intuition" as a bunch of helpful analogies borrowed from the everyday life or elsewhere. I argue in this Chapter that the changing mathematical intuition that I describe qualifies at least as intuition in Kant's sense of the term. The lack of discussion of Husserl's views is indeed a significant lacuna of this Chapter that I cannot easily fix. So I leave it for a future work.

Although I wholly share Lawvere's Hegelian view concerning the objective character of scientific logic (which perfectly squares with Cassirer's view on the place and the role of mathematics and logic expressed in the above epigraph) I also stress the role of the subjective intuition because it provides the necessary link that connects the pure mathematics to the individual sensual experience to the scientific empirical methods to the whole body of empirical science. Without such a link Hegel's objective dialectical logic too easily turns into a new form of speculative dogmatic metaphysics wholly detached from reality. One may suggest that since the dogmatic dialectics is an obvious oxymoron it cannot refer to anything real. But the dialectical logic quite rightly protects one from such naive conclusions made on abstract logical grounds: as a matter of painful historical fact the examples of dogmatic misuse of philosophical dialectics are abound.[1]

In Chap. **9** I discuss structuralism including its mathematical variety. Considering structuralism as a suggestive idea rather than a system of stable philosophical views I argue against the received view according to which category theory brings about a new variety of structuralism and provides a new framework for developing structural mathematics. I recognize the role of structural thinking in the development of category theory and describe this role in this Chapter. In particular, I elaborate on Eilenberg and Mac Lane's idea of category theory as a continuation of Klein's *Erlangen Program* (Eilenberg and MacLane 1945). This very analogy allows me to specify the crucial difference between Klein's structural thinking and new categorical thinking: when groups are generalized up to categories the notion of invariant structure is replaced by the notion of covariant or contravariant functor. I argue that the structuralist thinking about functoriality in terms of preservation of invariant structures is, generally, inappropriate; then I suggest a different philosophical view (or rather another suggestive idea) where the notion of functoriality (i.e., of co- and contravariance) becomes central. Although this conceptual development begins with a mere generalization of the structuralist *Erlangen Program* it brings about a new view, which is very unlike the structuralist view. In the end of this Chapter (Sect. **9.8**) I suggest a purely geometrical way of thinking about categories alternative to the more convenient way of thinking about categories as categories of structures. The basic idea here is thinking of geometrical objects as maps from types (of geometrical objects) to spaces. I demonstrate this approach with some elementary examples from the twentieth century geometry. Thus in my suggested post-structuralist picture the notion of object (this time understood as a map) becomes once again central.

The conceptual change described in Chap. **9** affects not only the choice of structures explored with the Formal Axiomatic Method but also this method itself. So in the concluding Chap. **10** I make the long-promised attempt to describe the New Axiomatic Method more systematically. I first describe the two basic functions of Axiomatic Method, which Lawvere calls the *unification* and the *concentration*. Here I contrast the unificatory strategy of the New Method to the more traditional unificatory strategy of Formal Axiomatic Method, which has a structuralist and a logicist underpinning. Then I describe the *concentration* part, which turns to be

[1] Unlike the older forms of dogmatism the more recent dialectical dogmatism does not use any fixed system of beliefs but enforces a permanent organized change of one's beliefs on changing pragmatic grounds (political, economical, etc.).

more traditional and in a new form reproduces some features of Euclid's Axiomatic Method. The most original part of the New Axiomatic Method is, of course, its logical part, which involves the notion of internal logic. Generalizing on works of Lawvere and Voevodsky I describe here in general terms a way of using the internal logic of some given category (which is construed in intuitive geometrical terms at the first step of the axiomatic construction) for improving upon the construction of this very category and providing it with some deductive structure. This way of using logic for building mathematical theories suggests a new way of thinking about the role of logic in mathematical theories, which is very unlike Hilbert's and Tarski's.

In my suggested approach logic is designed along with the rest of conceptual construction rather than used as a ready-made foundation for making further mathematical constructions. One may think that the freedom of making up logical calculi added to the freedom of making up new axiomatic mathematical theories (assured already by Hilbert) only reinforce the inflation of the "metaphysical world of thought". In fact the New Axiomatic Method prevents this inflation in two different ways. First, by taking into account the objective meaning of the category of interest (which can be, for example, a spatiotemporal category used in physics) and, second, by requiring the relevant logic to be the internal logic of this given category. While the former feature is at some degree also compatible with the standard Formal Axiomatic Method the latter feature is a genuinely original contribution of the New Method. The New Method no longer reduces the function of logical formalization to a logical censorship; instead logic is used here as a flexible tool for the internal conceptual reconstruction.

An important part of my argument consists of pointing to Lawvere's and Voevodsky's works as applications of this New Method, and stressing the fact that in both cases it allows for a remarkable conceptual simplification and clarification of otherwise difficult and conceptually problematic theories. Since in both cases the relevant logic is internal with respect to its base category this logic inherits the objective meaning of this base category. This allows me to suggest that the New Axiomatic Method may help to bridge the gap between mathematics and physics created and justified by the standard Formal Axiomatic Method and by the logicist view on mathematics that underpins this standard method. Notwithstanding my critique of Hilbert's version of Axiomatic Method developed throughout in this book, I believe (contra Arnold) that Hilbert was perfectly right when he described this method as "the basic instrument of all research" (Hilbert 1927, p. 467) and when he said that "[t]o proceed axiomatically means [..] nothing else than to think with consciousness" (Hilbert 1922, p. 1120).

Part I
A Brief History of the Axiomatic Method

In his famous address "Axiomatic Thought" delivered before the Swiss Mathematical Society in Zurich in 1917 Hilbert says:

> If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework. [..] These fundamental propositions can be regarded [..] as the axioms of the individual fields of knowledge: the progressive development of the individual field of knowledge then lies solely in the further logical construction of the already mentioned framework of concepts. This standpoint is especially predominant in pure mathematics. [.. A]nything at all that can be the object of scientific thought becomes dependent on the axiomatic method, an thereby indirectly on mathematics. (Hilbert 1918, pp. 1108–1115)

In a different paper the author makes a further epistemological claim:

> The axiomatic method is and remains the indispensable tool, appropriate to our minds, for all exact research in any field whatsoever: it is logically incontestable and at the same time fruitful. [..] To proceed axiomatically means in this sense nothing else than to think with consciousness. (Hilbert 1922, p. 1120)

Although Hilbert's enthusiasm about the Axiomatic Method and his high esteem of the role of this method in science may be not universally accepted today, the modern notion of axiomatic theory remains shaped by Hilbert's works; his *Grundlagen der Geometrie* (Foundations of Geometry) first published in 1899 still serves as a paradigm of axiomatic mathematical theory. As soon as this method is understood in the above general terms one may think that it has been practiced by mathematicians since the early days of their discipline. Indeed in the *Introduction* to his *Foundations of Geometry* of 1899 Hilbert states the following:

> Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space. (Hereafter Hilbert 1899 is quoted in English translation Hilbert 1950)

Notice Euclid's name is the above quote. Evidently Hilbert had in mind Euclid's *Elements* when he prepared his *Foundations of Geometry* for publication. Hilbert aims at developing Euclidean geometry on a wholly new conceptual basis.[1] In this sense Hilbert's *Foundations* of 1899 qualifies as a fairly revolutionary work. However one should not forget that rewriting geometrical chapters of Euclid's *Elements* in new terms is itself an old and well-establish tradition in the history of mathematical thought. Hilbert's *Foundations of Geometry* (as well as Bourbaki's open-ended *Elements of Mathematics* (Bourbaki 1939–1988) produced later in the twentieth century) make part of this long-term tradition and can be compared with such groundbreaking works of earlier generations as, for example, *Restored Euclid* by Borelli (1658), *New Elements of Geometry* by Arnauld (1683) and *Euclid Freed from All Flaws* by Saccheri (1733). Thus the Hilbertian revolution that still strongly influences today's mathematical practice is certainly not the first revolution of this sort and hopefully not the last one.

Hilbert thinks of his new version of Axiomatic Method as a development of and improvement over Euclid's method of theory-building. Surely Hilbert is aware about the fact that his method is not the same as Euclid's; we shall see that Hilbert in fact quite precisely points to the key difference (see Sect. **3.6**). The purpose of Chap. **2** is to describe this difference more precisely and more systematically. In Chap. **3**, I focus on Hilbert's work and compare Hilbert's approach to Euclid's. In Chap. **4**, I consider applications of Hilbert's Axiomatic Method in the twentieth century mathematics and, in particular, in Bourbaki. In Chap. **5**, I discuss Lawvere's work and show how some basic features of Euclid's approach deliberately ignored by Hilbert get a new life in today's categorical logic.

[1] I agree with David Rowe when he says that "The reform of geometry that [Hilbert] envisaged in *Grundlagen der Geometrie* was primarily conceived as a renewal of the fundamental structures of classical Euclidean geometry." (Rowe 2000, p. 71)

Chapter 2
Euclid: Doing and Showing

Reading older mathematical texts always involves a hermeneutical dilemma: in order to make sense of the mathematical content of a given old text one wants to interpret it in modern terms; in order to see the difference between the modern mathematical thinking and older ways of mathematical thinking one wants to avoid anachronisms and understand the old text in its own terms (Unguru 1975). Any scholar studying older mathematics needs to find a way between the Scylla of "antiquarianism" that seeks the scholar's conversion into a person living during a different historical epoch, and the Charybdis of radical "presentism" that finds in older texts nothing but a minor part of today's standard mathematical curricula and wholly ignores the historical change of basic patterns of mathematical thinking.[1] My way through the channel is the following. I read Euclid's text verbatim (relying on Heiberg's edition of the original Greek (Euclides 1883–1886) and using Fitzpatrick's new English translation Euclid 2011), consider its most important modern interpretations (including overtly anachronistic ones), criticize some of these interpretations on the basis of textual evidences, and finally suggest some alternative interpretations. In order to prevent the risk of losing the main argument behind the following historical details I formulate now my general conclusion. Contrary to popular opinion Euclid's geometry is not a system of propositions some of which have a special status of axioms while some other are derived from the axioms according to certain rules of logical inference. It can be rather described after Friedman as "a form of rational argument" (Friedman 1992, p. 94),[2] where some non-propositional content (including non-propositional first principles) is indispensable. Precipitating what follows (see particularly Sect. 3.6) let me mention

[1]Being between Scylla and Charybdis is an idiom deriving from Greek mythology. Scylla and Charybdis were mythical sea monsters noted by Homer. Scylla was rationalized as a rock shoal (described as a six-headed sea monster) on the Italian side of the strait and Charybdis was a whirlpool off the coast of Sicily. They were regarded as a sea hazard located close enough to each other that they posed an inescapable threat to passing sailors; avoiding Charybdis meant passing too close to Scylla and vice versa. (after WikipediA)

[2]See the full quote from Friedman in the end of Sect. 3.5.

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that certain non-propositional principles also make part of modern formal theories in the form of *syntactic rules*. As we shall now see in Euclid the non-propositional aspect of mathematical reasoning plays a more prominent role.

2.1 Demonstration and "Monstration"

All Propositions of Euclid's *Elements* (with few easily understandable exceptions) fit into the scheme described by Proclus in his *Commentary* (Proclus 1970) as follows:

> Every Problem and every Theorem that is furnished with all its parts should contain the following elements: an *enunciation*, an *exposition*, a *specification*, a *construction*, a *proof*, and a *conclusion*. Of these *enunciation* states what is given and what is being sought from it, a perfect *enunciation* consists of both these parts. The *exposition* takes separately what is given and prepares it in advance for use in the investigation. The *specification* takes separately the thing that is sought and makes clear precisely what it is. The *construction* adds what is lacking in the given for finding what is sought. The *proof* draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The *conclusion* reverts to the *enunciation*, confirming what has been proved. (Proclus 1970, p. 203, italic is mine)

It is appropriate to notice here that the term "proposition", which is traditionally used in translations as a common name of Euclid's problems and theorems, is not found in the original text of the *Elements*: Euclid numerates these things throughout each Book without naming them by any common name. (The reader will shortly see why this detail is important.) The difference between problems and theorems is explained in Sect. 2.4 below. Let me now show how Proclus' scheme applies to Proposition 5 of the First Book (Theorem 1.5), which is a well-known theorem about angles of the isosceles triangle. References in square brackets are added by the translator; some of them will be discussed later on. Words in round brackets are added by the translator for stylistic reason. Words in bold are borrowed from the above Proclus' quote. Throughout this Chap. 2 I write these words in italic when I use them in Proclus' specific sense (Fig. 2.1).

enunciation:

> For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

exposition:

> Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC*; and let the straight lines *BD* and *CE* have been produced further in a straight line with *AB* and *AC* (respectively). [Postulate 2].

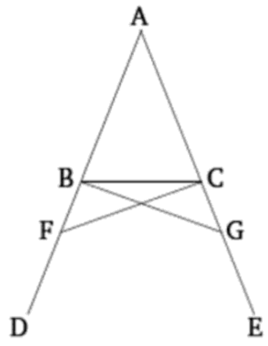

Fig. 2.1 Isosceles triangle theorem (Theorem 1.5 of Euclid's *Elements*)

specification:

I say that the angle *ABC* is equal to *ACB*, and (angle) *CBD* to *BCE*.

construction:

For let a point *F* be taken somewhere on *BD*, and let *AG* have been cut off from the greater *AE*, equal to the lesser *AF* [Proposition 1.3]. Also, let the straight lines *FC*, *GB* have been joined. [Postulate 1]

proof:

In fact, since *AF* is equal to *AG*, and *AB* to *AC*, the two (straight lines) *FA*, *AC* are equal to the two (straight lines) *GA*, *AB*, respectively. They also encompass a common angle *FAG*. Thus, the base *FC* is equal to the base *GB*, and the triangle *AFC* will be equal to the triangle *AGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Proposition 1.4]. (That is) *ACF* to *ABG*, and *AFC* to *AGB*. And since the whole of *AF* is equal to the whole of *AG*, within which *AB* is equal to *AC*, the remainder *BF* is thus equal to the remainder *CG* [Axiom 3]. But *FC* was also shown (to be) equal to *GB*. So the two (straight lines) *BF*, *FC* are equal to the two (straight lines) *CG*, *GB* respectively, and the angle *BFC* (is) equal to the angle *CGB*, while the base *BC* is common to them. Thus the triangle *BFC* will be equal to the triangle *CGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Proposition 1.4]. Thus *FBC* is equal to *GCB*, and *BCF* to *CBG*. Therefore, since the whole angle *ABG* was shown (to be) equal to the whole angle *ACF*, within which *CBG* is equal to *BCF*, the remainder *ABC* is thus equal to the remainder *ACB* [Axiom 3]. And they are at the base of triangle *ABC*. And *FBC* was also shown (to be) equal to *GCB*. And they are under the base.

conclusion:

> Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

An obvious difference between Proclus' analysis of the above theorem and its usual modern analysis is the following. For a modern reader the proof of this theorem begins with Proclus' *exposition* and includes Proclus' *specification*, *construction* and *proof*. Thus for Proclus the *proof* is only a part of what we call today the proof of this theorem. Also notice that Euclid's theorems conclude with the words "which ... was required to *show*" (as correctly translates Fitzpatrick) but not with the words "what it was required to *prove*" (as inaccurately translates Heath 1926). The standard Latin translation of this Euclid's formula as *quod erat demonstrandum* is also inaccurate. These inaccurate translations conflate two different Greek verbs: "apodeiknumi" (English "to prove", Latin "demonstrare") and "deiknumi" (English "to show", Latin "monstrare"). The difference between the two verbs can be clearly seen in the two Aristotle's *Analytics*: Aristotle uses the verb "apodeiknumi" and the derived noun "apodeixis" (proof) as technical terms in his syllogistic logic, and he uses the verb "deiknumi" in a broader and more informal sense when he discusses epistemological issues (mostly in the *Second Analytics*). Without trying to trace here the history of Greek logical and mathematical terminology and speculate about possible influences of some Greek writers on some other writers, I would like to stress the remarkable fact that Aristotle's use of verbs "deiknumi" and "apodeiknumi" agrees with Euclid's and Proclus'. In my view this fact alone is sufficient for taking seriously the difference between the two verbs and distinguishing between *proof* and "showing" (or otherwise between *demonstration* and *monstration*).[3]

One may think that the difference between the current meaning of the word "proof" in today's mathematics and logic and the meaning of Proclus' *proof* (Greek "apodeixis") is a merely terminological issue, which is due to difficulties of translation from Greek to English. I shall try now to show that this terminological difference points on a deeper problem, which is not merely linguistic. In today's logic the word "proof" stands for a logical inference of certain conclusion from some given premises. In fact this is what by and large was meant by proof also by Aristotle and Proclus. Indeed, looking at the *proof* (in Proclus' sense) of Euclids Theorem 1.5 we see that it also qualifies as a proof in the modern sense: we have here a number of premises (which I make explicit in the next Section) and certain

[3]As far as mutual influences are concerned two things are certain: (i) Proclus read Aristotle and (ii) Aristotle had at least a basic knowledge of the mathematical tradition, on which Euclid later elaborated in his *Elements* (as Aristotle's mathematical examples clearly show Heath 1949). It remains unclear whether Aristotle's work could influence Euclid. In my view this is unlikely. However Aristotle's logic certainly played an important role in later interpretations and revisions of Euclid's *Elements*. I leave this interesting issue outside of the scope of this book.

conclusions derived from those premises. It is irrelevant now whether or not this particular inference is valid according to today's logical standards; what I want to stress here is only the general setting that involves some premises, an inference (probably invalid) and some conclusions. This core meaning of the word "proof" (Greek "apodeixis") hardly changed since Proclus' times.

So we get a problem, which is clearly not only terminological: Is it indeed justified to describe the *exposition*, the *specification* and the *construction* as elements of the proof or one should rather follow Proclus and consider these things as independent constituents of a mathematical theorem?

The question of *logical significance* of the *exposition*, the *specification* and the *construction* in Euclid's geometry has been discussed in the literature; in what follows I shall briefly describe some tentative answers to it. However before doing this I would like to stress that this question may be ill-posed to begin with. As far as one assumes, first, that the theory of Euclid's *Elements* is (by and large) sound and, second, that any sound mathematical theory is an axiomatic theory in the modern sense, then, in order to make these two assumptions mutually compatible, one has to describe the *exposition*, the *specification* and the *construction* of each Euclid's theorem as parts of the proof of this theorem and specify their logical role and their logical status. I shall not challenge the usual assumption according to which Euclid's mathematics is by and large sound. (I say "by and large" in order to leave some room for possible revisions and corrections of Euclid's arguments and thus avoid controversies about the question whether a given interpretation of Euclid is authentic or not. Although I pay more attention to textual details than it is usually done in modern logical reconstructions of Euclid's reasoning, I am not going to criticize these reconstructions by pointing to their anachronistic character.) However I shall challenge the other assumption according to which any sound mathematical theory is an axiomatic theory in the modern sense. Since I do not take this latter assumption for granted I do not assume from the outset that the problematic elements of Euclid's reasoning (the *exposition*, the *specification* and the *construction*) play some *logical* role, which only needs to be made explicit and appropriately understood. In what follows I try to describe how these elements work without making about them any additional assumptions and only then decide whether the role of these elements qualifies as logical or not.

2.2 Are Euclid's Proofs Logical?

Let's look at Euclid's Theorem 1.5 more attentively. I begin its analysis with its *proof*. Among the premises of this *proof*, one may easily identify Axiom (Common Notion) 3 according to which

(Axiom 3): If equal things are subtracted from equal things then the remainders are equal documented in Aristotle's writings. In particular, Aristotle quotes Euclid's Axiom 3 (which, of course, Aristotle could know from another source) almost verbatim.[5]

However important Aristotle's argument in the history of Western thought may be, there is no reason to take it for granted every time when we try today to interpret Euclid's *Elements* or any other old mathematical text. Whatever is one's philosophical stance concerning the place of logical principles in human reasoning one can see what kind of harm can be made if Aristotle's assumption about the primacy of logical and ontological principles is taken straightforwardly and uncritically: one treats Euclid's Axioms on equal footing with premisses like **Con1–3** and **Hyp** and so misses the law-like character of the Axioms. Missing this feature doesn't allow one to see the relationships between Greek logic and Greek mathematics, which I just sketched.

Having said that I would like to repeat that Euclid's *proof* (apodeixis) is the part of Euclid's theorems, which more resembles what we today call proof (in logic) than other parts Euclid's theorems. For this reason in what follows I shall call inferences in Euclid's *proofs*, which are based on Axioms, *protological* inferences

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[5] Here are some quotes:

> By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind. (Met. 996b27-32, Heath's translation, corrected)

Here Aristotle refers to a logical principle as "common opinion". In the next quote he compares mathematical and logical axioms:

> We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28, my translation)

Here is the last quote where Aristotle refers to Axiom 3 explicitly:

> Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b, my translation)

The "science of philosopher" otherwise called the "first philosophy" is Aristotle's logic, which in his understanding is closely related to (if not indistinguishable from) what we call today ontology. After Alexandrian librarians we call today the relevant collection of Aristotle's texts by the name of *metaphysics* and also use this name for a relevant philosophical discipline.

and distinguish them from inferences of another type that I shall call *geometrical* inferences. This analysis is not incompatible with the idea (going back to Aristotle) that behind Euclid's protological and geometrical inferences there are inferences of a more fundamental sort, that can be called *logical* in the proper sense of the word. However I claim that Euclid's text as it stands provides us with no evidence in favor of this strong assumption. One can learn Euclid's mathematics and fully appreciate its rigor without knowing anything about logic just like Moliere's M. Jourdain could well express himself long before he learned anything about prose!

Whether or not the science of logic really helps one to improve on mathematical rigor – or this is rather the mathematical rigor that helps one to do logic rigorously – is a controversial question that I shall discuss throughout this book and suggest an answer only in the last Chapter. The purpose of my present reading of Euclid is at the same time more modest and more ambitious than the purpose of logical analysis. It is more modest because this reading doesn't purport to assess Euclid's reasoning from the viewpoint of today's mathematics and logic but aims at reconstructing this reasoning in its authentic archaic form. It is more ambitious because it doesn't take the today's viewpoint for granted but aims at reconsidering this viewpoint by bringing it into a historical perspective.

2.3 Instantiation, Objecthood and Objectivity

Let us now see where the premises **Hyp** and **Con1–3** come from. As I have already mentioned they actually come from two different sources: **Hyp** is assumed *by hypotheis* while **Con1–3** are assumed *by construction*. I shall consider here these two cases one after the other.

The notion of hypothetic reasoning is an important extension of the core notion of axiomatic theory outlined above; it is well-treated in the literature and I shall not cover it here in full. I shall consider only one particular aspect of hypothetical reasoning as it is present in Euclid. The hypothesis that validates **Hyp**, informally speaking, amounts to the fact that Theorem 1.5 tells us something about isosceles triangles (rather than about objects of another sort). The corresponding definition (Definition 1.20) tells us that two sides of the isosceles triangle are equal. However to get from here to **Hyp** one needs yet another step. The *enunciation* of Theorem 1.5 refers to isosceles triangles *in general*. But **Hyp** that is involved into the *proof* of this Theorem concerns only *particular* triangle ABC. Notice also that the *proof* concludes with the propositions $ABC = ACB$ and $FBC = GCB$ (where ABC, ACB, FBC and GCB are angles), which also concern only *particular* triangle ABC. This conclusion differs from the following *conclusion* (of the whole Theorem), which almost verbatim repeats the *enunciation* and once again refers to isosceles triangles and their angles in general terms.

The wanted step that allows Euclid to proceed from the *enunciation* to **Hyp** is made in the *exposition* of this Theorem, which introduces triangle ABC as an "arbitrary representative" of isosceles triangles (in general). In terms of modern logic this step can be described as the *universal instantiation*:

$$\forall x P(x) \Longrightarrow P(a/x)$$

where $P(a/x)$ is the result of the substitution of individual constant a at the place of all free occurrences of variable x in $P(x)$. The same notion of universal instantiation allows for interpreting Euclid's *specification* in the obvious way. The reciprocal backward step that allows Euclid to obtain the *conclusion* of the Theorem from the conclusion of the *proof* can be similarly described as the *universal generalization*:

$$P(a) \Longrightarrow \forall x P(x)$$

(which is a valid rule only under certain conditions that I skip here).

As long as the *exposition* and the *specification* are interpreted in terms of the universal instantiation these operations are understood as logical inferences and, accordingly, as element of proof in the modern sense of the word. A somewhat different – albeit not wholly incompatible – interpretation of Euclid's *exposition* and *specification* can be straightforwardly given in terms of Kant's *transcendental aesthetics* and *transcendental logic* developed in his *Critique of Pure Reason* (Kant 1999). Kant thinks of the traditional geometrical *exposition* not as a logical inference of one proposition from another but as a "general procedure of the imagination for providing a concept with its image"; a representation of such a general procedure Kant calls a *schema* of the given concept (A140). Thus for Kant any individual mathematical object (like triangle *ABC*) always comes with a specific *rule* that one follows constructing this object in one's imagination and that provides a link between this object and its corresponding concept (the concept of isosceles triangle in our example). According to Kant the representation of general concepts by imaginary individual objects (which Kant for short also describes as "construction of concepts") is the principal distinctive feature of mathematical thinking, which distinguishes it from a philosophical speculation.

> Philosophical cognition is rational cognition from concepts, mathematical cognition is that from the construction of concepts." But to construct a concept means to exhibit a priori the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition.... The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle.
>
> Philosophical cognition thus considers the particular only in the universal, but mathematical cognition considers the universal in the particular, indeed even in the individual... (A713-4/B741-2).

Kant's account can be understood as a further explanation of what the instantiation of mathematical concepts amounts to; then one may claim that the Kantian interpretation of Euclid's *exposition* and *specification* is compatible with its interpretation as the universal instantiation in the modern sense. However the Kantian interpretation doesn't suggest by itself to interpret the instantiation as a logical procedure, i.e., as an inference of a proposition from another proposition. As the above quote makes it clear Kant describes the instantiation as a cognitive procedure of a different sort.

Now coming back to Euclid we must first of all admit that the *exposition* and the *specification* of Theorem 1.5 as they stand are too concise for preferring one philosophical interpretation rather than another. Euclid introduces an isosceles triangle through Definition 1.20 providing no rule for constructing such a thing. (This example may serve as an evidence against the often-repeated claim that every geometrical object considered by Euclid is supposed to be constructed on the basis of Postulates beforehand.) Nevertheless given the important role of constructions in Euclid's geometry, which I explain in the next Section, the idea that every geometrical object in Euclid has an associated construction rule, appears very plausible. There is also another interesting textual feature of Euclid's *specification* that in my view makes the Kantian interpretation more plausible.

Notice the use of the first person in the *specification* of Theorem 1.5: "I say that ....". In *Elements* Euclid uses this expression systematically in the *specification* of every theorem. Interpreting the *specification* in terms of universal instantiation one should, of course, disregard this feature as merely rhetorical. However it may be taken into account through the following consideration. While the *enunciation* of a theorem is a general proposition that can be best understood á la Frege in the abstraction from any human or inhuman thinker, i.e., independently of any thinking *subject*, who might believe this proposition, assert it, refute it, or do anything else about it, the core of Euclid's theorems (beginning with their *exposition*) involves an individual thinker (individual subject) that cannot and should not be wholly abstracted away in this context. When Euclid *enunciates* a theorem this *enunciation* does not involve – or at least is not supposed to involve – any particularities of Euclid's individual thinking; the less this *enunciation* is affected by Euclid's (or anyone else's) individual writing and speaking style the better. However the *exposition* and the*specification* of the given theorem essentially involve an *arbitrary* choice of notation ("Let *ABC* be an isosceles triangle...”), which is an individual choice made by an individual mathematician (namely, made by Euclid on the occasion of writing his *Elements*). This individual choice of notation goes on par with what we have earlier described as *instantiation*, i.e. the choice of one individual triangle (triangle *ABC*) of the given type, which serves Euclid for proving the general theorem about *all* triangles of this type. The *exposition* can be also naturally accompanied by drawing a diagram, which in its turn involves the choice of a particular shape (provided this shape is of the appropriate type), to leave alone the choices of its further features like color, etc.

Thus when in the *specification* of Theorem 1.5 we read "I say that the angle *ABC* is equal to *ACB*" we indeed do have good reason to take Euclid's wording seriously.

For the sentence "angle *ABC* is equal to *ACB*" unlike the sentence "for isosceles triangles, the angles at the base are equal to one another" has a feature that is relevant only to one particular presentation (and to one particular diagram if any), namely the use of letters A, B, C rather than some other letters.[6] The words "I say that ..." in the given context stress this situational character of the following sentence "angle *ABC* is equal to *ACB*". What matters in these words is, of course, not Euclid's personality but the reference to a particular act of speech and cognition of an individual mathematician. Proving the same theorem on a different occasion Euclid or anybody else could use other letters and another diagram of the appropriate type.

A competent reader of Euclid is supposed to know that the choice of letters in Euclid's notation is arbitrary and that Euclid's reasoning does not depend of this choice. The arbitrary character of this notation should be distinguished from the general arbitrariness of linguistic symbols in natural languages. What is specific for the case of *exposition* and *specification* is the fact that here the arbitrary elements of reasoning (like notation) are sharply distinguished from its invariant elements. To use Kant's term we can say that behind the notion according to which the choice of Euclid's notation is arbitrary (at least at the degree that letters used in this notation are permutable) and according to which the same reasoning may work equally well with different diagrams (provided all of them belong to the same appropriate type) there is a certain invariant *schema* that sharply limits such possible choices. This schema not simply *allows* for making some arbitrary choices but *requires* every possible choice in the given reasoning to be wholly arbitrary. This requirement is tantamount to saying that subjective reasons behind choices made by an individual mathematician for presenting a given mathematical argument are strictly irrelevant to the "argument itself" (in spite of the fact that the argument cannot be formulated without making such choices). In general talks in natural languages there is no similar sharp distinction between arbitrary and invariant elements. When I write this paper I can certainly change some wordings without changing the sense of my argument but I am not in a position to describe precisely the scope of such possible changes and identify the intended "sense" of my argument with a mathematical rigor. This is because my present study is philosophical and historical but not purely mathematical.

Thus Euclid's *exposition* serves for the formulation of a given universal proposition in terms, which are suitable for a particular act of mathematical cognition made by an individual mathematician. This aspect of the *exposition* is not accounted for by the modern notion of universal instantiation. It may be argued that this aspect of the *exposition* needs not be addressed in a *logical* analysis of Euclid's mathematics that aims at explication of the *objective meaning* of Euclid's reasoning and may well leave aside cognitive aspects of this reasoning. I agree that this latter issue lies out of the scope of logical analysis in the usual sense of the term but I disagree that the objective meaning of Euclid's reasoning can be properly understood without addressing this issue. Euclid's mathematical reasoning is *objective* due

[6] Although the choice of letters in Euclid's notation is arbitrary the *system* of this notation is not. This traditional geometrical notation has a relatively stable and rather sophisticated syntax, which I briefly describe in what follows.

and thus "gets" these new objects. Under this interpretation Euclid's *constructions* turn into logical inferences of sort. As Hintikka and Remes emphasize in their paper the principal distinctive feature of Euclid's *constructions* (under their interpretation) is that these constructions introduce some *new* individuals; they call such individuals "new" in the sense that these individuals are not (and cannot be) introduced through the universal instantiation of hypotheses making part the *enunciation* of the given theorem.

The propositional interpretations of Euclid's Postulates are illuminating because they allow for analyzing traditional geometrical constructions in modern logical terms. However they require a paraphrasing of Euclid's wording, which from a logical point of view is far from being innocent. In order to see this let us leave aside the epistemic attitude expressed by the verb "postulate" and focus on the question of *what* Euclid postulates in his Postulates 1–3. Literally, he postulates the following:

P1: to draw a straight-line from any point to any point.
P2: to produce a finite straight-line continuously in a straight-line.
P3: to draw a circle with any center and radius.

As they stand expressions P1–3 don't qualify as propositions; they rather describe certain *operations*! And making up a proposition from something which is not a proposition is not a innocent step. My following analysis is based on the idea that Postulates are *not* primitive truths from which one may derive some further truths but are primitive operations that can be combined with each other and so produce into some further operations. In order to make my reading clear I paraphrase P1–3 as follows:

(OP1): drawing a (segment of) straight-line between its given endpoints
(OP2): continuing a segment of given straight-line indefinitely ("in a straight-line)"
(OP3): drawing a circle by given radius (a segment of straight-line) and center (which is supposed to be one of the two endpoints of the given radius).

Noticeably none of OP1–3 allows for producing geometrical constructions out of nothing; each of these fundamental operation produces a geometrical object out of some other objects, which are supposed to be *given* in advance. The table below specifies inputs (operands) and outputs (results) of OP1–3:

Operation	Input	Output
OP1	Two (different) points	Straight segment
OP2	Straight segment	(Bigger) Straight segment
OP3	Straight segment and one of its endpoints	Circle

PE1 as it stands does not imply that there exists at least one point or at least one line in Euclid's geometrical universe. If there are no points then there are no lines either. Similar remarks can be made about the existential interpretation of other Euclid's Postulates. Thus the existential interpretation of Postulates by itself does not turn these Postulates into existential axioms that guarantee that Euclid's universe is non-empty and contains all geometrical objects constructible by ruler and compass. To meet this purpose one also needs to postulate the existence of at least two different points – and then argue that the absence of any counterpart of such an above, the *exposition* describes reasoning of an individual mathematician rather than presents this reasoning in an objective form. That every complex construction must be performed through Postulates and earlier performed constructions is an epistemic requirement, which is on par with the requirement according to which every theorem must be proved rather than simply stated. Remind that the *expositions* of Euclid's Theorems have the form "I say that…". This indeed makes an apparent contrast with the *expositions* of Problems that have the form "it is required to ….". However this contrast doesn't seem me to be really sharp. Euclid's expression "I say that…" in the given context is interchangeable with the expression "it is required to show that…", which matches the closing formula of Theorems "(this is) the very thing it was required to show". Euclid's expression "it is required to…" that he uses in the *expositions* of Problems similarly matches the closing formula of Problems "(this is) the very thing it was required to do". The requirement according to which every Theorem must be "shown" or "monstrated" doesn't imply, of course, that the *enunciation* (statement) of this Theorem has a deontic meaning. The requirement according to which every Problem must be "done" doesn't imply either that the *enunciation* of this Problem has something to do with deontic modalities.

The analogy between axioms and theorems, on the one hand, and postulates and problems, on the other hand, may suggest that Euclid's geometry splits into two independent parts one of which is ruled by (proto)logical deduction while the other is ruled by geometrical production. However this doesn't happen and in fact problems and theorems turn to be mutually dependent elements of the same theory. The above example of Problem 1.1 and Theorem 1.5 show how the intertwining of problems and theorems works. Theorems, generally, involve *constructions* (called in this case auxiliary), which may depend (in the order of geometrical production) on earlier treated problems (as the *construction* of Theorem 1.5 depends on Problem 1.3.) Problems in their turn always involve appropriate *proofs* that prove that the *construction* of the given theorem indeed performs the operation described in the *enunciation* of this theorem (rather than performs some other operation). Such *proofs*, generally, depend (in the order of the protological deduction) on certain earlier treated theorems (just like in the case of *proofs* of theorems). Although this mechanism linking problems with theorems may look unproblematic it gives rise to the following puzzle. Geometrical production produces geometrical objects from some other objects. Protological deduction deduces certain propositions from some other propositions. How it then may happen that the geometrical production has an impact on the protological deduction? In particular, how the geometrical production may justify premises assumed "by construction", so these premises are used in following *proofs*?

In order to answer this question let's come back to the premise **Con3** ($AF = AG$) from Theorem 1.5 and see what if anything makes it true. $AF = AG$ because Euclid or anybody else following Euclid's instructions constructs this pair of straight segments in this way. How do we know that by following these instructions one indeed gets the desired result? This is because the *construction* of Problem 1.3 that contains the appropriate instruction is followed by a *proof* that proves that this

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form. Consider the following example taken from a standard mathematical textbook (Kolmogorov and Fomin 1976, p. 100, my translation into English):

Theorem 2.3. *Any closed subset of a compact space is compact*

Proof. Let F be a closed subset of compact space T and $\{F_\alpha\}$ be an arbitrary centered system of closed subsets of subspace $F \subset T$. Then every F_α is also closed in T, and hence $\{F_\alpha\}$ is a centered system of closed sets in T. Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that F is compact.

Although the above theorem is presented in the usual for today's mathematics form "proposition-proof", its Euclidean structure can be made explicit without re-interpretations and paraphrasing:

enunciation:

Any closed subset of a compact space is compact

exposition:

Let F be a closed subset of compact space T

specification: **absent**

construction:

[Let] $\{F_\alpha\}$ [be] an arbitrary centered system of closed subsets of subspace $F \subset T$.

proof:

[E]very F_α is also closed in T, and hence $\{F_\alpha\}$ is a centered system of closed sets in T. Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that F is compact.

conclusion: **absent**

The absent *specification* can be formulated as follows:
I say that F is a compact space

while the absent *conclusion* is supposed to be a literal repetition of the *enunciation* of this theorem. Clearly these latter elements can be dropped for parsimony reason. In order to better separate the *construction* and the *proof* of the above theorem the authors could first construct set $\cap F_\alpha$ and only then prove that it is non-empty. However this variation of the classical Euclidean scheme also seems me negligible. I propose the reader to check it at other modern examples that the Euclidean structure remains today at work.

Does this mean that the modern notion of axiomatic theory is inadequate to today's mathematical practice just like it is inadequate to Euclid's mathematics? Such a conclusion would be too hasty. Arguably, in spite of the fact that today's mathematics preserves some traditional outlook it is essentially different. So the