Undergraduate Lecture Notes in Physics

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Basic Concepts in Physics

From the Cosmos to Quarks

Springer

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Chapter 1
Gravitation and Newton's Laws

Our Sun is a star of intermediate size with a set of major planets describing closed orbits around it. These are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune. Pluto, considered the Solar System's ninth planet until 2006, was reclassified by the International Astronomical Union as a dwarf planet, due to its very small mass, together with other trans-Neptunian objects (Haumea, Makemake, Eris, Sedna, and others) recently discovered in that zone, called the Kuiper belt. Except for Mercury and Venus, all planets and even certain dwarf planets have satellites. Some of them, like the Moon and a few of the Jovian satellites, are relatively large. Between Mars and Jupiter, there are a lot of small planets or asteroids moving in a wide zone, the largest one being Ceres, classified as a dwarf planet. Other distinguished members of the Solar System are the comets, such as the well-known comet bearing the name of Halley. It seems that most comets originate in the Kuiper belt.

The Sun is located approximately 30,000 light-years (1 light-year $= 9.4\times 10^{12}$ km) from the Galactic Centre, around which it makes a complete turn at a speed of nearly 250 km/s in approximately 250 million years. The number of stars in our galaxy is estimated to be of the order of 10^{11}, classified by age, size, and state of evolution: young, old, red giants, white dwarfs, etc. (Fig. 1.1).

In fact, our galaxy, the Milky Way, is one member of a large family estimated to contain of the order of 10^{13} galaxies. These are scattered across what we call the visible Universe, which seems to be in expansion after some initial event. The galaxies are moving away from each other like dots painted on an inflating rubber balloon.

At the present time, our knowledge of the Universe and the laws governing it is increasing daily. Today we possess a vast knowledge of our planetary system, stellar evolution, and the composition and dynamics of our own galaxy, not to mention millions of other galaxies. Even the existence of several extra-solar planetary systems has been deduced from the discovery of planets orbiting around 51 Pegasi, 47 Ursae Majoris, and several other stars. But barely five centuries ago, we only knew about the existence of the Sun, the Moon, five planets (Mercury, Venus, Mars, Jupiter, and Saturn), some comets, and the visible stars. For thousands of years, people had gazed intrigued at those celestial objects, watching as they moved across the background of fixed stars, without knowing what they were, nor why they were moving like that.

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The discovery of the mechanism underlying the planetary motion, the starting point for our knowledge of the fundamental laws of physics, required a prolonged effort, lasting several centuries. Sometimes scientific knowledge took steps forward, but subsequently went back to erroneous concepts. However, fighting against the established dogma and sometimes going against their own prior beliefs, passionate scholars finally discovered the scientific truth. In this way, the mechanism guiding planetary motions was revealed, and the first basic chapter of physics began to be written: the science of mechanics.

1.1 From Pythagoras to the Middle Ages

Pythagoras of Samos (c. 580–c. 500 BCE) was the founder of a mystic school, where philosophy, science, and religion were blended together. For the Pythagorean school, numbers had a magical meaning. The Cosmos for Pythagoras was formed by the spherical Earth at the centre, with the Sun, the Moon and the planets fixed to concentric spheres which rotated around it. Each of these celestial bodies produced

Fig. 1.1 The Andromeda galaxy, at a distance of two million light-years from our own galaxy. They are similar in size.

a specific musical sound in the air, but only the master, Pythagoras himself, had the gift of hearing the music of the spheres.

Philolaus (c. 470–c. 385 BCE), a disciple of Pythagoras, attributed to the Earth one motion, not around its axis, but around some external point in space, where there was a central fire. Between the Earth and the central fire, Philolaus assumed the existence of an invisible planet, Antichthon, a sort of "counter-Earth". Antichthon revolved in such a way that it could not be seen, because it was always away from the Greek hemisphere. The central fire could not be seen from the Greek world either, and with its shadow Antichthon protected other distant lands from being burned. Antichthon, the Earth, the Sun, the Moon, and the other known planets Mercury, Venus, Mars, Jupiter, and Saturn revolved in concentric orbits around the central fire. The fixed stars were located on a fixed sphere behind all the above celestial bodies.

Heraclides of Pontus (c. 390–c. 310 BCE) took the next step in the Pythagorean conception of the Cosmos. He admitted the rotation of the Earth around its axis, and that the Sun and the Moon revolved around the Earth in concentric orbits. Mercury and Venus revolved around the Sun, and beyond the Sun, Mars, Jupiter, and Saturn also revolved around the Earth (Fig. 1.2).

Around the year when Heraclides died, Aristarchus (c. 310–c. 230 BCE) was born in Samos. From him, only a brief treatise has reached us: *On the Sizes and Distances from the Sun and the Moon*. In another book, Aristarchus claimed that the centre of the Universe was the Sun and not the Earth. Although this treatise has been lost, the ideas expressed in it are known through Archimedes and Plutarch. In one of his books Archimedes states: "He [Aristarchus] assumed the stars and the Sun as fixed, but that the Earth moves around the Sun in a circle, the Sun lying in the

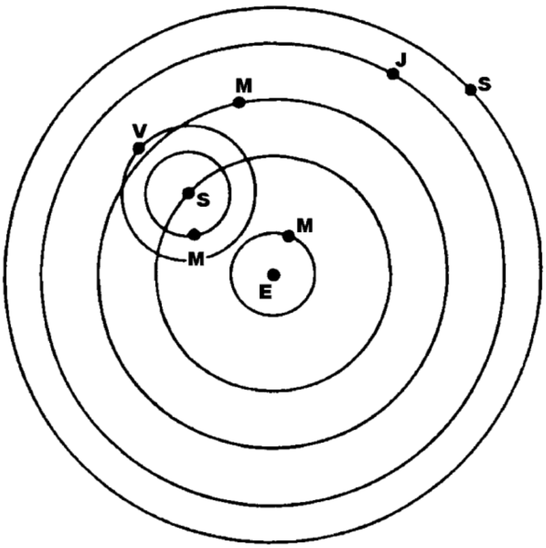

Fig. 1.2 The system of Heraclides.

middle of the orbit." Plutarch also quotes Aristarchus as claiming that: "The sky is quiet and the Earth revolves in an oblique orbit, and also revolves around its axis."

Aristarchus was recognized by posterity as a very talented man, and one of the most prominent astronomers of his day, but in spite of this, his heliocentric system was ignored for seventeen centuries, supplanted by a complicated and absurd system first conceived by Apollonius of Perga in the third century BCE, later developed by Hipparchus of Rhodes in the next century, and finally completed by Ptolemy of Alexandria (c. 70–c. 147 CE).

The Earth's sphericity was accepted as a fact from the time of Pythagoras, and its dimensions were estimated with great accuracy by another Greek scholar Eratosthenes of Cyrene, in the third century BCE. He read in a papyrus scroll that, in the city of Swenet (known nowadays as Aswan), almost on the Tropic of Cancer, in the south of Egypt, on the day corresponding to our 21 June (summer solstice), a rod nailed vertically on the ground did not cast any shadow at noon. He decided to see whether the same phenomenon would occur in Alexandria on that day, but soon discovered that this was not the case: at noon, the rod did cast some shadow. If the Earth had been flat, neither rods would have cast a shadow on that day, assuming the Sun rays to be parallel. But if in Alexandria the rod cast some shadow, and in Swenet not, the Earth could not be flat, but had to be curved.

It is believed that Eratosthenes paid some money to a man to measure the distance between Swenet and Alexandria by walking between the two cities. The result was equivalent to approximately 800 km. On the other hand, if we imagine the rods to extend down to the Earth's centre, the shadow indicated that the angle α between them was about 7° (Fig. 1.3). Then, establishing the proportionality

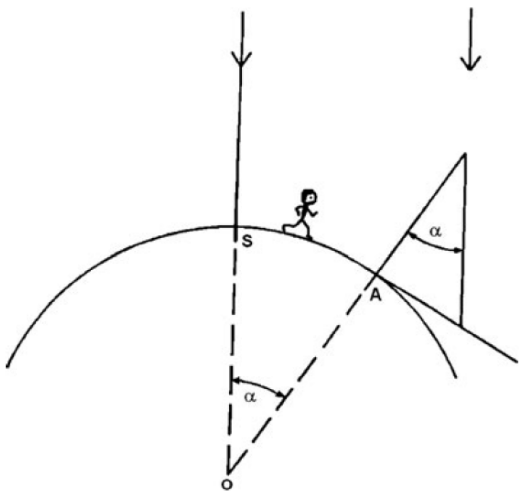

Fig. 1.3 Eratosthenes concluded that the shape of the Earth was a sphere. He used the fact that, when two rods were nailed vertically on the ground, one in the ancient Swenet and the other in Alexandria, at the noon of the day corresponding to our 21 June, the second cast a shadow while the first did not.

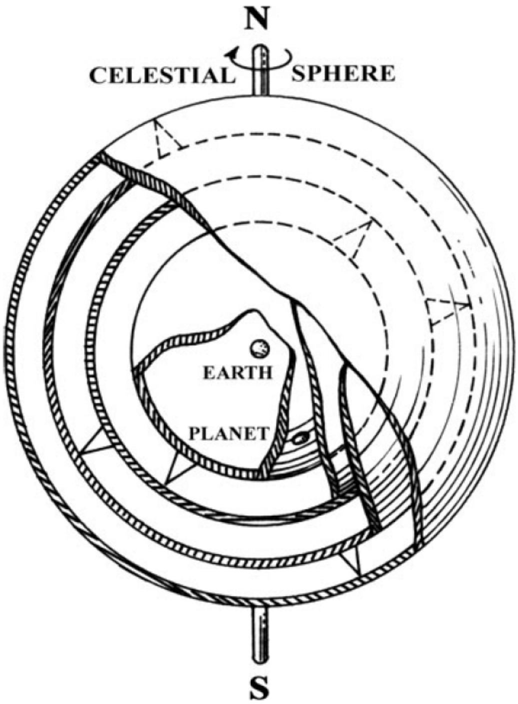

Fig. 1.4 The system of the world according to Ptolemy. The Earth was the centre of the Universe and the planets were fixed to spheres, each one rotating around some axis, which was supported on another sphere which in turn rotated around some axis, and so on.

$$\frac{360}{7} = \frac{x}{800} ,$$

the result is approximately $x = 40,000$ km, which would be the length of the circumference of the Earth if it were a perfect sphere. The value obtained by Eratosthenes was a little less (0.5% smaller).

It is astonishing that, using very rudimentary instruments, angles measured from the shadows cast by rods nailed on the ground, and lengths measured by the steps of a man walking a long distance (but having otherwise an exceptional interest in observation and experimentation), Eratosthenes was able to obtain such an accurate result for the size of the Earth, and so long ago, in fact, twenty-two centuries ago. He was the first person known to have measured the size of the Earth. We know at present that, due to the flattening of the Earth near the poles, the length of a meridian is shorter than the length of the equator. Later, Hipparchus measured the distance from the Moon to the Earth as 30.25 Earth diameters, making an error of only 0.3%.

But let us return to Ptolemy's system (Fig. 1.4). The reasons why it prevailed over Aristarchus' heliocentric system, are very complex. Some blame can probably be laid on Plato and Aristotle, but mainly the latter. Aristotle deeply influenced philosophical and ecclesiastic thinking up to modern times. Neither Plato nor Aristotle had a profound knowledge of astronomy, but they adopted the geocentric system because it was in better agreement with their philosophical beliefs, and their preference for a pro-slavery society. Their cosmology was subordinated to their political and philosophical ideas: they separated mind from matter and the Earth from the sky. And these ideas remained, and were adopted by ecclesiastic philosophy, until the work begun by Copernicus, Kepler, and Galileo and completed by Newton imposed a new way of thinking, where the angels who moved the spheres were no longer strictly necessary.

The system proposed by Ptolemy (Fig. 1.4) needed more than 39 wheels or spheres to explain the complicated motion of the planets and the Sun. When the king Alphonse X of Castile, nicknamed the Wise (1221–1284 CE), who had a deep interest in astronomy, learned about the Ptolemaic system, he exclaimed: "If only the Almighty had consulted me before starting the Creation, I would have recommended something simpler."

In spite of this, the tables deviced by Ptolemy for calculating the motion of the planets were very precise and were used, together with the fixed stars catalog of Hipparchus, as a guide for navigation by Christopher Columbus and Vasco da Gama. This teaches us an important lesson: an incorrect theory may be useful within the framework of its compatibility with the results of observation and experimentation.

In the Middle Ages, most knowledge accumulated by the Ancient Greeks had been forgotten, with very few exceptions, and even the idea of the Earth's sphericity was effaced from people's minds.

1.2 Copernicus, Kepler, and Galileo

In the fifteenth century, a Polish astronomer, Nicolaus Copernicus (1473–1543) brought Ptolemy's system to crisis by proposing a heliocentric system. Copernicus assumed the Sun (more exactly, a point near the Sun) to be the centre of the Earth's orbit and the centre of the planetary system. He considered that the Earth (around which revolved the Moon), as well as the rest of the planets, rotated around that point near the Sun describing circular orbits (Fig. 1.5). Actually, he rediscovered the system that Aristarchus had proposed in ancient times. Copernicus delayed the publication of his book containing the details of his system until the last few days of his life, apparently so as not to contradict the official science of the ecclesiastics. His system allowed a description of the planetary motion that was at least as good as the one which was based on Ptolemaic spheres. But his work irritated many of his contemporaries. The Catholic Church outlawed his book in 1616, and also Martin Luther rejected it, as being in contradiction with the Bible.

The next step was taken by Johannes Kepler, born in 1571 in Weil, Germany. Kepler soon proved to be gifted with a singular talent for mathematics and astronomy, and became an enthusiastic defender of the Copernican system. One day in the year of 1595, he got a sudden insight. From the Ancient Greeks, it was known that there are five regular polyhedra: tetrahedron, cube, octahedron, dodecahedron, and icosahedron – the so-called "Platonic solids" of antiquity. Each of these can be inscribed in a sphere. Similarly, there were five spaces among the known planets. Kepler guessed that the numbers might be related in some way. That idea became fixed in his mind and he started to work to prove it.

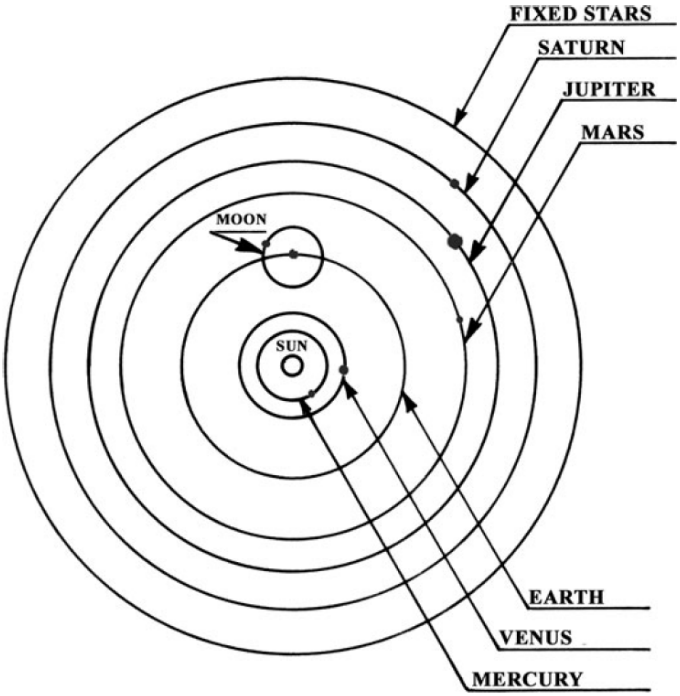

Fig. 1.5 The system of the world according to Copernicus. The Sun was at the centre of the planetary system, and around a point very near to it revolved the Earth and the rest of the planets, all describing circular orbits.

He conceived of an outer sphere associated with Saturn, and circumscribed in a cube. Between the cube and the tetrahedron came the sphere of Jupiter. Between the tetrahedron and the dodecahedron was the sphere of Mars. Between the dodecahedron and the icosahedron was the sphere of Earth. Between the icosahedron and the octahedron, the sphere of Venus. And finally, within the octahedron came the sphere of Mercury (Fig. 1.6). He soon started to compare his model with observational data. As it was known at that time that the distances from the planets to the Sun were not fixed, he imagined the planetary spheres as having a certain thickness, so that the inner wall corresponded to the minimum distance and the outer wall to the maximum distance.

Kepler was convinced a priori that the planetary orbits must fit his model. So when he started to do the calculations and realized that something was wrong, he attributed the discrepancies to the poor reliability of the Copernican data. Therefore he turned to the only man who had more precise data about planetary positions: the Danish astronomer Tycho Brahe (1546–1601), living at that time in Prague, who had devoted 35 years to performing exact measurements of the positions of the planets and stars.

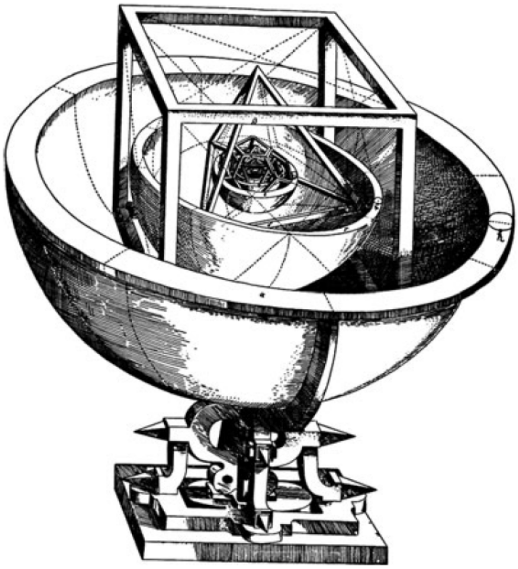

Fig. 1.6 Kepler's system of spheres and inscribed regular Platonic solids.

Tycho Brahe conceived of a system which, although geocentric, differed from that of Ptolemy and borrowed some elements from the Copernican system. He assumed that the other planets revolved around the Sun, but that the Sun and the Moon revolved around the Earth (Fig. 1.7).

In an attempt to demonstrate the validity of his model, he made very accurate observations of the positions of the planets with respect to the background of fixed stars. Brahe was a first-rate experimenter and observer. For more than 20 years he gathered the data of his observations, which were finally used by Kepler to deduce the laws of planetary motion.

Kepler believed in circular orbits, and to test his model, he used Brahe's observations of the positions of Mars. He found agreement with the circle up to a point, but the next observation did not fit that curve. So Kepler hesitated. The difference was 8 min of arc. What was wrong? Could it be his model? Could it be the observations made by Brahe? In the end, he accepted the outstanding quality of Brahe's measurements, and after many attempts, finally concluded that the orbit was elliptical. At this juncture, he was able to formulate three basic laws of planetary motion:

1. All planets describe ellipses around the Sun, which is placed at one of the foci;
2. The radius vector or imaginary line which joins a planet to the Sun sweeps out equal areas in equal intervals of time. Consequently, when the planet is nearest to the Sun (at the point called *perihelion*), it moves faster than when it is at the other extreme of the orbit, called *aphelion* (Fig. 1.8);
3. The squares of the periods of revolution of planets around the Sun are proportional to the cubes of the semi-major axis of the elliptical orbit.

Galileo Galilei (1564–1642) was a contemporary of Kepler and also a friend. At the age of 26, he became professor of mathematics at Pisa, where he stayed

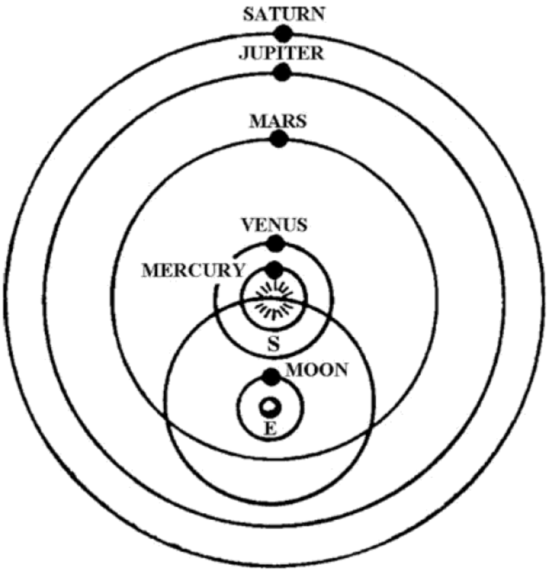

Fig. 1.7 Tycho Brahe's system. The Earth is the centre of the Universe, but the other planets rotate around the Sun, while this in turn moves around the Earth.

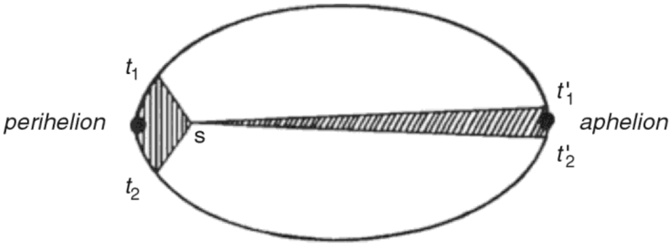

Fig. 1.8 The radius vector or imaginary line joining a planet with the Sun, sweeps out equal areas in equal intervals of time; when the planet is near the Sun, at perihelion, it moves faster than when it is at the other extreme of the orbit, at aphelion.

until 1592. His disagreement with Aristotle's ideas, and especially the claim that a heavy body falls faster than a light one, caused him some personal persecution, and he moved to the University of Padua as professor of mathematics. Meanwhile, his fame as a teacher spread all over Europe. In 1608, Hans Lippershey, a Dutch optician, invented a rudimentary telescope, as a result of a chance observation by an apprentice. Galileo learnt about this invention in 1609, and by 1610, he had already built a telescope. The first version had a magnifying factor of 3, but he improved it in time to a factor of 30. This enabled him to make many fundamental discoveries. He observed that the number of fixed stars was much greater than what could be seen by the naked eye, and he also found that the planets appeared as luminous disks.

In the case of Venus, Galileo discovered phases like those of the Moon. And he found that four satellites revolved around Jupiter. Galileo's observations with the telescope provided definite support for the Copernican system. He became famous also for his experiments with falling bodies and his investigations into the motion of a pendulum.

Galileo's work provoked a negative reaction, because it had brought Ptolemy's system into crisis. This left only two alternatives for explaining the phases of Venus: either Brahe's geocentric system or the Copernican system. The latter definitely went against the ecclesiastical dogma. The Church had created scholasticism, a mixture of religion and Aristotelian philosophy, which claimed to support the faith with elements of rational thinking.

But the Church also had an instrument of repression in the form of the Holy Inquisition, set up to punish any crime against the faith. When Galileo was 36, in 1600, the Dominican friar and outstanding scholar Giordano Bruno (1548–1600) was burned at the stake. He had committed the unforgivable crimes of declaring

Fig. 1.9 Nicolaus Copernicus. His model was presented in his book *De Revolutionibus Orbium Coelestium (On the Revolutions of Celestial Spheres)*, published thanks to the efforts of his collaborator Rheticus. This book was considered by the Church as heresy, and its publication was forbidden because it went against Ptolemy's system and its theological implications.

that he accepted the Copernican ideas of planetary motion, and holding opinions contrary to the Catholic faith.

When Galileo made his first astronomical discoveries, Bruno's fate was still fresh in his mind. Now he was becoming more and more convinced of the truth of the Copernican system, even though it was in conflict with official science, based on Ptolemy's system. The reaction of the Florentine astronomer Francesco Sizzi, when he learned about the discovery of Jupiter's satellites, was therefore no surprise: *The satellites are not visible to the naked eye, and for that reason they cannot influence the Earth. They are therefore useless, so they do not exist.*

On the one hand, Galileo's discoveries put him in a position of high prestige among many contemporaries, but on the other, he was attracting an increasing number of opponents. The support given by his discoveries to the Copernican theory and his attacks on Aristotelian philosophy aroused the anger of his enemies. In 1616, possibly under threat of imprisonment and torture, he was ordered by the Church "to relinquish altogether the said opinion that the Sun is the centre of the world and immovable [...] not henceforth to hold, teach or defend it in any way." Galileo acquiesced before the decrees and was allowed to return to Pisa. The Church

Fig. 1.10 Johannes Kepler was named "legislator of the firmament" for his laws of planetary motion, deduced as a result of long and patient work, using the extremely precise data gathered by Tycho Brahe.

Fig. 1.11 Tycho Brahe. Although his system of planetary motion was wrong, his very precise observations of the planetary positions enabled Kepler to formulate his laws.

was afraid to weaken its position by accepting facts opposed to the established Christian–Aristotelian–Ptolemaic doctrine.

In 1623, one of his friends, Cardinal Maffeo Barberini, became Pope Urban VIII, and Galileo received assurances of *pontifical good will*. Considering that the decree of 1616 would no longer be enforced, he wrote his book *Dialogues on the Ptolemaic and Copernican Systems*. But he faced an ever increasing number of enemies, and even the Pope became convinced that Galileo had tricked him. Galileo was called for trial under suspicion of heresy before the Inquisition at the age of 67. He was forced to retract under oath his beliefs about the Copernican system.

Later, a legend was concocted that Galileo, after abjuring, pronounced in low voice the words *And yet it moves*, referring to the Earth's motion around the Sun. That is, in spite of any court and any dogma, it was not possible to deny this physical fact, the objective reality of Earth's motion. However, it is interesting that Galileo never accepted the elliptical orbits discovered by Kepler; he believed only in circular orbits.

Among the most important achievements of Galileo, one must mention his laws of falling bodies, which can be resumed in two statements:

Fig. 1.12 Galileo Galilei. He discovered, among other things, four satellites of Jupiter and the phases of Venus, using a telescope of his own improved design. He enunciated the basic laws of falling bodies. His works stirred the antagonistic attitude of the ecclesiastical authorities, and he was forced to stand trial and to abjure his beliefs about the Copernican system.

1. All bodies fall in vacuum with the same acceleration. That is, if we let one sheet of paper, one ball of lead, and a piece of wood fall simultaneously in vacuum, they will fall with the same acceleration;
2. All bodies fall in vacuum with uniformly accelerated motion. This means that their acceleration is constant, that is, their velocity increases in proportion to the time elapsed from the moment the bodies started to fall.

The work initiated by Copernicus, Kepler, and Galileo was completed by Isaac Newton. He was born in 1642, the year in which Galileo died, and lived until 1727.

1.3 Newton and Modern Science

One day, Edmund Halley visited his friend Newton after a discussion with Robert Hooke and Christopher Wren, in which Hooke had claimed that he was able to explain planetary motions on the basis of an attractive force, inversely proportional to the square of the distance. When asked his opinion about it, Newton replied that he had already demonstrated that the trajectory of a body under such a central force was an ellipse.

Newton subsequently sent his calculations to Halley, and after looking through the manuscript, Halley convinced Newton to write in detail about the problem, since it could provide an explanation for the complicated motion of the whole planetary system. And this is how Newton started to write his *Philosophiae Naturalis Principia Mathematica*, a monograph which produced a revolution in modern science.

In the first book Newton stated his laws of motion, which owed much to Galileo, and laid their mechanical foundations. He deduced Kepler's laws by assuming a force inversely proportional to the square of the distance, and demonstrated that according to this law the mass of a homogeneous sphere can be considered as concentrated at its centre.

The second book is devoted to motion in a viscous medium, and it is the first known study of the motion of real fluids. In this book Newton dealt with wave motion and even with wave diffraction.

In the third book Newton studied the motion of the satellites around their planets, and of the planets around the Sun, due to the force of gravity. He estimated the density of the Earth as between 5 and 6 times that of water (the presently accepted value is 5.5), and with this value he calculated the masses of the Sun and the planets. He went on to give a quantitative explanation for the flattened shape of the Earth. Newton demonstrated that, for that shape of the Earth, the gravitational force exerted by the Sun would not behave as if all its mass were concentrated at its centre, but that its axis would describe a conical motion due to the action of the Sun: this phenomenon is known as the precession of the equinoxes.

Although Newton used the differential and integral calculus (which he invented himself, independently of Gottfried Leibniz) to get his results, he justified them in his book by using the methods of classical Greek geometry. One of the most practical consequences of his work was to supply a calculational procedure for determining the orbit of the Moon and the planets with much greater accuracy than ever before, using a minimum number of observations. Only three observations were enough to predict the future position of a planet over a long period of time. A confirmation of this was given by his friend Edmund Halley, who predicted the return of the comet which bears his name. Some other very important confirmations appeared in the nineteenth and twentieth centuries due to Le Verrier and Lowell, who predicted the existence of the then undiscovered planets Neptune and Pluto, deducing their existence from the perturbations they produced on other planetary motions.

The theory of gravitation conceived by Newton, together with all his other contributions to modern astronomy, marked the end of the Aristotelian world adopted by the scholastics and challenged by Copernicus. Instead of a Universe composed of perfect spheres moved by angels, Newton proposed a mechanism of planetary motion which was the consequence of a simple physical law, without need for the continuous application of direct holy action.

1.4 Newton's Laws

Newton established the following three laws as the basis of mechanics:

1. Every body continues in its state of rest or in uniform motion in a straight line unless it is compelled to change that state by forces acting on it;
2. The rate of change of momentum is proportional to the applied force, and it is in the direction in which the force acts;
3. To every action there is always opposed an equal reaction.

In the second law, momentum is defined as the product of the mass and the velocity of the body.

1.4.1 Newton's First Law

Newton's first law is known as the law or *principle of inertia*. It can only be verified approximately, since to do it exactly, a completely free body would be required (without external forces), and this would be impossible to achieve. But in any case it has a great value, since it establishes a limiting law, that is, a property which, although never exactly satisfied, is nevertheless satisfied more and more accurately, as the conditions of experimentation or observation approach the required ideal conditions.

As an example, an iron ball rolling along the street would move forward a little way, but would soon come to a stop. However, the same ball rolling on a polished surface like glass, would travel a greater distance, and in the first part of its trajectory, it would move uniformly along a straight line. Furthermore, the length of its trajectory would be longer if the friction between the ball and the surface (and between the ball and air) could be reduced. The only applied force is friction (acting in the opposite direction to the motion of the ball). The weight of the body acts perpendicular to the surface, and it is balanced by the reaction force of the surface.

1.4.2 Newton's Second Law

Newton's second law, known also as the *fundamental principle of dynamics*, states the proportionality between the acceleration a and the force F acting on a given body:

$$\mathbf{F} = m\mathbf{a}. \tag{1.1}$$

The constant of proportionality m is called mass. The mass can be interpreted as a measure of the inertia of the body. The larger the mass, the larger the force required to produce a given acceleration on a given body. The smaller the mass of a body, the larger the acceleration it would get when a given force is applied, and obviously, the

Fig. 1.13 Isaac Newton. His scientific work marks the beginning of physics as a modern science. His formulation of the laws of mechanics and universal gravitation laid the basis for explaining planetary motion and obtaining the Kepler laws. His work in optics, as well as in mathematics, was also remarkable, and he invented the differential and integral calculus independently of his contemporary Gottfried Leibniz.

more quickly it would reach high speeds. In modern physics this is observed with elementary particles: much less energy (and force) is required to accelerate electrons than to accelerate protons or heavy nuclei. On the other hand, photons (light quanta) move at the highest possible velocity (the speed of light, which is about 300,000 km/s), since they behave as massless particles (see Chap. 5).

But let us return to the second law. Its extraordinary value is due essentially to the fact that, if the interaction law is known for two bodies, from the mathematical expression for the mutual forces exerted it is possible to obtain their trajectories.

For instance, in the case of the Sun and a planet, as mentioned above, Newton established that a mutual force of attraction is exerted between them, a manifestation of universal gravitation. That force is directed along the line joining their centres, and it is proportional to the product of their masses and inversely proportional to the square of the distance between them. That is,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}_0, \tag{1.2}$$

where M and m are the masses of the Sun and planet, respectively, r is the distance between their centres, G is a constant whose value depends on the system of units used, and $\mathbf{r}_0$ is a unit vector along $\mathbf{r}$. $\mathbf{F}$ is a central force, that is, its direction always passes through a point which is the so-called centre of forces (in this case, it is a point inside the Sun).

Then, taking into account the fact that acceleration is a measure of the instantaneous rate of change of velocity with respect to time (the time derivative of velocity) and that in turn velocity is the rate of change of the position of the planet (time derivative of position), we have a mathematical problem that is easily solved (at least in principle) using differential calculus. Since acceleration is the second derivative with respect to time of the position vector of the planet with respect to the Sun, we can write:

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm\mathbf{r}_0}{r^2}. \tag{1.3}$$

This differential equation can be solved using the fact that the solar mass M is much greater than that of the planet m. The solution tells us that the orbit described by the planet is a conic section in which the Sun is placed at one of the foci. The type of orbit depends on the total energy of the body.

If the energy is negative, we have elliptical orbits (in the case of a minimum energy value, the ellipse degenerates into a circular orbit). If the energy is zero, the orbits are parabolic. Here we consider the total energy as the sum of the potential and kinetic energies, so that the zero corresponds to the case in which these are equal in absolute value; as we shall see later, in this case the potential energy is negative. Finally, for positive energies we have hyperbolic orbits.

The known planets describe elliptic orbits, but some comets coming from outer space describe parabolic or hyperbolic orbits. In that case, they get close to the Sun, move around it, and later disappear for ever. For most known comets, like Halley's, the orbit is elliptical but highly eccentric (i.e., very flattened).

As pointed out earlier, the application of Newtonian mechanics to the study of planetary motion gave astronomers an exceptionally important tool for the calculation of planetary orbits. But from the methodological point of view, Newtonian mechanics was of transcendental importance in modern science, since for the first time in physics a theory was established from which it was possible to predict consequences compatible with the results of observation. In that sense, Newton closed a circle which was initiated by Brahe, and which was continued by Kepler when he derived the laws of planetary motion from the data of Brahe's observations. Newton showed that such laws could be obtained by starting from very general physical principles: the equations of mechanics and the gravitational force between bodies.

For observers at rest or in uniform motion along a straight line, the laws of mechanics are the same. But the validity of Newton's laws depends on the acceleration of the observer: they do not hold equally for observers who are accelerated in different ways. For that reason it became necessary to introduce the concept of frame of reference, in particular, the concept of inertial frame, in which Newton's laws are valid. An inertial frame is something more than a system of reference; it includes the time, i.e., some clock. A simple geometrical change of coordinates does not change the frame of reference. We shall return to inertial frames in Sect. 1.7.

Vectors. We have already spoken about vectors indicating the position of the planets, and when discussing forces, velocities, and accelerations. Implicitly we have referred to the vectorial nature of these quantities. In order to characterize vectors, it is not sufficient to use simple numbers or scalars indicating their magnitude or absolute value. For vectors, besides the magnitude or modulus, we need to indicate their direction. Vectors are represented by arrows whose length and direction represent the magnitude and direction of the vector, respectively.

For instance, when referring to the velocity of a body, it is not enough to say how many meters per second it moves. We must also specify in which direction it is moving. A body that falls has a velocity which increases proportionally to the time elapsed, and its direction is vertical, from up to down. We represent that velocity as a vertical vector of increasing magnitude, with its end pointing downward.

Sometimes vectors are used to indicate the position of a point that moves with respect to another one taken as fixed. This is the case of the radius vector, to which we referred when describing Kepler's laws. The origin of the radius vector is at the Sun and the end is at the planet that moves.

Two parallel vectors, **A** and **B**, are simply summed, and the sum has the same direction as the added vectors. If they are parallel and of opposite directions, their sum is a vector of modulus equal to the difference of the moduli of the given vectors and its direction is that of the vector of larger modulus.

If two vectors **A** and **B** are not parallel, but have different orientations, their sum is geometrically a third vector obtained by displacing **B** parallel to itself so that its origin coincides with the end of **A**, and then, by joining the origin of **A** with the end of **B** we get the sum $\mathbf{A} + \mathbf{B}$ of the two vectors.

Given a system of orthogonal coordinates $Oxyz$, the vector **A** can be written in terms of its three components along the coordinate axes, $\mathbf{A} = (A_x, A_y, A_z)$, obtained from the projection of the vector on them. The modulus of **A** is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2},$$

where $A_x = A\cos\alpha$, $A_y = A\cos\beta$, $A_z = A\cos\gamma$, with α, β, γ being the angles between $\boldsymbol{A}$ and the axes Ox, Oy, and Oz, respectively. Thus, a vector in three dimensions is defined by an ordered set of three numbers, which are its components. Let us define the unit vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

One can write

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

In the same way,

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

and their sum is obviously

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}.$$

An important vector is the *position vector*,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

of any arbitrarily chosen point with respect to the system of coordinates $Oxyz$.

Mechanical quantities such as displacements, velocities, accelerations, forces, etc., are to be summed in accordance with this procedure of vectorial or geometrical sum.

If two forces have opposite directions but equal moduli, their vector sum is a null vector, that is, a vector of modulus zero. However, that does not necessarily mean that the physical effect is canceled: if the forces are applied at different points, both of them will have a mechanical effect. Opposite forces are responsible for static equilibrium – for instance, for a body having the weight $\mathbf{G}$ lying on a table. The weight $\mathbf{G}$ is applied to the table and the reaction of the table $\mathbf{R} = -\mathbf{G}$ is applied to the body. Opposite forces of equal modulus also appear in dynamics, as in the case of the Sun and a planet: their mutual action is expressed by opposite forces, but the forces are applied at different points, on the Sun and on the planet: the vector sum of the forces is zero, nevertheless they produce the motion of the bodies.

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, their *scalar product* is a number obtained by multiplying together the modulus of each vector by the cosine of the angle formed by their directions. Usually, the scalar product is represented by means of a dot between the two vectors:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\alpha. \tag{1.4}$$

The scalar product of two vectors can also be expressed as the product of the modulus of one of the vectors by the projection of the other on it. The scalar product is commutative, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. Moreover, $\mathbf{A} \cdot \mathbf{A} = A^2$, that is, the modulus squared of a vector is given by the scalar product of the vector with itself. If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A} \cdot \mathbf{B} = 0$. If c is a number, it is obvious that $(c\mathbf{A}) \cdot \mathbf{B} = c(\mathbf{A} \cdot \mathbf{B})$.

The unit vectors satisfy the properties

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

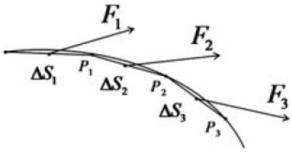

Fig. 1.14 The scalar product is used, for example, for calculating the work performed by a force.

Then one can write the scalar product in the form

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \tag{1.5}$$

The scalar product is particularly useful in expressing the work performed by a force on a particle that describes an arbitrary trajectory between two points, P_0 and P. At each point of the curve the force forms an angle with the tangent to the curve at the point. The total work performed by the force can be calculated in the following way: divide the curve into segments at the points 1, 2, 3, etc., and draw the corresponding chords $\Delta\mathbf{S}_1$, $\Delta\mathbf{S}_2$, $\Delta\mathbf{S}_3$ as vectors that join the points P_0, P_1, P_2, P_3, etc. Then take the value of the force at an arbitrary point inside each of these segments. Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$, etc., be the values of the force at such points (Fig. 1.14). Then take the sum of the scalar products:

$$\mathbf{F}_1 \cdot \Delta\mathbf{S}_1 + \mathbf{F}_2 \cdot \Delta\mathbf{S}_2 + \ldots + \mathbf{F}_n \cdot \Delta\mathbf{S}_n. \tag{1.6}$$

When the number of the points of the division tends to infinity, such that the modulus of the largest of the vectors $\Delta\mathbf{S}_i$ tends to zero, the work done by the force is obtained as

$$W = \lim_{\Delta\mathbf{S}_i \to 0} \sum_{i=1}^{\infty} \mathbf{F}_i \cdot \Delta\mathbf{S}_i. \tag{1.7}$$

This is represented by the symbol

$$W = \int_{P_0 P} \mathbf{F} \cdot d\mathbf{S}\,, \tag{1.8}$$

which is called the line integral between P_0 and P.

The *vector product* (or *cross product*) of two vectors is a new vector, obtained by performing a mathematical operation on them. To illustrate it, let **A** and **B** be two vectors in a plane (Fig. 1.15). Decompose **B** into two other vectors, $\mathbf{B}_1$ and $\mathbf{B}_2$ (whose sum is **B**). The vector $\mathbf{B}_1$ is in the direction of **A**, while the vector $\mathbf{B}_2$ is perpendicular to **A**. We now define a third vector that we call the vector product of **A** by **B**, denoted by $\mathbf{A} \times \mathbf{B}$, whose characteristics are:

1. Its modulus is the product of the moduli of **A** and $\mathbf{B}_2$. In other words, it is equal to the product of the moduli of **A** and **B** with the sine of the angle between them, $AB \sin\alpha$;
2. Its direction is perpendicular to the plane spanned by **A** and **B** and is determined as follows. If the direction of rotation to superpose **A** on **B** is indicated by the index, middle, ring, and little fingers of the right hand (as shown in Fig. 1.15),

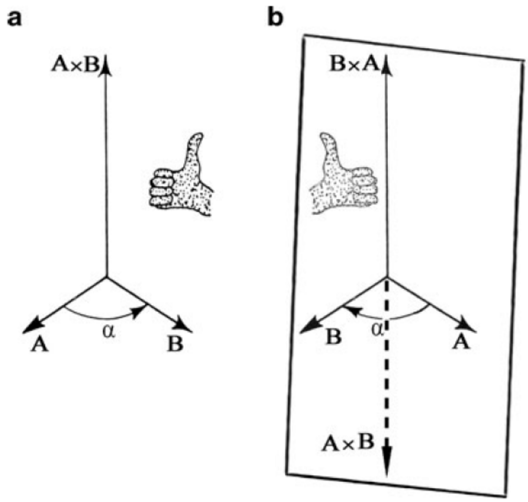

Fig. 1.15 (**a**) The vector product of two vectors **A** and **B** is a third vector, perpendicular to **A** and **B**, whose modulus is the product of the moduli of **A** and **B** with the sine of the angle between them, or equivalently, the product of the modulus of one of them with the projection of the other on the direction perpendicular to the first. The direction of the vector product is given by the right-hand rule as shown in the figure. (**b**) The mirror image does not satisfy the definition for the vector product of two vectors, but obeys a left-hand rule, since the image of the right hand is the left hand.

then the thumb indicates the direction of $\mathbf{A} \times \mathbf{B}$ (assuming that the angle α between the vectors is smaller than 180°).

Strictly speaking, the vector product of two vectors is not a true vector, but a *pseudovector*, since the mirror image does not satisfy the previous definition, but the *left-hand* rule, which is obviously not equivalent to it: the mirror image of the right hand is the *left* hand.

Consequently, the product $\mathbf{B} \times \mathbf{A}$ gives a vector of the same modulus but opposite direction to $\mathbf{A} \times \mathbf{B}$. This is an interesting result: the vector product is not commutative, but rather one can write $\mathbf{B} \times \mathbf{A} + \mathbf{A} \times \mathbf{B} = 0$, meaning that the vector product is *anticommutative*. In particular, $\mathbf{A} \times \mathbf{A} = 0 = \mathbf{B} \times \mathbf{B}$. This property can be generalized to higher dimensional spaces, and leads to the definition of exterior algebras or Grassmann algebras (see Chap. 7).

For the unit vectors, we have the properties:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

and

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}\,.$$

Since the product is anticommutative, if we exchange the pair on the left, the sign is changed on the right. In terms of components, we get

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}. \qquad (1.9)$$

It is easily seen that the vector product vanishes if the vectors are parallel.

Transformations of vectors. Vector components transform like coordinates. For instance, under a rotation of the system of coordinates, the components A_x, A_y, A_z transform like the coordinates x, y, z. Under a positive (counterclockwise) rotation of angle θ around the z-axis, the position vector of a point P, expressed as $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in the original system, is transformed in the rotated system to $\mathbf{r}' = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}$, where the new coordinates x', y', z' are given by the product of the rotation matrix $\mathbf{R}$ with the initial vector $\mathbf{r}$. The unit vectors in the rotated system are $\mathbf{i}', \mathbf{j}'$, whereas $\mathbf{k}$ does not change. The rotation matrix is an array of 3×3 numbers in three rows and three columns. The components of a matrix are labeled by two indices (i, j), where the first identifies the row and the second indicates the column. The rotated vector $\mathbf{r}'$ is the product of the rotation matrix $\mathbf{R}$ with the original vector $\mathbf{r}$. For the particular rotation of angle θ around the z-axis, we write this product as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad (1.10)$$

Under this rotation, the components of a vector $\mathbf{A}$ transform as

$$\begin{aligned} A'_x &= A_x \cos\theta + A_y \sin\theta, \\ A'_y &= -A_x \sin\theta + A_y \cos\theta, \\ A'_z &= A_z. \end{aligned}$$

Under an inversion of the coordinate axis, $(x, y, z) \rightarrow (x, y, -z)$, the vector $\mathbf{A}$ transforms as $(A_x, A_y, A_z) \rightarrow (A_x, A_y, -A_z)$. A pseudovector $\mathbf{P}$ transforms under rotations like the coordinates, but under an inversion, it remains the same, $(P_x, P_y, P_z) \rightarrow (P_x, P_y, P_z)$.

There is an alternative way of writing the previous 'vector' rotation. If we now denote the indices of components along x, y, z by $i = 1, 2, 3$, respectively, we may write the vector components as A_i. Further, we shall write the matrix $\mathbf{R}$ in terms of its elements as R_{ij} (row i and column j). Then, for instance,

$$A'_3 = \sum_j R_{3j} A_j = R_{31} A_1 + R_{32} A_2 + R_{33} A_3.$$

In what follows, we adopt Einstein's summation convention: if a term contains the same index twice, the summation over all values of that index is to be understood. Thus, $A'_3 = R_{3j} A_j$ means the sum over j, as j ranges over $1, 2, 3$. (From now on, we shall use the indices x, y, z as an alternative to $1, 2, 3$, understanding the correspondence $x \rightarrow 1, y \rightarrow 2, z \rightarrow 3$.)

Tensors. The *dyadic product* **AB** of two vectors **A** and **B** is a quantity with the property that

$$(\mathbf{AB})\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}). \tag{1.11}$$

The result is a vector in the direction **A**, since $\mathbf{B}\cdot\mathbf{C}$ is a scalar. As

$$(\mathbf{AB})\cdot(c\mathbf{C}) = c[(\mathbf{AB})\cdot\mathbf{C})]$$

and

$$(\mathbf{AB})\cdot(\mathbf{C}+\mathbf{D}) = (\mathbf{AB})\cdot\mathbf{C} + (\mathbf{AB})\cdot\mathbf{D},$$

the quantity **AB** is called a linear operator, or *tensor*, and (1.11) is a linear function of **C**. A tensor is a quantity whose components transform as a product of the coordinates. For instance, the component T_{xy} of a tensor **T** transforms as the product xy. The unit tensor is the dyadic $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$. It is easy to check that $\mathbf{I}\cdot\mathbf{A} = \mathbf{A}$. In three-dimensional space, a second rank tensor can be written in the form

$$\begin{aligned}\mathbf{T} = {} & T_{xx}\mathbf{ii} + T_{xy}\mathbf{ij} + T_{xz}\mathbf{ik} \\ & + T_{yx}\mathbf{ji} + T_{yy}\mathbf{jj} + T_{yz}\mathbf{jk} \\ & + T_{zx}\mathbf{ki} + T_{zy}\mathbf{kj} + T_{zz}\mathbf{kk}.\end{aligned} \tag{1.12}$$

However, it is simpler to write it in the matrix form

$$\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}. \tag{1.13}$$

By using the numerical indices $i, j = 1, 2, 3$, we may write the general component of **T** as T_{ij}. A tensor is symmetric if $T_{ij} = T_{ji}$, and antisymmetric if $T_{ij} = -T_{ji}$. An arbitrary tensor can be written as the sum of a symmetric and an antisymmetric tensor.

In a similar way we can define tensors of third rank as T_{ijk}, etc. For us, the most interesting third rank tensor is the completely antisymmetric unit tensor ϵ_{ijk}, called the Levi-Civita tensor. (Actually, it is a pseudotensor, because it behaves as a tensor except under the inversion of coordinates.) Its components are as follows: zero, if at least two indices are equal; $+1$, if the permutation of the (unequal) indices ijk is even (i.e. $123, 312, 231$), and -1, if the permutation of the indices is odd (i.e., $213, 321, 132$).

Let us consider two vectors represented by their components A_j and B_k. If we write the product of ϵ_{ijk} with these vectors, and sum over j and k, we get

$$C_i = \epsilon_{ijk} A_j B_k, \tag{1.14}$$

i.e.,

$$\begin{aligned} C_1 &= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2, \\ C_2 &= \epsilon_{231} A_3 B_1 + \epsilon_{213} A_1 B_3, \\ C_3 &= \epsilon_{312} A_1 B_2 + \epsilon_{213} A_2 B_1. \end{aligned}$$

Hence, $C_i = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1)$; in other words, C_i with $i = 1, 2, 3$ are the components of the vector product $\mathbf{A} \times \mathbf{B}$.

Thus, the vector product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ can be written in components as $C_i = \frac{1}{2}\epsilon_{ijk}T_{jk}$, where T_{jk} are the components of the antisymmetric tensor $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} 0 & C_2 & -C_3 \\ -C_2 & 0 & C_1 \\ C_3 & -C_1 & 0 \end{pmatrix}. \tag{1.15}$$

The pseudovector $\mathbf{C}$ is called the dual pseudovector of the tensor $\mathbf{T}$.

Very important physical quantities are usually expressed as vector products. Examples are the angular momentum $\mathbf{L}$, or the magnetic field $\mathbf{B}$. The vector product will also be used in the expression (1.27) for the velocity written in a rotating system of coordinates, and it is useful to remember in these cases that it is a pseudovector, i.e., the dual of an antisymmetric tensor.

For a satellite of mass m that moves around the Earth, we can assume its velocity at each instant to be the sum of two vectors: a component along the radius vector and another perpendicular to it, contained in the plane of the orbit. The angular momentum of the satellite around the Earth (Fig. 1.16) is given by the cross product of the radius vector $\mathbf{r}$ of the satellite with respect to the Earth with the momentum $\mathbf{p} = m\mathbf{v}$ of the satellite:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1.16}$$

1.4.3 Planetary Motion in Newton's Theory

It is instructive to analyze the motion of a planet around the Sun (or of the Moon around the Earth) as a consequence of Newton's second law.

Assume that at a given instant the momentum of the planet is $\mathbf{p} = m\mathbf{v}$ around the Sun. If the gravitational attraction could be switched off at that precise moment, the planet would continue to move uniformly in a straight line, that is, with a constant momentum $\mathbf{p}$. In the time interval Δt elapsed between two adjacent positions 1 and 2, the planet suffers a change in its momentum due to the action of the Sun's attractive force $\mathbf{F}$.

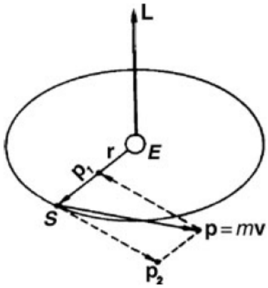

Fig. 1.16 Angular momentum of a satellite S that moves around the Earth E. The angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

intermediate agent for the interaction, propagating with finite speed. This will be discussed in more detail in later chapters.

Newton's third law should be interpreted with care in the atomic world, because of the finite velocity of propagation of the interactions. For instance, two charged particles in motion exert mutual forces of attraction or repulsion, but at a given instant the force exerted on one of the particles is determined by the position of the other at some previous instant, and the effect of its new position will be felt some time later.

Newton's third law can be considered as a manifestation of a much more general law. In fact, the expression action $\leftrightarrow$ reaction does not necessarily establish their equality, but rather the relation stimulus–response, in which the second is opposed to the first one. This can be found in all fields of physics. For instance, in electromagnetic theory, it is found that, when a magnetic field varies near a conductor, an electric current is induced on the latter (Faraday's law). But this electric current in turn creates a magnetic field which acts oppositely to the applied field (Lenz law). Furthermore, an electric charge in a medium creates an electric field, attracting charges of opposite sign, and the net effect is a screening of the charge and the field created by it.

In thermodynamics, that general law is expressed by Le Chatelier's principle: if some external actions are applied to a system in equilibrium, and if these tend to alter it, some reactions originate in the system which tend to compensate the external actions and take the system to a new state of equilibrium. For instance, if we heat a jar with a match at some point, the jar alters its state of equilibrium. However, the heat spreads across its mass cooling the hot point, and after some time, the jar reaches a new state of equilibrium at a uniform temperature higher than before because of the absorbed heat.

1.5 Conservation Laws

Starting from Newton's laws, and on the basis of a simple hypothesis about the interaction forces between the particles, it is possible to establish three conservation laws:

1. Conservation of linear momentum;
2. Conservation of angular momentum;
3. Conservation of energy.

The conservation of these quantities is usually accepted as valid in all fields of physics, and they can be derived as a consequence of the basic symmetry properties of space and time. Thus, the conservation of linear momentum is a consequence of the homogeneity of space, the conservation of the angular momentum is due to the isotropy of space (meaning that all the directions of space are equivalent for a given physical system, i.e., its properties do not vary when it is rotated as a whole), and the conservation of energy is a consequence of the homogeneity of time (in other words, the evolution of a system with respect to time, starting from an initial instant t_0, is the same for any value of t_0). This correspondence between symmetry properties and conservation laws is extremely general and crops up again in other theories, particularly in microscopic physics. The ultimate understanding of these relations was given by the German Jewish mathematician Emmy Noether (1882–1935), in the theorem which bears her name and which turned out to be one of the most influential works for the development of theoretical physics in the twentieth century.

1.5.1 Conservation of Linear Momentum

It is easy to demonstrate that for a system of particles under no external influences, the total linear momentum (the sum of the linear momenta of all the particles) is conserved when Newton's third law is satisfied. Put another way, the total linear momentum is conserved if the action and the reaction are equal in modulus but act in opposite directions.

It may happen that one or both interacting particles emit some radiation. In that case one must attribute some momentum to the radiation field in order that the linear momentum be conserved. When the radiation is assumed to be composed of quantum particles (for example, photons), the law of conservation of linear momentum is restored by including newly created particles carrying a certain amount of linear momentum.

We shall refer to an example from macroscopic physics. If we shoot a gun, the bullet, having a small mass, leaves the gun at a speed of several meters per second. The gun moves back in the opposite direction at a lower speed (at the moment of shooting we can neglect the action of the force exerted by the Earth's gravitational field). If m is the mass and $\mathbf{v}$ the speed of the bullet, and if M and $\mathbf{V}$ are the mass and speed of the gun, we find that $M\mathbf{V} = -m\mathbf{v}$. So the momentum acquired by the bullet is the same (but of opposite sign) as that acquired by the gun. The sum of two quantities equal in modulus but with opposite directions is zero, which was the initial value of the total linear momentum.

But what happens if we fix the gun to a solid wall? In this case the gun does not move back, the wall stops it. But now the conditions have changed. An external force is exerted on the gun, since it has been fixed to the wall, and this in turn is fixed to the Earth.

This means that the Earth should move back with a speed which, when multiplied by its mass, yields a momentum equal in modulus but opposite to that carried by the bullet. Let us suppose that the bullet has a mass of 100 g, and that its speed is 100 m/s $= 10^4$ cm/s. The mass M of the Earth is 5.98×10^{27} g. From the equation $M\mathbf{V} = -m\mathbf{v}$, we find that, after the gun is fired, the Earth should recoil with a speed of the order of 10^{-20} cm/s. For all practical purposes, this is zero.

Something similar happens if we throw a rubber ball against a wall. The ball bounces and comes back with a velocity of approximately the same modulus, but in the opposite direction. Apparently, the linear momentum is not conserved, but the ball has subtracted a certain amount of momentum from the wall, or from the Earth, which recoils with insignificant velocity.

We should emphasize that, in the previous example, the velocity of the ball bouncing off the wall has opposite direction to what it had before the collision, but its modulus is actually somewhat smaller. The wall did not *give back* all the incident momentum, but absorbed a part of it. An extreme case occurs if we throw a ball of clay against the wall. In this case the ball does not rebound. All the linear momentum of the ball is transmitted to the wall, and as it is fixed to the Earth, its resulting change of motion is not perceptible. But if the wall were supported by wheels that could move without friction, it would start to move with the colliding ball of clay stuck to it. Its speed would be easily obtained: if M is its mass, and m and $\mathbf{v}$ are the mass and velocity of the ball of clay, we conclude that the modulus of the velocity of the wall V would be

$$V = \mathbf{v}\frac{m}{m+M}. \tag{1.17}$$

Conservation of Linear Momentum and the Mössbauer Effect. The previous example of the gun fixed to the Earth (that does not recoil) has an interesting analogy in nuclear physics, in the so-called Mössbauer effect. In this case, the gun is an atomic nucleus, and the bullet is the gamma radiation emitted by it. The gamma radiation emitted by a nucleus has a constant frequency, but when the nucleus is able to move, as happens in a gas, we have a case similar to the first example of the recoiling gun. The nucleus recoils when emitting the gamma radiation. This causes a range of frequencies to be observed, within a certain bandwidth $\Delta\omega$, that is, there are many values of the frequency in such an interval and a continuous set of frequencies is observed due to the different values of the energy lost by the recoil of the nucleus. The frequency no longer has a precise value, but lies in an interval of possible values, which we may call the imprecision or error.

However, in certain crystals (for example, iridium 197 and iron 57) phenomena occur as in the example of the gun fixed to the Earth, since the emitting nucleus is effectively fixed to the crystal (which does not recoil significantly). Then the frequency of the emitted radiation has an extraordinarily narrow band width $\Delta\omega$. In the case of iron 57, the band width divided by the frequency ω is of the order of $\Delta\omega/\omega \sim 3\times10^{-13}$. This is equivalent in units of time to an error of one second in an interval of 30,000 years.

As can be seen from this, the Mössbauer effect can be used to make very precise measurements of frequency.

1.5.2 Conservation of Angular Momentum

For the case of motion under the action of a central force (directed along the radius vector joining the planet with the Sun), angular momentum is conserved: it does not vary with time. Referring again to Fig. 1.16 of a satellite around the Earth, the change $\Delta\mathbf{p}$ in the linear momentum that the satellite acquires by the action of the terrestrial gravity force is always directed along **r**. For that reason it does not contribute to the angular momentum, which is due only to the component $\mathbf{p}_2$ perpendicular to **r**.

If **r** decreases, $\mathbf{p}_2$ increases so that the product $L = rp_2$ remains constant. This is equivalent to the statement of Kepler's second law, which is nothing but an expression of the conservation of angular momentum. The planets move faster when they approach the Sun (the radius vector diminishes) than when they are more distant.

In addition to the angular momentum due to the orbital motion around the Sun, the planets have an angular momentum due to the rotation around their axis. This creates a magnetic field due to the rotation of electric charges inside them.

Something similar takes place in the atomic world. An electron in an atom has some intrinsic angular momentum or spin that is retained even if it moves outside the atom, although it would not be correct to imagine the electron as a sphere that rotates around its axis.

The spin angular momentum is measured in terms of a unit $\hbar$ which is the Planck constant h divided by 2π, and whose value is 1.05×10^{-27} erg · s. Electrons, protons, neutrons, neutrinos, and other particles have spin equal to 1/2 of this unit. Photons have spin equal to 1 and π mesons have spin 0.

Particles with spin 1/2 (or any half-integer) are called *fermions*, in honour of the Italian physicist Enrico Fermi (1901–1954), and they obey Pauli's exclusion principle, formulated in 1925 by Wolfgang Pauli (1900–1958). On the other hand, if a particle has integer spin, it is called a *boson*, in honour of the Indian physicist Satyendra Nath Bose (1894–1974). Bosons do not obey the Pauli principle.

The angular momentum of an isolated system of particles is also conserved if the particles exert equal and opposite forces on one another.

As in the case of the linear momentum, it may happen that a particle loses a certain amount of angular momentum, which is carried by a newly created particle. This is the case of an electron in an atom: upon jumping from some level of energy to another one, it loses a certain amount of angular momentum, but the emitted photon carries precisely the missing angular momentum.

When neutron decay was investigated, it was observed that the resulting particles were a proton and an electron. Since the neutron had a spin angular momentum equal to 1/2, the same as the proton and the electron, it was a mystery why the total spin of the resulting particles was not 1/2. Furthermore, the energy was not conserved either. Then, in 1931, Pauli proposed the existence of a neutral particle that carries the missing spin and energy. This particle was called the neutrino and was assumed to have spin 1/2. Although it took more than 20 years, the existence of the neutrino was finally demonstrated in the laboratory. It took so long because neutrinos are particles whose interaction with matter is very weak. Neutrinos and weak interactions will be discussed in Chaps. 9, 10, and 11.

1.5.3 Conservation of Energy

For a body of mass m that moves with speed $\mathbf{v}$, its kinetic energy is $\frac{1}{2}mv^2$. Unlike the linear momentum and the angular momentum, which are vectorial, the kinetic energy does not depend on the direction of motion.

Another form of energy is *potential* energy. If the same body of mass m is placed at a certain height h above the Earth, we say that it has a potential energy mgh with respect to the surface of the Earth (g is the acceleration due to the Earth's gravity, with the value 9.8 m/s^2), that is to say, this potential energy is equal to the product of the weight by the height. If the object falls freely, the potential energy diminishes due to the decreasing height with respect to the floor. But on the other hand, the body acquires an increasing speed: the decrease in potential energy produces an increase in kinetic energy, in such way that their sum is constant:

$$\frac{1}{2}mv^2 + mgz = \text{const.} = mgh,$$

where z is the height at any instant between the moment when the object was released and the moment when it touched the Earth.

For a planet of mass m that moves around the Sun, for example, its kinetic energy, denoted by T, is:

$$T = \frac{1}{2}m(v_r^2 + v_l^2) \ , \tag{1.18}$$

where v_r is its radial speed, directed along the radius that joins the planet with the Sun, and v_l is the velocity perpendicular to the radius vector. The potential energy (which is equal to the energy required to bring the planet from infinity to the point where it is located) is denoted by V and is equal to

$$V = -\frac{GMm}{r}. \tag{1.19}$$

Here, G is the universal constant of gravitation, M is the solar mass, r is the distance from the planet to the Sun (or from the centre of the planet to the centre of mass of the Sun–planet system, which is a point located in the Sun). The negative sign of the potential energy is due to the fact that the force between the planet and the Sun is attractive and the energy required to bring it from infinity to its present position is negative: it is not necessary to waste energy, because the Sun gives up this energy through its attractive force. The total energy would thus be

$$E = T + V = \frac{1}{2}m(v_r^2 + v_l^2) - \frac{GMm}{r}. \tag{1.20}$$

But as the angular momentum $L = mv_l r$ is constant, one can write

$$v_l = L/mr, \tag{1.21}$$

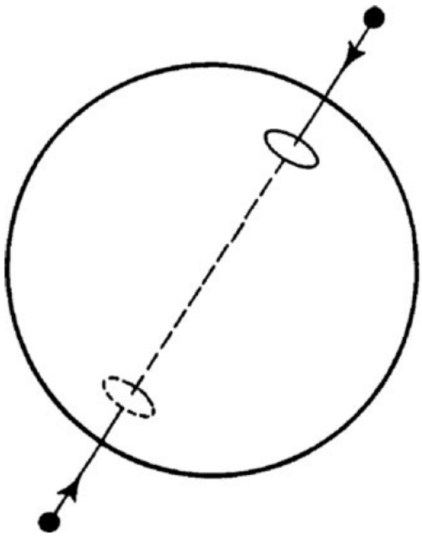

Fig. 1.22 If it were possible to make a hole right through the Earth, passing through its centre, a particle thrown down the hole would oscillate permanently between the two ends of the diameter.

Returning to our example, if the angular momentum is zero, $U = V$. This is the case for a body thrown vertically upward. If r_0 is the Earth radius, the body could reach a height r_1, and then fall back to the Earth's surface.

Let us imagine what would happen if a hypothetical hole were dug through the Earth, along one of its diameters. Then the body could pass through the centre of the Earth, where it would arrive with the maximum kinetic energy. After crossing the centre, it would exit through the opposite end, reaching a position entirely symmetrical in the land of the antipodes. In principle, it would then come back to its starting point and thereafter oscillate indefinitely (Fig. 1.22). Its orbit would be a linear oscillator. At a point inside the Earth at a distance r from its centre, the force of gravity exerted on a body of mass m falling down the hole is the force produced by a sphere of radius r (due to Gauss's law). If ρ is the average Earth density, this mass is $M' = 4\pi r^3 \rho/3$, leading to a force $F = 4\pi G m \rho r/3$.

Of course, not only is it technically impossible to make such a hole through the Earth, but other factors must be considered. For instance, among other things, friction could not be avoided, and this would damp the oscillations.

If the body comes from infinity with an energy greater than or equal to zero (and $L = 0$), it will move toward the centre of forces (for example, the Earth) until it is stopped by the Earth's surface, at a distance r from the centre of forces.

What would happen in the case of a repulsive force? If the force is repulsive, the potential energy is positive, resulting in an effective potential that looks like the one depicted in Fig. 1.23. The total energy is always positive and the resulting orbits are hyperbolas.

A problem of this type occurs in the case of relative motion of electric charges of equal sign. This is interesting in connection with the famous experiment performed by Ernest Rutherford (1871–1937), in which a sheet of gold was bombarded

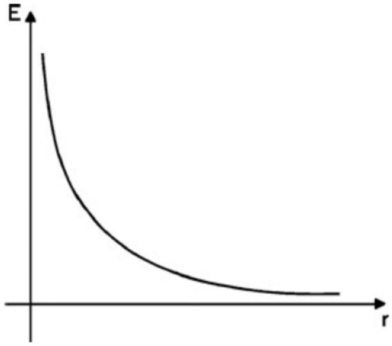

Fig. 1.23 For a positive potential, the total energy can be only positive. A positive potential occurs when particles repel each other, as in the case of an atomic nucleus interacting with alpha particles.

with alpha particles (positively charged helium nuclei). By studying their deviations (assuming that the particles describe hyperbolic orbits), Rutherford proposed a *planetary model* of the atom in which the nucleus was positively charged. We shall return to this point in Chap. 6.

If the sum of the kinetic and potential energies is constant, we say that the energy is conserved or that the system is conservative. This is the case for planets in their motion around the Sun, or for a falling object, until some instant when it hits the Earth. At the moment of impact, all the kinetic energy of the body is dissipated in the form of vibrations (for example, sound), elastic deformations, friction, and heat produced by friction.

1.6 Degrees of Freedom

A particle moving freely has three degrees of freedom – it can move independently in the three directions of space.

A pendulum oscillating in a plane has only one degree of freedom, which is the angle formed between the suspending cord and the vertical (Fig. 1.24).

Two free particles have six degrees of freedom, three for each. But if the particles are fixed to the ends of a bar, they lose one degree of freedom, and retain five: the three directions of space in which the bar can move, and the two angles which indicate its inclination say, around its midpoint (Fig. 1.25).

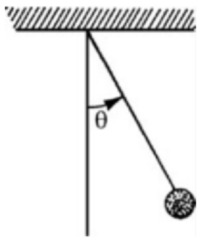

Fig. 1.24 A pendulum that oscillates in a plane has only one degree of freedom: the angle θ.

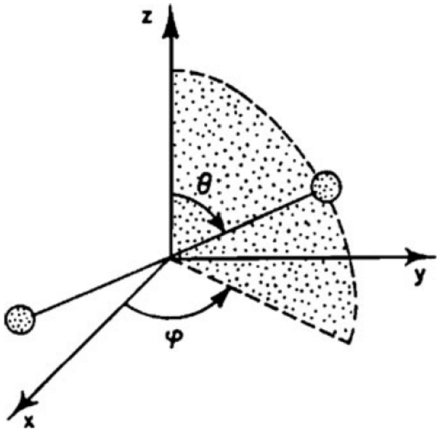

Fig. 1.25 Two particles joined by a rigid bar have five degrees of freedom: the three directions of space in which their centre of mass can move, and the two angles determining the positions of the particles with respect to it. This is almost the case for diatomic molecules, although for them the "bar" is not completely rigid, and can oscillate longitudinally. Therefore, diatomic molecules have six degrees of freedom.

We shall discuss the mechanism of energy dissipation by using the example of the pendulum. We consider the pendulum as a mass hanging by a thread tied to a nail. When the pendulum oscillates, an enormous number of molecules of air (each of them having three degrees of freedom) collide with it. When the thread moves relative to the nail at the point of contact, it collides with a very large number of constituent particles of the nail (atoms and ions forming the lattice of the metal, and electrons). Energy dissipation in the pendulum (and in other physical systems) is related to the energy transfer from a system with very few degrees of freedom to other systems with a very large number of degrees of freedom, and the energy is in this case *disordered*.

The energy absorbed by a system with a very large number of degrees of freedom increases its *internal energy*. We shall consider this problem in more detail in the next chapter.

1.7 Inertial and Non-inertial Systems

As mentioned earlier, in order to describe the position and motion of a body, classical mechanics needs the concept of frames of reference. Such a frame could be a system of three perpendicular axes and a clock to measure the time. The origin O could be fixed to some body (Fig. 1.26).

For instance, in order to describe the Earth's motion around the Sun, the origin of the system of coordinates could be at the centre of the Sun. A frame of reference is said to be *inertial* if a free particle (on which no force acts) is at rest or moves with constant velocity along a straight line with respect to the frame (assuming that the

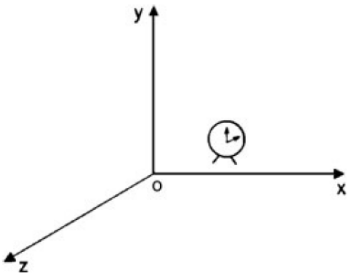

Fig. 1.26 A frame of reference is characterized by three perpendicular axes along which the three spatial coordinates x, y, z are measured, and a clock with which the time t is measured.

other two laws of Newton are also valid). This definition is not free from difficulties, but it is very useful. Given an inertial frame of reference S, all the frames S', S'', etc. moving with respect to S with uniform motion along a straight line are also inertial.

When we travel in a car and it accelerates abruptly, we feel a force pushing us back. If we brake, a force pushes us forward. If we follow a curved road in the car, a force pushes us outward (centrifugal force). All these are so-called *inertial forces* which appear in non-inertial frames of reference.

If the car accelerates, a pendulum will move away from the vertical line to some angle α. The same deviation α would occur in other cars if pendulums were placed in them when these cars move with respect to the first with the same acceleration, although different speeds. We thus conclude that the laws of mechanics (indicated by the verticality of the thread of the pendulum at rest with respect to the car) are not satisfied in a non-inertial frame, because fictitious forces, called *inertial forces*, appear. Furthermore, the laws of mechanics would not be valid in any frame of reference moving with constant velocity with respect to the first, non-inertial frame.

For such frames, some other set of mechanical laws is valid, modified by the inertial forces. Then the question arises: does a frame of reference exist in which the laws of mechanics are actually satisfied, if in fact one could have an enormous variety of frames of references?

In classical mechanics, we assume the existence of an *absolute frame* of reference in which Newton's laws are satisfied. They would also be satisfied for all the systems in uniform motion with regard to the absolute frame. Furthermore, an absolute time is assumed: in all the inertial frames the time is measured with the same universal clock.

Moreover, classical mechanics assumes that the interaction between particles takes place instantaneously. In such conditions, Galileo's relativity principle is satisfied. This states that the laws of mechanics are the same in all inertial systems. The principle implies that Galileo's transformations (1.24) are used when we want to describe the position of a particle with respect to two different inertial systems (Fig. 1.27).

If (x, y, z) are the numbers giving the position of a particle at some time t in the system S, and (x', y', z') the position at the same time $t = t'$ in the system S' moving with respect to S with the speed V along the x-axis, as shown in the figure,

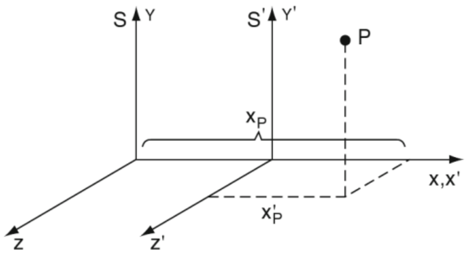

Fig. 1.27 Position of a particle with respect to two systems of coordinates S and S'.

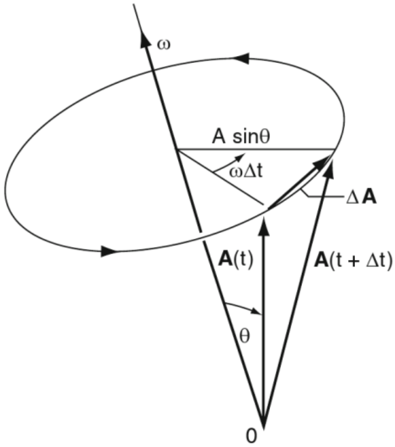

Fig. 1.28 Variation of a vector rotating around an axis with angular velocity $\boldsymbol{\omega}$ in the time interval Δt.

then the following equations hold between the two sets of numbers:

$$x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t. \tag{1.24}$$

This is a Galilean transformation. As we shall see in Chap. 5, these transformations are not satisfied by electromagnetic phenomena (in other words, Maxwell's equations are not covariant with respect to these transformations), nor indeed by any kind of physical phenomenon, although they are approximately valid for bodies moving at small velocities.

The need for a new principle of relativity in physics led to Einstein's principle of relativity, as we shall see in Chap. 5.

It is important to be able to compare two reference systems, the first S at rest and the second, non-inertial, S' rotating around an arbitrary axis at angular velocity ω radians per second (Fig. 1.28). The angular velocity is represented by a vector $\boldsymbol{\omega}$ directed along the axis of rotation, as specified by the right-hand rule (Fig. 1.28). If we consider a vector $\mathbf{A}$ fixed to S' and forming an angle θ with the rotation axis, as shown in the figure, the change in $\mathbf{A}$ during the time interval from t to $t + \Delta t$ is given by

$$\Delta\mathbf{A} = \mathbf{A}(t + \Delta t) - \mathbf{A}(t) = A\omega\Delta t \sin\theta \, \mathbf{u} \, , \tag{1.25}$$

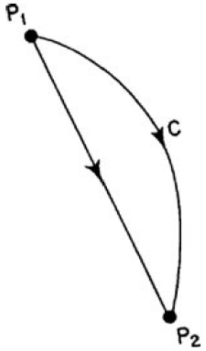

Fig. 1.29 The action takes a minimum value along the dynamic trajectory P_1CP_2. For another trajectory, like the straight line P_1P_2, its value would be larger.

$$m\frac{dv}{dt} = 0\,, \tag{1.33}$$

with the solution v = constant.

Consider two positions of a planet, P_1 and P_2, on its orbit around the Sun S. If the variation of the action is calculated between P_1 and P_2 when the planet follows the usual elliptical trajectory and when it is constrained to follow the straight line P_1P_2, one finds that the action is smaller for the elliptical trajectory P_1CP_2. Moreover, it remains smaller than any other value obtained by varying the trajectories between P_1 and P_2 (Fig. 1.29).

The principle of least action is of the utmost importance in all branches of physics, and it was first formulated by Pierre-Louis Moreau de Maupertuis (1698–1759), and in more complete form by William Rowan Hamilton (1805–1865).

A similar principle exists in optics for the trajectory followed by light when it propagates through a medium. This is Fermat's principle, established by Pierre Fermat (1601–1665). The principle states that, when light travels from one point to another in a medium, it does it in such a way that the required *time* has an extremum value, generally a minimum, although it could be a maximum.

Principles of least action have great importance in modern theoretical physics.

Lagrange Equations and Planetary Motion. Consider again the case of a planet moving around the Sun. To describe its motion, it is convenient to use polar coordinates with the pole located at the Sun. Remember that the force exerted by the Sun on the planet is a vector $\mathbf{F} = -GMm\mathbf{r}_0/r^2$. The polar coordinates are

$$x = r\cos\theta, \qquad y = r\sin\theta\,, \tag{1.34}$$

where r, θ are the radius vector and the angle with respect to the polar axis, respectively. The velocities along and perpendicular to $\mathbf{r}$ are given by

$$v_r = \dot{r} = dr/dt, \qquad v_l = r\dot{\theta} = rd\theta/dt. \tag{1.35}$$

The Lagrangian for the planet is the difference between the kinetic energy T and the potential energy V, that is,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}. \tag{1.36}$$

Since θ does not appear explicitly in L (only $\dot{\theta}$ appears), the first Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0,$$

leads to the following equation:

$$\frac{d(mr^2\dot{\theta})}{dt} = 0 , \tag{1.37}$$

that is to say, $mr^2\dot{\theta} = C =$ const., which is the law of conservation of angular momentum (in this example only, we denote the value of the conserved angular momentum by C, not to confuse it with the Lagrangian). If $\dot{\theta} = C/mr^2$ is substituted into the expression for the total energy, $T + V$, we find that

$$E = \frac{1}{2}m\dot{r}^2 + \frac{C^2}{2mr^2} - \frac{GMm}{r}. \tag{1.38}$$

This in turn implies that

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E + \frac{GMm}{r} - \frac{C^2}{2mr^2}\right]}.$$

By combining $\dot{r}$ and $\dot{\theta}$, one obtains an equation for r as a function of θ, which is the parametric equation of the orbit:

$$d\theta = \frac{C dr/r^2}{\sqrt{2m[E + GMm/r - C^2/2m\,r^2]}} ,$$

leading, upon integration, to

$$\theta = \arccos\frac{C/r - GMm^2/C}{\sqrt{2mE + G^2M^2m^4/C^2}}. \tag{1.39}$$

If we make the notations $d = C^2/GMm^2$, $\epsilon = \sqrt{1 + 2EC^2/G^2M^2m^3}$, we can finally write the equation for the orbit as the typical equation of a conic:

$$r = \frac{d}{1 + \epsilon\cos\theta} , \tag{1.40}$$

where ϵ is the eccentricity. If $E < 0$, then $\epsilon < 1$ and the orbit is elliptic. If $E = 0$, then $\epsilon = 1$ and the orbit is parabolic. If $E > 0$, then $\epsilon > 1$ and the orbit is hyperbolic.

1.9 Hamilton Equations

Instead of using generalized coordinates and velocities to describe the motion of a physical system with N degrees of freedom, it is sometimes easier to use coordinates and momenta, as an independent set of N pairs of *canonical coordinates*. If L is the Lagrangian of a system, considered as a function of the coordinates q_i, the velocities $\dot{q}_i$, and the time t, where $i = 1, 2, \ldots, N$, one can write the Lagrangian's total differential in terms of the generalized coordinates and velocities as

$$dL = \sum_{i=1}^{N} \frac{\partial L}{\partial q_i} dq_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt.$$

By definition,

$$p_i = \partial L / \partial \dot{q}_i$$

are the generalized momenta (also called canonically conjugated momenta). Recalling the Euler–Lagrange equations, it follows that

$$dL = \sum_{i=1}^{N} \dot{p}_i \, dq_i + \sum_{i=1}^{N} p_i \, d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{1.41}$$

We can write $\sum p_i \, d\dot{q}_i = d\Big(\sum p_i \dot{q}_i\Big) - \sum \dot{q}_i \, dp_i$. Then, reorganizing the terms and defining

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - L$$

as the *Hamiltonian* function, such that

$$dH = -\sum_{i=1}^{N} \dot{p}_i \, dq_i + \sum_{i=1}^{N} \dot{q}_i \, dp_i - \frac{\partial L}{\partial t} dt, \tag{1.42}$$

we conclude that

$$\dot{q}_i = \partial H / \partial p_i, \qquad \dot{p}_i = -\partial H / \partial q_i, \quad i = 1, 2, \ldots, N. \tag{1.43}$$

We also obtain the equation

$$\partial H / \partial t = -\partial L / \partial t.$$

If the Lagrangian does not depend explicitly on time, the Hamiltonian does not depend on it either. Then H is a constant of motion, similar to the total energy. Equations (1.43) are called Hamilton's equations. They constitute a set of $2N$ first order differential equations, equivalent to the set of N Euler–Lagrange equations of second order.

We consider as an example the harmonic oscillator of mass m and elastic constant k, described by the Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2.$$

The velocity can be written as $\dot{q} = p/m$. The Hamiltonian is then

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 ,$$

which is the expression for the total energy. The Hamilton equations are

$$\dot{q} = p/m, \qquad \dot{p} = -kq.$$

Taking the derivative with respect to time of the first and substituting the result into the second, we get the equation

$$\ddot{q} = -\frac{k}{m}q ,$$

whose general solution is

$$q = A\cos(\omega t + \varphi),$$

where $\omega = \sqrt{k/m}$, A is the amplitude, and φ is an arbitrary angle (initial conditions must be given for fixing the values of A and φ). The same equation and solution can be obtained from the Euler–Lagrange equations.

The Hamiltonian formalism is of exceptional importance, mainly in connection with the transformation of a set of canonical coordinates p_i, q_i to another P_i, Q_i. We can consider the mechanical motion of a system as a canonical transformation of coordinates from some initial conditions to the set of canonical coordinates at some arbitrary instant t. It is possible to obtain a fundamental differential equation for the action S, the so-called Hamilton–Jacobi equation, whose solution allows us to find the equations of motion.

One can define a *phase space*, determined by the set p_i, q_i. The volume in phase space is invariant under canonical transformations. In this way, the phase space is an essential tool when dealing with systems having a very large number of particles, as it happens in statistical mechanics. The Hamiltonian formalism is also essential in quantum theory (see Chaps. 6 and 7).

Poisson Brackets. It is often necessary to define functions of the coordinates, the momenta, and the time, $f = f(q_i, p_i, t)$. The total derivative with respect to time is