

CALCULUS MADE EASY

SILVANUS P. THOMPSON
AND
MARTIN GARDNER



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PREFACE TO THE 1998 EDITION

Introductory courses in calculus are now routinely taught to high school students and college freshmen. For students who hope to become mathematicians or to enter professions that require a knowledge of calculus, such courses are the highest hurdle they have to jump. Studies show that almost half of college freshmen who take a course in calculus fail to pass. Those who fail almost always abandon plans to major in mathematics, physics, or engineering—three fields where advanced calculus is essential. They may even decide against entering such professions as architecture, the behavioral sciences, or the social sciences (especially economics) where calculus can be useful. They exit what they fear will be too difficult a road to consider careers where entrance roads are easier.

One reason for such a high dropout rate is that introductory calculus is so poorly taught. Classes tend to be so boring that students sometimes fall asleep. Calculus textbooks get fatter and fatter every year, with more multicolor overlays, computer graphics, and photographs of eminent mathematicians (starting with Newton and Leibniz), yet they never seem easier to comprehend. You look through them in vain for simple, clear exposition and for problems that will hook a student's interest. Their exercises have, as one mathematician recently put it, "the dignity of solving crossword puzzles." Modern calculus textbooks often contain more than a thousand pages—heavy enough to make excellent doorstops—and more than a thousand frightening exercises! Their prices are rapidly approaching \$100.

"Why do calculus books weigh so much?" Lynn Arthur Steen asked in a paper on "Twenty Questions for Calculus

Reformers” that is reprinted in *Toward a Lean and Lively Calculus* (Mathematical Association of America, 1986), edited by Ronald Douglas. Because, he answers, “the economics of publishing compels authors ... to add every topic that anyone might want so that no one can reject the book just because some particular item is omitted. The result is an encyclopaedic compendium of techniques, examples, exercises and problems that more resemble an overgrown workbook than an intellectually stimulating introduction to a magnificent subject.”

“The teaching of calculus is a national disgrace,” Steen, a mathematician at St. Olaf College, later declared. “Too often calculus is taught by inexperienced instructors to ill-prepared students in an environment with insufficient feedback.”

Leonard Gillman, writing on “The College Teaching Scandal” (*Focus*, Vol. 8, 1988, page 5), said: “The calculus scene has been execrable for many years, and given the inertia of our profession is quite capable of continuing that way for many more.”

Calculus has been called the topic mathematicians most love to hate. One hopes this is true only of teachers who do not appreciate its enormous power and beauty. Howard Eves is a retired mathematician who actually enjoyed teaching calculus. In his book *Great Moments in Mathematics* I found this paragraph:

Surely no subject in early college mathematics is more exciting or more fun to teach than the calculus. It is like being the ringmaster of a great three-ring circus. It has been said that one can recognize the students on a college campus who have studied the calculus—they are the students with no eyebrows. In utter astonishment at the incredible applicability of the subject, the eyebrows of the calculus students have receded higher and higher and finally vanished over the backs of their heads.

Recent years have seen a great hue and cry in

mathematical circles over ways to improve calculus teaching. Endless conferences have been held, many funded by the federal government. Dozens of experimental programs are underway here and there.

Some leaders of reform argue that while traditional textbooks get weightier, the need for advanced calculus is actually diminishing. In his popular *Introduction to the History of Mathematics*, Eves sadly writes: “Today the larger part of mathematics has no, or very little connection with calculus or its extensions.”

Why is this? One reason is obvious. Computers! Today’s digital computers have become incredibly fast and powerful. Continuous functions which once could be handled only by slow analog machines can now be turned into discrete functions which digital computers handle efficiently with step-by-step algorithms. Hand-held calculators called “graphers” will instantly graph a function much too complex to draw with a pencil on graph paper. The trend now is away from continuous math to what used to be called finite math, but now is more often called discrete math.

Calculus is steadily being downgraded to make room for combinatorics, graph theory, topology, knot theory, group theory, matrix theory, number theory, logic, statistics, computer science, and a raft of other fields in which continuity plays a relatively minor role.

Discrete mathematics is all over the scene, not only in mathematics but also in science and technology. Quantum theory is riddled with it. Even space and time may turn out to be quantized. Evolution operates by discrete mutation leaps. Television is on the verge of replacing continuous analog transmission by discrete digital transmission which greatly improves picture quality. The most accurate way to preserve a painting or a symphony is by converting it to discrete numbers which last forever without deteriorating.

When I was in high school I had to master a pencil-and-paper way to calculate square roots. Happily, I was not forced to learn how to find cube and higher roots! Today it

would be difficult to locate mathematicians who can even recall how to calculate a square root. And why should they? They can find the n th root of any number by pushing keys in less time than it would take to consult a book with tables of roots. Logarithms, once used for multiplying huge numbers, have become as obsolete as slide rules.

Something similar is happening with calculus. Students see no reason today why they should master tedious ways of differentiating and integrating by hand when a computer will do the job as rapidly as it will calculate roots or multiply and divide large numbers. Mathematica, a widely used software system developed by Stephen Wolfram, for example, will instantly differentiate and integrate, and draw relevant graphs, for any calculus problem likely to arise in mathematics or science. Calculators with keys for finding derivatives and integrals now cost less than most calculus textbooks. It has been estimated that more than ninety percent of the exercises in the big textbooks can be solved by using such calculators.

Leaders of calculus reform are not suggesting that calculus no longer be taught, what they recommend is a shift of emphasis from problem solving, which computers can do so much faster and more accurately, to an emphasis on understanding what computers are doing when they answer calculus questions. A knowledge of calculus is even essential just to know what to ask a computer to do. Above all, calculus courses should instill in students an awareness of the great richness and elegance of calculus.

Although suggestions are plentiful for ways to improve calculus teaching, a general consensus is yet to emerge. Several mathematicians have proposed introducing integral calculus before differential calculus. A notable example is the classic two-volume *Differential and Integral Calculus* (1936–37) by Richard Courant. However, differentiating is so much easier to master than integrating that this switch has not caught on.

Several calculus reformers, notably Thomas W. Tucker (See

his “Rethinking Rigor in Calculus,” in *American Mathematical Monthly*, (Vol. 104, March 1997, pp. 231–240) have recommended that calculus texts replace the important mean value theorem (MVT) with an increasing function theorem (IFT). (On the mean value theorem see my [Postscript to Thompson’s Chapter 10](#).) The IFT states that if the derivative on a function’s interval is equal to or greater than zero, then the function is increasing on that interval. For example, if a car’s speedometer always shows a number equal to or greater than zero, during a specified interval of time, then during that interval the car is either standing still or moving forward. Stated geometrically, it says that if the curve of a continuous function, during a given interval, has a tangent that is either horizontal or sloping upward, the function on that interval is either unchanging or increasing. This change also has not caught on.

Many reformers want to replace the artificial problems in traditional textbooks with problems about applications of calculus in probability theory, statistics, and in the biological and social sciences. Unfortunately, for beginning students not yet working in these fields, such “practical” problems can seem as dull and tedious as the artificial ones.

More radical reformers believe that calculus should no longer be taught in high school, and not even to college freshmen unless they have decided on a career for which a knowledge of calculus is required. And there are opponents of reform who find nothing wrong in the way calculus has been traditionally taught, assuming, of course, it is taught by competent teachers.

In its February 28, 1992, issue, *Science* investigated Project Calc, a computer oriented calculus course offered at Duke University. Only 57 percent of its students continued on with a second course in calculus as compared with 68 percent who continued on after taking a more traditional course. A few students liked the experimental course. Most did not. One student called it “the worst class I ever took.” Another described it as “a big exercise in confusion.” Still another is

quoted in *Science* as saying: “I am very jealous of my friends in normal calculus. I would do anything to have taken a regular calculus with pencil and paper.”

Efforts are now underway to combine continuous and discrete mathematics in a single textbook. An outstanding example is *Concrete Mathematics* (1984, revised 1989), an entertaining textbook by Ronald Graham, Donald Knuth, and Oren Patashnik. The authors coined the term “concrete” by taking “con” from the start of “continuous,” and “crete” from the end of “discrete.” However, even this exciting textbook presupposes a knowledge of calculus.

The American philosopher and psychologist William James, in an 1893 letter to Theodore Flournoy, a Geneva psychologist, asked “Can you name me any simple book on the differential calculus which gives an insight into the philosophy of the subject?”

In spite of the current turmoil over fresh ways to teach calculus, I know of no book that so well meets James’s request as the book you are now holding. Many similar efforts have been made, with such titles as *Calculus for the Practical Man*, *The ABC of Calculus*, *What Is Calculus About?*, *Calculus the Easy Way*, and *Simplified Calculus*. They tend to be either too elementary, or too advanced. Thompson strikes a happy medium. It is true that his book is old-fashioned, intuitive, and traditionally oriented. Yet no author has written about calculus with greater clarity and humor. Thompson not only explains the “philosophy of the subject,” he also teaches his readers how to differentiate and integrate simple functions.

Silvanus Phillips Thompson was born in 1851, the son of a school teacher in York, England. From 1885 until his death in 1916 he was professor of physics at the City and Guilds Technical College, in Finsbury. A distinguished electrical engineer, he was elected to the Royal Society in 1891, and served as president of several scientific societies.

Thompson wrote numerous technical books and manuals on electricity, magnetism, dynamos, and optics, many of

which went through several editions. He also authored popular biographies of scientists Michael Faraday, Philipp Reis, and Lord Kelvin. A devout Quaker and an active Knight Templar, he wrote two books about his faith: *The Quest for Truth* (1915), and *A Not Impossible Religion* (published posthumously 1918). He was much in demand as a lecturer, and said to be a skillful painter of landscapes. He also wrote poetry. In 1920 two of his four daughters, Jane Smeal Thompson and Helen G. Thompson, published a book about their father titled *Silvanus Phillips Thompson: His Life and Letters*.

Calculus Made Easy was first issued by Macmillan in England in 1910 under the pseudonym of F.R.S.—initials that stand for Fellow of the Royal Society. The identity of the author was not revealed until after his death. The book was reprinted three times before the end of 1910. Thompson revised the book considerably in 1914, correcting errors and adding new material. The book was further revised and enlarged posthumously in 1919, and again in 1945 by F. G. W. Brown. Some of these later additions, such as the chapter on partial fractions, are more technical than chapters in Thompson's original work. Curiously, Thompson's first edition, with its great simplicity and clarity, is in a way closer to the kind of introductory book recommended today by reformers who wish to emphasize the basic ideas of calculus, and to downplay tedious techniques for solving problems that today can be solved quickly by computers. Readers who wish merely to grasp the essentials of calculus can skip the more technical chapters, and need not struggle with solving all the exercises. The book has never been out of print. St. Martin's Press published its paperback edition in 1970.

Almost all reviews of the book's first edition were favorable. A reviewer for *The Athenaeum* wrote:

It is not often that it falls to the lot of a reviewer of mathematical literature to read such a gay and boisterous book as this "very simplest introduction to

those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the integral calculus.” As a matter of fact, professional mathematicians will give a warm welcome to a book which is so orthodox in its teaching and so vigorous in its exposition.

Professor E. G. Coker, a colleague, said in a letter to Thompson:

I am very pleased to hear that your little book on the calculus is likely to be available for general use. As you know, I have been teaching the elements of this subject to the junior classes here for some years, and I do not know of any other book so well adapted to give fundamental ideas. One of the great merits of the book is that it dispels the mysteries with which professional mathematicians envelope the subject. I feel sure that your little book, with its common sense way of dealing with elementary ideas of the calculus, will be a great success.

Many of today’s eminent mathematicians and scientists first learned calculus from Thompson’s book. Morris Kline, himself the author of a massive work on calculus, always recommended it as the best book to give a high school student who wants to learn calculus. The late economist and statistician Julian Simon, sent me his paper, not yet published, titled “Why Johnnies (and Maybe You) Hate Math and Statistics.” It contains high praise for Thompson’s book:

Calculus Made Easy has been damned by every professional mathematician I have asked about it. So far as I know, it is not used in any calculus courses anywhere. Nevertheless, almost a century after its first publication, it still sells briskly in paperback even in college bookstores. It teaches a system of approximation that makes quite clear the central idea of calculus—the idea that is extraordinarily difficult to

comprehend using the mathematician's elegant method of limits.

Later on Simon asks:

Question: why don't high school and college kids get to learn calculus the Thompson way? Answer: Thompson's system has an unremediable fatal flaw: It is ugly in the eyes of the world-class mathematicians who set the standards for the way mathematics is taught all down the line; the run-of-the-mill college and high school teachers, and ultimately their students, are subject to this hegemony of the aesthetic tastes of the great. Thompson simply avoids the deductive devices that enthrall mathematicians with their beauty and elegance.

I mentioned earlier a book titled *Toward a Lean and Lively Calculus*. It contains papers by mathematicians who participated in a 1986 conference at Tulane University on how to improve calculus teaching. Most of the contributors urged cutting down on techniques for problem solving, stressing the understanding of ideas, integrating calculus with the use of calculators, and scaling down textbooks to leaner and livelier forms. Now, the leanest and liveliest introduction to calculus ever written is Thompson's *Calculus Made Easy*, yet Peter Renz was the only mathematician at the conference who had the courage to praise the book and list it as a reference.

The two most important concepts in calculus are functions and limits. Because Thompson more or less assumes that his readers understand both notions, I have tried in two preliminary chapters to make clearer what they both mean. And I have added a brief chapter about derivatives. Here and there throughout *Calculus Made Easy* I have inserted footnotes where I think something of interest can be said about the text. These notes are initialed M.G. to distinguish them from Thompson's notes.

Where Thompson speaks of British currency I have changed the values to dollars and cents. Terminology has been updated. Thompson uses the obsolete term “differential coefficient.” I have changed it to “derivative.” The term “indefinite integral” is still used, but it is rapidly giving way to “antiderivative,” so I have made this substitution.

Thompson followed the British practice of raising a decimal point to where it is easily confused with the dot that stands for multiplication. I have lowered every such point to conform to American custom. Where Thompson used a now discarded sign for factorials, I have changed it to the familiar exclamation mark. Where Thompson used the Greek letter for epsilon, I have changed it to the english e . Where Thompson used the symbol \log_e , I have replaced it with \ln . Finally, in a lengthy appendix, I have thrown together a variety of calculus-related problems that have a recreational flavor.

I hope my revisions and additions for this newly revised edition of *Calculus Made Easy* will render it even easier to understand—not just for high school and college students, but also for older laymen who, like William James, long to know what calculus is all about. Most mathematics deals with static objects such as circles and triangles and numbers. But the great universe “out there,” not made by us, is in a constant state of what Newton called flux. At every microsecond it changes magically into something different.

Calculus is the mathematics of change. If you are not a mathematician or scientist, or don't intend to become one, there is no need for you to master the techniques for solving calculus problems by hand. But if you avoid acquiring some insight into the essentials of calculus, into what James called its philosophy, you will miss a great intellectual adventure. You will miss an exhilarating glimpse into one of the most marvelous, most useful creations of those small and mysterious computers inside our heads.

I am indebted to Dean Hickerson, Oliver Selfridge, and Peter Renz for looking over this book's manuscript and providing a raft of corrections and welcome suggestions.

—MARTIN GARDNER
January, 1998

Preliminary Chapter 1

WHAT IS A FUNCTION?



No concept in mathematics, especially in calculus, is more fundamental than the concept of a function. The term was first used in a 1673 letter written by Gottfried Wilhelm Leibniz, the German mathematician and philosopher who invented calculus independently of Isaac Newton. Since then the term has undergone a gradual extension of meaning.

In traditional calculus a function is defined as a relation between two terms called variables because their values vary. Call the terms x and y . If every value of x is associated with exactly one value of y , then y is said to be a function of x . It is customary to use x for what is called the *independent variable*, and y for what is called the *dependent variable* because its value depends on the value of x .

As Thompson explains in [Chapter 3](#), letters at the end of the alphabet are traditionally applied to variables, and letters elsewhere in the alphabet (usually first letters such as a, b, c, \dots) are applied to constants. Constants are terms in an equation that have a fixed value. For example, in $y = ax + b$, the variables are x and y , and a and b are constants. If $y = 2x + 7$, the constants are 2 and 7. They remain the same as x and y vary.

A simple instance of a geometrical function is the dependence of a square's area on the length of its side. In this case the function is called a one-to-one function because the dependency goes both ways. A square's side is also a function

of its area.

A square's area is the length of its side multiplied by itself. To express the area as a function of the side, let y be the area, x the side, then write $y = x^2$. It is assumed, of course, that x and y are positive.

A slightly more complicated example of a one-to-one function is the relation of a square's side to its diagonal. A square's diagonal is the hypotenuse of an isosceles right triangle. We know from the Pythagorean theorem that the square of the hypotenuse equals the sum of the squares of the other two sides. In this case the sides are equal. To express the diagonal as a function of the square's side, let y be the diagonal, x the side, and write $y = \sqrt{2x^2}$, or more simply $y = x\sqrt{2}$ to express the side as a function of the diagonal, let

y be the side, x the diagonal, and write $y = \sqrt{\frac{x^2}{2}}$, or more simply $y = \frac{x}{\sqrt{2}}$.

The most common way to denote a function is to replace y , the dependent variable, by $f(x)$ — f being the first letter of “function.” Thus $y = f(x) = x^2$ means that y , the dependent variable, is the square of x . Instead of, say, $y = 2x - 7$, we write $y = f(x) = 2x - 7$. This means that y , a function of x , depends on the value of x in the expression $2x - 7$. In this form the expression is called an *explicit* function of x . If the equation has the equivalent form of $2x - y - 7 = 0$, it is called an *implicit* function of x because the explicit form is implied by the equation. It is easily obtained from the equation by rearranging terms. Instead of $f(x)$, other symbols are often used.

If we wish to give numerical values to x and y in the example $y = f(x) = 2x - 7$, we replace x by any value, say 6, and write $y = f(6) = (2 \cdot 6) - 7$, giving the dependent variable y a value of 5.

If the dependent variable is a function of a single

independent variable, the function is called a function of one variable. Familiar examples, all one-to-one functions, are:

The circumference or area of a circle in relation to its radius.

The surface or volume of a sphere in relation to its radius.

The log of a number in relation to the number.

Sines, cosines, tangents, and secants are called trigonometric functions. Logs are logarithmic functions. Exponential functions are functions in which x , the independent variable, is an exponent in a equation, such as $y = 2^x$. There are, of course, endless other examples of more complicated one-variable functions which have been given names.

Functions can depend on more than one variable. Again, there are endless examples. The hypotenuse of a right triangle depends on its two sides, not necessarily equal. (The function of course involves three variables, but it is called a two-variable function because it has two independent variables.) If z is the hypotenuse, we know from the Pythagorean theorem that $z = \sqrt{x^2 + y^2}$. Note that this is not a one-to-one function. Knowing x and y gives z a unique value, but knowing z does not yield unique values for x and y .

Two other familiar examples of a two-variable function, neither one-to-one, are the area of a triangle as a function of its altitude and base, and the area of a right circular cylinder as a function of its radius and height.

Functions of one and two variables are ubiquitous in physics. The period of a pendulum is a function of its length. The distance covered by a dropped stone and its velocity are each functions of the elapsed time since it was dropped. Atmospheric pressure is a function of altitude. A bullet's energy is a two-variable function dependent on its mass and velocity. The electrical resistance of a wire depends on the length of the wire and the diameter of its circular cross section.

Functions can have any number of independent variables. A simple instance of a three-variable function is the volume of a rectangular room. It is dependent on the room's two sides and height. The volume of a four-dimensional hyper-room is a function of four variables.

A beginning student of calculus must be familiar with how equations with two variables can be modeled by curves on the Cartesian plane. (The plane is named after the French mathematician and philosopher René Descartes who invented it.) Values of the independent variable are represented by points along the horizontal x axis. Values of the dependent variable are represented by points along the vertical y axis. Points on the plane signify an ordered pair of x and y numbers. If a function is linear—that is, if it has one form $y = ax + b$ —the curve representing the ordered pairs is a straight line. If the function does not have the form $ax + b$ the curve is not a straight line.

Figure 1 is a Cartesian graph of $y = x^2$. The curve is a parabola. Points along each axis represent real numbers (rational and irrational), positive on the right side of the x axis, negative on the left; positive at the top of the y axis, negative at the bottom. The graph's origin point, where the axes intersect, represents zero. If x is the side of a square, we assume it is neither zero nor negative, so the relevant curve would be only the right side of the parabola. Assume the square's side is 3. Move vertically up from 3 on the x axis to the curve, then go left to the y axis where you find that the square of 3 is 9. (I apologize to readers for whom all this is old hat.)

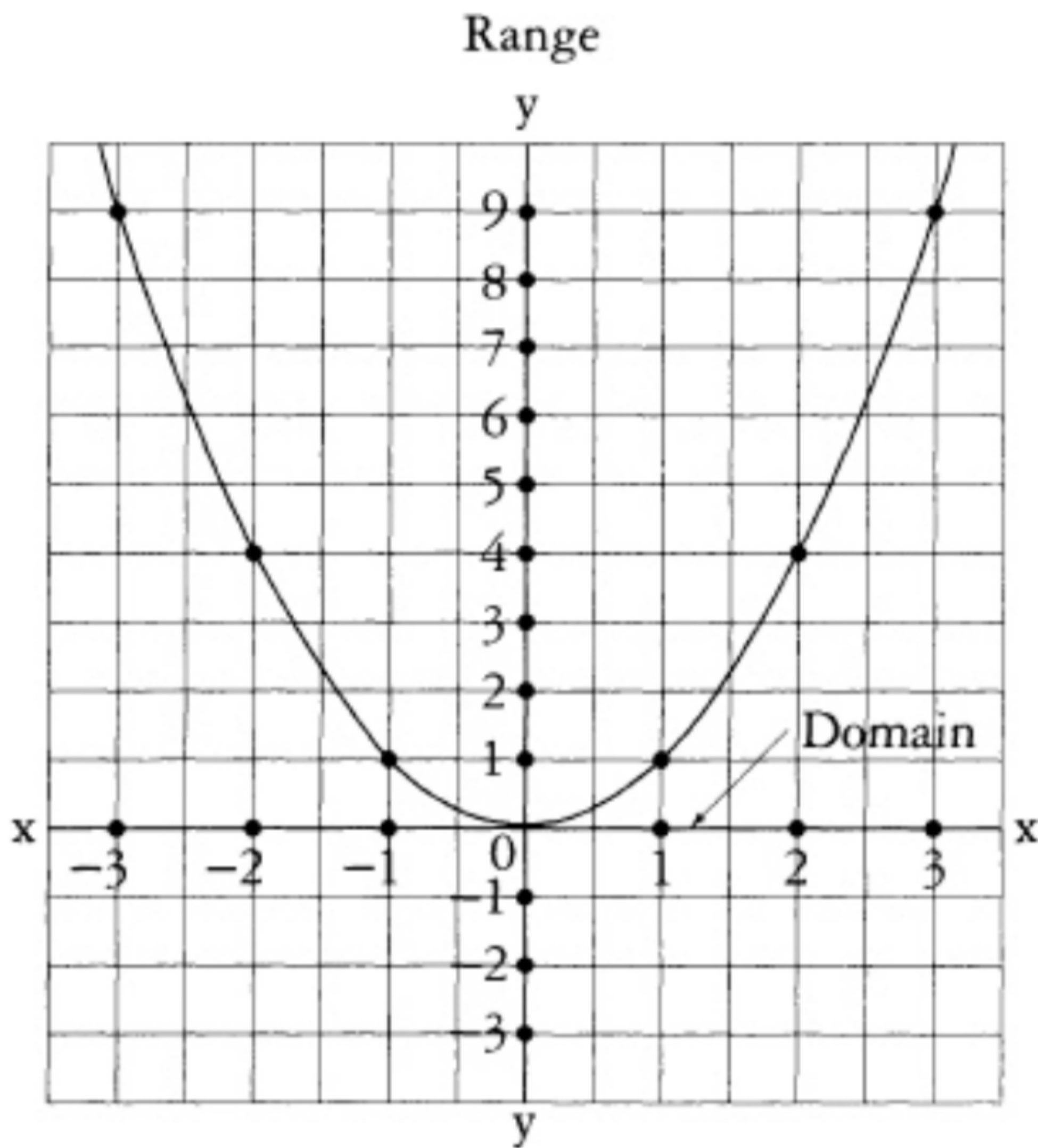


FIG. 1. $y = x^2$ or $f(x) = x^2$

Note that the scales are different on the two axes.

If a function involves three independent variables, the Cartesian graph must be extended to a three-dimensional space with axes x , y , and z . I once heard about a professor, whose name I no longer recall, who liked to dramatize this space to his students by running back and forth while he exclaimed “This is the x axis!” He then ran up and down the center aisle shouting “This is the y axis!”, and finally hopped up and down while shouting “This is the z axis!” Functions of more than three variables require a Cartesian space with more than three axes. Unfortunately, a professor cannot dramatize axes higher than three by running or jumping.

Note the labels “domain” and “range” in [Figure 1](#). In recent decades it has become fashionable to generalize the definition of function. Values that can be taken by the independent variable are called the variable’s *domain*. Values

that can be taken by the dependent variable are called the *range*. On the Cartesian plane the domain consists of numbers along the horizontal (x) axis. The range consists of numbers along the vertical (y) axis.

Domains and ranges can be infinite sets, such as the set of real numbers, or the set of integers; or either one can be a finite set such as a portion of real numbers. The numbers on a thermometer, for instance, represent a finite interval of real numbers. If used to measure the temperature of water, the numbers represent an interval between the temperatures at which water freezes and boils. Here the height of the mercury column relative to the water's temperature is a one-to-one function of one variable.

In modern set theory this way of defining a function can be extended to completely arbitrary sets of numbers for a function that is described not by an equation but by a set of rules. The simplest way to specify the rules is by a table. For example, the table in [Figure 2](#) shows a set of arbitrary numbers that constitute the domain on the left. The corresponding set of arbitrary numbers in the range is on the right. The rules that govern this function are indicated by arrows. These arrows show that every number in the domain correlates to a single number on the right. As you can see, more than one number on the left can lead to the same number on the right, but not vice versa. Another example of such a function is shown in [Figure 3](#), along with its graph, consisting of 6 isolated points in the plane.

Because every number on the left leads to exactly one number on the right, we can say that the numbers on the right are a function of those on the left. Some writers call the numbers on the right “images” of those on the left. The arrows are said to furnish a “mapping” of domain to range. Some call the arrows “correspondence rules” that define the function.

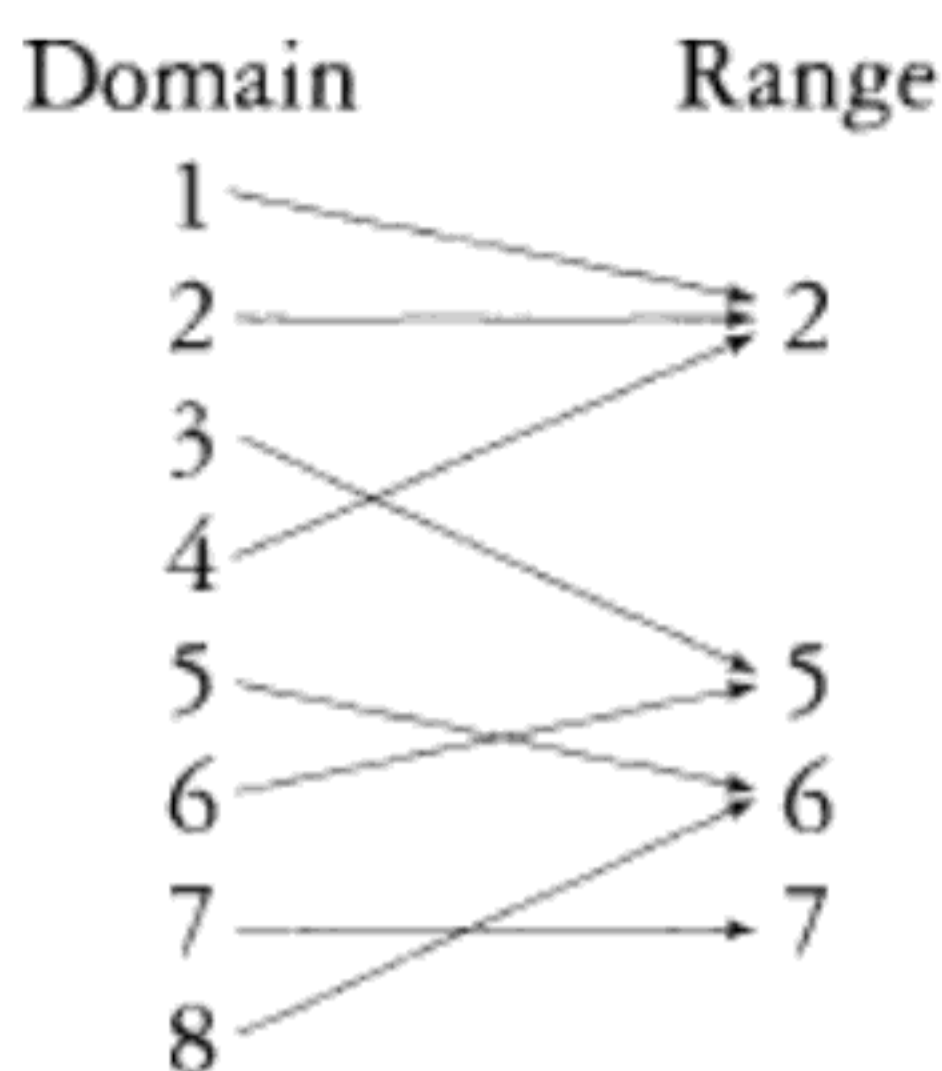


FIG. 2. An arbitrary function.

For most of the functions encountered in calculus, the domain consists of a single interval of real numbers. The domain might be the entire x axis, as it is for the function $y = x^2$. Or it might be an interval that's bounded; for example, the domain of $y = \arcsin x$ consists of all x such that $-1 \leq x \leq 1$. Or it might be bounded on one side and unbounded on the other; for example, the domain $y = \sqrt{x}$ consists of all $x \geq 0$. We call such a function "continuous" if its graph can be drawn without lifting the pencil from the paper, and "discontinuous" otherwise. (The complete definition of continuity, which is also applicable to functions with more complicated domains, is beyond the scope of this book.)

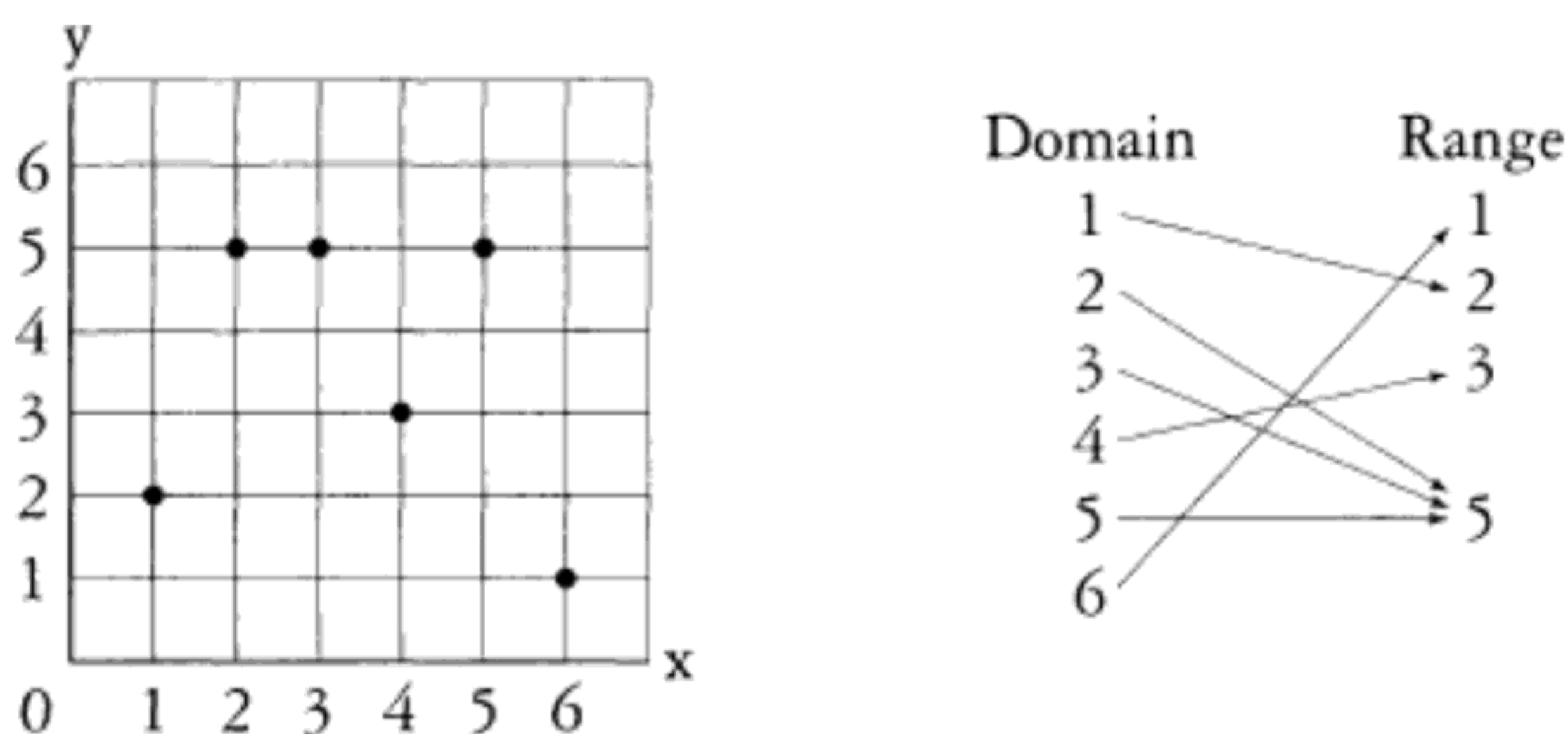


FIG. 3. How another arbitrary discrete function of integers is graphed.

For example, the three functions just mentioned are all continuous. [Figure 4](#) shows an example of a discontinuous function. Its domain consists of all real numbers, but its graph has infinitely many pieces that aren't connected to each other. In this book we will be concerned almost entirely with continuous functions.

Note that if a vertical line from the x axis intersects more than one point on a curve, the curve cannot represent a function because it maps an x number to more than one y number. [Figure 5](#) is a graph that clearly is not a function because vertical lines, such as the one shown dotted, intersect the graph at three spots. (It should be noted that Thompson did not use the modern definition of “function.” For example the graph shown in [Figure 30](#) of [Chapter XI](#) fails this vertical line test, but Thompson considers it a function.)

In this generalized definition of function, a one-variable function is any set of ordered pairs of numbers such that every number in one set is paired with exactly one number of the other set. Put differently, in the ordered pairs no x number can be repeated though a y number can be.

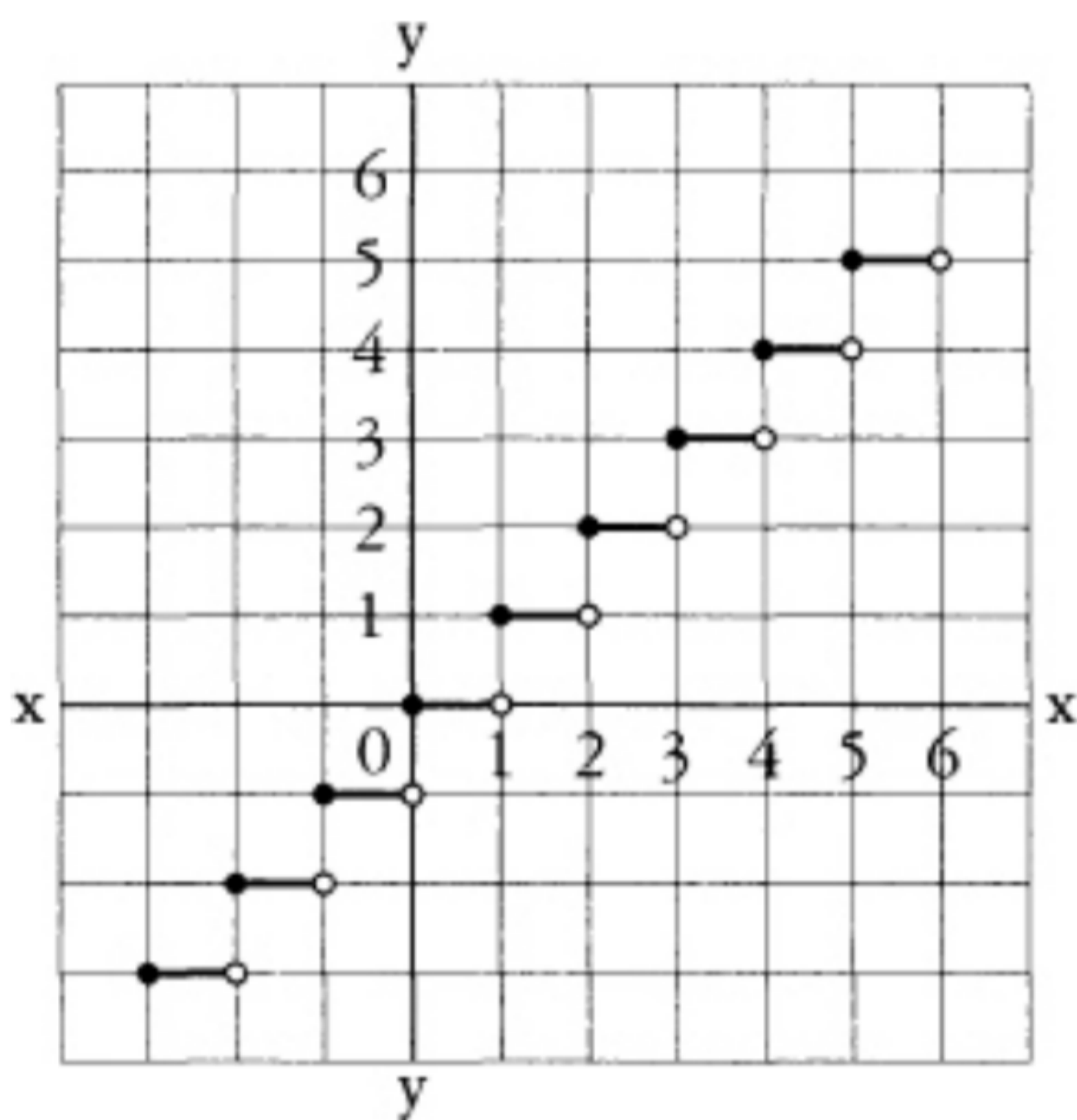


FIG. 4. This function is called the greatest integer function because it maps each real number (on the x axis) to the

largest integer on the y axis that is equal to or less than the real number.

In this broad way of viewing functions, the arbitrary combination of a safe or the sequence of buttons to be pushed to open a door, are functions of counting numbers. To open a safe you must turn the knob back and forth to a random set of integers. If the safe's combination is, say, 2-19-3-2-19, then those numbers are a function of 1,2,3,4,5. They represent the order in which numbers must be taken to open the safe, or the order in which buttons must be pushed to open a door. In a similar way the heights of the tiny "peaks" along a cylinder lock's key are an arbitrary function of positions along the key's length.

In recent years mathematicians have widened the notion of function even further to include things that are not numbers. Indeed, they can be anything at all that are elements of a set. A function is simply the correlation of each element in one set to exactly one element of another set. This leads to all sorts of uses of the word function that seem absurd. If Smith has red hair, Jones has black hair, and Robinson's hair is white, the hair color is a function of the three men. Positions of towns on a map are a function of their positions on the earth. The number of toes in a normal family is a function of the number of persons in the family. Different persons can have the same mother, but no person has more than one mother. This allows one to say that mothers are a function of persons. Elephant mothers are a function of elephants, but not grandmothers because an elephant can have two grandmothers. As one mathematician recently put it, functions have been generalized "up to the sky and down into the ground."

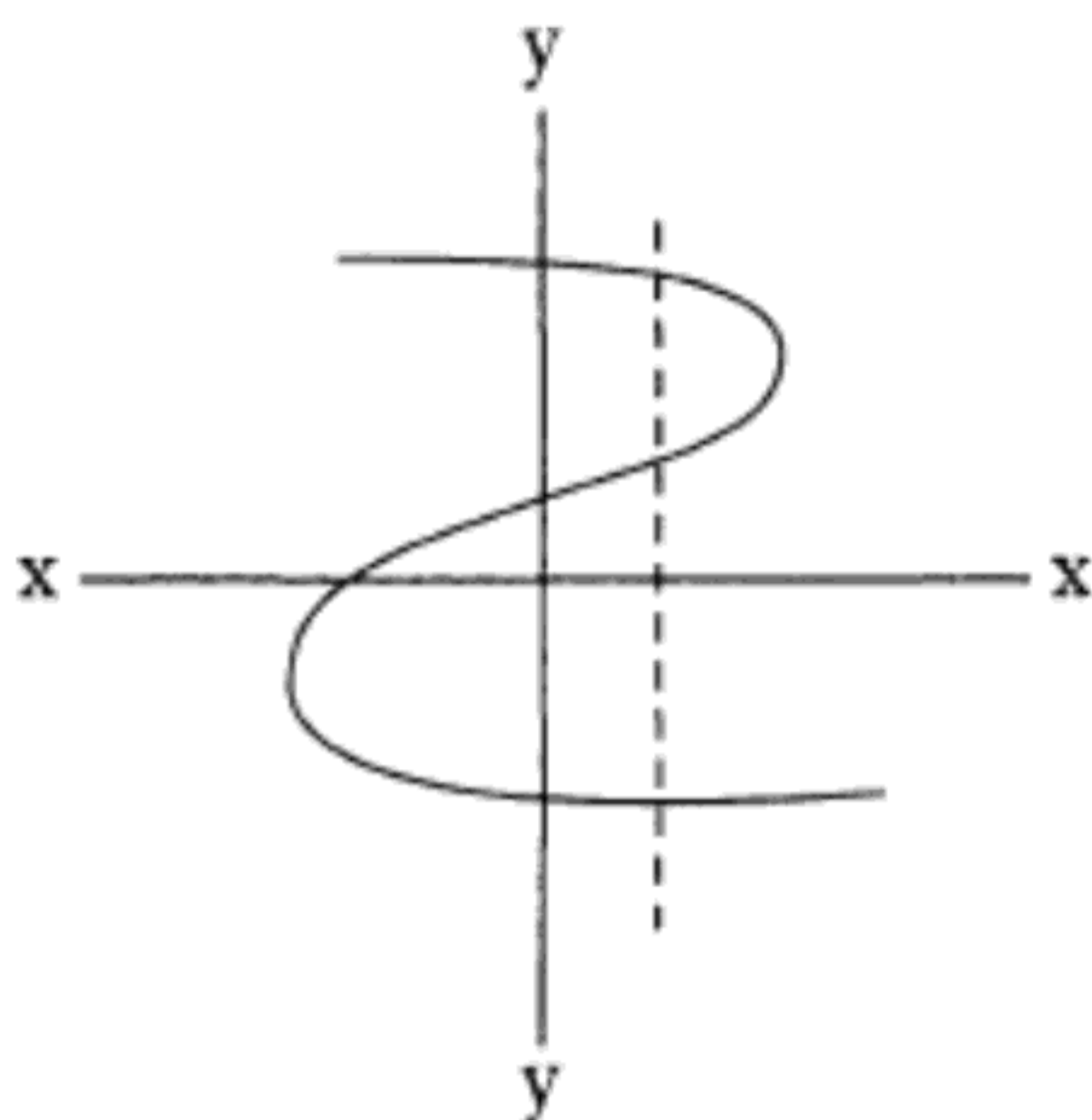


FIG. 5. A graph that does not represent a function.

A useful way to think of functions in this generalized way is to imagine a black box with input and output openings. Any element in a domain, numbers or otherwise, is put into the box. Out will pop a single element in the range. The machinery inside the box magically provides the correlations by using whatever correspondence rules govern the function. In calculus the inputs and outputs are almost always real numbers, and the machinery in the black box operates on rules provided by equations.

Because the generalized definition of a function leads to bizarre extremes, many educators today, especially those with engineering backgrounds, think it is confusing and unnecessary to introduce such a broad definition of functions to beginning calculus students. Nevertheless, an increasing number of modern calculus textbooks spend many pages on the generalized definition. Their authors believe that defining a function as a mapping of elements from any set to any other set is a strong unifying concept that should be taught to all calculus students.

Opponents of this practice think that calculus should not be concerned with toes, towns, mothers, and elephants. Its domains and ranges should be confined, as they have always been, to real numbers whose functions describe continuous change.

It is a fortunate and astonishing fact that the fundamental laws of our fantastic fidgety universe are based on relatively simple equations. If it were otherwise, we surely would know less than we know now about how our universe behaves, and Newton and Leibniz would probably never have invented (or discovered?) calculus.

Preliminary Chapter 2

WHAT IS A LIMIT?



It is possible, though difficult, to understand calculus without a firm grasp on the meaning of a limit. A derivative, the fundamental concept of differential calculus, is a limit. An integral, the fundamental concept of integral calculus, is a limit.

To explain what is meant by a limit, we will be concerned in this chapter only with limits of discrete functions because limits are easier to understand in discrete terms. When you read *Calculus Made Easy* you will learn how the limit concept applies to what are called functions of a continuous variable because their variables have real number values that vary continuously. Functions of discrete variables have variables whose values jump from one value to another. There are also functions of complex variables in which the values are complex numbers—numbers based on the imaginary square root of minus one. Complex variables are outside the scope of Thompson's book.

A sequence is a set of numbers in some order. The numbers don't have to be different and they need not be integers. Consider the sequence 1,2,3,4, This is just the positive integers. It is an infinite sequence because it continues without stopping. If it stopped it would be a finite sequence.

If the terms of a finite sequence are added to obtain a finite sum, it is called a series. If a series is infinite, the sum up to any specified term is called a "partial sum." If the partial

sums of an infinite series get closer and closer to a number k , so that by continuing the series you can make the sum as close to k as you please, then k is called the limit of the partial sums, or the limit of the infinite series. The terms are said to “converge” on k . If there is no convergence, the series is said to “diverge.”

The limit of an infinite series is sometimes called its “sum at infinity,” but of course this is not a sum in the usual arithmetical sense when the number of terms is finite. You can't obtain the “sum” of an infinite series by adding because the number of terms to be added is infinite. When we speak of the “sum” of an infinite series, this is just a short way of naming its limit.

An infinite series can converge on its limit in three different ways:

1. The partial sums get ever closer to the limit without actually reaching it, but they never go beyond the limit.
2. The partial sums reach the limit.
3. The partial sums go beyond the limit before they converge.

Let's look at examples of types 1 and 3.

The fifth century B.C. Greek philosopher Zeno of Elia invented several famous paradoxes intended to show that there is something extremely mysterious about motion. One of them imagines a runner going from A to B . He first runs half the distance, then half the remaining distance, then again half the remaining distance, and so on. The distances he runs get smaller and smaller in the halving series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n}$. Distances from B approach zero as their limit while the distances from A form a series that converges on 1. The runner, of course, models a point moving along a line from A to B . Does the runner ever reach the goal?

It depends.

Assume that after each step in the series the runner pauses to rest for a second. We can model this with a pawn (representing a point) that you push across a table from one edge to the edge opposite. First you push the pawn half the distance, then pause for a second. You push it half the remaining distance and again pause for a second. If this procedure continues, the pawn (point) will get closer and closer to the limit, but will never reach it.

There is an old joke based on this. A mathematics professor places a male student at one side of an empty room and a gorgeous female student at the opposite wall. On command, the boy walks half the distance toward the girl, waits a second, then goes half the rest of the way, and so on, always pausing a second before he cuts the remaining distance in half. The girl says, “Ha ha, you'll never reach me!” The boy replies, “True, but I can get close enough for all practical purposes.”

Suppose, now, that instead of waiting a second after each pawn push, the pawn is moved at a steady rate. Assume that the constant speed is such that the pawn goes half the distance in one second, half the remaining distance in half a second, and so on. No pauses. A discrete process has been transformed into a continuous one. In two seconds the pawn has reached the table's far edge. Zeno's runner, if he goes at a steady rate, will reach the goal in a finite period of time. The halving series, modeled in this fashion, converges exactly on the limit.

Zeno's runner leads to a variety of amusing paradoxes involving what are called “infinity machines.” A simple example is a lamp that is turned off at the end of one minute, then turned on at the end of half a minute, off after a quarter minute, and so on in an infinite series of ons and offs. The time series converges on two minutes. At the end of two minutes is the lamp on or off? This of course is a thought experiment. It can't be performed with an actual lamp, but can it be answered in the abstract? No, because there is no

last operation in an infinite series of on and off. It is like asking if the last digit of pi is odd or even.*

An easy way to “see” that the limit of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is 1 is to mark off the fractional lengths along a number line as Thompson does in his [Figure 46](#). A similar “look-see” proof that the series converges on 1 is shown by the dissected unit square in [Figure 6](#). The partial sums of this series are generated by the discrete function $1 - \frac{1}{2^n}$, where n takes the integral values 1,2,3,4,5,

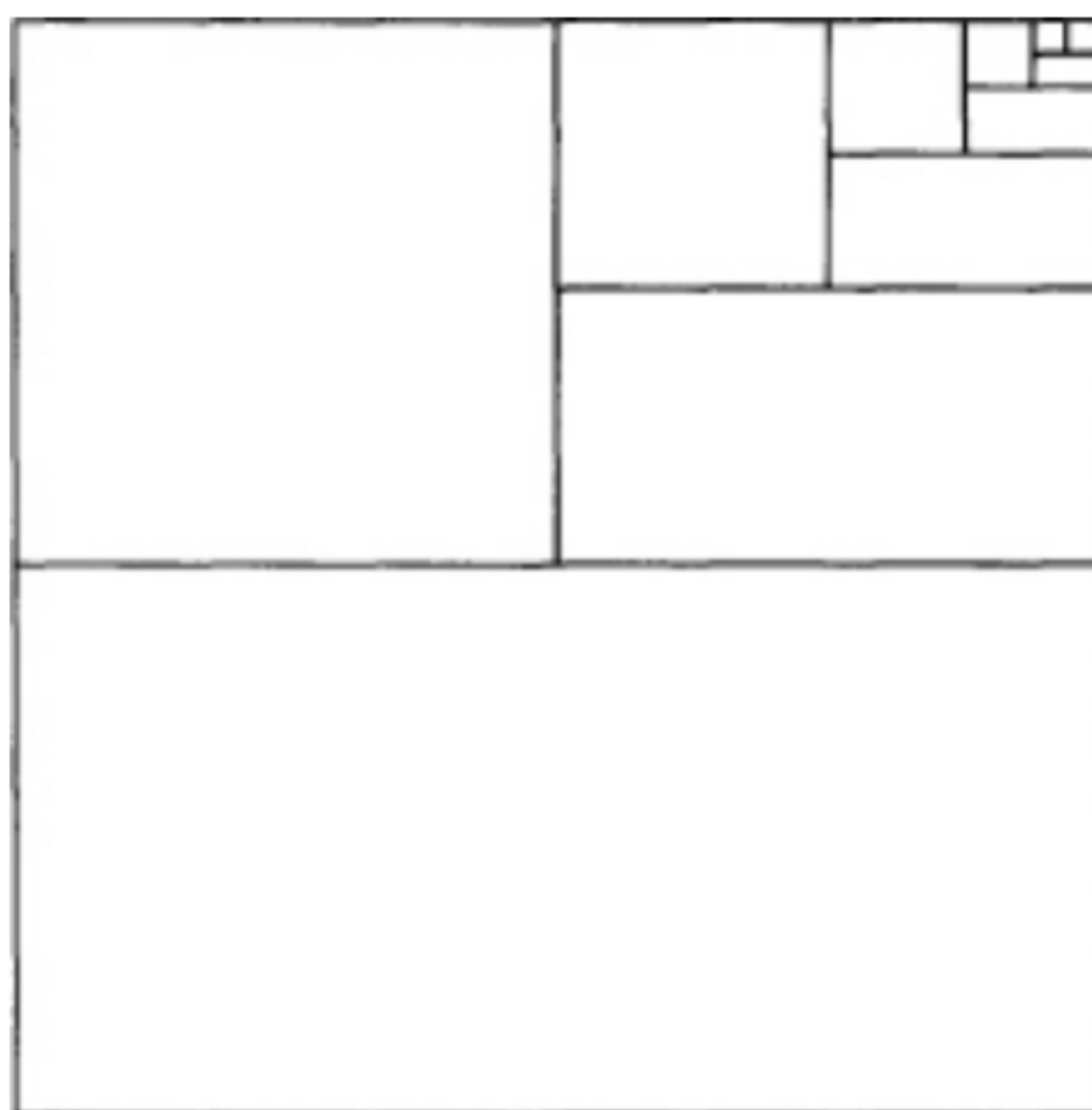


FIG. 6. A two-dimensional “look-see” proof that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1.$$

We turn now to an infinite series that goes past its limit before finally converging. An example is provided by changing every other sign in the halving series to a minus sign: $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$. The partial sums of this “alternating series” are alternately above and below the limit of $\frac{1}{3}$. The difference from $\frac{1}{3}$ can be made as small as you please, but every other partial sum is larger than the limit.

As an infinite series approaches but never reaches its limit, the differences between a partial sum and the limit get closer and closer to zero. Indeed they get so close that you can assume they *are* zero and therefore, as Thompson likes to say, they can be “thrown away.” In early books on calculus, *terms*

said to become infinitely close to zero were called “infinitesimals.” Clearly there is something spooky about numbers living in a neverland that is infinitely close to zero, yet somehow not zero. In the halving series, for example, the fractions approaching zero never become infinitesimals because they always remain a finite portion of 1. Infinitesimals are an infinitely small part of 1. They are smaller than any finite fraction you can name, yet never zero. Are they legitimate mathematical entities, or should they be banished from mathematics?

The most outspoken opponent of infinitesimals was the eighteenth-century British philosopher Bishop George Berkeley who attacked them in a 1734 book titled *The Analyst, Or a Discourse Addressed To an Infidel Mathematician*. The infidel was the astronomer Edmond Halley, for whom Halley’s comet is named, and the man who persuaded Newton to publish his famous *Principia*.

Here are some of Bishop Berkeley’s complaints about infinitesimals. (“Fluxion” was Newton’s term for a derivative.)

And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?

And of the aforesaid fluxions there be other fluxions, which fluxions of fluxions are called second fluxions. And the fluxions of these second fluxions are called third fluxions: and so on, fourth, fifth, sixth, etc., *ad infinitum*. Now, as our Sense is strained and puzzled with the perception of objects extremely minute, even so the Imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particle of time, or the least increment generated therein: and much more to comprehend the

moments, or those increments of the flowing quantities in *status nascenti*, in their first origin or beginning to exist, before they become finite particles. And it seems still more difficult to conceive the abstracted velocities of such nascent imperfect entities. But the velocities of the velocities, the second, third, fourth, and fifth velocities, etc., exceed, if I mistake not, all human understanding. The further the mind analyseth and pursueth these fugitive ideas the more it is lost and bewildered; the objects, at first fleeting and minute, soon vanishing out of sight. Certainly, in any sense, a second or third fluxion seems an obscure Mystery. The incipient celerity of an incipient celerity, the nascent augment of a nascent augment, i. e. of a thing which hath no magnitude; take it in what light you please, the clear conception of it will, if I mistake not, be found impossible; whether it be so or no I appeal to the trial of every thinking reader. And if a second fluxion be inconceivable, what are we to think of third, fourth, fifth fluxions, and so on without end.

He who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in Divinity.

Johann Bernoulli, a Swiss mathematician who did pioneering work in developing calculus, expressed the paradox of infinitesimals crisply. They are so tiny, he said, that “if a quantity is increased or decreased by an infinitesimal, then that quantity is neither increased nor decreased.”

For two centuries most mathematicians agreed with Berkeley and refused to use the term. You won't find it in *Calculus Made Easy*. Bertrand Russell, in *Principles of Mathematics* (1903, Chapters 39 and 40) has a vigorous attack on infinitesimals. He calls them “mathematically useless,” “unnecessary, erroneous, and self-contradictory.” As late as 1941 the noted mathematician Richard Courant

wrote: “Infinitely small quantities are now definitely and dishonorably discarded.” Like Russell and others, he believed that calculus should replace infinitesimals by the concept of limits.

Charles Peirce (1839-1914), America’s great mathematician and philosopher, and friend of William James, strongly disagreed. He was almost alone in his day in siding with Leibniz, who believed that infinitesimals were as real and as legitimate as imaginary numbers. Here are some typical remarks by Peirce that I found by checking “infinitesimal” in the indexes of the volumes that make up Peirce’s *Collected Papers* and his *New Elements of Mathematics*.

Infinitesimals may exist and be highly important for philosophy, as I believe they are.

The doctrine of infinitesimals is far simpler than the doctrine of limits.

Is it consistent ... freely to admit of imaginaries while rejecting infinitesimals as inconceivable?

Infinitesimals, in the strict and literal sense, are perfectly intelligible, contrary to the teaching of the great body of modern textbooks on the calculus.

There is nothing contradictory about the idea of such quantities. ... As a mathematician, I prefer the method of infinitesimals to that of limits, as far easier and less infested with snares.

Peirce would have been delighted had he lived to see the work of Abraham Robinson, of Yale University. In 1960, to the vast surprise of mathematicians everywhere, Robinson found a way to reintroduce Leibniz’s infinitesimals as legitimate, precisely defined mathematical entities! His way of using them in calculus is known as “nonstandard analysis.” (Analysis is a term applied to calculus and all higher mathematics that use calculus.) Nonstandard analysis has produced simpler solutions than standard analysis to many calculus problems, and of course it is closer to an intuitive

way of interpreting infinite converging series. Robinson's achievement is too difficult to go into here, but you will find a good introduction to it in "Nonstandard Analysis," by Martin Davis and Reuben Hersh, in *Scientific American*, June 1972.

Mathematician and science fiction writer Rudy Rucker, in his book *Infinity and the Mind* (1982) vigorously defends infinitesimals:

So great is the average person's fear of infinity that to this day calculus all over the world is being taught as a study of *limit processes* instead of what it really is: *infinitesimal analysis*.

As someone who has spent a good portion of his adult life teaching calculus courses for a living, I can tell you how weary one gets of trying to explain the complex and fiddling theory of limits to wave after wave of uncomprehending freshmen. ...

But there is hope for a brighter future. Robinson's investigations of the hyperreal numbers have put infinitesimals on a logically unimpeachable basis, and here and there calculus texts based on infinitesimals have appeared.

Which is preferable? To talk about quantities so infinitely small that you can, as Thompson says, "throw them away," or to talk of values approaching a limit? Debate over the infinitesimal versus the limit language goes nowhere because they are two ways of saying the same thing. It's like choosing between calling a triangle a polygon with three sides or a polygon with three angles. Calculations in differentiating or integrating are exactly the same regardless of your preference for how to describe what you are doing. Now that infinitesimals have become respectable again, thanks to nonstandard analysis, you needn't hesitate, if you like, to use the term.

You might suppose that if the terms of an infinite series get

smaller and smaller, the series must converge. This is far from true. The most famous example is $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$. Known as the “harmonic series,” it has countless applications in physics as well as in mathematics. Although its fractions get progressively smaller, converging on zero, its partial sums grow without limit! The growth is infuriatingly slow. After a hundred terms the partial sum is only a bit higher than 5. To reach a sum of 100 requires more than 10^{43} terms!

If we eliminate all terms in the harmonic series that have even denominators, will it converge? Amazingly, it will not, though its rate of growth is much slower. If we eliminate from the series all terms whose denominators contain a specific digit one or more times, the series *will* then converge. The following table gives to two decimal places the limit for each omitted digit:

<i>Omitted digit</i>	<i>Sum</i>
1	16.18
2	19.26
3	20.57
4	21.33
5	21.83
6	22.21
7	22.49
8	22.73
9	22.92
0	23.10

Limits of infinite series can be expressed by unending decimal fractions. For example, $.33333\dots$ is the limit of the series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots = \frac{1}{3}$. Incidentally, there is a ridiculously easy way to determine the integral limit of any repeating decimal. The trick is to divide the repetend (the repeated sequence of digits) by a number consisting of the same number of nines as there are digits in the repetend.

Thus .3333 ... reduces to $\frac{3}{9} = \frac{1}{3}$. If the repeating decimal is, say, .123123123. ... the limit is $\frac{123}{999}$ which reduces to $\frac{41}{333}$.

Irrational numbers such as irrational roots, and transcendental numbers such as pi and e, are limits of many infinite series. Pi, for example, is the limit of such highly patterned series as: $\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots$. The number e (you will encounter it in Thompson's [Chapter 14](#)) is the limit of $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$.

Although Archimedes did not know calculus, he anticipated integration by calculating pi as the limit of the perimeters of regular polygons as their number of sides increases. In the language of infinitesimals, a circle can be viewed as the perimeter of a regular polygon with an infinity of sides, its perimeter consisting of an infinity of straight line segments each of infinitesimal length.

Many ingenious techniques have been found for determining if an infinite series converges or diverges, as well as ways, sometimes not easy, of finding the limit. If the terms of a series decrease in a geometric progression (each term is the same fraction of the preceding one) finding the limit is easy. Here is how it works on the halving series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$. Let x equal the entire series. Multiply each side of the equation by 2:

$$2x = 2 + \frac{2}{2} + \frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \dots$$

Reduce the terms:

$$2x = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Note that the series beyond 2 is the same as the original halving series which we took as x. This enables us to substitute x for the sequence and write $2x = 2 + x$. Rearranging terms to $2x - x = 2$ gives x, the limit of the series, a value of 2.

The same trick will show that $\frac{1}{2}$ is the limit of

$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$; it works on any series in which terms decrease in geometric progression.

Bouncing ball problems are common in the literature on limits. They assume that an ideally elastic ball is dropped a specified distance to a hard floor. After each bounce it rises a constant fraction of the previous height. Here is a typical example.

The ball is dropped from a height of four feet. Each bounce takes it to $\frac{3}{4}$ the previous height. In practice, of course, a rubber ball bounces only a finite number of times, but the idealized ball bounces an infinite number of times. The rises approach zero as a limit, but because the times of each bounce also approach a limit of zero, the ball (like Zeno's runner) finally reaches the limit. After an infinity of bounces, it comes to rest after a finite period of time. When the ball ceases to bounce, how far has it traveled?

We can solve this problem by using the same trick used on the halving series. Ignoring for a moment the initial drop of four feet, the ball will rise three feet then fall three feet for a total of six feet. After that, each bounce (rise plus fall) is three-fourths the previous bounce. Letting x be the total distance the ball travels after the first drop of four feet, we write the equation:

$$x = 6 + \frac{18}{4} + \frac{54}{16} + \frac{162}{64} + \frac{486}{256} + \dots$$

Reducing the fractions:

$$x = 6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \frac{243}{128} + \dots$$

Because each term is $\frac{4}{3}$ its following term, we multiply each side by $\frac{4}{3}$ to get:

$$\frac{4x}{3} = 8 + 6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \dots$$

Observe that after 8 the sequence is the same as x , so we can substitute x for it:

$$\frac{4x}{3} = 8 + x$$

$$4x = 24 + 3x$$

$$x = 24$$

This is the distance the ball bounces after the initial drop of 4 feet. The total distance traveled by the ball is $24 + 4 = 28$ feet.

Sam Loyd, America's great puzzlemaker, in his *Cyclopedia of Puzzles* (p. 23), and his British counterpart Henry Ernest Dudeney, in *Puzzles and Curious Problems* (Problem 223), each give the following ball bounce problem. A ball is dropped 179 feet from the Tower of Pisa. Each bounce is one-tenth the height of the previous bounce. How far does the ball travel after an infinity of bounces before it finally comes to rest? (See [Figure 7](#).)

We can solve this by the trick used before, but because each fraction is one-tenth the previous one, we can find the answer by an even faster method.

After the initial drop of 179 feet, the height of the first bounce is 17.9. Succeeding bounces have heights of 1.79, .179, .0179, and so on, with the decimal point moving one position left after each bounce. Adding these heights gives a total of 19.8888. ... We now double this distance to obtain the up plus down distances for each bounce to get 39.7777. ... Finally, we add the initial drop of 179 feet to obtain the total distance the ball travels: 218.7777 ..., or exactly 218 and $\frac{7}{9}$ feet.



FIG. 7. Sam Loyd's bouncing ball puzzle.

Converging series that do not decrease by a geometric progression often can be solved by other clever methods. Here is an interesting example.

$$x = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \frac{11}{32} + \dots$$

Note that the numerators are odd numbers in sequence, and the denominators are a doubling series. Here is a simple way to find the limit.

First divide each term by 2:

$$\frac{x}{2} = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \dots$$

Subtract this sequence from the original sequence:

$$\begin{array}{r} x = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \dots \\ \frac{x}{2} = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \dots \\ \hline \frac{x}{2} = 1 + \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right] \end{array}$$

Observe that after the 1 inside the brackets, the sequence that follows is our old friend the halving series which we know converges on 1. Adding 2 to the initial 1 gives the series a limit of 3. Since 3 is half of x , x must be 6, the limit of the original series.

Thompson does not spend much time on series and their limits. I have done so in this chapter for two reasons: they are the best way to become comfortable with the limit concept, and modern calculus textbooks now usually include chapters on infinite series and their usefulness in many aspects of calculus.

*On infinity machines, see “Alephs and Supertasks,” Chapter 4, in my *Wheels, Life, and Other Mathematical Amusements* (W. H. Freeman, 1983), and the references cited in that chapter’s bibliography.—M.G.

Preliminary Chapter 3

WHAT IS A DERIVATIVE?



In [Chapter 3](#) Thompson makes crystal clear what a derivative is, and how to calculate it. However, it seemed to me useful to make a few introductory remarks about derivatives that may make Thompson's chapter even easier to understand.

Let's start with Zeno's runner. Assume that he runs ten meters per second on a path from zero to 100 meters. The independent variable is time, represented by the x axis of a Cartesian graph. The dependent variable y is the runner's distance from his starting spot. It is represented on the y axis. Because the function is linear, the runner's motion graphs as an upward tilted straight line from zero, the graph's origin, to the point that is ten seconds on the time axis and 100 meters on the distance axis. ([Figure 8](#)) If by distance we mean distance *from* the goal, the line on the graph tilts the other way ([Figure 9](#)).

Given any point in time, how fast is the runner moving? Because we are dealing with a simple linear function we don't need calculus to tell us that at every instant he is going ten meters per second. The function's equation is $y = 10x$. Note that the slope of the line on the graph, as measured by the height in meters at any point divided by the elapsed time in seconds at any point, is 10. At each instant the runner has gone in meters ten times the elapsed number of seconds. His instantaneous speed throughout the run clearly is ten meters per second.

Consider any moment of time along the x axis, then go vertically up the graph to the distance traveled in meters. You will find that the distance is always ten times the elapsed time. As you will learn when you read *Calculus Made Easy*, the derivative of a function is simply another function that describes the rate at which a dependent variable changes with respect to the rate at which the independent variable changes. In this case the runner's speed never changes, so the derivative of $y = 10x$ is simply the number 10. It tells you two things: (1) that at any time the runner's speed is ten meters per second, and (2) that at any point on the line that graphs this function, the slope of the line is 10. This generalizes to all linear functions in which the variable y changes with respect to variable x at a constant rate. If a function is $y = ax$, its derivative is simply the constant a .

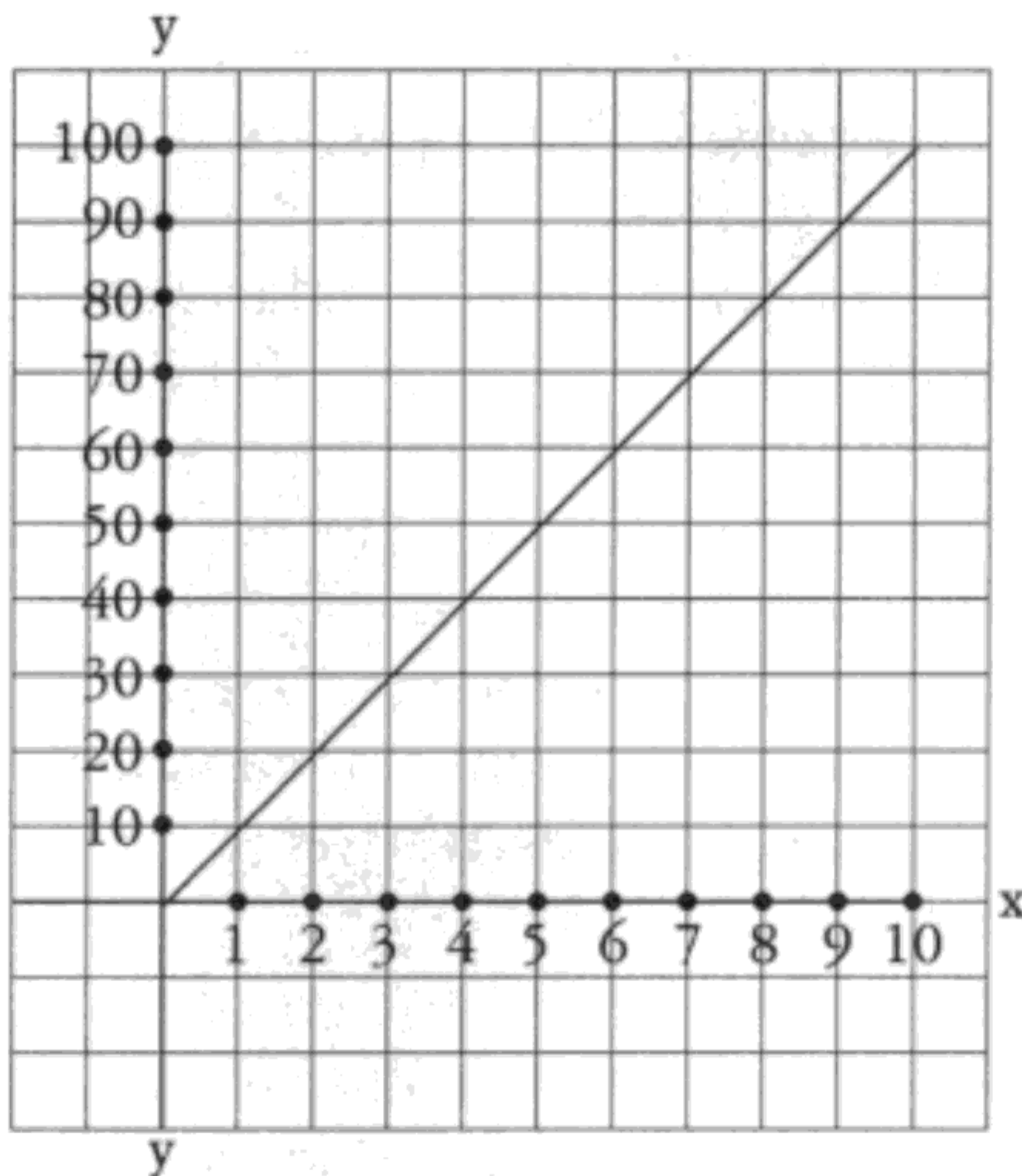


FIG. 8. Graph of Zeno's runner.

The x axis is time, the y axis is the distance from the start of the run.

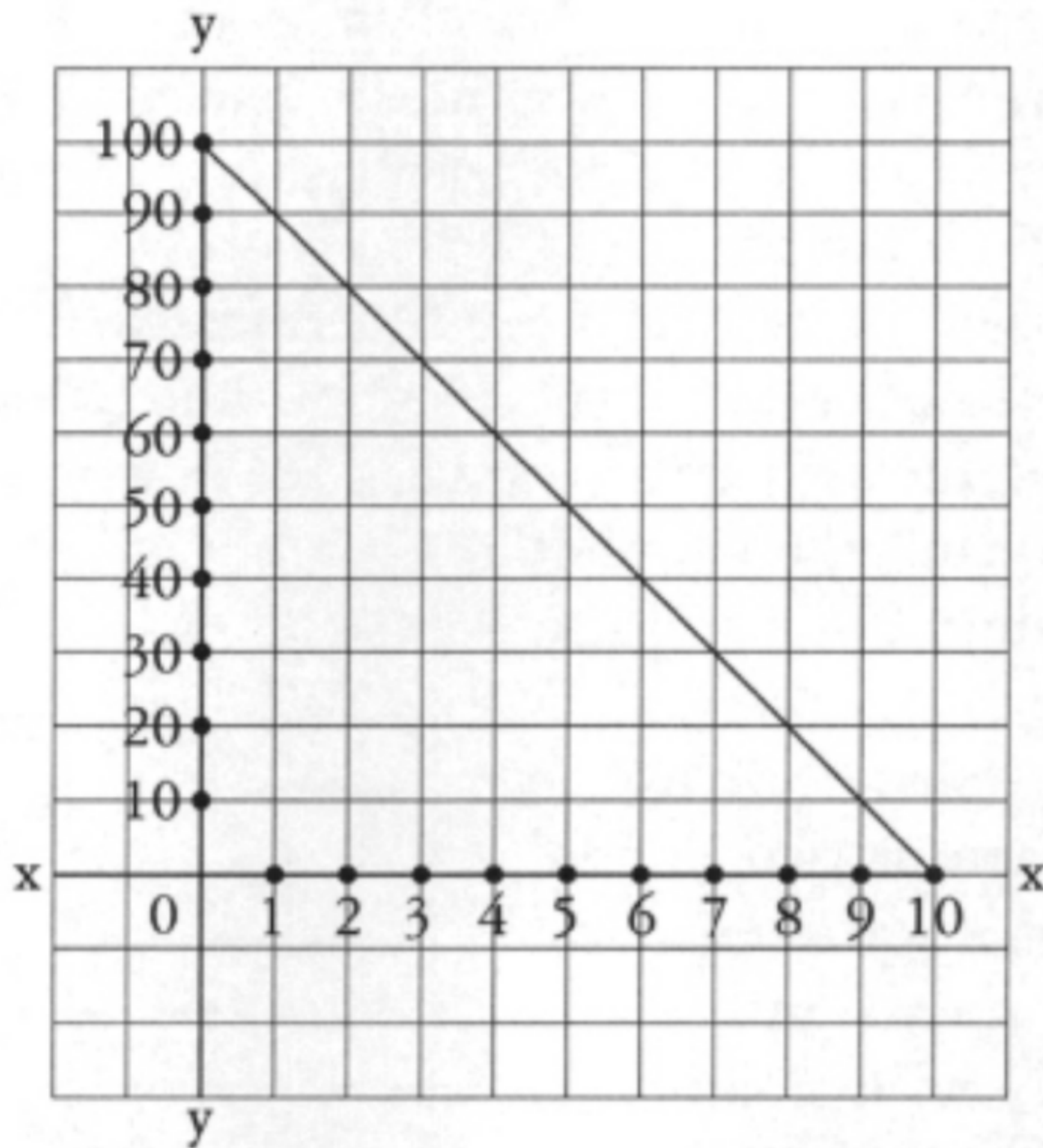


FIG. 9. Graph of Zeno's runner showing distance from goal.
The equation is $y = 10(10 - x)$.

As I said, you don't need calculus to tell you all this, but it is good to know that calculating derivatives gives the correct result even when functions are linear.

An even simpler case of a derivative, too obvious to require any thought, let alone demanding calculus, is the case of a runner who stands perfectly still. Let's say he stops running after going ten meters. The function is $y = 10$. The graph becomes a horizontal straight line as shown in [Figure 10](#). Its slope is zero which is the same as saying that the rate at which the runner's distance from the start changes, relative to changes of time, is zero. The function's derivative is zero. Even in this extreme case it is comforting to know that calculus still applies. In general, the derivative of any constant is zero.

Calculus ceases to be trivial when functions are nonlinear. Consider the simple nonlinear function $y = x^2$, which

Thompson uses to open his chapter on derivatives. Let's see how it applies to the growth of a square, the simplest geometrical interpretation of this function.

Imagine a monster living on Flatland, a plane of two dimensions. It is born a square of side 1 and area 1, then grows at a steady rate. We wish to know, at any instant of time, how fast its area grows with respect to the growth of its side.

The monster's area, of course, is the square of its side, so the function we have to consider is $y = x^2$, where y is the area and x the side. (It graphs as the parabola shown in [Figure 1](#) of the first preliminary chapter.) As you will learn from Thompson, the function's derivative is $2x$. What does this tell us? It tells us that at any given moment the monster's area is growing at a rate that is $2x$ times as fast as its side is lengthening.

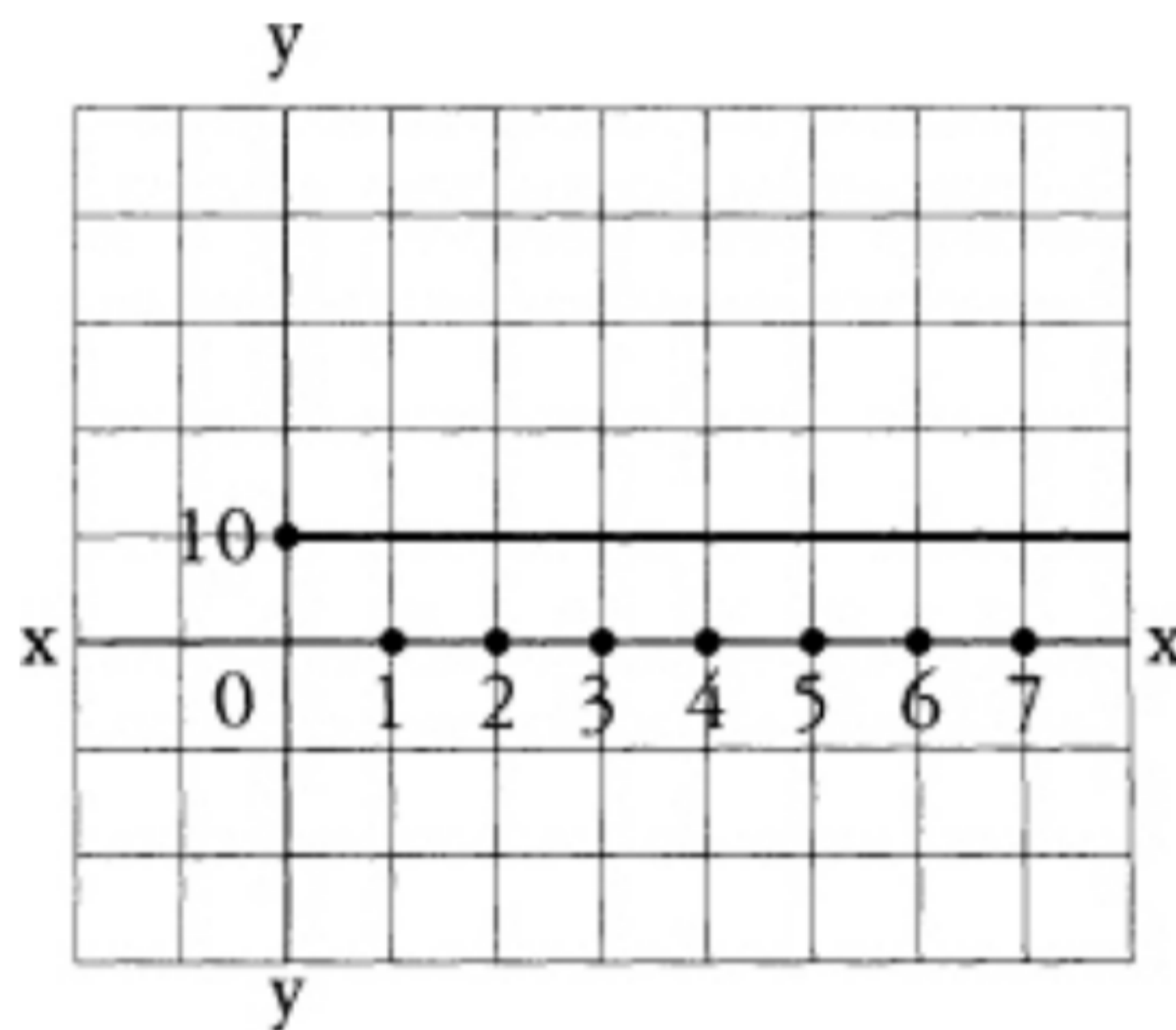


FIG. 10. Graph of a runner who stands still at a distance of ten units from the start.

For example, let's say the monster's side is growing at a rate of 3 units per second. Starting with a side of one unit, at the end of ten seconds its side will have reached 31 units. The value of x at this point is 31. The derivative says that when the monster's side is 31, its area is increasing with respect to its side at a rate of $2x$, or $2 \times 31 = 62$ units.

When the square reaches 100 on its side, its area will be increasing with respect to its side by $2 \times 100 = 200$ units.

These figures express the rate of the square's growth with respect to its side. For the square's growth rate with respect to *time* we have to multiply these values by 3. Thus when the square has a side of 31 (after ten seconds), it is growing at a rate of $3 \times 2 \times 31 = 186$ square units per second. When the side is 100, its growth rate per second is $3 \times 2 \times 100 = 600$.

Suppose the monster is a cube of edge x which increases at a steady rate of 2 units per second. The cube's volume, y , is x^3 . The derivative of the function $y = x^3$ is $3x^2$. This tells you that the cube's volume in cubical units grows $3x^2$ times as fast as its edge grows. Thus when the cube's side reaches, say, 10, the value of x , its volume, is growing $3 \times 10^2 = 300$ square inches as fast as its side. Its growth rate per second is $2 \times 3 \times 10^2 = 600$.

Although Thompson avoids defining a derivative as the limit of a sequence of ratios, this clearly is the case. Suppose, for instance, that our growing square has sides that increase at one unit per second. We can tabulate the area's growth at times slightly larger than 2 seconds as follows:

<i>Time</i>	<i>Side</i>	<i>Area</i>
2	3	9
2.1	3.1	9.61
2.01	3.01	9.0601
2.001	3.001	9.006001

The average rate of growth from time 2 to time 2.1 is:

$$\frac{9.61 - 9}{2.1 - 2} = 6.1$$

And from time 2 to time 2.01:

$$\frac{9.0601 - 9}{2.01 - 2} = 6.01$$

And from time 2 to time 2.001:

$$\frac{9.006001 - 9}{2.001 - 2} = 6.001$$

The averages obviously approach a limit of 6. Thus the derivative of the area with respect to time is the limit of an infinite sequence of ratios that converge on 6. Put simply, a derivative is the rate at which a function's dependent variable grows with respect to the growth rate of the independent variable. In geometrical terms, it determines the exact slope of the tangent to a function's curve at any specified point along the curve. This equivalence of the algebraic and geometrical definitions of a derivative is one of the most beautiful aspects of calculus.

I hope this and the previous two preliminary chapters will help prepare you for understanding *Calculus Made Easy*.

CALCULUS MADE EASY

What one fool can do, another can.

—*Ancient Simian proverb*

PUBLISHER'S NOTE ON THE THIRD

EDITION



Only once in its long and useful life in 1919, has this book been enlarged and revised. But in twenty-six years much progress can be made, and the methods of 1919 are not likely to be the same as those of 1945. If, therefore, any book is to maintain its usefulness, it is essential that it should be overhauled occasionally so that it may be brought up-to-date where possible, to keep pace with the forward march of scientific development.

For the new edition the book has been reset, and the diagrams modernised. Mr. F. G. W. Brown has been good enough to revise the whole of the book, but he has taken great care not to interfere with the original plan. Thus teachers and students will still recognise their well-known guide to the intricacies of the calculus. While the changes made are not of a major kind, yet their significance may not be inconsiderable. There seems no reason now, even if one ever existed, for excluding from the scope of the text those intensely practical functions, known as the hyperbolic sine, cosine and tangent, whose applications to the methods of integration are so potent and manifold. These have, accordingly, been introduced and applied, with the result that some of the long cumbersome methods of integrating have been displaced, just as a ray of sunshine dispels an obstructing cloud.

The introduction, too, of the very practical integrals:

$$\int e^{pt} \sin kt \cdot dt \quad \text{and} \quad \int e^{pt} \cos kt \cdot dt$$

has eliminated some of the more ancient methods of “Finding Solutions” ([Chapter XXI](#)). By their application, shorter and more intelligible ones have grown up naturally instead.

In the treatment of substitutions, the whole text has been tidied up in order to render it methodically consistent. A few examples have also been added where space permitted, while the whole of the exercises and their answers have been carefully revised, checked and corrected. Duplicated problems have thus been removed and many hints provided in the answers adapted to the newer and more modern methods introduced.

It must, however, be emphatically stated that the plan of the original author remains unchanged; even in its more modern form, the book still remains a monument to the skill and the courage of the late Professor Silvanus P. Thompson. All that the present reviser has attempted is to revitalize the usefulness of the work by adapting its distinctive utilitarian bias more closely in relation to present-day requirements.

PROLOGUE



Considering how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

COMMON GREEK LETTERS USED AS SYMBOLS

Capital Small English Name

<i>A</i>	α	Alpha
<i>B</i>	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
<i>E</i>	ϵ	Epsilon
<i>H</i>	η	Eta
Θ	θ	Thēta
<i>K</i>	κ	Kappa

Λ	λ	Lambda
M	μ	Mu
Ξ	ξ	Xi
Π	π	Pi
ρ	ρ	Rho
Σ	σ	Sigma
Φ	φ	Phi
Ω	ω	Omega

Chapter I

TO DELIVER YOU FROM THE PRELIMINARY TERRORS



The preliminary terror, which chokes off most high school students from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means “a little bit of”.

Thus dx means a little bit of x ; or du means a little bit of u . Ordinary mathematicians think it more polite to say “an element of”, instead of “a little bit of”. Just as you please. But you will find that these little bits (or elements) may be considered to be infinitely small.

(2) \int which is merely a long S , and may be called (if you like) “the sum of”.

Thus $\int dx$ means the sum of all the little bits of x ; or $\int dt$ means the sum of all the little bits of t . Ordinary mathematicians call this symbol “the integral of”. Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx , if you add them all up together you get the sum of all the dx 's (which is the same thing as the whole of x). The word “integral” simply means “the whole”. If you think of the duration of time for one

hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

Chapter II

ON DIFFERENT DEGREES OF SMALLNESS



We shall find that in our processes of calculation we have to deal with small quantities of various degrees of smallness.

We shall have also to learn under what circumstances we may consider small quantities to be so minute that we may omit them from consideration. Everything depends upon relative minuteness.

Before we fix any rules let us think of some familiar cases. There are 60 minutes in the hour, 24 hours in the day, 7 days in the week. There are therefore 1440 minutes in the day and 10,080 minutes in the week.

Obviously 1 minute is a very small quantity of time compared with a whole week. Indeed, our forefathers considered it small as compared with an hour, and called it “one minute”, meaning a minute fraction—namely one sixtieth—of an hour. When they came to require still smaller subdivisions of time, they divided each minute into 60 still smaller parts, which, in Queen Elizabeth’s days, they called “second minutes” (*i.e.* small quantities of the second order of minuteness). Nowadays we call these small quantities of the second order of smallness “seconds”. But few people know *why* they are so called.

Now if one minute is so small as compared with a whole

day, how much smaller by comparison is one second!

Again, think of a hundred dollars compared with a penny: it is worth only a $\frac{1}{100}$ part. A penny is of precious little importance compared with a hundred dollars: it may certainly be regarded as a *small* quantity. But compare a penny with ten thousand dollars: relative to this greater sum, a penny is of no more importance than a hundredth of a penny would be to a hundred dollars. Even a hundred dollars is relatively a negligible quantity in the wealth of a millionaire.

Now if we fix upon any numerical fraction as constituting the proportion which for any purpose we call relatively small, we can easily state other fractions of a higher degree of smallness. Thus if, for the purpose of time, $\frac{1}{60}$ be called a *small* fraction, then $\frac{1}{60}$ of $\frac{1}{60}$ (being a *small* fraction of a *small* fraction) may be regarded as a *small quantity of the second order of smallness*.*

Or, if for any purpose we were to take 1 percent (*i.e.* $\frac{1}{100}$) as a *small* fraction then 1 percent of 1 percent (*i.e.* $\frac{1}{10,000}$) would be a small fraction of the second order of smallness; and $\frac{1}{1,000,000}$ would be a small fraction of the third order of smallness, being 1 percent of 1 percent of 1 percent.

Lastly, suppose that for some very precise purpose we should regard $\frac{1}{1,000,000}$ as “small”. Thus, if a first-rate chronometer is not to lose or gain more than half a minute in a year, it must keep time with an accuracy of 1 part in 1,051,200. Now if, for such a purpose, we regard $\frac{1}{1,000,000}$ (or one millionth) as a small quantity, then $\frac{1}{1,000,000}$ of $\frac{1}{1,000,000}$, that is, $\frac{1}{1,000,000,000,000}$ will be a small quantity of the second order of smallness, and may be utterly disregarded, by comparison.

Then we see that the smaller a small quantity itself is, the more negligible does the corresponding small quantity of the second order become. Hence we know that *in all cases we are justified in neglecting the small quantities of the second—or third*

(or higher)—*orders*, if only we take the small quantity of the first order small enough in itself.

But it must be remembered that small quantities, if they occur in our expressions as factors multiplied by some other factor, may become important if the other factor is itself large. Even a penny becomes important if only it is multiplied by a few hundred.

Now in the calculus we write dx for a little bit of x . These things such as dx , and du , and dy , are called “differentials”, the differential of x , or of u , or of y , as the case may be. [You read them as *dee-eks*, or *dee-you*, or *dee-wy*.] If dx be a small bit of x , and relatively small of itself, it does not follow that such quantities as $x \cdot dx$, or $x^2 dx$, or $a^x dx$ are negligible. But $dx \times dx$ would be negligible, being a small quantity of the second order.

A very simple example will serve as illustration. Consider the function $f(x) = x^2$

Let us think of x as a quantity that can grow by a small amount so as to become $x + dx$, where dx is the small increment added by growth. The square of this is $x^2 + 2x \cdot dx + (dx)^2$. The second term is not negligible because it is a first-order quantity; while the third term is of the second order of smallness, being a bit of a bit of x^2 . Thus if we took dx to mean numerically, say, $\frac{1}{60}$ of x , then the second term would be $\frac{2}{60}$ of x^2 , whereas the third term would be $\frac{1}{3,600}$ of x^2 . This last term is clearly less important than the second. But if we go further and take dx to mean only $\frac{1}{1000}$ of x , then the second term will be $\frac{2}{1000}$ of x^2 , while the third term will be only $\frac{1}{1,000,000}$ of x^2 .

Geometrically this may be depicted as follows: Draw a square (Fig. 1) the side of which we will take to represent x . Now suppose the square to grow by having a bit dx added to its size each way. The enlarged square is made up of the original square x^2 , the two rectangles at the top and on the

right, each of which is of area $x \cdot dx$ (or together $2x \cdot dx$), and a little square at the top right-hand corner which is $(dx)^2$. In [Fig. 2](#) we have taken dx as quite a big fraction of x —about $\frac{1}{5}$. But suppose we had taken it only $\frac{1}{100}$ —about the thickness of an inked line drawn with a fine pen (See [Figure 3](#)). Then the little corner square will have an area of only $\frac{1}{10,000}$ of x^2 , and be practically invisible. Clearly $(dx)^2$ is negligible if only we consider the increment dx to be itself small enough.

Let us consider a simile.

Suppose a millionaire were to say to his secretary: next week I will give you a small fraction of any money that comes in to me. Suppose that the secretary were to say to his boy: I will give you a small fraction of what I get. Suppose the fraction in each case to be $\frac{1}{100}$ part. Now if Mr. Millionaire received during the next week \$1,000, the secretary would receive \$10 and the boy 1 dime. Ten dollars would be a small quantity compared with \$1,000; but a dime is a small small quantity indeed, of a very secondary order. But what would be the disproportion if the fraction, instead of being $\frac{1}{100}$, had been settled at $\frac{1}{1000}$ part? Then, while Mr. Millionaire got his \$1,000, Mr. Secretary would get only \$1.00, and the boy only a tenth of a penny!

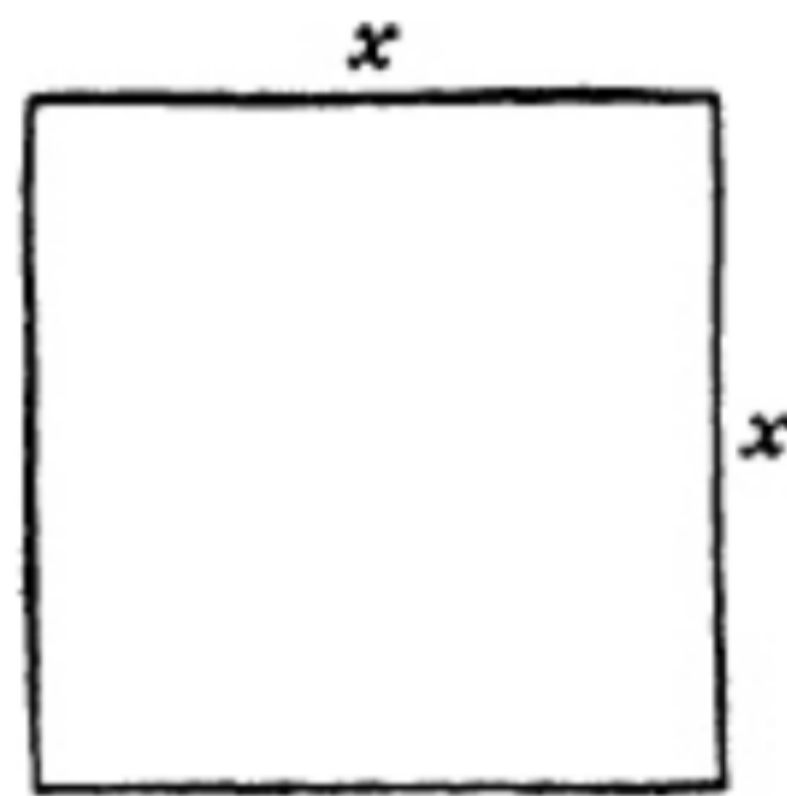


FIG. 1.

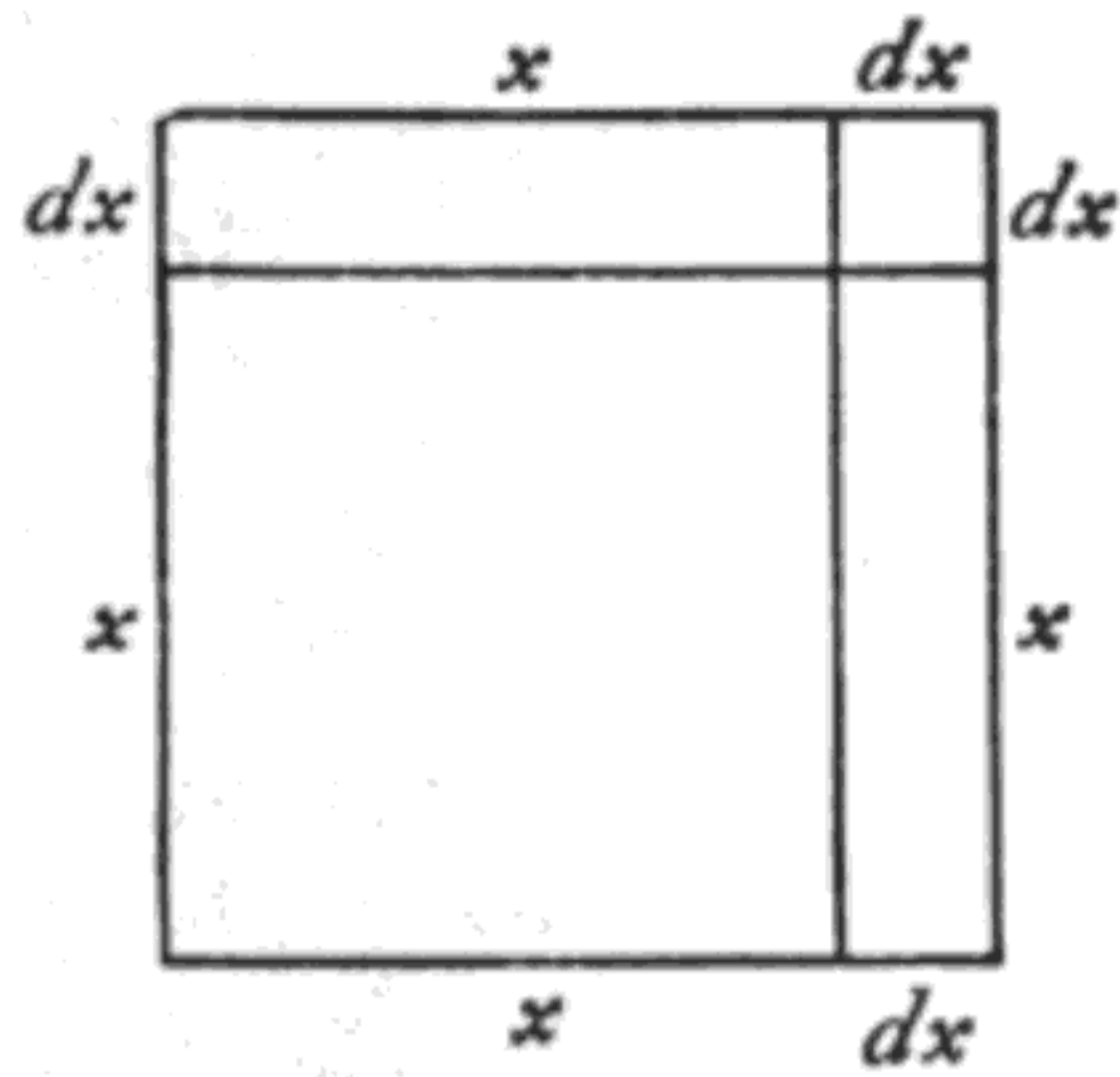


FIG. 2.

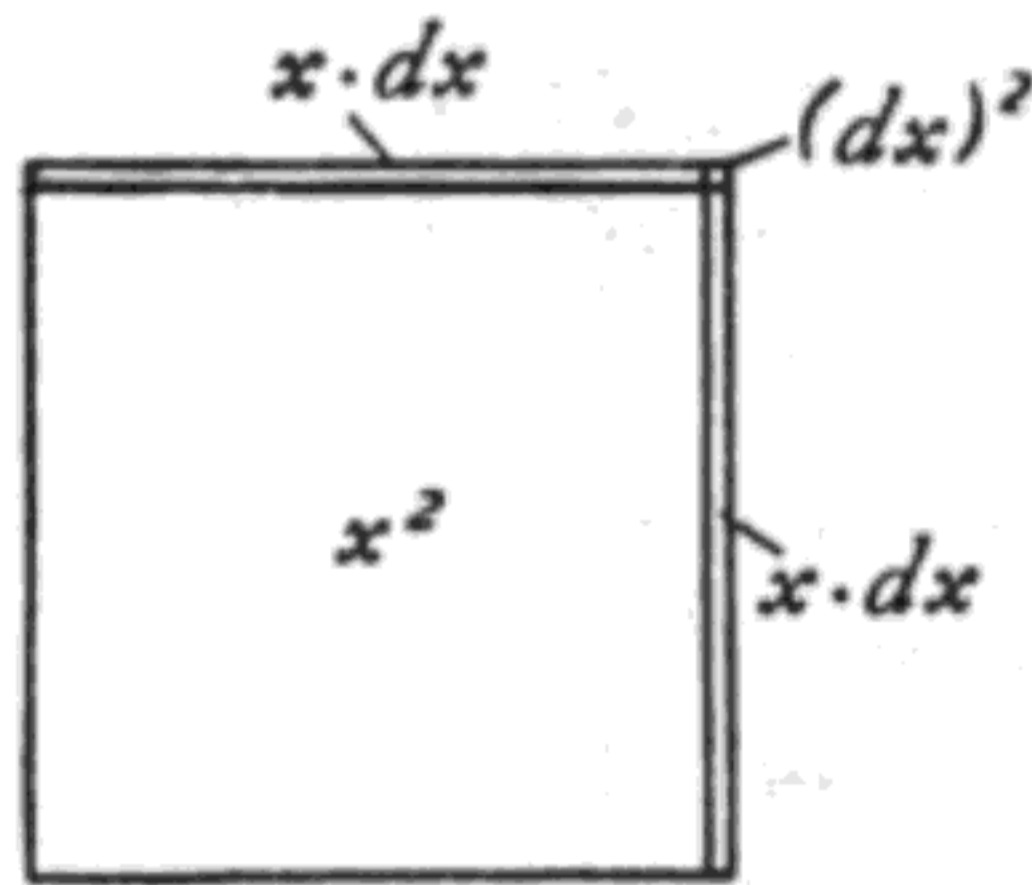


FIG. 3.

The witty Dean Swift once wrote:

So, Nat'ralists observe, a Flea
 Hath smaller Fleas that on him prey.
 And these have smaller Fleas to bite 'em.
 And so proceed *ad infinitum*.

An ox might worry about a flea of ordinary size—a small creature of the first order of smallness. But he would probably not trouble himself about a flea's flea; being of the second order of smallness, it would be negligible. Even a gross of fleas' fleas would not be of much account to the ox.

*The mathematicians may talk about the second order of "magnitude" (*i.e.* greatness) when they really mean second order of *smallness*. This is very

confusing to beginners.

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