Category Theory and Applications

A Textbook for Beginners

Marco Grandis

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Introduction

0.1 Categories

Category Theory originated in an article of Eilenberg and Mac Lane [EiM], published in 1945. It has now developed into a branch of mathematics, with its own internal dynamics. Here we want to present its elementary part, and its strong links with the origins in Algebra and Topology.

Many mathematical theories deal with a certain kind of mathematical objects, like groups, or ordered sets, or topological spaces, etc. Each kind has its own privileged mappings, that preserve the structure in some sense, like homomorphisms of groups, or order-preserving mappings, or continuous mappings.

We have thus the 'category' Gp of groups and their homomorphisms, Ord of ordered sets and monotone mappings, Top of topological spaces and continuous mappings, and so on. More elementarily, we have the category Set of sets and their mappings.

In all these instances, the privileged mappings are called morphisms or arrows of the category; an arrow from the object X to the object Y is written as $f\colon X \to Y$.

Two consecutive morphisms, say $f\colon X \to Y$ and $g\colon Y \to Z$, can be composed giving a morphism $gf\colon X \to Z$; this partial composition law is 'as regular as it can be', which means that it is associative (when legitimate) and has a partial identity $\mathrm{id}X\colon X \to X$ (written also as 1_X) for every object, which acts as a unit for every legitimate composition.

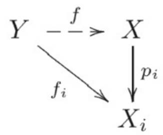
'Concrete categories' are often associated with mathematical structures, in this way; but we shall see that categories are not limited to these instances, by far.

0.2 Universal properties

In these categories (and many 'similar' ones) we have cartesian products, constructed by forming the cartesian product of the underlying sets and putting on it the 'natural' structure of the kind we are considering, be it of algebraic character, or an ordering, or a topology, or something else.

All these procedures can be unified, so that we can better understand what we are doing: we have a family $(X_i)_{i \in I}$ of objects of a category, indexed by a set I, and we want to find an object X equipped with a family of morphisms $p_i \colon X \to X_i$ $(i \in I)$, called *cartesian projections*, which satisfies the following *universal property*:

- for every object Y and every family of morphisms $f_i \colon Y \to X_i$ $(i \in I)$ in the given category



(0.1)

there exists precisely one morphism $f \colon Y \to X$ such that $p_i f = f_i$, for all $i \in I$.

We shall see that this property determines the solution *up to isomorphism*, i.e. an invertible morphism of the given category. In fact, the proof is quite easy and some hints can be useful as of now: given two solutions $(X, (p_i))$, $(Y, (q_i))$, we can determine two morphisms $f \colon X \to Y$ and $g \colon Y \to X$ and prove that their composites coincide with the identities of X and Y.

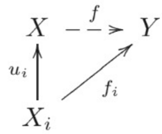
(Let us remark, incidentally, that the product topology is much less obvious than the other product structures we have mentioned; the universal property tells us that it is indeed the 'right' choice.)

These facts bring to light a crucial aspect: a categorical definition (as the previous one) *is based on morphisms and their composition*, while the objects only step in as domains and codomains of arrows. If we want to understand what unifies the product of a family (X_i) of sets, or groups, or ordered sets, or topological spaces we should forget the nature of the objects, and think of the family of cartesian projections $p_i \colon X \to X_i$, together with the previous property. Then – in each category we are interested in – we come back to the objects in order to prove that a solution exists (and also to fix a particular solution, when this can be useful).

From a structural point of view, a category only 'knows' its objects by their morphisms and composition.

This is even more evident if we think of another procedure, which in category theory is called a 'sum', or 'coproduct'. We start again from a family $(X_i)_{i \in I}$ of objects; its (categorical) *sum* is an object X equipped with a family of morphisms $u_i \colon X_i \to X$ $(i \in I)$, called *injections*, which satisfies the following universal property:

- for every object Y and every family of morphisms $f_i \colon X_i \to Y$ $(i \in I)$ in the given category



(0.2)

there exists precisely one morphism $f \colon X \to Y$ such that $fu_i = f_i$, for all $i \in I$.

Again, the solution is determined up to isomorphism; its existence depends on the category.

The sum of a family of objects is easy to construct in the category Set of sets, by a disjoint union. We have similar solutions in Ord and Top. But in Gp the categorical sum of a family of groups is called the *free product* of the family; its construction is rather complex, and its underlying set has little to do with the disjoint union of the sets underlying the groups. The categorical approach highlights the fact that we are 'solving the same problem' and – finally – makes clear what we are doing.

It is also important to note that the categorical definitions of product and coproduct are *dual* to each other: each of them is obtained from the other by reversing all arrows and compositions. This only makes sense – in general – within category theory, because the dual of a given category, formed by reversing its arrows and partial composition law, is a formal construction: the dual of a category of structured sets and mappings is *not* a category of the same kind (even though, *in certain cases*, it may essentially be, as a result of some duality theorem).

0.3 Diagrams in a category

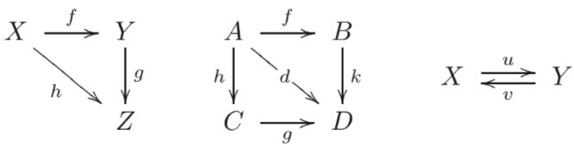
In a category the objects and morphisms that we are considering are often represented by vertices and arrows in a *diagram*, as above in (0.1) and (0.2), to make evident their relationship and which compositions are legitimate.

As an important property, satisfied in the previous cases, we say that a diagram in a category is *commutative* if:

(i) whenever we have two 'paths' of consecutive arrows, from a certain object to another, the two composed morphisms are the same,

(ii) whenever we have a 'loop' of consecutive arrows, from an object to itself, then the composed morphism is the identity of that object.

Let us consider some other instances



(0.3)

The first diagram above is commutative if and only if $gf = h$. For the second, commutativity means that $kf = d = gh$. For the third, it means that $vu = \mathrm{id}X$ and $uv = \mathrm{id}Y$; let us note that in this case the morphisms u and v are inverse to each other, and each of them is an isomorphism of our category.

Formal definitions of these notions, diagrams and commutative diagrams, will be given in 1.4.9.

0.4 Functors

It becomes now possible to view on the same level, so to say, mathematical theories of different branches, and formalise their links. A well-behaved mapping between categories is called a *functor*.

Among the simplest examples there are the *forgetful functors*, that forget the structure (or part of it), like:

$$\mathsf{Gp} \to \mathsf{Set}, \qquad \mathsf{Ord} \to \mathsf{Set}, \qquad \mathsf{Top} \to \mathsf{Set}, \qquad (0.4)$$

For instance, $U\colon \mathsf{Gp} \to \mathsf{Set}$ takes a group G to its underlying set $U(G)$, and a homomorphism $f\colon G \to G'$ to its underlying mapping $U(f)\colon U(G) \to U(G')$. The whole procedure is well-behaved, in the sense that it preserves composition and identities.

These functors are so obvious that they are often overlooked, in mathematics; but here it will be important to keep trace of them. In particular, we shall see that they often determine other functors 'backwards', which are much less obvious: like the *free-group* functor $F\colon \mathsf{Set} \to \mathsf{Gp}$, 'left adjoint' to $U\colon \mathsf{Gp} \to \mathsf{Set}$ (of which more will be said below).

In another range of examples, a reader with some knowledge of Algebraic Topology will know that the core of this discipline is constructing *functors from a category of topological spaces to a category of algebraic structures*, and using them to reduce topological problems to simpler algebraic ones. We have thus the sequence of *singular homology functors*, with values in the category of abelian groups

$$H_n\colon \mathsf{Top} \to \mathsf{Ab} \qquad (n \geqslant 0), \tag{0.5}$$

and the fundamental group functor $\pi_1\colon \mathsf{Top}_\bullet \to \mathsf{Gp}$, defined on the category of pointed topological spaces and pointed continuous mappings.

0.5 Natural transformations and adjunctions

The third basic element of category theory is a *natural transformation* $\varphi\colon F \to G$ between two functors $F, G\colon \mathsf{C} \to \mathsf{D}$ with the same domain C and the same codomain D. We also write $\varphi\colon F \to G\colon \mathsf{C} \to \mathsf{D}$.

This simply amounts to a family of morphisms of D, indexed by the objects X of C

$$\varphi_X\colon F(X) \to G(X), \tag{0.6}$$

under a condition of 'naturality' which will be made explicit in the text.

Here we just give an example, based on groups and free groups (but the reader can replace groups with semigroups, or abelian groups, or R-modules, or any algebraic structure defined by 'equational axioms'). We have mentioned in the previous point the forgetful functor $U\colon \mathsf{Gp} \to \mathsf{Set}$ and the free-group functor $F\colon \mathsf{Set} \to \mathsf{Gp}$, that turns a set X into the free group generated by X.

The insertion of X in $F(X)$, as its basis, is a canonical mapping in Set

$$\eta_X\colon X \to U(F(X)). \tag{0.7}$$

All of them give a natural transformation

$$\eta\colon \mathrm{id} \to UF\colon \mathsf{Set} \to \mathsf{Set}, \tag{0.8}$$

where id is the identity functor of the category Set (turning objects and arrows into themselves) and $UF\colon \mathsf{Set} \to \mathsf{Set}$ is the composed functor (turning each set into the underlying set of the free group on it). This natural transformation is essential in linking the functors U, F and making the functor $F\colon \mathsf{Set} \to \mathsf{Gp}$ *left adjoint to* $U\colon \mathsf{Gp} \to \mathsf{Set}$.

The link is represented by the *universal property* of the insertion of the basis, namely the fact that every mapping $f\colon X \to U(G)$ with values in a group G (more precisely in its underlying set) can be uniquely extended to a homomorphism $g\colon F(X) \to G$. Formally:

- for every morphism $f\colon X \to U(G)$ in Set, there exists precisely one morphism $g\colon F(X) \to G$ in Gp such that $U(g).\eta_X = f$.

One can reformulate this link in an equivalent presentation of the adjunction, which may be more familiar to the reader: there is a (natural) bijective correspondence

$$\begin{aligned} \varphi_{XG} \colon \mathsf{Gp}(F(X), G) &\to \mathsf{Set}(X, U(G)), \\ \varphi_{XG}(g) &= U(g).\eta_X, \end{aligned} \tag{0.9}$$

between the set of group-homomorphisms $F(X) \to G$ and the set of mappings $X \to U(G)$.

We shall see many constructions of free algebraic structures, or more generally of left (or right) adjoints of given functors (in particular in Chapters 3, 4 and 6). Many of them are 'real constructions', which give a good idea of the backward procedure; others are so complicated that one can doubt of their constructive character. In such cases, one can prefer to prove the existence of the adjoint, by the Adjoint Functor Theorem (in Section 3.5) or some other general statement: then the result is determined up to isomorphism, and its universal property allows its use. Much in the same way as we can define a real function as the solution of a certain differential equation with initial data, as soon as we know that the solution exists and is unique.

0.6 A brief outline

The first chapter deals with the basic tools: categories, functors and natural transformations. Ordered sets can be seen as categories of a special kind; adjunctions in this particular case are well known, as 'covariant Galois connections'; studying them (in Section 1.7) will also serve as an elementary introduction to general adjunctions.

Chapter 2 introduces the limits in a category (including products and the classical projective limits) and their dual notion, the colimits (including sums and the classical inductive limits); universal arrows with respect to a functor are a general formulation of universal properties (in Section 2.7).

Chapter 3 studies the crucial notion of adjunction: many important constructions in mathematics can be described as an adjoint functor of some obvious functor: from free algebraic structures to Stone-Čech compactification, metric completion etc. We also explore the related notion of monad and its algebras, which investigates when a category can be thought of as a category of 'algebras' over another category, in a very wide sense.

The next three chapters explore applications of category theory in Algebra, Topology and Algebraic Topology, Homological Algebra.

Finally, Chapter 7 is a brief introduction to higher dimensional category theory, with some of its applications. The starting point is the fact that 'small' categories, functors and natural transformations – with all their composition laws – form a 'two-dimensional category'. Another aspect of the power of category theory is that it is able to 'study itself'.

0.7 Classes of categories

It is a common feature of Mathematics to look for the natural framework where certain properties should be studied: for instance, the properties of ordinary polynomials with real coefficients are usually examined in general polynomial rings, or in more general algebras. Besides yielding more general results, the natural framework gives a deeper comprehension of what we are studying.

Category theory makes a further step in this sense. For instance, categories of modules are certainly important, but – since Buchsbaum's *Appendix* [Bu2] and Grothendieck's paper [Gt] in the 1950's – a consistent part of Homological Algebra finds its natural framework in *abelian categories* and their generalisations (see Chapter 6). Similarly, the categories of structured sets can be viewed as particular *concrete categories* (defined in 1.4.8), and the categories of equational algebras as particular *monadic categories* (see Sections 3.6 and 4.4).

In other words, when studying certain 'classes' of categories, we may (or perhaps had better) look for a *structural definition* including this class, rather than some general way of constructing the important examples we want to study.

This leads again to considering categories of (small) categories, and to higher dimensional category theory.

0.8 Our approach

Notions will be presented in a concrete way, starting from examples taken from elementary mathematical theories. Then their theory is developed, with new examples and many exercises; the latter are generally endowed with a solution, or a partial solution, or suitable hints. Three chapters are devoted to applications.

We hope that a beginner can get, from these examples and applications, a concrete grasp of a theory which might risk of being quite abstract.

Many examples and applications are standard. But some of them may come out as unexpected, and hopefully intriguing, like those devoted to distributive lattices in 1.2.7, or to chains of adjunctions in 1.7.7 and 3.2.8, or to networks in 1.8.9.

The author's comments on some possibly unclear or controversial points are expressed:

- in 1.1.5, on the relationship between mathematical structures and categories,

- in 1.1.6 and 3.6.1, on varieties of algebras and artificial exclusions,

- in 1.5.9, 3.3.7 and 3.6.6, on favouring notions invariant up to categorical equivalence,

- in the introduction to Chapter 7, on the relationship between (strict and weak) 2-categories and double categories,

- in 7.3.3, about the 'naive view' of enrichment of a category over a monoidal category.

The foundational setting we use is based on standard set theory, assuming the existence of Grothendieck universes to formalise some necessary 'smallness' conditions. This aspect, presented in 1.1.3, will be mostly left as understood.

0.9 Literature

For further study of general category theory there are excellent texts, like Mac Lane [M4], Borceux [Bo1, Bo2, Bo3], Adámek, Herrlich and Strecker [AHS], Freyd and Scedrov [FrS]. References for higher dimensional category theory can be found in Chapter 7.

Sheaf categories and toposes, which are not treated here, are presented in Mac Lane–Moerdijk [MaM] and [Bo3], with extensive references to more advanced texts like Johnstone's [Jo3, Jo4]. A peculiar, conceptual introduction to category theory can be found in Lawvere and Schanuel [LwS].

As already warned, the range of applications of category theory is much wider than what will be seen here, and can be presented in an elementary text.

For instance, we only explore a few categorical properties of topological vector spaces and Banach spaces, for which the reader is referred to Semadeni's book [Se]. The applications in Homological Algebra examined here follow a particular approach, discussed in Chapter 6, where other approaches are cited. Some old and new applications of categories to the theory of networks are briefly presented in 1.8.9. Morita equivalence is only mentioned, in 1.5.6.

We do not examine the deep relationship among category theory, Logic and theoretical Computer Science, which is explored in texts like Makkai and Reyes [MkR], Lambek and Scott [LaS], Barr and Wells [BarW], Crole

[Cr], nor the growing influence of categories and higher dimensional categories in parts of theoretical Physics, for which one can see Baez and Lauda [BaL].

Differential Geometry is studied in a 'synthetic' way in Kock [Ko], making formal use of infinitesimals. A new book by Bunge, Gago and San Luis [BunGS] extends this subject to Synthetic Differential Topology. Different forms of Galois Theory are explored in Borceux and Janelidze [BoJ]. A categorical view of Set Theory can be found in Lawvere and Rosebrugh [LwR].

The relationship of category theory with Algebraic Geometry is perhaps too complex to be simply referred to.

0.10 Notation

The symbol $\subset$ denotes *weak* inclusion.

As usual, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural, integral, rational, real or complex numbers; $\mathbb{N}^*$ is the subset of the positive integers. Open and semi-open real intervals are denoted as $]a, b[$, $[a, b[$, etc. This notation, loosely taken from Bourbaki, has the advantage of distinguishing the interval $]a, b[$ from the pair (a, b).

A singleton set is often written as $\{*\}$. The equivalence class of an element x, with respect to an assigned equivalence relation, is generally written as $[x]$. A bullet in a diagram stands for an arbitrary object.

Categories, 2-categories and bicategories are generally denoted as A, B...; strict or weak double categories as $\mathbb{A}, \mathbb{B}$...

A section, subsection or part marked with * deals with some topic out of the main line of this book, and is often addressed to readers having some knowledge of the subject.

0.11 Acknowledgements

Many points have been discussed with my colleagues and friends: in particular with Bob Paré and George Janelidze, during a long collaboration.

1
Categories, functors and natural transformations

Categories were introduced by Eilenberg and Mac Lane [EiM] in 1945, together with the other basic terms of category theory.

1.1 Categories

We start by considering concrete categories, associated with mathematical structures. But categories are not restricted to these instances, and the theory must be developed in a general way.

Given a mathematical discipline, it may not be obvious which category or categories are best suited for its study. This questionable point is discussed in 1.1.5, 1.1.6.

1.1.1 Some examples

Loosely speaking, before giving a precise definition, a category C consists of *objects* and *morphisms* together with a (partial) *composition law*: given two 'consecutive' morphisms $f: X \to Y$ and $g: Y \to Z$ we have a composed morphism $gf: X \to Z$. This partial operation is associative (whenever composition is legitimate) and every object X has an *identity*, written as $\mathrm{id}X: X \to X$ or 1_X, which acts as a unit for legitimate compositions.

The prime example is the category Set *of sets* (and mappings), where:

- an object is a set,

- the morphisms $f: X \to Y$ between two given sets X and Y are the (set-theoretical) mappings from X to Y,

- the composition law is the usual composition of mappings, where $(gf)(x) = g(f(x))$.

The following categories of structured sets and structure-preserving mappings (with the usual composition) will often be used and analysed:

- the category Top *of topological spaces* (and continuous mappings),

- the category Hsd *of Hausdorff spaces* (and continuous mappings),

- the category Gp *of groups* (and their homomorphisms),

- the category Ab *of abelian groups* (and homomorphisms),

- the category Mon *of monoids*, i.e. unitary semigroups (and homomorphisms),

- the category Abm *of abelian monoids* (and homomorphisms),

- the category Rng *of rings*, understood to be associative and unitary (and homomorphisms),

- the category CRng *of commutative rings* (and homomorphisms),

- the category $R\,\mathsf{Mod}$ *of left modules* on a fixed unitary ring R (and homomorphisms),

- the category $K\,\mathsf{Vct}$ ($= K\,\mathsf{Mod}$) *of vector spaces* on a commutative field K (and homomorphisms),

- the category $R\mathsf{Alg}$ *of unitary algebras* on a fixed commutative unitary ring R (and homomorphisms),

- the category Ord of *ordered sets* (and monotone mappings),

- the category pOrd of *preordered sets* (and monotone mappings),

- the category $\mathsf{Set}_\bullet$ *of pointed sets* (and pointed mappings),

- the category $\mathsf{Top}_\bullet$ *of pointed topological spaces* (and pointed continuous mappings),

- the category Ban of *Banach spaces and continuous linear mappings.*

- the category Ban_1 of *Banach spaces and linear weak contractions* (with norm $\leqslant 1$).

A homomorphism of a 'unitary structure', like a monoid or a unitary ring, is always assumed to preserve units.

For $\mathsf{Set}_\bullet$ we recall that a *pointed set* is a pair (X, x_0) consisting of a set X and a *base*-element $x_0 \in X$, while a pointed mapping $f\colon (X, x_0) \to (Y, y_0)$ is a mapping $f\colon X \to Y$ such that $f(x_0) = y_0$.

Similarly, a *pointed topological space* (X, x_0) is a space with a base-point, and a pointed map $f\colon (X, x_0) \to (Y, y_0)$ is a continuous mapping from X to Y such that $f(x_0) = y_0$. The reader may know that the category $\mathsf{Top}_\bullet$ is important in Algebraic Topology: for instance, the fundamental group $\pi_1(X, x_0)$ is defined for a pointed topological space.

For the categories Ban and Ban_1 it is understood that we have chosen *either* the real *or* the complex field; when using both one can write $\mathbb{R}\mathsf{Ban}$ and $\mathbb{C}\mathsf{Ban}$.

When a category is named after its objects alone (e.g. the 'category of groups'), this means that the morphisms are understood to be the obvious ones (in this case the homomorphisms of groups), with the obvious composition law. Different categories with the same objects are given different names, like Ban and Ban_1 above.

1.1.2 Definition

A *category* C consists of the following data:

(a) a set $\mathrm{Ob}\mathsf{C}$, whose elements are called *objects* of C,

(b) for every pair X, Y of objects, a set $\mathsf{C}(X, Y)$ (called a *hom-set*) whose elements are called *morphisms* (or *maps*, or *arrows*) of C from X to Y and denoted as $f\colon X \to Y$,

(c) for every triple X, Y, Z of objects of C, a mapping of *composition*

$$\mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \to \mathsf{C}(X, Z), \qquad (f, g) \mapsto gf,$$

where gf is also written as $g.f$.

These data must satisfy the following axioms.

(i) *Associativity.* Given three consecutive arrows, $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, one has: $h(gf) = (hg)f$.

(ii) *Identities.* Given an object X, there exists an *endo*map $e\colon X \to X$ which acts as an identity whenever composition makes sense; in other words if $f\colon X' \to X$ and $g\colon X \to X''$, one has: $ef = f$ and $ge = g$. One shows, in the usual way, that e is determined by X; it is called the *identity* of X and written as 1_X or $\mathrm{id}X$.

We generally assume that the following condition is also satisfied:

(iii) *Separation.* For all X, X', Y, Y' objects of C, if $\mathsf{C}(X, Y) \cap \mathsf{C}(X', Y') \neq \emptyset$ then $X = X'$ and $Y = Y'$.

Therefore a map $f\colon X \to Y$ has a well-determined *domain* and *codomain*

$$\mathrm{Dom}\,(f) = X, \qquad \mathrm{Cod}\,(f) = Y.$$

Concretely, when constructing a category, one can forget about condition (iii), since one can always satisfy it by *redefining* a morphism $\hat{f}\colon X \to Y$ as a triple (X, Y, f) where f is a morphism from X to Y in the original sense (possibly not satisfying the Separation axiom).

$\mathrm{Mor}\mathsf{C}$ denotes the set of all the morphisms of C, i.e. the disjoint union of its hom-sets. Two morphisms f, g are said to be *parallel* when they have the same domain and the same codomain.

If C is a category, the *opposite* (or *dual*) category, written as C^{op}, has the same objects as C, reversed arrows and reversed composition

$$\begin{gathered}\mathsf{C}^{\mathrm{op}}(X,Y)=\mathsf{C}(Y,X),\\ g*f=fg, \qquad\qquad \mathrm{id}^{\mathrm{op}}(X)=\mathrm{id}X.\end{gathered} \tag{1.1}$$

Every topic of category theory has a dual instance, which comes from the opposite category (or categories). A dual notion is generally distinguished by the prefix 'co-'.

A set X can be viewed as a *discrete* category: its objects are the elements of X, and the only arrows are their (formal) identities; here $X^{\mathrm{op}}=X$.

As usual in category theory, the term *graph* will be used to denote a simplified structure, with objects (or *vertices*) and morphisms (or arrows) $f\colon x\to y$, but no assigned composition nor identities. (This is called a *directed multigraph* in graph theory, or also a *quiver*). A *morphism of graphs* preserves objects, arrows, domain and codomain. Every category has an *underlying graph*.

1.1.3 Small and large categories

We shall not insist on set-theoretical foundations. Yet some caution is necessary, to avoid speaking of 'the set of all sets', or requiring of a category properties of completeness that are 'too large for its size' (as we shall see in 2.2.3).

We assume the existence of a (Grothendieck) *universe* $\mathcal{U}$, which is fixed throughout; its axioms – recalled below, in 1.1.7 – say that we can perform inside it the usual operations of set theory. Its elements are called *small* sets (or $\mathcal{U}$-small sets, if necessary).

A category is understood to have objects and arrows belonging to this universe, and is said to be *small* if its set of morphisms belongs to $\mathcal{U}$, *large* if it does not (and is just a subset of $\mathcal{U}$). As a consequence, in a small category the set of objects (which is in bijective correspondence with the set of identities) also belongs to $\mathcal{U}$. A category C is said to *have small hom-sets* if all its sets $\mathsf{C}(X,Y)$ are small; in this case C is small if and only if its set of objects is.

The categories of structured sets that we consider are generally large $\mathcal{U}$-categories, like the category Set of small sets (and mappings), or Top of small topological spaces (and continuous mappings), or Ab of small abelian groups (and homomorphisms); *in such cases, the term 'small' (referred to these structured sets) will be generally understood*, and we speak – as usual – of the 'category of sets', and so on.

*In fact one often needs a hierarchy of universes. For instance, Cat will denote the category of small categories and functors, introduced in 1.4.1. In order to view the (large) categories Set, Top, Ab, etc. in a similar structure we should assume the existence of a second universe $\mathcal{V}$, with $\mathcal{U} \in \mathcal{V}$, and use the category $\mathsf{Cat}_{\mathcal{V}}$ (also written as CAT) of $\mathcal{V}$-small categories. In a more complex situation one may need a longer chain of universes. Most of the time *these aspects will be left understood.**

1.1.4 Isomorphisms and groupoids

In a category C a morphism $f\colon X \to Y$ is said to be *invertible*, or an *isomorphism*, if it has an inverse, i.e. a morphism $g\colon Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. The latter is uniquely determined; it is called the *inverse* of f and written as f^{-1}.

In the categories listed in 1.1.1 this definition gives the usual isomorphisms of the various structures – called 'homeomorphisms' in the case of topological spaces.

The *isomorphism relation* $X \cong Y$ between objects of C (meaning that there exists an isomorphism $X \to Y$) is an equivalence relation.

A *groupoid* is a category where every map is invertible; it is interesting to recall that this structure was introduced before categories, by H. Brandt in 1927 [Bra].

The fundamental groupoid of a space X is an important structure, that contains all the fundamental groups $\pi_1(X, x)$ for $x \in X$. It will be reviewed in 5.2.9.

1.1.5 A digression on mathematical structures and categories

When studying a mathematical structure with the help of category theory, it is crucial to choose the 'right' kind of structure and the 'right' kind of morphisms, so that the result is sufficiently general and 'natural' to have good properties with respect to the goals of our study – even if we are interested in more particular situations.

(a) A first point to be verified is that the isomorphisms of the category (i.e. its invertible arrows) preserve the structure we are interested in, or we risk of studying something different from our purpose.

As a trivial example, the category T of topological spaces and *all* mappings between them has little to do with topology: an isomorphism of T is any bijection between topological spaces. Indeed T is 'equivalent' to the category of sets (as we shall see in 1.5.5), and is a 'deformed' way of looking at the latter.

Less trivially, the category M of metric spaces and continuous mappings misses crucial properties of metric spaces, since its invertible morphisms need not preserve completeness: e.g. the real line is homeomorphic to any non-empty open interval. In fact, M is equivalent to the category of *metrisable topological spaces* and continuous mappings (by 1.5.5, again), and should be replaced with the latter.

A 'reasonable' category of metric spaces should be based on *Lipschitz* maps, or – more particularly – on weak contractions, so that its isomorphisms (bi-Lipschitz or isometric bijections, respectively) do preserve metric properties, like being complete or bounded (see 5.1.7).

(b) Other points will become clearer below. For instance, the category Top of topological spaces and continuous mappings is a classical framework for studying Topology. Among its good properties there is the fact that all 'categorical products' and 'categorical sums' (studied in Section 2.1, but already sketched in the Introduction) exist, and are computed *as in* Set, then equipped with a suitable topology determined by the structural maps.

(More generally, this is true of all limits and colimits, and – as we shall see – is a consequence of the fact that the forgetful functor Top $\to$ Set has a left *and* a right adjoint, corresponding to discrete and chaotic topologies).

Hausdorff spaces are certainly important, but it is often better to view them *in* Top, as their category Hsd is less well behaved: colimits exist, but are not computed as in Set, and the simplest way to compute them – generally – is to take the colimit in Top and 'make it Hausdorff' (see 5.1.4(b)).

*(c) Many category theorists would agree with Mac Lane, saying that even Top is not sufficiently good ([M4], Section VII.8), because it is not a cartesian closed category (see 5.1.1), and prefer – for instance – the category of compactly generated Hausdorff spaces (see 2.6.3(d)). However, researchers interested in Homotopy Theory and Algebraic Topology might be satisfied with the fact that the standard interval (with its cartesian powers) is exponentiable in Top, as we shall exploit in Section 5.2.

We shall also hint, in 5.1.8, at another approach called 'pointless topology', which is based on the category of locales and is favoured in topos theory.

(d) Finally we remark that artificial exclusions 'most of the time' give categories of poor properties, like the category of *non-abelian groups*, or *non-empty semigroups*. The latter case needs some further comment.

1.1.6 Variety of algebras and **horror vacui**

In Universal Algebra, a 'variety of algebras' includes all the algebraic structures of a given signature (i.e. with a certain family of operations, of assigned arity), that satisfy a given set of equational axioms (or universally quantified identities): e.g. all groups, or all rings; but not all fields, because multiplicative inverses only exist for non-zero elements, and cannot be given by a 'general' unary operation satisfying some *universal* identities.

Here a *variety of algebras* will mean a category of objects defined in this way, with their homomorphisms (as made precise in Section 4.3). *We do not follow the convention* that the underlying set should be non empty; a convention which has unlucky consequences for any theory without 0-ary operations, like semigroups: for instance two subalgebras of an algebra may not have a meet (as subalgebras).

This convention is rather usual in Universal Algebra (cf. Grätzer [Gr1]), but is not followed in Cohn's book [Coh]. (Of course, generally speaking, results can be easily translated from one setting to the other.)

For a reader with some knowledge of limits and colimits, dealt with in Chapter 2, we can add that a variety of algebras (in the present sense) has all limits and colimits, while the category of non-empty semigroups lacks certain limits (like equalisers and pullbacks) and certain colimits (as an initial object), precisely because we have artificially taken out a solution.

Other comments in this sense can be found in 3.6.1; but there is also a comment *in the opposite sense*, in 5.3.9, for categories used in an 'auxiliary way'.

*1.1.7 *Grothendieck universes*

For the interested reader, we recall the definition of a *universe* as given in [M4], Section I.6. It is a set $\mathcal{U}$ satisfying the following (redundant) properties:

(i) $x \in u \in \mathcal{U}$ implies $x \in \mathcal{U}$,

(ii) $u, v \in \mathcal{U}$ implies that $\{u, v\}$, (u, v) and $u \times v$ belong to $\mathcal{U}$,

(iii) $x \in \mathcal{U}$ implies that $\mathcal{P}x$ and $\bigcup x$ belong to $\mathcal{U}$,

(iv) the set $\mathbb{N}$ of finite ordinals belongs to $\mathcal{U}$,

(v) if $f\colon x \to y$ is a surjective mapping with $x \in \mathcal{U}$ and $y \subset \mathcal{U}$, then $y \in \mathcal{U}$.

Here $\mathcal{P}x$ is the set of subsets of x and $\bigcup x = \{y \mid y \in z \text{ for some } z \in x\}$.

1.2 Monoids and preordered sets as categories

Monoids and preordered sets can be viewed as categories of two simple kinds, providing intuition and models for some aspects of category theory.

We also review here some basic facts about the theory of lattices, to be used later on. A reader interested in this beautiful domain will find pleasure in browsing, or studying, the classical texts of Birkhoff and Grätzer [Bi, Gr2].

1.2.1 Monoids and categories

As we have seen, monoids (i.e. semigroups with unit) and their homomorphisms form a category, which we write as Mon. But we deal here with a different aspect.

A single monoid M can (and will often) be viewed as a category with one formal object $*$. The morphisms $x\colon * \to *$ are the elements of M, composed by the multiplication xy of the monoid, with identity $\mathrm{id}(*) = 1$, the unit of the monoid. If M is a group, the associated category is a groupoid.

On the other hand, in every category C, the *endomorphisms* $X \to X$ of any object form a (possibly large) monoid, under the composition law

$$\mathrm{End}(X) = \mathsf{C}(X, X), \tag{1.2}$$

and the invertible ones form the group $\mathrm{Aut}(X)$ of *automorphisms* of X.

In this way a monoid is essentially the same as a category on a single object, while a category can be thought to be a 'multi-object generalisation' of a monoid. Groups and groupoids have a similar relationship.

The theory of regular, orthodox and inverse semigroups (see [ClP, Ho, Law]) has a strong interplay with the categories of relations and their applications in Homological Algebra, which is investigated in [G9].

1.2.2 Preordered and ordered sets

We shall use the following terminology for orderings. A *preorder* relation $x \prec x'$ is reflexive and transitive. An *order* relation, generally written as $x \leqslant x'$, is also anti-symmetric: if $x \leqslant x' \leqslant x$ then $x = x'$.

The category of *ordered sets and increasing mappings* (the order preserving ones, also called *monotone*) will be written as Ord, while we write as pOrd the category of *preordered sets and monotone mappings.*

An order relation is said to be *total* if for all x, x' we have $x \leqslant x'$ or $x' \leqslant x$.

(An ordered set is often called a '*partially* ordered set', abbreviated to 'poset', to stress that the ordering is not assumed to be total. Accordingly, the reader can find the notation Pos for the category Ord.)

A preordered set X has an associated equivalence relation $x \sim x'$ defined by the conjunction: $x \prec x'$ and $x' \prec x$. The quotient $X/\!\sim$ has an induced order: $[x] \leqslant [x']$ if $x \prec x'$.

If X is a preordered set, X^{op} is the opposite one (with reversed preorder). If $a \in X$, the symbols $\downarrow a$ and $\uparrow a$ denote the downward or upward closed subsets of X generated by the element a

$$\downarrow a = \{x \in X \mid x \prec a\}, \qquad \uparrow a = \{x \in X \mid a \prec x\}. \tag{1.3}$$

It will be important to note that every hom-set $\mathsf{Ord}(X, Y)$ is canonically ordered by the *pointwise order* relation, defined as follows for $f, g \colon X \to Y$

$$f \leqslant g \quad \text{if for all } x \in X \text{ we have } f(x) \leqslant g(x) \text{ in } Y. \tag{1.4}$$

Similarly, every hom-set $\mathrm{p}\mathsf{Ord}(X, Y)$ has a canonical preorder $f \prec g$.

In a preordered set X, the *infimum* (or *meet*) of a subset A, written as $\inf A$ or $\wedge A$, is defined as the greatest element of X smaller than all the elements of A, and is determined up to the associated equivalence relation (provided it exists). Dually, the *supremum* (or *join*) $\sup A = \vee A$ is the infimum of A in X^{op}. In an ordered set these results are uniquely determined – if extant.

1.2.3 Preorders and categories

A preordered set X will often be viewed as a category, where the objects are the elements of X and the set $X(x, x')$ contains precisely one arrow if $x \prec x'$, which can be written as $(x, x') \colon x \to x'$, and no arrow otherwise. Composition and units are (necessarily) as follows

$$(x', x'').(x, x') = (x, x''), \qquad\qquad \mathrm{id}(x) = (x, x).$$

In this way a preordered set is essentially the same as a category where each hom-sets has at most one element. All diagrams in these categories commute. Two elements x, x' are isomorphic objects if and only if $x \sim x'$.

In particular, each ordinal defines a category, written as $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$

- $\mathbf{0}$ is the empty category,

- $\mathbf{1}$ is the *singleton category*, i.e. the discrete category on one object,

- $\mathbf{2}$ is the *arrow category*, with two objects (0 and 1), and precisely one non-identity arrow, $0 \to 1$.

Here **2** should not be confused with the cardinal $2 = \{0, 1\}$, which is viewed as a discrete category.

1.2.4 Lattices

Classically a lattice is defined as an ordered set X such that every pair x, x' of elements has a *join* $x \vee x' = \sup\{x, x'\}$ (the least element of X greater than both) and a *meet* $x \wedge x' = \inf\{x, x'\}$ (the greatest element of X smaller than both).

Here we follow a different convention, usual in category theory: *lattice* will always mean an ordered set with *finite* joins and meets, which amounts to the existence of binary joins and meets *together with* the least element $0 = \vee\emptyset$ and the greatest element $1 = \wedge\emptyset$. (This structure is called a 'bounded lattice' in Lattice Theory.)

The bounds 0 and 1 (the empty join and the empty meet) are equal in the one-point lattice $\{*\}$, and only there.

Consistently with this terminology, a *lattice homomorphism* has to preserve finite joins and meets; a *sublattice* of a lattice X is closed under such operations (and has the same bounds as X). The category of lattices and homomorphisms will be written as Lth.

Occasionally we speak of a *quasi lattice* when we only assume the existence of binary joins and meets; a homomorphism of quasi lattices only has to preserve them. A *quasi sublattice* Y of a quasi lattice X is closed under binary joins and meets in X; when X is a lattice, Y may have different bounds, or lack some of them.

For instance, if X is a lattice and $a \in X$, the downward and upward closed subsets $\downarrow a$, $\uparrow a$ of X generated by a (see 1.2.2) are quasi sublattices of X, and lattices in their own right.

Let us note that the *free lattice* (see 2.7.3) generated by an element x *has three elements*: $0 < x < 1$, while the *free quasi lattice* L generated by x is just the singleton $\{x\}$: in fact every mapping $\{x\} \to X$ with values in a lattice (resp. in a quasi lattice) has a unique extension to a homomorphism $L \to X$ (resp. $\{x\} \to X$). In the same way, the free lattice L generated by a set S can be obtained from the corresponding free quasi-lattice M by adding a (new) minimum and a (new) maximum, even when M is *already* bounded – as above.

1.2.5 Exercises and complements

(a) *Lattices as algebras.* The reader may know, or be interested to prove, that a lattice can be equivalently presented as a set X equipped with two operations, $x \vee y$ and $x \wedge y$, called *join* and *meet*, that satisfy the following axioms:

(i) both operations are associative, commutative and idempotent:

$$x \vee (y \vee z) = (x \vee y) \vee z, \qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z,$$
$$x \vee y = y \vee x, \qquad x \wedge y = y \wedge x,$$
$$x \vee x = x, \qquad x \wedge x = x,$$

(ii) each operation has a unit, written as 0 and 1, respectively

$$x \vee 0 = x = x \wedge 1,$$

(iii) the following *absorption laws* hold:

$$x \vee (x \wedge y) = x = x \wedge (x \vee y).$$

Given this presentation, one defines the ordering by letting $x \leqslant y$ if $x \vee y = y$, or equivalently $x \wedge y = x$. Again, one should be aware that $0 = 1$ is not excluded; we already know that in this case all the elements coincide. The reader will easily guess how the opposite lattice X^{op} is defined here.

(b) *Complete lattices.* The reader may know, or easily prove, that a preordered set has all infima (of its subsets) if and only if it has all suprema. In this case – if it is an ordered set – it is called a complete lattice. This agrees with the notion of a 'complete category', as we shall see in 2.2.5(e).

Frames are particular complete lattices, related with topological spaces and 'pointless topology', and will be briefly introduced in 5.1.8.

1.2.6 Distributive and modular lattices

A lattice is said to be *distributive* if the meet operation distributes over the join operation, or equivalently if the join distributes over the meet. In fact, if we assume that meets distribute over joins, we have:

$$(x \vee y) \wedge (x \vee z) = ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z)$$
$$= x \vee (x \wedge z) \vee (y \wedge z) = x \vee (y \wedge z).$$

A *boolean algebra* is a distributive (bounded) lattice where every element x has a (necessarily unique) *complement* x', defined by the properties:

$$x \wedge x' = 0, \qquad x \vee x' = 1.$$

The subsets of a set X form the classical boolean algebra $\mathcal{P}X$, which is a complete lattice.

The (complete) lattice Sub$A$ of subgroups of an abelian group (or submodules of a module) is not distributive, generally (see 1.2.7(d)); but one can easily check that it always satisfies a weaker, restricted form of distributivity, called modularity.

Namely, a lattice is said to be *modular* if it satisfies the following selfdual property (for all elements x, y, z)

(i) if $x \leqslant z$ then $(x \vee y) \wedge z = x \vee (y \wedge z)$.

The category of modular (resp. distributive) lattices and their homomorphisms will be written as Mlh (resp. Dlh).

By Birkhoff's representation theorem ([Bi] III.5, Theorem 5) the free distributive lattice on n generators is finite and isomorphic to a lattice of subsets. The reader may also be interested to know that the free modular lattice on three elements is finite and (obviously!) not distributive (see [Bi], III.6, Fig. 10), while four generators already give an infinite free modular lattice (see the final Remark in [Bi], III.6).

*1.2.7 *Exercises and complements* (Distributive lattices)

The goal of this point is to show that the (complete) lattice $X = \text{Sub}(\mathbb{Z})$ of subgroups of the abelian group $\mathbb{Z}$ (or ideals of the ring $\mathbb{Z}$) is distributive – a fact that will have unexpected links with Homological Algebra, in Section 6.6.

The interested reader is invited to directly investigate the problem, before considering the layout given below. The first step is showing that X is anti-isomorphic to the divisibility lattice of natural numbers (a well-known, nearly obvious point). Then one proves that the latter is distributive, a (hopefully amusing) exercise based on our school knowledge – prime factorisation.

(a) The reader likely knows, or should prove, that each subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, for a unique $n \in \mathbb{N}$ (and is an ideal of the ring of integers).

This gives us an isomorphism $X \to Y^{\text{op}}$, where $Y = (\mathbb{N}, |)$ is the set of natural numbers ordered by the divisibility relation $m|n$. Therefore Y is a (complete) lattice as well, and we recognise its operations $m \vee n$ and $m \wedge n$ as fairly well-known. We also note that $1 = \min Y$ and $0 = \max Y$.

(b) Now, each $n \in \mathbb{N}^*$ has a unique decomposition $n = \prod_p p^{n_p}$ as a product of powers of non-invertible prime numbers p; of course the natural exponents n_p are quasi-null (i.e. all of them are 0, out of a finite number of prime indices p), so that the factorisation is essentially finite.

The reader will use this fact to prove that Y can be embedded in the cartesian product $\prod_p \overline{\mathbb{N}}$ of countably many copies of the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, with the natural order. One can now show that Y is distributive, and X as well.

(c) As a consequence, the lattice $\mathrm{Sub}(A)$ of any cyclic group A is also distributive.

(d) The reader can easily show that this property fails for the abelian group $\mathbb{Z}^2$, or for any non-trivial abelian group $A \oplus A$. (Again, this will be of use in Section 6.6.)

On the other hand, $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$ has a distributive lattice of subgroups.

(e) A reader acquainted with *principal ideal domains* may like to rethink the whole thing.

1.3 Monomorphisms and epimorphisms

In a category, monomorphisms and epimorphisms (monos and epis for short) are defined by cancellation properties with respect to composition.

For categories of structured sets, they represent an 'approximation' to the injective and surjective mappings of the category.

1.3.1 Main definitions

In a category C a morphism $f\colon X \to Y$ is said to be a *monomorphism*, or *mono*, if it satisfies the following cancellation property: for every pair of maps $u, v\colon X' \to X$ (defined on an arbitrary object X') such that $fu = fv$, one has $u = v$ (see the left diagram below)

$$X' \underset{v}{\overset{u}{\rightrightarrows}} X \xrightarrow{f} Y \qquad\qquad X \xrightarrow{f} Y \underset{v}{\overset{u}{\rightrightarrows}} Y' \tag{1.5}$$

Dually, the morphism $f\colon X \to Y$ is said to be an *epimorphism*, or *epi*, if it satisfies the dual cancellation property: for every pair of maps $u, v\colon Y \to Y'$ such that $uf = vf$, one has $u = v$ (see the right diagram above).

An arrow $\rightarrowtail$ will always denote a monomorphism, while $\twoheadrightarrow$ stands for an epimorphism.

Every isomorphism is mono and epi. A category is said to be *balanced* if the converse holds: every morphism which is mono and epi is invertible.

Suppose now that we have, in a category C, two maps $m\colon A \to X$ and $p\colon X \to A$ such that $pm = \mathrm{id}A$. It follows that m is a monomorphism (called a *section*, or a *split monomorphism*), while p is an epimorphism (called a *retraction*, or a *split epi*); A is said to be a *retract* of X.

A family of morphisms $f_i\colon X \to Y_i$ ($i \in I$) with the same domain is said to be *jointly mono* if for every pair of maps $u, v: X' \to X$ such that $f_i u = f_i v$ (for all indices i) one has $u = v$. Dually a family $f_i: X_i \to Y$ is *jointly epi* if for all $u, v: Y \to Y'$ such that $uf_i = vf_i$ (for all i) one has $u = v$.

The general properties of monos, epis and retracts will be examined in 1.3.6. Related notions, like regular monos and epis, strong monos and epis, subobjects and quotients, will be seen in the next chapter.

1.3.2 Comments

In a category of structured sets and structure-preserving mappings, an injective mapping (of the category) is obviously a monomorphism, while a surjective one is an epimorphism. The converse may require a non-trivial proof, or even fail. This can only be understood by working out the examples below.

Interestingly, a divergence appears between monos and epis: the theory of categories is self-dual, but our frameworks of structured sets are not! When we classify monos in Set, this tells us everything about the epis of $\mathsf{Set}^{\mathrm{op}}$ but nothing about the epis of Set.

In fact, in all the examples below it will be easy to prove that the monomorphisms coincide with the injective morphisms. Later we shall see, in 2.7.4(d), some conditions that ensure this fact, and hold true in all the 'usual categories of structured sets'.

On the other hand, various problems occur with epimorphisms: classifying them in various categories of algebraic structures leads to difficult problems with no elementary solution (and no real need of it).

1.3.3 Exercises and complements, I

(a) The first point is to prove that in Set every mono is an injective mapping and every epi is surjective. We write down the proof, but a beginner should try to give an independent solution; this is quite easy for monos and slightly less for epis.

If $f: X \to Y$ is a monomorphism in Set, let us suppose that $f(x) = f(x')$, for $x, x' \in X$. We consider the mappings u, v

$$\{*\} \underset{v}{\overset{u}{\rightrightarrows}} X \xrightarrow{f} Y \qquad\qquad u(*) = x, \quad v(*) = x'. \tag{1.6}$$

Now we have $fu = fv$, whence $u = v$ and $x = x'$, which shows that f is injective. Note that the proof works by simulating an element of X with a map $\{*\} \to X$.

On the other hand, if $f\colon X \to Y$ is an epimorphism in Set, we define two mappings u, v with values in the set $\{0, 1\}$

$$X \xrightarrow{f} Y \underset{v}{\overset{u}{\rightrightarrows}} \{0,1\} \tag{1.7}$$

where u is the characteristic function of the subset $f(X) \subset Y$ (with $u(y) = 1$ if and only if $y \in f(X)$) and v is the constant map $v(y) = 1$. Then $uf = vf$, whence $u = v$ and $f(X) = Y$. (A different proof can be based on the set $Y \times \{0, 1\}$, the disjoint union of two copies of Y; or a quotient of the latter.)

Since the invertible morphisms in Set are obviously the same as the bijective mappings, we remark that the category Set is balanced.

(b) Now the reader should prove that also in Top and Ab monos and epis coincide with the injective and surjective mappings of the category, respectively.

For monos the proof is quite similar to the previous one, making use of the singleton in Top and of the group $\mathbb{Z}$ in Ab. Note that the latter allows us to simulate an element $x \in X$ by a homomorphism $u\colon \mathbb{Z} \to X$, sending the generator 1 to x.

To prove that an epi is surjective, one can use a two-point codiscrete space in Top and the quotient group $Y/f(X)$ in Ab. Note that the last point follows a different pattern: constructing arrows is fairly free in Set, somewhat less in Top, much less in categories of algebraic structures.

We conclude here that Ab is balanced, while Top is not: a bijective continuous mapping need not be invertible in Top, i.e. a homeomorphism.

(c) The reader should prove that, in the categories Mon of monoids and Rng of rings, the monomorphisms coincide again with the injective homomorphisms. Then one can easily show that the inclusion $\mathbb{N} \to \mathbb{Z}$ (of additive monoids) is mono and *epi* in Mon, which is not balanced. The same holds for the inclusion $\mathbb{Z} \to \mathbb{Q}$ in Rng.

In fact epimorphisms in Mon and Rng have no elementary characterisation and the 'regular epimorphisms' (namely the surjective homomorphisms) are more important, as we shall see in Section 4.4.

(d) In a preordered set, viewed as a category, all arrows are mono and epi. The category is balanced under a precise condition on the preordering.

1.3.4 Exercises and complements, II

(a) In the category Gp of groups all epimorphisms are surjective: a non-obvious fact, whose proof can be found in [M4], Section I.5, Exercise 5.