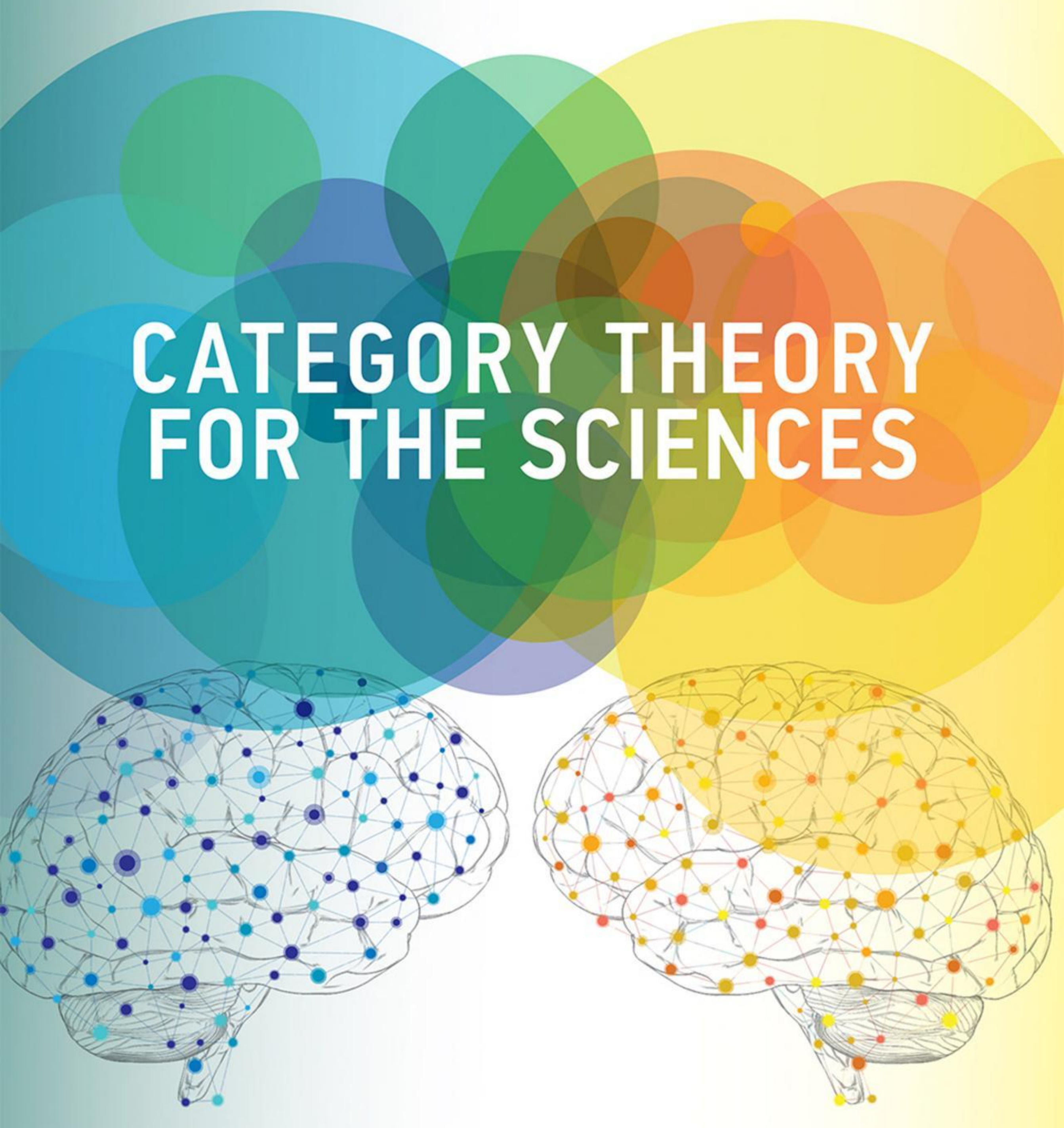
CATEGORY THEORY FOR THE SCIENCES



David I. Spivak

Copyrighted material

Category Theory for the Sciences

David I. Spivak

The MIT Press
Cambridge, Massachusetts
London, England

Copyrighted material

© 2014 Massachusetts Institute of Technology

All rights reserved. No part of this book may be reproduced in any form by any electronic or mechanical means (including photocopying, recording, or information storage and retrieval) without permission in writing from the publisher.

MIT Press books may be purchased at special quantity discounts for business or sales promotional use. For information, please email special_sales@mitpress.mit.edu.

Library of Congress Cataloging-in-Publication Data

Spivak, David I., 1978– author.
Category theory for the sciences / David I. Spivak.
pages cm
Includes bibliographical references and index.
ISBN 978-0-262-02813-4 (hardcover : alk. paper) 1.
Science—Mathematical models. 2. Categories (Mathematics) I. Title.
Q175.32.M38S65 2014
512'.62—dc23
2014007215

10 9 8 7 6 5 4 3

Copyrighted material

Contents

Copyrighted material

Copyrighted material

Acknowledgments

I would like to express my deep appreciation to the many scientists with whom I have worked over the past six years. It all started with Paea LePendu, who first taught me about databases when I was naively knocking on doors in the University of Oregon computer science department. This book would never have been written if Tristan Nguyen and Dave Balaban had not noticed my work and encouraged me to continue. Dave Balaban and Peter Gates have been my scientific partners since the beginning, working hard to understand what I am offering and working just as hard to help me understand all that I am missing. Peter Gates has deepened my understanding of data in profound ways.

I have also been tremendously lucky to know Haynes Miller, who made it possible for me to settle at MIT, with the help of Clark Barwick and Jacob Lurie. I knew that MIT would be the best place in the world for me to pursue this type of research, and it consistently lives up to expectation. Researchers like Markus Buehler and his graduate students Tristan Giesa and Dieter Brommer have been a pleasure to work with, and the many materials science examples scattered throughout this book are a testament to how much our work together has influenced my thinking.

I would also like to thank the collaborators and conversation partners with whom I have discussed subjects written about in this book. Besides the people mentioned previously, these include Steve Awodey, Allen Brown, Adam Chlipala, Carlo Curino, Dan Dugger, Henrik Forssell, David Gepner, Jason Gross, Bob Harper, Ralph Hutchison, Robert Kent, Jack Morava, Scott Morrison, David Platt, Joey Perricone, Dylan Rupel, Guarav Singh, Sam Shames, Nat Stapleton, Patrick Schultz, Ka Yu Tam, Ryan Wisnesky, Jesse Wolfson, and Elizabeth Wood.

I would like to thank Peter Kleinhenz and Peter Gates for reading an earlier version of this book and providing invaluable feedback before I began teaching the 18-S996 class at MIT in spring 2013. In particular, the first figure of the book, Figure 1.1, is a slight alteration of a diagram Gates sent me to help motivate the book for scientists. I would also like to greatly thank the 18-S996 course grader Darij Grinberg, who not only was

Copyrighted material

the best grader I have had in my 14 years of teaching, but gave me more comments than anyone else on the book itself. I would like to thank the students from the 18-S996 class at MIT who found typos, pointed out unclear explanations, and generally helped improve the book in many ways: Aaron Brookner, Leon Dimas, Dylan Erb, Deokhwan Kim, Taesoo Kim, Owen Lewis, Yair Shenfeld, and Adam Strandberg, among others. People outside the class, V. Galchin, K. Hofmeyr, D. McAdams, D. Holmes, C. McNally, P. O'Neill, and R. Harper, also contributed to finding errata and making improvements.

I'd also like to thank Marie Lufkin Lee, Marc Lowenthal, Katherine Almeida, and everyone else at MIT Press who helped get this book ready for publication. And thanks to Laura Baldwin, who helped me work through some painful LaTeX issues. The book is certainly far better than when I originally submitted it. I also appreciate the willingness of the Press to work with me in making a copy of this book publicly available.

Thanks also to my teacher Peter Ralston, who taught me to repeatedly question the obvious. My ability to commit to a project like this one and to see it to fruition has certainly been enhanced since I studied with him.

Finally, I acknowledge my appreciation for support from the Office of Naval Research and Air Force Office of Scientific Research[1] without which this book would not have been remotely possible. I believe that the funding of basic research is an excellent way of ensuring that the United States remains a global leader in the years to come.

[1] Grant numbers: N000140910466, N000141010841, N000141310260, FA9550-14-1-0031.

Copyrighted material

Chapter 1

Introduction

The diagram in Figure 1.1 is intended to evoke thoughts of the scientific method.

Copyrighted image

Figure 1.1

An observation analyzed by a person yields a hypothesis, which analyzed by a person produces a prediction, which motivates the specification of an experiment, which when executed results in an observation.

Copyrighted material

Its statements look valid, and a good graphic can be very useful for leading a reader through a story that the author wishes to tell.

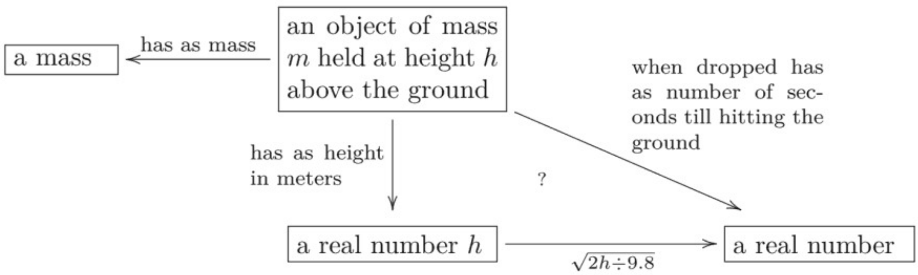
But a graphic has the power to evoke feelings of understanding without really meaning much. The same is true for text: it is possible to use a language like English to express ideas that are never made rigorous or clear. When someone says, "I believe in free will," what does she believe in? We may all have some concept of what she's saying—something we can conceptually work with and discuss or argue about. But to what extent are we all discussing the same thing, the thing she intended to convey?

Science is about agreement. When we supply a convincing argument, the result of this convincing is agreement. When, in an experiment, the observation matches the hypothesis—success!—that is agreement. When my methods make sense to you, that is agreement. When practice does not agree with theory, that is disagreement. Agreement is the good stuff in science; it is the celebratory moment.

But it is easy to think we are in agreement, when we really are not. Modeling our thoughts on heuristics and graphics may be convenient for quick travel down the road, but we are liable to miss our turnoff at the first mile. The danger is in mistaking convenient conceptualizations for what is actually there. It is imperative that we have the ability at any time to ground in reality. What does that mean?

Data. Hard evidence. The physical world. It is here that science is grounded and heuristics evaporate. So let's look again at Figure 1.1. It is intended to evoke an idea of how science is performed. Do hard evidence and data back up this theory? Can we set up an experiment to find out whether science is actually performed according to such a protocol? To do so we have to shake off the impressions evoked by the diagram and ask, What does this diagram intend to communicate?

In this book I will use a mathematical tool called *ologs*, or ontology logs, to give some structure to the kinds of ideas that are often communicated in graphics. Each olog inherently offers a framework in which to record data about the subject. More precisely, it encompasses a *database schema*, which means a system of interconnected tables that are initially empty but into which data can be entered. For example, consider the following olog:



Copyrighted material

center of the mathematical universe. He explained how whole algebraic theories can be viewed as examples of a single system. He and others went on to show that higher-order logic was beautifully captured in the setting of category theory (more specifically toposes). It is here also that Grothendieck and his school worked out major results in algebraic geometry.

In 1980, Joachim Lambek showed that the types and programs used in computer science form a specific kind of category. This provided a new semantics for talking about programs, allowing people to investigate how programs combine and compose to create other programs, without caring about the specifics of implementation. Eugenio Moggi brought the category-theoretic notion of monads into computer science to encapsulate ideas that up to that point were considered outside the realm of such theory.

It is difficult to explain the clarity and beauty brought to category theory by people like Daniel Kan and André Joyal. They have each repeatedly extracted the essence of a whole mathematical subject to reveal and formalize a stunningly simple yet extremely powerful pattern of thinking, revolutionizing how mathematics is done.

All this time, however, category theory was consistently seen by much of the mathematical community as ridiculously abstract. But in the twenty-first century it has finally come to find healthy respect within the larger community of pure mathematics. It is the language of choice for graduate-level algebra and topology courses, and in my opinion will continue to establish itself as the basic framework in which to think about and express mathematical ideas.

As mentioned, category theory has branched out into certain areas of science as well. Baez and Dolan [6] have shown its value in making sense of quantum physics, it is well established in computer science, and it has found proponents in several other fields as well. But to my mind, we are at the very beginning of its venture into scientific methodology. Category theory was invented as a bridge, and it will continue to serve in that role.

1.2 Intention of this book

The world of *applied mathematics* is much smaller than the world of *applicable mathematics.* As mentioned, this book is intended to create a bridge between the vast array of mathematical concepts that are used daily by mathematicians to describe all manner of phenomena that arise in our studies and the models and frameworks of scientific disciplines such as physics, computation, and neuroscience.

For the pure mathematician I try to prove that concepts such as categories, functors, natural transformations, limits, colimits, functor categories, sheaves, monads, and operads—concepts that are often considered too abstract even for math majors—can be communicated to scientists with no math background beyond linear algebra. If this material is as teachable as I think, it means that category theory is not esoteric but well

Copyrighted material

aligned with ideas that already make sense to the scientific mind. Note, however, that this book is example-based rather than proof-based, so it may not be suitable as a reference for students of pure mathematics.

For the scientist I try to prove the claim that category theory includes a formal treatment of conceptual structures that the scientist sees often, perhaps without realizing that there is well-oiled mathematical machinery to be employed. A major topics is the structure of information itself: how data is made meaningful by its connections, both internal and outreaching, to other data.[2] Note, however, that this book should certainly not be taken as a reference on scientific matters themselves. One should assume that any account of physics, materials science, chemistry, and so on, has been oversimplified. The intention is to give a flavor of how category theory may help model scientific ideas, not to explain those ideas in a serious way.

Data gathering is ubiquitous in science. Giant databases are currently being mined for unknown patterns, but in fact there are many (many) known patterns that simply have not been catalogued. Consider the well-known case of medical records. In the early twenty-first century, it is often the case that a patient's medical history is known by various doctor's offices but quite inadequately shared among them. Sharing medical records often means faxing a handwritten note or a filled-in house-created form from one office to another.

Similarly, in science there exists substantial expertise making brilliant connections between concepts, but this expertise is conveyed in silos of English prose known as journal articles. Every scientific journal article has a methods section, but it is almost impossible to read a methods section and subsequently repeat the experiment—the English language is inadequate to precisely and concisely convey what is being done.

The first thought I wish to convey in this book is that reusable methodologies can be formalized and that doing so is inherently valuable. Consider the following analogy. Suppose one wants to add up the area of a region in space (or the area under a curve). One breaks the region down into small squares, each with area A, and then counts the number of squares, say n. One multiplies these numbers together and says that the region has an area of about nA. To obtain a more precise and accurate result, one repeats the process with half-size squares. This methodology can be used for any area-finding problem (of which there are more than a first-year calculus student generally realizes) and thus it deserves to be formalized. But once we have formalized this methodology, it can be taken to its limit, resulting in integration by Riemann sums. Formalizing the problem can lead

[2] The word *data* is generally considered to be the plural form of the word *datum*. However, individual datum elements are only useful when they are organized into structures (e.g., if one were to shuffle the cells in a spreadsheet, most would consider the data to be destroyed). It is the whole organized structure that really houses the information; the data must be in formation in order to be useful. Thus I use the word *data* as a collective noun (akin to *sand*); it bridges the divide between the *individual datum elements* (akin to grains of sand) and the *data set* (akin to a sand pile).

Copyrighted material

to powerful techniques that were unanticipated at the outset.

I intend to show that category theory is incredibly efficient as a language for experimental design patterns, introducing formality while remaining flexible. It forms a rich and tightly woven conceptual fabric that allows the scientist to maneuver between different perspectives whenever the need arises. Once she weaves that fabric into her own line of research, she has an ability to think about models in a way that simply would not occur without it. Moreover, putting ideas into the language of category theory forces a person to clarify her assumptions. This is highly valuable both for the researcher and for her audience.

What must be recognized in order to find value in this book is that conceptual chaos is a major problem. Creativity demands clarity of thinking, and to think clearly about a subject requires an organized understanding of how its pieces fit together. Organization and clarity also lead to better communication with others. Academics often say they are paid to think and understand, but that is not the whole truth. They are paid to think, understand, and *communicate their findings*. Universal languages for science, such as calculus and differential equations, matrices, or simply graphs and pie charts, already exist, and they grant us a cultural cohesiveness that makes scientific research worthwhile. In this book I attempt to show that category theory can be similarly useful in describing complex scientific understandings.

1.3 What is requested from the student

The only way to learn mathematics is by doing exercises. One does not get fit by merely looking at a treadmill or become a chef by merely reading cookbooks, and one does not learn math by watching someone else do it. There are about 300 exercises in this book. Some of them have solutions in the text, others have solutions that can only be accessed by professors teaching the class.

A good student can also make up his own exercises or simply play around with the material. This book often uses databases as an entry to category theory. If one wishes to explore categorical database software, FQL (functorial query language) is a great place to start. It may also be useful in solving some of the exercises.

1.4 Category theory references

I wrote this book because the available books on category theory are almost all written for mathematicians (the rest are written for computer scientists). One book, *Conceptual Mathematics* by Lawvere and Schanuel [24], offers category theory to a wider audience,

Copyrighted material

but its style is not appropriate for a course or as a reference. Still, it is very well written and clear.

The bible of category theory is *Categories for the Working Mathematician* by Mac Lane [29]. But as the title suggests, it was written for working mathematicians and would be opaque to my target audience. However, once a person has read the present book, Mac Lane's book may become a valuable reference.

Other good books include Awodey's *Category theory* [4], a recent gentle introduction by Simmons [37], and Barr and Wells's *Category Theory for Computing Science*, [11]. A paper by Brown and Porter, "Category Theory: an abstract setting for analogy and comparison" [9] is more in line with the style of this book, only much shorter. Online, I find Wikipedia [46] and a site called *nLab* [34] to be quite useful.

This book attempts to explain category theory by examples and exercises rather than by theorems and proofs. I hope this approach will be valuable to the working scientist.

Copyrighted material

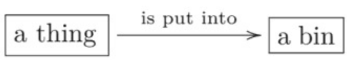
Chapter 2

The Category of Sets

The theory of sets was invented as a foundation for all of mathematics. The notion of sets and functions serves as a basis on which to build intuition about categories in general. This chapter gives examples of sets and functions and then discusses commutative diagrams. Ologs are then introduced, allowing us to use the language of category theory to speak about real world concepts. All this material is basic set theory, but it can also be taken as an investigation of the *category of sets*, which is denoted **Set**.

2.1 Sets and functions

People have always found it useful to put things into bins.



The study of sets is the study of things in bins.

2.1.1 Sets

You probably have an innate understanding of what a set is. We can think of a set X as a collection of *elements* $x \in X$, each of which is recognizable as being in X and such that for each pair of named elements $x, x' \in X$ we can tell if $x = x'$ or not.[1] The set of pendulums is the collection of things we agree to call pendulums, each of which is

[1] Note that the symbol x', read "x-prime," has nothing to do with calculus or derivatives. It is simply notation used to name a symbol that is somehow like x. This suggestion of kinship between x and x' is meant only as an aid for human cognition, not as part of the mathematics.

Copyrighted material

Figure 2.2 A function from a set X to a set Y.

Application 2.1.2.2. In studying the mechanics of materials, one wishes to know how a material responds to tension. For example, a rubber band responds to tension differently than a spring does. To each material we can associate a force-extension curve, recording how much force the material carries when extended to various lengths. Once we fix a methodology for performing experiments, finding a material's force-extension curve would ideally constitute a function from the set of materials to the set of curves.

◊◊

Exercise 2.1.2.3.

Here is a simplified account of how the brain receives light. The eye contains about 100 million photoreceptor (PR) cells. Each connects to a retinal ganglion (RG) cell. No PR cell connects to two different RG cells, but usually many PR cells can attach to a single RG cell.

Let PR denote the set of photoreceptor cells, and let RG denote the set of retinal ganglion cells.

a. According to the above account, does the connection pattern constitute a function $RG \to PR$, a function $PR \to RG$, or neither one?

b. Would you guess that the connection pattern that exists between other areas of the brain are function-like? Justify your answer.

◊

Copyrighted material

Solution 2.1.2.3.

a. To every element of PR we associate an element of RG, so this is a function $PR \to RG$.

b. (Any justified guess is legitimate.) With no background in the subject, I might guess this happens in any case of immediate perception being translated to neural impulses.

♦

Example 2.1.2.4. Suppose that X is a set and $X' \subseteq X$ is a subset. Then we can consider the function $X' \to X$ given by sending every element of X' to "itself" as an element of X. For example, if $X = \{a, b, c, d, e, f\}$ and $X' = \{b, d, e\}$, then $X' \subseteq X$. We turn that into the function $X' \to X$ given by $b \mapsto b, d \mapsto d, e \mapsto e$.[2]

As a matter of notation, we may sometimes say the following: Let X be a set, and let $i\colon X' \subseteq X$ be a subset. Here we are making clear that X' is a subset of X, but that i is the name of the associated function.

Exercise 2.1.2.5.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be the function that sends every natural number to its square, e.g., $f(6) = 36$. First fill in the blanks, then answer a question.

a. $2 \mapsto$ ________

b. $0 \mapsto$ ________

c. $-2 \mapsto$ ________

d. $5 \mapsto$ ________

e. Consider the symbol $\to$ and the symbol $\mapsto$. What is the difference between how these two symbols are used so far in this book?

◊

Solution 2.1.2.5.

a. 4

b. 0

c. The function does not apply to -2 because -2 is not an element of $\mathbb{N}$.

[2]This kind of arrow, $\mapsto$, is read "maps to." A function $f\colon X \to Y$ means a rule for assigning to each element $x \in X$ an element $f(x) \in Y$. We say that "x maps to $f(x)$" and write $x \mapsto f(x)$.

Copyrighted material

d. 25

e. The symbol $\to$ is used to denote a function from one set to another. For example, the arrow in $g\colon X \to Y$ is a symbol that tells us that g is the name of a function from set X to set Y. The symbol $\mapsto$ is used to tell us where the function sends a specific element of the domain. So in our squaring function $f\colon \mathbb{N} \to \mathbb{N}$, we write $5 \mapsto 25$ because the function f sends 5 to 25.

♦

Given a function $f\colon X \to Y$, the elements of Y that have at least one arrow pointing to them are said to be *in the image* of f; that is, we have

$$\mathrm{im}(f) := \{y \in Y \mid \exists x \in X \text{ such that } f(x) = y\}. \tag{2.3}$$

The image of a function f is always a subset of its codomain, $\mathrm{im}(f) \subseteq Y$.

Exercise 2.1.2.6.

If $f\colon X \to Y$ is depicted by Figure 2.2, write its image, $\mathrm{im}(f)$ as a set. ◊

Solution 2.1.2.6.

The image is $\mathrm{im}(f) = \{y_1, y_2, y_4\}$. ♦

Given a function $f\colon X \to Y$ and a function $g\colon Y \to Z$, where the codomain of f is the same set as the domain of g (namely, Y), we say that f and g are *composable*

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

The *composition of f and g* is denoted by $g \circ f\colon X \to Z$. See Figure 2.3.

Slogan 2.1.2.7.

> *Given composable functions $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a way of putting every thing in X into a bin in Y, and we have a way of putting each bin from Y into a larger bin in Z. The composite, $g \circ f\colon X \to Z$, is the resulting way that every thing in X is put into a bin in Z.*

Exercise 2.1.2.8.

If $A \subseteq X$ is a subset, Example 2.1.2.4 showed how to think of it as a function $i\colon A \to X$. Given a function $f\colon X \to Y$, we can compose $A \xrightarrow{i} X \xrightarrow{f} Y$ and get a function $f \circ i\colon A \to Y$. The image of this function is denoted

$$f(A) := \mathrm{im}(f \circ i),$$

Copyrighted material

Figure 2.3 Functions $f\colon X \to Y$ and $g\colon Y \to Z$ compose to a function $g \circ f\colon X \to Z$ (follow the arrows).

see (2.3) for the definition of image.

Let $X = Y := \mathbb{Z}$, let $A := \{-1, 0, 1, 2, 3\} \subseteq X$, and let $f\colon X \to Y$ be given by $f(x) = x^2$. What is the image set $f(A)$? ◊

Solution 2.1.2.8.

By definition of image (see (2.3), we have

$$f(A) = \{y \in \mathbb{Z} \mid \exists a \in A \text{ such that } f \circ i(a) = y\}.$$

Since $A = \{-1, 0, 1, 2, 3\}$ and since $i(a) = a$ for all $a \in A$, we have $f(A) = \{0, 1, 4, 9\}$. Note that an element of a set can only be in the set once; even though $f(-1) = f(1) = 1$, we need only mention 1 once in $f(A)$. In other words, if a student has an answer such as $\{1, 0, 1, 4, 9\}$, this suggests a minor confusion. ♦

Notation 2.1.2.9. Let X be a set and $x \in X$ an element. There is a function $\{☺\} \to X$ that sends $☺ \mapsto x$. We say that this function *represents* $x \in X$. We may denote it $x\colon \{☺\} \to X$.

Copyrighted material

Exercise 2.1.2.10.

Let X be a set, let $x \in X$ be an element, and let $x\colon \{☺\} \to X$ be the function representing it. Given a function $f\colon X \to Y$, what is $f \circ x$? ◊

Solution 2.1.2.10.

It is the function $\{☺\} \to Y$ that sends ☺ to $f(x)$. In other words, it represents the element $f(x)$. ♦

Remark 2.1.2.11. Suppose given sets A, B, C and functions $A \xrightarrow{f} B \xrightarrow{g} C$. The *classical order* for writing their composition has been used so far, namely, $g \circ f\colon A \to C$. For any element $a \in A$, we write $g \circ f(a)$ to mean $g(f(a))$. This means "do g to whatever results from doing f to a."

However, there is another way to write this composition, called *diagrammatic order*. Instead of $g \circ f$, we would write $f;g\colon A \to C$, meaning "do f, then do g." Given an element $a \in A$, represented by $a\colon \{☺\} \to A$, we have an element $a; f; g$.

Let X and Y be sets. We write $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ to denote the set of functions $X \to Y$.[3] Note that two functions $f, g\colon X \to Y$ are equal if and only if for every element $x \in X$, we have $f(x) = g(x)$.

Exercise 2.1.2.12.

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y\}$.

a. How many elements does $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ have?

b. How many elements does $\mathrm{Hom}_{\mathbf{Set}}(B, A)$ have?

◊

Solution 2.1.2.12.

a. 32. For example, $1 \mapsto x, 2 \mapsto x, 3 \mapsto x, 4 \mapsto y, 5 \mapsto x$.

b. 25. For example, $x \mapsto 1, y \mapsto 4$.

♦

[3] The notation $\mathrm{Hom}_{\mathbf{Set}}(-, -)$ will make more sense later, when it is seen in a larger context. See especially Section 5.1.

Copyrighted material

Application 2.1.2.16. There is an isomorphism between the set Nuc_{DNA} of nucleotides found in DNA and the set Nuc_{RNA} of nucleotides found in RNA. Indeed, both sets have four elements, so there are 24 different isomorphisms. But only one is useful in biology. Before we say which one it is, let us say there is also an isomorphism $\text{Nuc}_{\text{DNA}} \cong \{A, C, G, T\}$ and an isomorphism $\text{Nuc}_{\text{RNA}} \cong \{A, C, G, U\}$, and we will use the letters as abbreviations for the nucleotides.

The convenient isomorphism $\text{Nuc}_{\text{DNA}} \xrightarrow{\cong} \text{Nuc}_{\text{RNA}}$ is that given by RNA transcription; it sends

$$A \mapsto U, \;\; C \mapsto G, \;\; G \mapsto C, \;\; T \mapsto A.$$

(See also Application 5.1.2.21.) There is also an isomorphism $\text{Nuc}_{\text{DNA}} \xrightarrow{\cong} \text{Nuc}_{\text{DNA}}$ (the matching in the double helix), given by

$$A \mapsto T, \;\; C \mapsto G, \;\; G \mapsto C, \;\; T \mapsto A.$$

Protein production can be modeled as a function from the set of 3-nucleotide sequences to the set of eukaryotic amino acids. However, it cannot be an isomorphism because there are $4^3 = 64$ triplets of RNA nucleotides but only 21 eukaryotic amino acids.

◊◊

Exercise 2.1.2.17.

Let $n \in \mathbb{N}$ be a natural number, and let X be a set with exactly n elements.

a. How many isomorphisms are there from X to itself?

b. Does your formula from part (a) hold when $n = 0$?

◊

Solution 2.1.2.17.

a. There are $n!$, pronounced "n factorial." For example, if $X = \{a, b, c, d\}$, then we have $4! = 4 * 3 * 2 * 1 = 24$ isomorphisms $X \xrightarrow{\cong} X$. One such isomorphism is $a \mapsto a, b \mapsto d, c \mapsto b, d \mapsto b$. The heuristic reason that the answer is 4! is that there are four ways to pick where a goes, but then only three remaining ways to pick where b goes, then only two remaining ways to pick were c goes, and then only one remaining way to pick where d goes. To really understand this answer, list all the isomorphisms $\{1, 2, 3, 4\} \xrightarrow{\cong} \{1, 2, 3, 4\}$ for yourself.

b. Yes, there is one function $\varnothing \to \varnothing$ and it is an isomorphism.

♦

Copyrighted material

Proposition 2.1.2.18. *The following facts hold about isomorphism.*

1. *Any set A is isomorphic to itself; i.e., there exists an isomorphism $A \xrightarrow{\cong} A$.*
2. *For any sets A and B, if A is isomorphic to B, then B is isomorphic to A.*
3. *For any sets A, B, and C, if A is isomorphic to B, and B is isomorphic to C, then A is isomorphic to C.*

Proof.

1. The identity function $\mathrm{id}_A \colon A \to A$ is invertible; its inverse is id_A because $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$.
2. If $f \colon A \to B$ is invertible with inverse $g \colon B \to A$, then g is an isomorphism with inverse f.
3. If $f \colon A \to B$ and $f' \colon B \to C$ are each invertible with inverses $g \colon B \to A$ and $g' \colon C \to B$, then the following calculations show that $f' \circ f$ is invertible with inverse $g \circ g'$:

$$(f' \circ f) \circ (g \circ g') = f' \circ (f \circ g) \circ g' = f' \circ \mathrm{id}_B \circ g' = f' \circ g' = \mathrm{id}_C$$
$$(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f = g \circ \mathrm{id}_B \circ f = g \circ f = \mathrm{id}_A$$

□

Exercise 2.1.2.19.

Let A and B be these sets:

A:= B:

Copyrighted image
Copyrighted image

Note that the sets A and B are isomorphic. Suppose that $f \colon B \to \{1, 2, 3, 4, 5\}$ sends "Bob" to 1, sends ♣ to 3, and sends $r8$ to 4. Is there a canonical function $A \to \{1, 2, 3, 4, 5\}$ corresponding to f?[4] ◊

[4] Canonical, as used here, means something like "best choice," a choice that stands out as the only reasonable one.

Copyrighted material

Solution 2.1.2.19.

No. There are a lot of choices, and none is any more reasonable than any other, i.e., none are canonical. (In fact, there are six choices; do you see why?)

The point of this exercise is to illustrate that even if one knows that two sets are isomorphic, one cannot necessarily treat them as the same. To treat them as the same, one should have in hand a specified *isomorphism* $g\colon A \xrightarrow{\cong} B$, such as $a \mapsto r8, \quad 7 \mapsto$ "*Bob*", $\quad Q \mapsto \clubsuit$. Now, given $f\colon B \to \{1,2,3,4,5\}$, there is a canonical function $A \to \{1,2,3,4,5\}$ corresponding to f, namely, $f \circ g$. ♦

Exercise 2.1.2.20.

Find a set A such that for any set X, there is an isomorphism of sets

$$X \cong \mathrm{Hom}_{\mathbf{Set}}(A, X).$$

Hint: A function $A \to X$ points each element of A to an element of X. When would there be the same number of ways to do that as there are elements of of X? ◊

Solution 2.1.2.20.

Let $A = \{☺\}$. Then to point each element of A to an element of X, one must simply point ☺ to an element of X. The set of ways to do that can be put in one-to-one correspondence with the set of elements of X. For example, if $X = \{1,2,3\}$, then ☺ $\mapsto 3$ is a function $A \to X$ representing the element $3 \in X$. See Notation 2.1.2.9. ♦

Notation 2.1.2.21. For any natural number $n \in \mathbb{N}$, define a set

$$\underline{n} := \{1, 2, 3, \ldots, n\}. \tag{2.4}$$

We call $\underline{n}$ the *numeral set* of size n. So, in particular, $\underline{2} = \{1,2\}, \underline{1} = \{1\}$, and $\underline{0} = \varnothing$.

Let A be any set. A function $f\colon \underline{n} \to A$ can be written as a length n sequence

$$f = (f(1), f(2), \ldots, f(n)). \tag{2.5}$$

We call this the *sequence notation* for f.

Exercise 2.1.2.22.

a. Let $A = \{a, b, c, d\}$. If $f\colon \underline{10} \to A$ is given in sequence notation by $(a, b, c, c, b, a, d, d, a, b)$, what is $f(4)$?

b. Let $s\colon \underline{7} \to \mathbb{N}$ be given by $s(i) = i^2$. Write s in sequence notation.

◊

Copyrighted material

Solution 2.1.2.22.

a. c

b. $(1, 4, 9, 16, 25, 36, 49)$

♦

Definition 2.1.2.23 (Cardinality of finite sets). Let A be a set and $n \in \mathbb{N}$ a natural number. We say that A *has cardinality* n, denoted

$$|A| = n,$$

if there exists an isomorphism of sets $A \cong \underline{n}$. If there exists some $n \in \mathbb{N}$ such that A has cardinality n, then we say that A is *finite*. Otherwise, we say that A is *infinite* and write $|A| \geqslant \infty$.

Exercise 2.1.2.24.

a. Let $A = \{5, 6, 7\}$. What is $|A|$?

b. What is $|\{1, 1, 2, 3, 5\}|$?

c. What is $|\mathbb{N}|$?

d. What is $|\{n \in \mathbb{N} \mid n \leqslant 5\}|$?

◊

Solution 2.1.2.24.

a. $|5, 6, 7| = 3$.

b. $|\{1, 1, 2, 3, 5\}| = 4$. As explained in Solution 2.1.2.8, a set contains each of its elements only once. So we have $\{1, 1, 2, 3, 5\} = \{1, 2, 3, 5\}$, which has cardinality 4.

c. $|\mathbb{N}| \geqslant \infty$.

d. $|\{n \in \mathbb{N} \mid n \leqslant 5\}| = |\{0, 1, 2, 3, 4, 5\}| = 6$.

♦

We will see in Corollary 3.4.5.6 that for any $m, n \in \mathbb{N}$, there is an isomorphism $\underline{m} \cong \underline{n}$ if and only if $m = n$. So if we find that A has cardinality m and that A has cardinality n, then $m = n$.

Copyrighted material

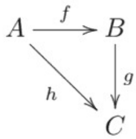
Proposition 2.1.2.25. *Let A and B be finite sets. If there is an isomorphism of sets $f\colon A \to B$, then the two sets have the same cardinality, $|A| = |B|$.*

Proof. If $f\colon A \to B$ is an isomorphism and $B \cong \underline{n}$, then $A \cong \underline{n}$ because the composition of two isomorphisms is an isomorphism.

□

2.2 Commutative diagrams

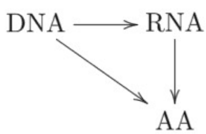
At this point it is difficult to precisely define diagrams or commutative diagrams in general, but we can get a heuristic idea.[5] Consider the following picture:



(2.6)

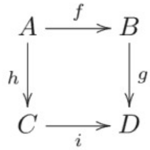
We say this is a *diagram of sets* if each of A, B, C is a set and each of f, g, h is a function. We say this diagram *commutes* if $g \circ f = h$. In this case we refer to it as a commutative triangle of sets, or, more generally, as a *commutative diagram* of sets.

Application 2.2.1.1. In its most basic form, the central dogma of molecular biology is that DNA codes for RNA codes for protein. That is, there is a function from DNA triplets to RNA triplets and a function from RNA triplets to amino acids. But sometimes we just want to discuss the translation from DNA to amino acids, and this is the composite of the other two. The following commutative diagram is a picture of this fact



◊◊

Consider the following picture:



[5]Commutative diagrams are precisely defined in Section 6.1.2.

Copyrighted material

2.3.1.1 Types with compound structures

Many types have compound structures, i.e., they are composed of smaller units. Examples include

a man and a woman	a food portion f and a child c such that c ate all of f	Copyrighted image	(2.9)

It is good practice to declare the variables in a compound type, as in the last two cases of (2.9). In other words, it is preferable to replace the first box in (2.9) with something like

a man m and a woman w	or	a pair (m, w), where m is a man and w is a woman

so that the variables (m, w) are clear.

Rules of good practice 2.3.1.2. A type is presented as a text box. The text in that box should

(i) begin with the word *a* or *an*;

(ii) refer to a distinction made and recognizable by the olog's author;

(iii) refer to a distinction for which instances can be documented;

(iv) be the common name that each instance of that distinction can be called; and

(v) declare all variables in a compound structure.

The first, second, third, and fourth rules ensure that the class of things represented by each box appears to the author to be a well defined set, and that the class is appropriately named. The fifth rule encourages good readability of arrows (see Section 2.3.2).

I do not always follow the rules of good practice throughout this book. I think of these rules being as followed "in the background," but I have nicknamed various boxes. So ⌜Steve⌝ may stand as a nickname for ⌜a thing classified as Steve⌝ and ⌜arginine⌝ as a nickname for ⌜a molecule of arginine⌝. However, one should always be able to rename each type according to the rules of good practice.

Copyrighted material

2.3.2 Aspects

An aspect of a thing x is a way of viewing it, a particular way in which x can be regarded or measured. For example, a woman can be regarded as a person; hence "being a person" is an aspect of a woman. A molecule has a molecular mass (say in daltons), so "having a molecular mass" is an aspect of a molecule. In other words, when it comes to ologs, the word *aspect* simply means function. The domain A of the function $f\colon A \to B$ is the thing we are measuring, and the codomain is the set of possible answers or results of the measurement.

$$\boxed{\text{a woman}} \xrightarrow{\text{is}} \boxed{\text{a person}} \tag{2.10}$$

$$\boxed{\text{a molecule}} \xrightarrow{\text{has as molecular mass (Da)}} \boxed{\text{a positive real number}} \tag{2.11}$$

So for the arrow in (2.10), the domain is the set of women (a set with perhaps 3 billion elements); the codomain is the set of persons (a set with perhaps 6 billion elements). We can imagine drawing an arrow from each dot in the "woman" set to a unique dot in the "person" set, just as in Figure 2.2. No woman points to two different people nor to zero people—each woman is exactly one person—so the rules for a function are satisfied. Let us now concentrate briefly on the arrow in (2.11). The domain is the set of molecules, the codomain is the set $\mathbb{R}_{>0}$ of positive real numbers. We can imagine drawing an arrow from each dot in the "molecule" set to a single dot in the "positive real number" set. No molecule points to two different masses, nor can a molecule have no mass: each molecule has exactly one mass. Note, however, that two different molecules can point to the same mass.

2.3.2.1 Invalid aspects

To be valid an aspect must be a functional relationship. Arrows may on their face appear to be aspects, but on closer inspection they are not functional (and hence not valid as aspects).

Consider the following two arrows:

$$\boxed{\text{a person}} \xrightarrow{\text{has}} \boxed{\text{a child}} \tag{2.12*}$$

$$\boxed{\text{a mechanical pencil}} \xrightarrow{\text{uses}} \boxed{\text{a piece of lead}} \tag{2.13*}$$

Copyrighted material

A person may have no children or may have more than one child, so the first arrow is invalid: it is not a function. Similarly, if one drew an arrow from each mechanical pencil to each piece of lead it uses, one would not have a function.

Warning 2.3.2.2. The author of an olog has a worldview, some fragment of which is captured in the olog. When person A examines the olog of person B, person A may or may not agree with it. For example, person B may have the following olog

Copyrighted image

which associates to each marriage a man and a woman. Person A may take the position that some marriages involve two men or two women and thus see B's olog as wrong. Such disputes are not "problems" with either A's olog or B's olog; they are discrepancies between worldviews. Hence, a reader R may see an olog in this book and notice a discrepancy between R's worldview and my own, but this is not a problem with the olog. Rules are enforced to ensure that an olog is structurally sound, not to ensure that it "correctly reflects reality," since worldviews can differ.

Consider the aspect $\ulcorner\text{an object}\urcorner \xrightarrow{\text{has}} \ulcorner\text{a weight}\urcorner$. At some point in history, this would have been considered a valid function. Now we know that the same object would have a different weight on the moon than it has on earth. Thus, as worldviews change, we often need to add more information to an olog. Even the validity of $\ulcorner\text{an object on earth}\urcorner \xrightarrow{\text{has}} \ulcorner\text{a weight}\urcorner$ is questionable, e.g., if I am considered to be the same object on earth before and after I eat Thanksgiving dinner. However, to build a model we need to choose a level of granularity and try to stay within it, or the whole model would evaporate into the nothingness of truth. Any level of granularity is called *a stereotype*; e.g., we stereotype objects on earth by saying they each have a weight. A stereotype is a lie, more politely a conceptual simplification, that is convenient for the way we want to do business.

Remark 2.3.2.3. In keeping with Warning 2.3.2.2, the arrows in (2.12*) and (2.13*) may not be wrong but simply reflect that the author has an idiosyncratic worldview or vocabulary. Maybe the author believes that every mechanical pencil uses exactly one piece of lead. If this is so, then $\ulcorner\text{a mechanical pencil}\urcorner \xrightarrow{\text{uses}} \ulcorner\text{a piece of lead}\urcorner$ is indeed a valid aspect. Similarly, suppose the author meant to say that each person *was once* a child, or that a person has an inner child. Since every person has one and only one inner child (according to the author), the map $\ulcorner\text{a person}\urcorner \xrightarrow{\text{has as inner child}} \ulcorner\text{a child}\urcorner$ is a valid as-

Copyrighted material

pect. We cannot fault the olog for its author's view, but note that we have changed the name of the label to make the intention more explicit.

2.3.2.4 Reading aspects and paths as English phrases

Each arrow (aspect) $X \xrightarrow{f} Y$ can be read by first reading the label on its source box X, then the label on the arrow f, and finally the label on its target box Y. For example, the arrow

$$\boxed{\text{a book}} \xrightarrow{\text{has as first author}} \boxed{\text{a person}} \tag{2.14}$$

is read "a book has as first author a person."

Remark 2.3.2.5. Note that the map in (2.14) is a valid aspect, but a similarly benign-looking map $\ulcorner\text{a book}\urcorner \xrightarrow{\text{has as author}} \ulcorner\text{a person}\urcorner$ would not be valid, because it is not functional. When creating an olog, one must be vigilant about this type of mistake because it is easy to miss, and it can corrupt the olog.

Sometimes the label on an arrow can be shortened or dropped altogether if it is obvious from context (see Section 2.3.3). Here is a common example from the way I write ologs.

(2.15)

Copyrighted image

Neither arrow is readable by the preceding protocol (e.g., "a pair (x, y), where x and y are integers x an integer" is not an English sentence), and yet it is clear what each map means. For example, given $(8, 11)$ in A, arrow x would yield 8 and arrow y would yield 11. The label x can be thought of as a nickname for the full name "yields as the value of x," and similarly for y. I do not generally use the full name, so as not to clutter the olog.

One can also read paths through an olog by inserting the word *which* (or *who*) after each intermediate box. For example, olog (2.16) has two paths of length 3 (counting

Copyrighted material

arrows in a chain):

(2.16)

Copyrighted image

The top path is read "a child is a person, who has as parents a pair (w, m), where w is a woman and m is a man, which yields, as the value of w, a woman." The reader should read and understand the content of the bottom path, which associates to every child a year.

2.3.2.6 Converting nonfunctional relationships to aspects

There are many relationships that are not functional, and these cannot be considered aspects. Often the word *has* indicates a relationship—sometimes it is functional, as in ⌜a person⌝ $\xrightarrow{\text{has}}$ ⌜a stomach⌝, and sometimes it is not, as in ⌜a father⌝ $\xrightarrow{\text{has}}$ ⌜a child⌝. Clearly, a father may have more than one child. This one is easily fixed by realizing that the arrow should go the other way: there is a function ⌜a child⌝ $\xrightarrow{\text{has}}$ ⌜a father⌝.

What about ⌜a person⌝ $\xrightarrow{\text{owns}}$ ⌜a car⌝. Again, a person may own no cars or more than one car, but this time a car can be owned by more than one person too. A quick fix would be to replace it by ⌜a person⌝ $\xrightarrow{\text{owns}}$ ⌜a set of cars⌝. This is okay, but the relationship between ⌜a car⌝ and ⌜a set of cars⌝ then becomes an issue to deal with later. There is another way to indicate such nonfunctional relationships. In this case it would look like this:

Copyrighted image

Copyrighted material

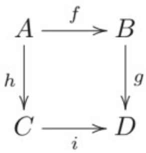
example, consider the two paths from A to C in the olog

(2.17)

Copyrighted image

We know as English speakers that a woman parent is called a mother, so these two paths $A \to C$ should be equivalent. A mathematical way to say this is that the triangle in olog (2.17) *commutes*. That is, path equivalences are simply commutative diagrams, as in Section 2.2. In the preceding example we concisely say "a woman parent is equivalent to a mother." We declare this by defining the diagonal map in (2.17) to be *the composition* of the horizontal map and the vertical map.

I generally prefer to indicate a commutative diagram by drawing a check mark, ✓, in the region bounded by the two paths, as in olog (2.17). Sometimes, however, one cannot do this unambiguously on the two-dimensional page. In such a case I indicate the commutative diagram (fact) by writing an equation. For example, to say that the diagram



commutes, we could either draw a check mark inside the square or write the equation

$${}_A[f, g] \simeq {}_A[h, i]$$

above it.[6] Either way, it means that starting from A, "doing f, then g" is equivalent to "doing h, then i."

[6]We defined function composition in Section 2.1.2, but here we are using a different notation. There we used *classical order*, and our path equivalence would be written $g \circ f = i \circ h$. As discussed in Remark 2.1.2.11, category theorists and others often prefer the *diagrammatic order* for writing compositions, which is $f; g = h; i$. For ologs, we roughly follow the latter because it makes for better English sentences, and for the same reason, we add the source object to the equation, writing ${}_A[f, g] \simeq {}_A[h, i]$.

Copyrighted material

Here is another example:

Copyrighted image

Note how this diagram gives us the established terminology for the various ways in which DNA, RNA, and protein are related in this context.

Exercise 2.3.3.1.

Create an olog for human nuclear biological families that includes the concepts of person, man, woman, parent, father, mother, and child. Make sure to label all the arrows and that each arrow indicates a valid aspect in the sense of Section 2.3.2.1. Indicate with check marks (✓) the diagrams that are intended to commute. If the 2-dimensionality of the page prevents a check mark from being unambiguous, indicate the intended commutativity with an equation. ◊

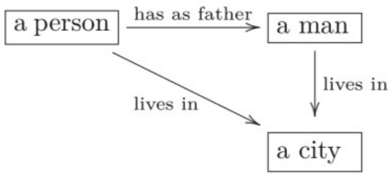
Solution 2.3.3.1.

Copyrighted image

Note that neither of the two triangles from child to person commute. To say that they did commute would be to say that “a child and its mother are the same person” and that “a child and its father are the same person.” ♦

Copyrighted material

Example 2.3.3.2 (Noncommuting diagram). In my conception of the world, the following diagram does not commute:



(2.18)

The noncommutativity of diagram (2.18) does not imply that no person lives in the same city as his or her father. Rather it implies that it is not the case that *every* person lives in the same city as his or her father.

Exercise 2.3.3.3.

Create an olog about a scientific subject, preferably one you think about often. The olog should have at least five boxes, five arrows, and one commutative diagram. ◊

Solution 2.3.3.3.

Copyrighted image

Copyrighted material

To be clear, the check mark in the lower half of the diagram indicates that the square including W commutes; the square that includes W' does not. The reason is that the space of possible definitions for a word includes the meanings for that word in all contexts. The space of definitions for a word in context is smaller than the space of definitions for a word taken out of context, because out of context the meaning of a word is more ambiguous.

In my conception, a word has a space rather than simply a set of meanings. For example, consider the sentence, "He wore a large hat." Here, the word *large* has a space of meanings, though I might say that the space is connected in that the meaning of *large* is fluid but not ambiguous. On the other hand, in the ambiguous sentence, "Kids make nutritious snacks," the word *make* has two disconnected spaces of meanings: either the kids assemble snacks or they are themselves considered to be snacks. ♦

2.3.3.4 A formula for writing facts as English

Every fact consists of two paths, say, P and Q, that are to be declared equivalent. The paths P and Q will necessarily have the same source, say, s, and target, say, t, but their lengths may be different, say, m and n respectively.[7] We draw these paths as

$$P: \quad \overset{a_0=s}{\bullet} \xrightarrow{f_1} \overset{a_1}{\bullet} \xrightarrow{f_2} \overset{a_2}{\bullet} \xrightarrow{f_3} \cdots \xrightarrow{f_{m-1}} \overset{a_{m-1}}{\bullet} \xrightarrow{f_m} \overset{a_m=t}{\bullet} \tag{2.19}$$

$$Q: \quad \overset{b_0=s}{\bullet} \xrightarrow{g_1} \overset{b_1}{\bullet} \xrightarrow{g_2} \overset{b_2}{\bullet} \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} \overset{b_{n-1}}{\bullet} \xrightarrow{g_n} \overset{b_n=t}{\bullet}$$

Every part ℓ of an olog (i.e., every box and every arrow) has an associated English phrase, which we write as $\langle\langle \ell \rangle\rangle$. Using a dummy variable x, we can convert a fact into English too. The following general formula may be a bit difficult to understand (see Example 2.3.3.5). The fact $P \simeq Q$ from (2.19) can be Englished as follows:

(2.20)

Given $x, \langle\langle s \rangle\rangle$ consider the following.
We know that x is $\langle\langle s \rangle\rangle$,
which $\langle\langle f_1 \rangle\rangle$ $\langle\langle a_1 \rangle\rangle$, which $\langle\langle f_2 \rangle\rangle$ $\langle\langle a_2 \rangle\rangle$, which $\ldots$ $\langle\langle f_{m-1} \rangle\rangle$ $\langle\langle a_{m-1} \rangle\rangle$, which $\langle\langle f_m \rangle\rangle$ $\langle\langle t \rangle\rangle$,
that we call $P(x)$.
We also know that x is $\langle\langle s \rangle\rangle$,
which $\langle\langle g_1 \rangle\rangle$ $\langle\langle b_1 \rangle\rangle$, which $\langle\langle g_2 \rangle\rangle$ $\langle\langle b_2 \rangle\rangle$, which $\ldots$ $\langle\langle g_{n-1} \rangle\rangle$ $\langle\langle b_{n-1} \rangle\rangle$, which $\langle\langle g_n \rangle\rangle$ $\langle\langle t \rangle\rangle$,
that we call $Q(x)$.
Fact: Whenever x is $\langle\langle s \rangle\rangle$, we will have $P(x) = Q(x)$.

[7] If the source equals the target, $s = t$, then it is possible to have $m = 0$ or $n = 0$, and the ideas that follow still make sense.

Copyrighted material

Example 2.3.3.5. Consider the olog

(2.21)

Copyrighted image

To put the fact that diagram (2.21) commutes into English, we first English the two paths: F="a person has an address which is in a city" and G="a person lives in a city." The source of both is s = "a person" and the target of both is t = "a city." Write:

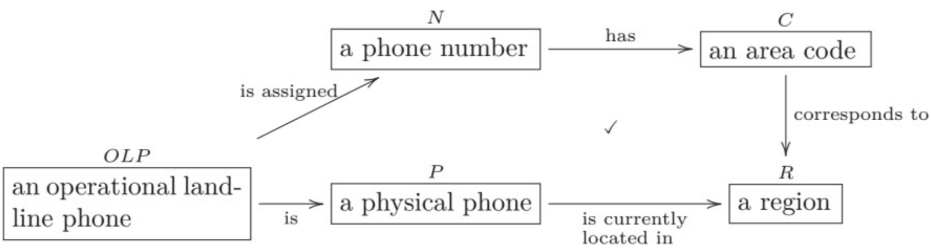
> Given x, a person, consider the following.
> We know that x is a person,
> who has an address, which is in a city,
> that we call $P(x)$.
> We also know that x is a person,
> who lives in a city
> that we call $Q(x)$.
> Fact: Whenever x is a person, we will have $P(x) = Q(x)$.

More concisely, one reads olog 2.21 as

> A person x has an address, which is in a city, and this is the city x lives in.

Exercise 2.3.3.6.

This olog was taken from Spivak [38].



(2.22)

It says that a landline phone is physically located in the region to which its phone number is assigned. Translate this fact into English using the formula from (2.20). ◊

Copyrighted material

assuming each person has a primary residence.

The point is that the notion of image creates new types out of existing aspects, or functions. This connection puts the function first and derives the type from it as its image. A bicycle owner is not a type of person until we have the function that assigns ownership. ♦

Copyrighted material

Chapter 3

Fundamental Considerations in Set

In this chapter we continue to pursue an understanding of sets. We begin by examining how to combine sets in various ways to get new sets. To that end, products and coproducts are introduced, and then more complex limits and colimits, with the aim of conveying a sense of their *universal properties*. The chapter ends with some additional interesting constructions in **Set**.

3.1 Products and coproducts

This section introduces two concepts that are likely to be familiar, although perhaps not by their category-theoretic names: product and coproduct. Each is an example of a large class of ideas that exist far beyond the realm of sets (see Section 6.1.1).

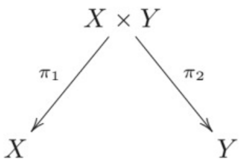
3.1.1 Products

Definition 3.1.1.1. Let X and Y be sets. The *product of* X *and* Y, denoted $X \times Y$, is defined as the set of ordered pairs (x, y), where $x \in X$ and $y \in Y$. Symbolically,

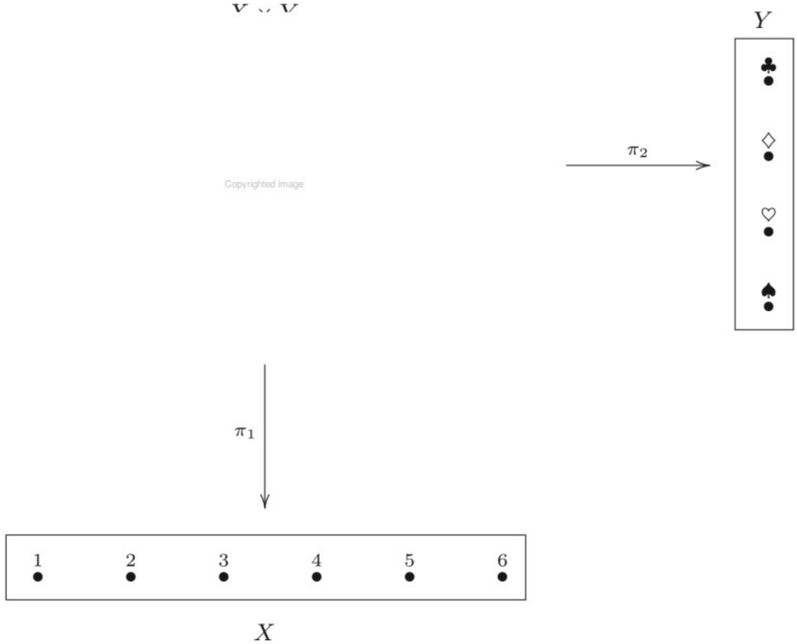
$$X \times Y = \{(x, y) \mid x \in X, \;\; y \in Y\}.$$

Copyrighted material

There are two natural *projection functions*, $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$.



Example 3.1.1.2 (Grid of dots). Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Then we can draw $X \times Y$ as a 6 by 4 grid of dots, and the projections as projections



(3.1)

Application 3.1.1.3. A traditional (Mendelian) way to predict the genotype of offspring based on the genotype of its parents is by the use of Punnett squares. If F is the set of possible genotypes for the female parent, and M is the set of possible genotypes of the male parent, then $F \times M$ is drawn as a square, called a Punnett square, in which every combination is drawn. ◊◊

Exercise 3.1.1.4.

How many elements does the set $\{a, b, c, d\} \times \{1, 2, 3\}$ have? ◊

Copyrighted material

Solution 3.1.1.4.

$4 \times 3 = 12.$ ♦

Application 3.1.1.5. Suppose we are conducting experiments about the mechanical properties of materials, as in Application 2.1.2.2. For each material sample we will produce multiple data points in the set $\ulcorner\text{extension}\urcorner \times \ulcorner\text{force}\urcorner \cong \mathbb{R} \times \mathbb{R}$.

◊◊

Remark 3.1.1.6. It is possible to take the product of more than two sets as well. For example, if A, B, and C are sets, then $A \times B \times C$ is the set of triples

$$A \times B \times C := \{(a, b, c) \mid a \in A, b \in B, c \in C\}.$$

This kind of generality is useful in understanding multiple dimensions, e.g., what physicists mean by ten-dimensional space. It comes under the heading of *limits* (see Section 6.1.3).

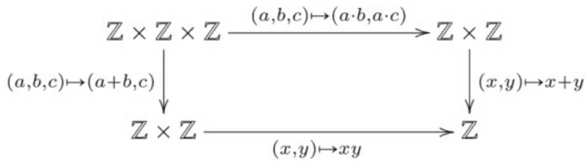
Example 3.1.1.7. Let $\mathbb{R}$ be the set of real numbers. By $\mathbb{R}^2$ we mean $\mathbb{R} \times \mathbb{R}$. Similarly, for any $n \in \mathbb{N}$, we define $\mathbb{R}^n$ to be the product of n copies of $\mathbb{R}$.

According to Penrose [35], Aristotle seems to have conceived of space as something like $S := \mathbb{R}^3$ and of time as something like $T := \mathbb{R}$. Space-time, had he conceived of it, would probably have been $S \times T \cong \mathbb{R}^4$. He, of course, did not have access to this kind of abstraction, which was probably due to Descartes. (The product $X \times Y$ is often called *Cartesian product*, in his honor.)

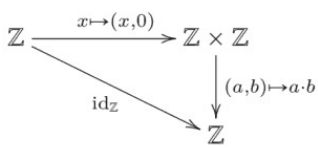
Exercise 3.1.1.8.

Let $\mathbb{Z}$ denote the set of integers, and let $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denote the addition function and $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denote the multiplication function. Which of the following diagrams commute?

a.

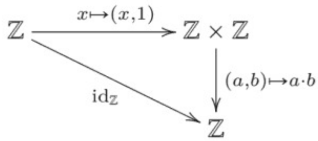


b.



Copyrighted material

c.



◊

Solution 3.1.1.8.

a. This diagram does not commute because $a \cdot b + a \cdot c \neq (a+b) \cdot c$, at least for some integers $a, b, c \in \mathbb{Z}$, e.g., $a = 0, b = 1, c = 1$.

b. This diagram does not commute because $x \cdot 0 \neq x$, at least for some integers $x \in \mathbb{Z}$.

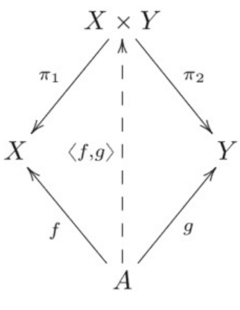
c. This diagram commutes. For every integer $x \in \mathbb{Z}$, we have $x \cdot 1 = x$.

♦

3.1.1.9 Universal property for products

A universal property is an abstract quality that characterizes a given construction. For example, the following proposition says that the product construction is characterized as possessing a certain quality.

Proposition 3.1.1.10 (Universal property for product). *Let X and Y be sets. For any set A and functions $f\colon A \to X$ and $g\colon A \to Y$, there exists a unique function $A \to X \times Y$ such that the following diagram commutes:*



(3.2)

We say this function is induced by *f and g, and we denote it*

$$\langle f, g\rangle\colon A \to X \times Y, \qquad \text{where} \qquad \langle f, g\rangle(a) = (f(a), g(a)).$$

Copyrighted material

b. Consider the map $\langle p_2, p_1\rangle\colon Y \times X \to X \times Y$. Let $s\colon X \times Y \to X \times Y$ be the composite $\langle p_2, p_1\rangle \circ \langle \pi_2, \pi_1\rangle$. We have $\pi_1 \circ s = \pi_1$ by the following calculation:

$$\begin{aligned}\pi_1 \circ s &= \pi_1 \circ \langle p_2, p_1\rangle \circ \langle \pi_2, \pi_1\rangle \\ &= p_2 \circ \langle \pi_2, \pi_1\rangle = \pi_1,\end{aligned}$$

and by a similar calculation, $\pi_2 \circ s = \pi_2$. But we also have $\pi_1 \circ \mathrm{id}_{X\times Y} = \pi_1$ and $\pi_2 \circ \mathrm{id}_{X\times Y} = \pi_2$. Thus the universal property (Proposition 3.1.1.10) implies that $s = \mathrm{id}_{X\times Y}$.

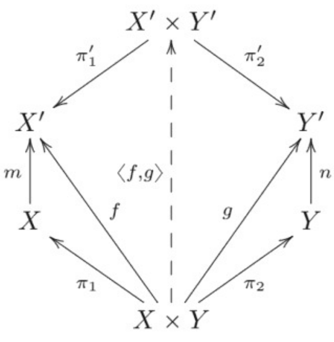
By similar reasoning, if $t\colon Y \times X \to Y \times X$ is the composite $\langle \pi_2, \pi_1\rangle \circ \langle p_2, p_1\rangle$, we can show that $t = \mathrm{id}_{Y\times X}$. By Definition 2.1.2.14, the functions s and t constitute an isomorphism $X \times Y \to Y \times X$.

♦

Example 3.1.1.15. Suppose given sets X, X', Y, Y' and functions $m\colon X \to X'$ and $n\colon Y \to Y'$. We can use the universal property for products to construct a function $s\colon X \times Y \to X' \times Y'$.

The universal property (Proposition 3.1.1.10) says that to get a function from any set A to $X' \times Y'$, we need two functions, namely, some $f\colon A \to X'$ and some $g\colon A \to Y'$. Here we want to use $A := X \times Y$.

What we have readily available are the two projections $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$. But we also have $m\colon X \to X'$ and $n\colon Y \to Y'$. Composing, we set $f := m \circ \pi_1$ and $g := n \circ \pi_2$.



The dotted arrow is often called the *product* of $m\colon X \to X'$ and $n\colon Y \to Y'$. Here it is denoted $\langle f, g\rangle$, but f and g were not given variables. Since writing $\langle m \circ \pi_1, n \circ \pi_2\rangle$ is clunky notation, we instead denote this function

$$m \times n\colon X \times Y \to X' \times Y'.$$

Copyrighted material

3.1.1.16 Ologging products

Given two objects c, d in an olog, there is a canonical label $\langle\langle c \times d \rangle\rangle$ for their product $c \times d$, written in terms of the labels $\langle\langle c \rangle\rangle$ and $\langle\langle d \rangle\rangle$. Namely,

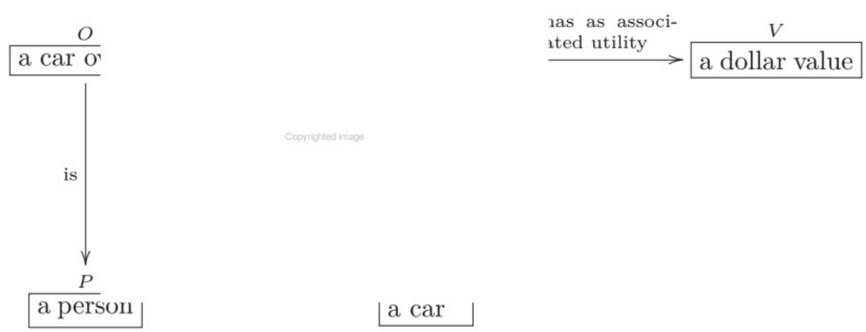
$$\langle\langle c \times d \rangle\rangle := \text{"a pair } (x, y)\text{, where } x \text{ is } \langle\langle c \rangle\rangle \text{ and } y \text{ is } \langle\langle d \rangle\rangle\text{."}$$

The projections $c \leftarrow c \times d \rightarrow d$ can be labeled "yields, as x," and "yields, as y," respectively.

Suppose that e is another object, and $p\colon e \rightarrow c$ and $q\colon e \rightarrow d$ are two arrows. By the universal property for products (Proposition 3.1.1.10), p and q induce a unique arrow $e \rightarrow c \times d$, making the evident diagrams commute. This arrow can be labeled

$$\text{"yields, insofar as it } \langle\langle p \rangle\rangle \langle\langle c \rangle\rangle \text{ and } \langle\langle q \rangle\rangle \langle\langle d \rangle\rangle\text{,"} .$$

Example 3.1.1.17. Every car owner owns at least one car, but there is no obvious function ⌜a car owner⌝ → ⌜a car⌝ because he or she may own more than one. One good choice would be the car that the person drives most often, which can be called his or her primary car. Also, given a person and a car, an economist could ask how much utility the person would get out of the car. From all this we can put together the following olog involving products:



The composite map $O \rightarrow V$ tells us the utility a car owner gets out of their primary car.

3.1.2 Coproducts

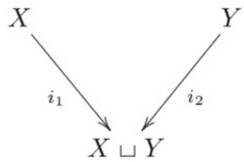
We can characterize the coproduct of two sets with its own universal property.

Definition 3.1.2.1. Let X and Y be sets. The *coproduct of* X *and* Y, denoted $X \sqcup Y$, is defined as the disjoint union of X and Y, i.e., the set for which an element is either an

Copyrighted material

element of X or an element of Y. If something is an element of both X and Y, then we include both copies, and distinguish between them, in $X \sqcup Y$. See Example 3.1.2.2.

There are two natural inclusion functions, $i_1 : X \to X \sqcup Y$ and $i_2 : Y \to X \sqcup Y$.



(3.3)

Example 3.1.2.2. The coproduct of $X := \{a, b, c, d\}$ and $Y := \{1, 2, 3\}$ is

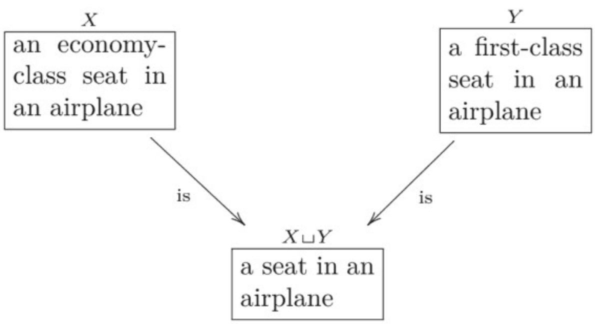
$$X \sqcup Y \cong \{a, b, c, d, 1, 2, 3\}.$$

The coproduct of X and itself is

$$X \sqcup X \cong \{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\}.$$

The names of the elements in $X \sqcup Y$ are not so important. What is important are the inclusion maps i_1, i_2 from (3.3), which ensure that we know where each element of $X \sqcup Y$ came from.

Example 3.1.2.3 (Airplane seats).



(3.4)

Exercise 3.1.2.4.

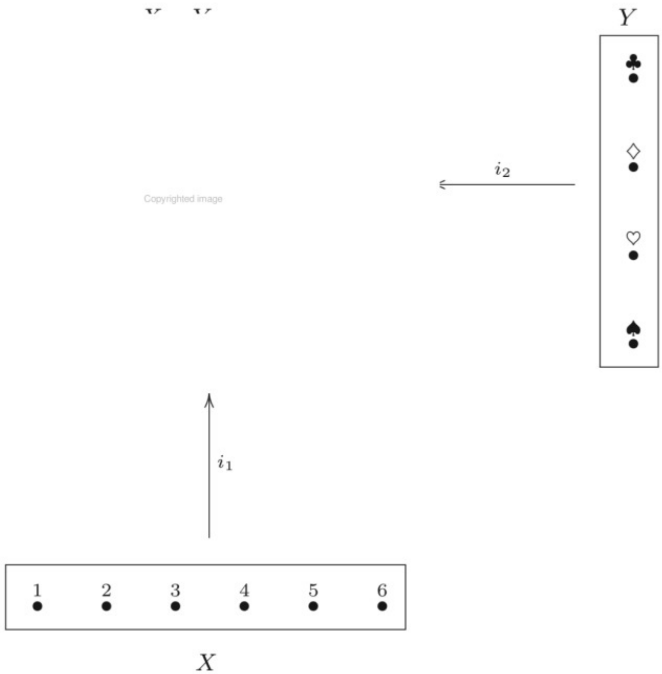
Would you say that ⌜a phone⌝ is the coproduct of ⌜a cell phone⌝ and ⌜a landline phone⌝? ◊

Copyrighted material

Solution 3.1.2.4.

Let's make the case that ⌜a phone⌝ is the coproduct ⌜a cell phone⌝⊔⌜a landline phone⌝. First, there is no overlap between cell phones and landline phones (nothing is both). But is it true that every phone is either a cell phone or a landline? There used to be something called car phones, which were mobile in that they worked from any location but were immobile in the sense that the said location had to be within a given car. So, if at the time this solution is being read, there are phones that are neither landlines nor cell phones, then the answer to this question is no. But if every phone is either a cell phone or a landline, then the answer to this question is yes. ♦

Example 3.1.2.5 (Disjoint union of dots). Below, X and Y are sets, having six and four elements respectively, and $X \sqcup Y$ is their coproduct, which has ten elements.



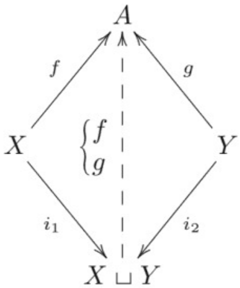
(3.5)

3.1.2.6 Universal property for coproducts

Proposition 3.1.2.7 (Universal property for coproduct). *Let X and Y be sets. For any set A and functions $f\colon X \to A$ and $g\colon Y \to A$, there exists a unique function $X \sqcup Y \to A$*

Copyrighted material

such that the following diagram commutes:



We say this function is induced by f *and* g*, and we denote it* [1]

$$\begin{cases} f \\ g \end{cases} : X \sqcup Y \to A.$$

That is, we have $\begin{cases} f \\ g \end{cases} \circ i_1 = f$ *and* $\begin{cases} f \\ g \end{cases} \circ i_2 = g$*, and* $\begin{cases} f \\ g \end{cases}$ *is the only function for which that is so.*

Proof. Suppose given f, g as in the proposition statement. To provide a function $\ell \colon X \sqcup Y \to A$ is equivalent to providing an element $f(m) \in A$ for each $m \in X \sqcup Y$. We need such a function $\ell = \begin{cases} f \\ g \end{cases}$ such that $\begin{cases} f \\ g \end{cases} \circ i_1 = f$ and $\begin{cases} f \\ g \end{cases} \circ i_2 = g$. But each element $m \in X \sqcup Y$ is either of the form $i_1 x$ or $i_2 y$ and cannot be of both forms. So we assign

$$\begin{cases} f \\ g \end{cases} (m) = \begin{cases} f(x) & \text{if } m = i_1 x, \\ g(y) & \text{if } m = i_2 y. \end{cases} \tag{3.6}$$

This assignment is necessary and sufficient to make all relevant diagrams commute. □

Slogan 3.1.2.8.

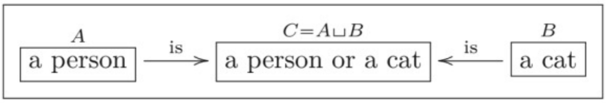
Any time behavior is determined by cases, there is a coproduct involved.

[1] We are using a two-line symbol, which is a bit unusual. A certain function $X \sqcup Y \to A$ is being denoted by the symbol $\begin{cases} f \\ g \end{cases}$, called *case notation*. The reasoning for this will be clear from the proof, especially (3.6).

Copyrighted material

To assign an A value to each element of $X \sqcup Y$, you can delegate responsibility: have one person assign an A value to each element of X, and have another person assign an A value to each element of Y. One function is equivalent to two. ♦

Example 3.1.2.13. In the following olog the types A and B are disjoint, so the coproduct $C = A \sqcup B$ is just the union.



Example 3.1.2.14. In the following olog A and B are not disjoint, so care must be taken to differentiate common elements.

Copyrighted image

Since ducks can both swim and fly, each duck is found twice in C, once labeled "A", a flyer, and once labeled "B", a swimmer. The types A and B are kept disjoint in C, which justifies the name disjoint union.

Exercise 3.1.2.15.

Following Section 3.1.1.16, devise a naming system for coproducts, the inclusions, and the universal maps. Try it out by making an olog (involving coproducts) that discusses the idea that both a .wav file and an .mp3 file can be played on a modern computer. Be careful that your arrows are valid (see Section 2.3.2.1). ◊

Solution 3.1.2.15.

Given two objects c, d in an olog, there is a canonical label "$c \sqcup d$" for their coproduct $c \sqcup d$, written in terms of the labels "c" and "d." Namely,

$$\langle\langle c \sqcup d \rangle\rangle := \text{"}\langle\langle c \rangle\rangle \text{ (indicated as being "}\langle\langle c \rangle\rangle\text{") or } \langle\langle d \rangle\rangle \text{ (indicated as being "}\langle\langle d \rangle\rangle\text{")."}$$

The inclusions $c \to c \sqcup d \leftarrow d$ can be labeled "after being tagged "c" is" and "after being tagged "d" is" respectively.

Copyrighted material

For example,

Copyrighted image

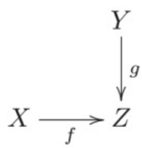
♦

3.2 Finite limits in Set

This section discusses *limits* of variously shaped diagrams of sets. This is made more precise in Section 6.1.3, which discusses arbitrary limits in arbitrary categories.

3.2.1 Pullbacks

Definition 3.2.1.1 (Pullback). Suppose given the following diagram of sets and functions:

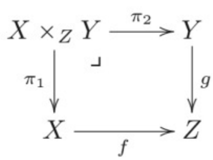


(3.8)

Its *fiber product* is the set

$$X \times_Z Y := \{(x, z, y) \mid f(x) = z = g(y)\}.$$

There are obvious projections $\pi_1 \colon X \times_Z Y \to X$ and $\pi_2 \colon X \times_Z Y \to Y$ (e.g., $\pi_2(x, z, y) = y$). The following diagram commutes:



(3.9)

Copyrighted material

Given the setup of diagram (3.8), we define a *pullback of* X *and* Y *over* Z to be any set W for which we have an isomorphism $W \xrightarrow{\cong} X \times_Z Y$. The corner symbol $\lrcorner$ in diagram (3.9) indicates that $X \times_Z Y$ is a pullback.

Exercise 3.2.1.2.

Let X, Y, Z be as drawn and $f\colon X \to Z$ and $g\colon Y \to Z$ the indicated functions.

What is the fiber product of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$? ◊

Solution 3.2.1.2.

It is the five-element set

$$X \times_Z Y = \{(x_1, z_1, y_1), (x_2, z_2, y_2), (x_2, z_2, y_4), (x_3, z_2, y_2), (x_3, z_2, y_4)\}.$$

♦

Exercise 3.2.1.3.

a. Draw a set X with five elements and a set Y with three elements. Color each element of X and each element of Y red, blue, or green,[2] and do so in a random-looking way. Considering your coloring of X as a function $X \to C$, where $C = \{\text{red, blue, green}\}$, and similarly obtaining a function $Y \to C$, draw the fiber product $X \times_C Y$.

b. The universal property for products guarantees a function $X \times_C Y \to X \times Y$, which will be an injection. This means that the drawing you made of the fiber product can be embedded into the 5×3 grid. Draw the grid and indicate this subset.

◊

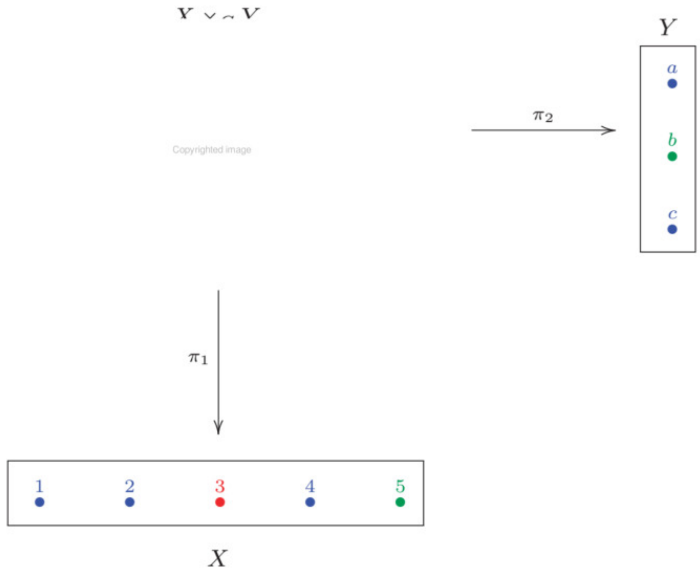
[2]You may use shadings rather than coloring, if you prefer.

Copyrighted material

Solution 3.2.1.3.

a. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c\}$. The fiber product is shown in part (b).

b.



(3.10)

Note that inside the set of $X \times Y = \underline{15}$ possible (x, y) pairs is the set of pairs that agree on color—this is $X \times_C Y$. The grid $X \times Y$ is not drawn, but it includes the drawn dots, $X \times_C Y \subseteq X \times Y$, as well as eight nondrawn dots such as $(3, a)$, which "couldn't agree on a color."

♦

Remark 3.2.1.4. Some may prefer to denote the fiber product in (3.8) by $f \times_Z g$ rather than $X \times_Z Y$. The former is mathematically better notation, but human-readability is often enhanced by the latter, which is also more common in the literature. We use whichever is more convenient.

Exercise 3.2.1.5.

Let $f \colon X \to Z$ and $g \colon Y \to Z$ be functions.

Copyrighted material

a. Suppose that $Y = \varnothing$; what can you say about $X \times_Z Y$?

b. Suppose now that Y is any set but that Z has exactly one element; what can you say about $X \times_Z Y$?

◊

Solution 3.2.1.5.

a. If $Y = \varnothing$, then $X \times_Z Y = \varnothing$ regardless of X, Y, Z, f, and g.

b. We always have that $X \times_Z Y$ is the set of all triples (x, z, y), where $x \in X, y \in Y, z \in Z$ and $f(x) = z = g(y)$. If Z has only one element, say, $Z = \{\smiley\}$, then for all $x \in X$ and $y \in Y$, we have $f(x) = \smiley = g(y)$. So $X \times_{\{\smiley\}} Y = \{(x, \smiley, z) \mid x \in X, y \in Y\}$. But this set is isomorphic to the set $\{(x, y) \mid x \in X, y \in Y\}$. In other words, if Z has one element, then $X \times_Z Y \cong X \times Y$. One way of seeing this is by looking at Exercise 3.2.1.3 and thinking about what happens when there is only one color.

♦

Exercise 3.2.1.6.

Let $S = \mathbb{R}^3, T = \mathbb{R}$, and think of them as (Aristotelian) space and time, with the origin in $S \times T$ given by the center of mass of MIT at the time of its founding. Let $Y = S \times T$, and let $g_1 : Y \to S$ be one projection and $g_2 : Y \to T$ the other projection. Let $X = \{\smiley\}$ be a set with one element, and let $f_1 : X \to S$ and $f_2 : X \to T$ be given by the origin in both cases.

a. What are the fiber products W_1 and W_2:

b. Interpret these sets in terms of the center of mass of MIT at the time of its founding.

◊

Solution 3.2.1.6.

Let $(s_0, t_0) \in S \times T$, where $s_0 \in S$ and $t_0 \in T$, be the center of mass of MIT at the time of its founding.

Copyrighted material

Solution 3.2.1.10.

a. This is appropriate.

b. This is appropriate.

c. This is misleading. If a piece of furniture has precisely the same width as a space in our house, it is not a good fit—it is terribly frustrating but not a fit.

♦

Exercise 3.2.1.11.

Consider your olog from Exercise 2.3.3.1. Are any of the commutative squares in it actually pullback squares? ◊

Solution 3.2.1.11.

Yes, both commutative squares are pullbacks. That is, a mother is a parent who is a woman, and a father is a parent who is a man. ♦

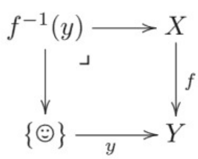
Definition 3.2.1.12 (Preimage). Let $f\colon X \to Y$ be a function and $y \in Y$ an element. The *preimage of* y *under* f, denoted $f^{-1}(y)$, is the subset $f^{-1}(y) := \{x \in X \mid f(x) = y\}$. If $Y' \subseteq Y$ is any subset, the *preimage of* Y' *under* f, denoted $f^{-1}(Y')$, is the subset $f^{-1}(Y') = \{x \in X \mid f(x) \in Y'\}$.

Exercise 3.2.1.13.

Let $f\colon X \to Y$ be a function and $y \in Y$ an element. Draw a pullback diagram in which the fiber product is isomorphic to the preimage $f^{-1}(y)$. ◊

Solution 3.2.1.13.

It is often useful to think of an element $y \in Y$ as a function $y\colon \{☺\} \to Y$, as in Notation 2.1.2.9. Then the following diagram is a pullback:

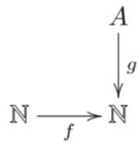


♦

Copyrighted material

Exercise 3.2.1.14.

Consider the function $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n) = n + 3$. Let $A = \{i \in \mathbb{N} \mid i \geqslant 7\}$, and let $g\colon A \to \mathbb{N}$ be the inclusion, e.g., $g(17) = 17$. What is the pullback of the following diagram?



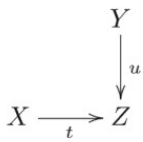
◊

Solution 3.2.1.14.

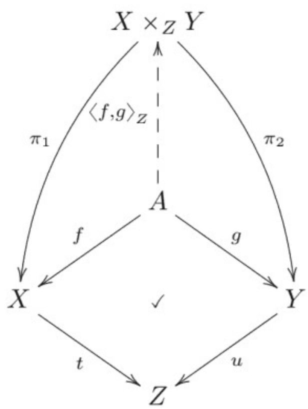
The pullback is isomorphic to the set $\{(n, i) \in \mathbb{N}\times\mathbb{N} \mid n{+}3 = i \geqslant 7\} \cong \{n \in \mathbb{N} \mid n \geqslant 4\}$.

♦

Proposition 3.2.1.15 (Universal property for pullback). *Suppose given the diagram of sets and functions as below:*



For any set A and the following commutative solid-arrow diagram (i.e., functions $f\colon A \to X$ and $g\colon A \to Y$ such that $t \circ f = u \circ g$), there is a unique function $A \to X \times_Z Y$ such that the diagram commutes:



(3.12)

Copyrighted material

Exercise 3.2.1.16.

a. Create an olog whose underlying shape is a commutative square. Now add the fiber product so that the shape is the same as that of diagram (3.12).

b. Use your result to devise English labels to the object $X \times_Z Y$, to the projections π_1, π_2, and to the dotted map $A \xrightarrow{\langle f,g\rangle_Z} X \times_Z Y$, such that these labels are as canonical as possible.

◊

Solution 3.2.1.16.

a.

Copyrighted image

b. The answer to part (a) is not as general as possible, so we proceed in two steps. First we will give a good guess for the answer to part (a), then we will give the general answer.

What makes this example particularly nice is that the function $u\colon Y \to Z$ is labeled "is", suggesting that Y is a subset of Z. In this case, the map π_1 is labeled the same as u (namely, "is"), the map π_2 is labeled the same as t, and the map $\langle f, g\rangle_Z\colon A \to X \times_Z Y$ is labeled the same as f. If $\langle\langle X\rangle\rangle, \langle\langle Y\rangle\rangle, \langle\langle Z\rangle\rangle, \langle\langle t\rangle\rangle$, and $\langle\langle u\rangle\rangle$ are the labels for X, Y, Z, t, and u respectively, then the object $X \times_Z Y$ is labeled "$\langle\langle X\rangle\rangle$, which $\langle\langle t\rangle\rangle\langle\langle Y\rangle\rangle$." For the part (a) example, it would be ⌜a person who has as favorite color blue⌝. See Example 3.4.5.10.

But, in general, we cannot expect either t or u to be an "is." In general, $X \times_Z Y$ should be labeled "a pair (x, z), where x is $\langle\langle X\rangle\rangle$, y is $\langle\langle Y\rangle\rangle$, and $x\langle\langle t\rangle\rangle\langle\langle Z\rangle\rangle$ that is the same

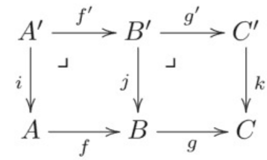
Copyrighted material

as $\langle\langle y\rangle\rangle\langle\langle u\rangle\rangle$." The maps π_1 and π_2 should simply be labeled "yields, as x" and "yields, as y." The map $\langle f, g\rangle_Z$ should be labeled "yields, insofar as it $\langle\langle f\rangle\rangle\langle\langle X\rangle\rangle$ and $\langle\langle g\rangle\rangle\langle\langle Y\rangle\rangle$ and these agree as $\langle\langle X\rangle\rangle\langle\langle t\rangle\rangle\langle\langle Z\rangle\rangle$ and $\langle\langle Y\rangle\rangle\langle\langle u\rangle\rangle\langle\langle Z\rangle\rangle$."

♦

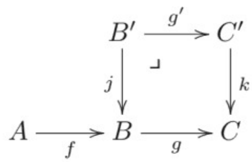
3.2.1.17 Pasting diagrams for pullback

Consider the following diagram, which includes a left-hand square, a right-hand square, and a big rectangle:



The right-hand square has a corner symbol indicating that $B' \cong B \times_C C'$ is a pullback. But the corner symbol in the leftmost corner is ambiguous; it might be indicating that the left-hand square is a pullback, or it might be indicating that the big rectangle is a pullback. It turns out not to be ambiguous because the left-hand square is a pullback if and only if the big rectangle is. This is the content of the following proposition.

Proposition 3.2.1.18. *Consider the diagram:*



where $B' \cong B \times_C C'$ is a pullback. Then there is an isomorphism $A \times_B B' \cong A \times_C C'$. In other words, there is an isomorphism

$$A \times_B (B \times_C C') \cong A \times_C C'.$$

Proof. We first provide a map $\phi\colon A \times_B (B \times_C C') \to A \times_C C'$. An element of $A \times_B (B \times_C C')$ is of the form $(a, b, (b, c, c'))$ such that $f(a) = b, g(b) = c$ and $k(c') = c$. But this implies that $g \circ f(a) = c = k(c')$ so we put $\phi(a, b, (b, c, c')) := (a, c, c') \in A \times_C C'$. Now we provide a proposed inverse, $\psi\colon A \times_C C' \to A \times_B (B \times_C C')$. Given (a, c, c') with $g \circ f(a) = c = k(c')$, let $b = f(a)$ and note that (b, c, c') is an element of $B \times_C C'$. So we can define $\psi(a, c, c') = (a, b, (b, c, c'))$. It is easy to see that ϕ and ψ are inverse.

□

Copyrighted material

Proposition 3.2.1.18 can be useful in authoring ologs. For example, the type ⌜a cell phone that has a bad battery⌝ is vague, but we can lay out precisely what it means using pullbacks:

(3.13)

Copyrighted image

The category-theoretic fact described here says that since $A \cong B \times_D C$ and $C \cong D \times_F E$, it follows that $A \cong B \times_F E$. That is, we can deduce the definition "a cell phone that has a bad battery is defined as a cell phone that has a battery which remains charged for less than one hour."

Exercise 3.2.1.19.

a. Create an olog that defines two people to be "of approximately the same height" if and only if their height difference is less than half an inch, using a pullback. Your olog can include the box ⌜a real number x such that $-.5 < x < .5$⌝.

b. In the same olog, use pullbacks to make a box for those people whose height is approximately the same as a person named "Mary Quite Contrary."

◊

Copyrighted material