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Category Theory in Physics, Mathematics, and Philosophy

INTERNATIONAL CENTER FOR FORMAL ONTOLOGY

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Chapter 1
Why Categories?

Marek Kuś, Bartłomiej Skowron and Krzysztof Wójtowicz

The aim of philosophy, abstractly formulated, is to understand how things in the broadest possible sense of the term hang together in the broadest possible sense of the term. [35]

Mathematical tools are much richer than our everyday intuitions and purely verbal distinctions; they are able to reveal unexpected aspects of reality. [13]

We did not then regard it as a field for further research efforts, but just as a language and an orientation—a limitation which we followed for a dozen years or so, till the advent of adjoint funtors. [26]

Abstract In this article we answer the question of why categories are becoming more and more popular in physics, mathematics and philosophy. The article presents a review of the role of categories in the philosophy of mathematics, in the foundations of mathematics, in metaphysics and in quantum mechanics. Our claim is that category theory is a formal ontology that captures the relational aspects of the given domain in question.

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1.1 Introduction

Eilenberg and Mac Lane—the founders of category theory (CT)—initially treated their invention as a useful language for certain mathematical problems. Its intensive and unexpected development in the twentieth century led to the fact that it was an increasingly popular theory among mathematicians, computer scientists, physicists, engineers and philosophers alike. What has made CT so popular and why is it being used more and more widely in so many different fields of knowledge? This review article is supposed to answer—at least partially—this question.

In the second section (following the introduction) we describe the beginnings of CT. The third section deals with relations between CT and philosophy, and in particular with philosophical problems concerning the foundations and unity of mathematics, as well as the problem of structuralism in the philosophy of mathematics. The third section also describes how CT affects contemporary metaphysics. The fourth section deals with the role of CT in contemporary physics, in particular quantum mechanics.

1.2 The Beginnings of Category Theory

Category theory is a joint work of Samuel Eilenberg and Saunders Mac Lane. Their collaboration in the 1940s led to its creation. More specifically, the emergence of category theory was determined by the combination of Mac Lane's algebraic talent and Eilenberg's topological talent. Here, Mac Lane himself recalls the origins of CT [2, p. 20–1]:

> In the spring of 1941 Michigan invited me to give a series of five or six lectures, so I talked about group extensions. This was a subject on which I had done some work and it came out of my earlier work on valuations with [O. F. G.] Schilling. I had calculated a particular group extension for *p*-adic solenoids. Eilenberg was in the audience, except at the last lecture, and made me give the last lecture to him ahead of time. Then he said, "Well, now that calculation smells like something we do in topology, in a paper of [Norman] Steenrod." So we stayed up all night trying to figure out what the connection was and we discovered one. We wrote our first joint paper on group extensions in homology, which exploited precisely that connection. It so happened that this was a time when more sophisticated algebraic techniques were coming into algebraic topology. Sammy knew much more than I did about the topological background, but I knew about the algebraic techniques and had practice in elaborate algebraic calculations. So our talents fitted together. That's how our collaboration got started. And so it went on for fifteen major papers.

At first, it did not seem that CT would constitute a separate and independent subject of mathematical research. The notion of category was only an auxiliary notion, which was needed for other purposes—it was just an abstract basis for research on the phenomenon of natural equivalences. In their joint article *General Theory of Natural Equivalences* in 1945, which now serves as a classic reference, Eilenberg and Mac Lane [10, p. 247] claimed:

> It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a *functor* and of a natural transformation (...). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Hom" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves.

Eilenberg and Mac Lane introduced very abstract tools into mathematics, which seemed even too abstract. Nevertheless, they motivated their work with both technical merits, which allow for an effective study of the phenomenon of naturality, and conceptual advantages. They noted that the proposed conception is so general that it allows for the detection of the same structures in fundamentally different fields of mathematics. By finding new analogies between different fields of mathematics it suggests new results. Thanks to the fact that categorical glasses allow for the observation of the same structures in both topology and algebra, these glasses allow for a unifying view of mathematics. Already in 1945 it was clear that CT had the power to unify mathematics.

From an ontological point of view, it can be said that Eilenberg and Mac Lane have made a certain shift. Well, mathematical objects in practice are considered as if they were autonomous, separate from other objects. As if they existed as independent substances, whose interior determines what they really are. It is enough to look inside to know what properties they have. This is a standard and natural cognitive approach to mathematical objects. Eilenberg and Mac Lane did it differently, contrary to this natural and widespread attitude. They suggested that mathematical objects should always be considered with their surroundings. If we consider groups, we should consider them together with all homomorphisms, if we consider topological space, we should consider all homeomorphisms. Therefore, we do not consider objects in themselves, but consider them simultaneously with morphisms; in other words, we do not consider individual objects, but *categories* [10, p. 236].

The ontological shift proposed by the fathers of CT has many consequences. Group theory in the categorical approach becomes a study of the invariants of the respective functors. Group theory explores constructions that are covariant or contravariant under homomorphisms. In their words: "group theory studies functors defined on well specified categories of groups, with values in another such category" [10, p. 237]. It was not a completely new approach. The authors themselves have noticed that this is actually an extension of Klein's Erlanger Programme. Geometric space was considered by Klein together with its transformation group, while Eilenberg and Mac Lane suggested that one should consider the categories together with its algebra of morphisms.

The proposal of such a general conceptual framework was not obvious at that time. Mac Lane himself wondered whether categorical concepts had been introduced too early. Here, again, in his own words [26, p. 334–5]:

> It was perhaps a rash step to introduce so quickly such a sweeping generality—an evident piece of what was soon to be called "general abstract nonsense." One of our good friends (an admirer of Eilenberg) read the paper and told us privately that he thought the paper was without any content. Eilenberg took care to see to it that the editor of the *Transactions* sent the manuscript to a young referee (perhaps one who might be gently bullied). The paper was accepted by *Transactions*. I have sometimes wondered what could have happened had the same paper been submitted by a couple of wholly unknown authors. At any rate, we did think it was good, and that it provided a handy language to be used by topologists and others, and that it offered a conceptual view of parts of mathematics, in some way analogous to Felix Klein's "Erlanger programme." We did not then regard it as a field for further research efforts, but just as a language and an orientation—a limitation which we followed for a dozen years or so, till the advent of adjoint funtors.

Nevertheless, such abstract concepts as category hung in the air somewhere at the time. What if Eilenberg and Mac Lane hadn't introduced CT? Mac Lane speculated that other mathematicians would have done it, unless they were afraid of the excessive abstraction of the emerging concepts (cf. [28, p. 210]). Among the potential creators of category theory he listed Claude Chevalley, Heinz Hopf, Norman Steenrod, Henri Cartan, Charles Ehresmann, and John von Neumann (cf. [18, p. 3]).

Category theory developed very quickly and intensively. As one of the breakthrough years for the development of CT Mac Lane [26, p. 346] indicates the year 1963. It was then that Lawvere's groundbreaking dissertation appeared, which contained categorical descriptions of algebraic theories and many other important ideas. That year also marked the first public presentation of the adjoint functor theorem by Freyd, Ehresmann published his paper on what we call internal categories, Mac Lane's first coherence theorem also appeared in 1963. And SGA IV was also published—the seminar notes from the *Séminaire de Géométrie Algébrique du Bois Marie* run by Alexander Grothendieck. As Mac Lane [26, p. 347] estimates between 1962 and 1967 around 60 people started working in category theory. Already in 1965 in California, at the CT conference, Lawvere delivered the talk "The category of categories as a foundation of mathematics" [26, p. 351].

The classic (still advanced) textbook in category theory *Categories for the Working Mathematician* was published by Mac Lane for the first time in 1976. In 1992 Mac Lane, jointly with Ieke Moerdijk, published *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. In 2017, a textbook for philosophers was published: *Categories for the Working Philosopher*, edited by Elaine Landry [19]. In 2006 Steve Awodey, Mac Lane's last Ph.D. student, wrote a textbook *Category Theory* that was easier to read then the textbook by Mac Lane. That's why Awodey felt the need to write a new textbook [3, p. v]:

> Why write a new textbook on Category Theory, when we already have Mac Lane's *Categories for the Working Mathematician*? Simply put, because Mac Lane's book is for the working (and aspiring) mathematician. What is needed now, after 30 years of spreading into various other disciplines and places in the curriculum, is a book for everyone else.

Let's see what CT looks like in Poland. Due to the intensive development of category theory, students of the first years of mathematical studies can now study CT in some departments in Poland. Categorical terms are often introduced in other courses, e.g. in topological courses. The first (and so far the only) CT textbook in Polish was *Wstęp do teorii kategorii i funktorów* published by Zbigniew Semadeni and Antoni Wiweger in 1972.

1.3 Category Theory and Philosophy

Category theory, as well as set theory and, unlike algebraic topology, aroused the interest of philosophers from the very beginning. Already in the classic paper [10, p. 247] there are comments on the foundations of category theory (but not yet the foundations of mathematics!), in particular there are references to an unramified theory of types or to the Fraenkel–von Neumann–Bernays' system—these systems are mentioned as possible solutions for the ontological foundation of categories.

Multidimensional relations occur between category theory and philosophy.[1] Below we will restrict ourselves to a brief discussion of four selected themes:

1. the discussion between mathematical structuralism and object realism;
2. the problem of foundations of mathematics;
3. the problem of the unity of mathematics;
4. the role of CT in contemporary metaphysics.

1.3.1 Structuralism Versus Object Realism

One of the fundamental philosophical question concerns the nature of mathematical concepts—which, under the realistic interpretation, is formulated as the ontological question concerning the status and properties of mathematical objects. Speaking in very general terms, the question is, whether mathematical objects have any intrinsic properties, or whether their properties are purely relative. Earlier we pointed to this issue when we mentioned the ontological shift of Eilenberg and Mac Lane. A related question concerns the identity criterion for mathematical objects: is identity determined by some immanent properties of the objects, or rather purely by the relations in which this object stands to other mathematical objects?

Mathematical structuralism rejects the view, that there are intrinsic properties. According to the structuralist point of view, an object is constituted by the relations to other objects. For instance, the function of a president (of a country or a company) is defined regardless of the particular person holding the office—and only due to the place within the whole structure (country or company).

Consider natural numbers. According to the structuralist picture, natural numbers can be characterized only by their role in the natural numbers structure (i.e. in the ω-sequence). They have no intrinsic properties, and what really matters are only

[1]In this volume, see [8, 17, 23, 37, 44].

relationships between natural numbers. Indeed, the question, whether the number 5 has its properties regardless of the existence of other natural numbers seems strange—for instance, whether it is still prime. The notion of primeness would not make sense if only one number existed. This way of thinking is then extended to other mathematical objects: according to this view, the only properties of mathematical objects are relational properties—so their identity is determined only via the roles they play in mathematical structures. We can say, that mathematical objects are like vertices in a giant graph or—as Mac Lane (see [24, Chap. XII entitled *The Mathematical Network*]) would probably say—network.

The argument between structuralist and non-structuralists is a prototypical example of an relativist-essentialist argument in metaphysics [34, p. 86]:

> Quine has captured the incompleteness of mathematical objects in his doctrine of ontological relativity: there is no fact of the matter as to whether the ontology of one theory is included in that of another except relative to an interpretation of the former in the latter. What I have tried to do so far is show that Quine's surprising doctrine is what we would expect to hold in mathematics.

The categorical point of view seems to be particularly well-suited to express the intuitions of mathematical structuralism. Take a typical expression of the structuralist position [34, p. 84]:

> For me, mathematical objects have no distinguishing characteristics except those they have by virtue of their relationships to other positions in the structures to which they belong. In short, I take the geometrical point, the paradigm position, as a paradigmatic mathematical object.

The familiar illustration of categories in terms of graphs (with black dots and arrows) suits this view very well. All the black dots look alike—and the only things, that determines their identity is the place within the graph, in particular the arrows which connect the dot with the other dots. The properties of objects within a category are defined only via the respective morphisms. For instance, the "essence" of being the initial or terminal object is captured by the morphisms—not by any immanent properties, which are secondary. Also the notion of identity in some sense loses its meaning, as we only speak of isomorphisms within a category (and this is obviously a category-relative notion).

Objects and morphisms within the category live on the first level of abstraction. They might be considered to be natural environment(s) for different kinds of mathematical objects. But what is much more interesting and conceptually fertile are the relationships between categories, expressed in terms of functors (and higher-order constructions, like natural transformations). Categories—in a sense—become "dots" and are viewed from a higher level.

Coming back to the natural numbers example we can not only think of the particular numbers as points ("dots") within a structure (so that the "dots" inherit their properties from the structure they live in). We can also think of the natural number structure in terms of its universal properties within a "higher-order structure", i.e. the Peano category, as Mazur [29, p. 231–2] claims:

> This strategy of defining the Natural Numbers as "an" initial object in a category of (what amounts to) discrete dynamical systems, as we have just done, is revealing, I think; it isolates, as Peano himself had done, the fundamental role of mere succession in the formulation of the natural numbers.

A paradigm example of a structure with purely relational properties is a group. It does not make sense to ask, what the neutral element "really is": of course, a group can have many representations (as number, matrices, linear transformations—and many others), but the "essence" of the neutral element is exactly being neutral with respect to the operation (and not being a 15×15 identity matrix or a certain function).

1.3.2 The Foundations of Mathematics

The problem of finding a suitable foundation for the mathematical edifice has been discussed extensively for at least 150 years—and the crisis in foundations of mathematics around 1900 made the topic very hot.[2] There are many mathematical disciplines, which *prima facie* seem very different—like geometry and algebra in the historical sense of these terms (today they are of course "entangled" in a profound way, and the term "algebraic geometry" illustrates that). Do these different disciplines have common roots?

According to the most widespread view, set theory can serve the role of a foundational theory. Indeed, set theory is so strong and general that virtually all mathematical notions previously used by mathematicians in an informal way (e.g. the concepts of: natural, rational, real, complex numbers; real-valued function; probability; differentiation in the real and complex sense; Banach space; differential manifold, etc.) can be formally reconstructed in the language of set theory. But there is a tension between the fact, that mathematical notions can be formally represented in set theory—but on the other hand, that they have a meaning outside the context of set theory, and lead a happy life without ever noticing, that (according to some foundationalists) they are really sets. In fact, mathematical practice does not really need set theory—apart from some elementary textbook facts.[3] More advanced set-theoretic notions (large cardi-

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[2]Perhaps today the discussions are not so emotional, and there is no crisis in sight: even if some philosophical doubts can be formulated, mathematics seems to be doing well even without any (official) foundations. But this is more of a pragmatic than a fundamental issue.

[3]The problem of how much set theory is needed, and whether this set theory is really "set-theoretic set theory" or perhaps "category-theoretic set theory" is discussed for instance by Colin McLarty in [30, p. 2]:

> All support the claim that mathematicians know and use the concepts and axioms of the Elementary Theory of the Category of Sets (ETCS), often without knowing or caring that they are the ETCS axioms.

He examines the example of two standard and influential textbooks on topology and algebra. McLarty also claims:

nal, Boolean-valued model) are very remote from everyday mathematical practice and play virtually no role there.

A standard illustration are natural numbers. Usually, on the pre-theoretic level, we consider them to be objects per se—and the computations performed on them have a "self-contained character". But from the point of view of the set-theoretic foundations of mathematics, natural numbers are just sets. For instance, in the (rather standard) von Neumann representation, natural numbers are identified with finite ordinal numbers.[4] But there are many representations of this kind, and all of them are admissible, from the logical point of view.[5] But from the point of view of the working mathematician (for instance, a number theorist), the set-theoretic reduction of numbers plays no role, as mathematicians obviously do not think of natural numbers as being sets obtained from an empty set of set-theoretic operations! Indeed, the question, whether $0 \in 2$ seems rather awkward, and is surely irrelevant for, say, the Twin Primes Conjecture. So the problem of set-theoretic reduction seems artificial from the point of view of mathematical practice: no mathematician is really worried about the problem of translating the results into some awkward logical notation (into the language with one two-place predicate $\in$, i.e. into set theory).

The foundational enterprise has a theoretic character, just like the formalizability postulate: no mathematician bothers with formalizing proofs to the full extent, however, this postulate (the common belief, that proofs can—in principle—be formalized) serves as a kind of methodological warrant. Complaints about the artificial character of the reductions can be countered with the observations, that our everyday habits are not a warrant for methodological correctness. We are free to use informal, and even metaphorical language even in science—as long as we are aware of the fact, and are able to provide rigorous paraphrases.

Category theory is believed (perhaps not very widely—but very firmly!) to be a good candidate for a foundational theory. But the notion of "foundation" is different than in the case of set theory. It is not about isolating one single notion or a theory (like "set" and "set theory") which appears in the ultimate definiendum of all mathematical notions. This point of view is typical for set-theoretic foundationalism. Adherents of category-theoretic foundations stress the fact, that CT takes mathematics "at face value": mathematical notions are not forced into the Procrustean bed of set theory (or some other formal theory we choose), but are free to live within their natural conceptual environment: groups populate the category Grp; topological spaces live within Top; partial orders also have their own place to live. Sets are

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the category of sets described in ETCS is a closer fit to the practical needs of most mathematicians than is the cumulative hierarchy of sets described in ZFC.

[4]The empty set becomes 0; the singleton $\{\emptyset\}$ becomes 1; $\{\emptyset, \{\emptyset\}\}$ plays the role of 2, *etc.* The successor functions is defined as a set-theoretic operation: $n + 1 = n \cup \{n\}$, and the resulting sequence of natural numbers is: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$

[5]An example of a different representation is: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\{\{\emptyset\}\}\}\}, \ldots$—i.e. the empty set is 0, and the rule for successor is: $n + 1 = \{n\}$. This leads to the famous multiple-reduction problem in philosophy of mathematics (the *locus classicus* is [5]). As there are many possible reductions, it is not clear, that any of them is the proper one. So perhaps identifying numbers with sets is not legitimate.

indeed important—they have their category of their own—but this is just one of many categories, by far neither the only, nor the most important one. The notion of set is one of many natural mathematical notions—just like the notion of continuity. And even the logic is not absolute—a topos might have some internal logic, and could serve as a way of interpreting mathematical notions within it.[6]

So this way of thinking is not quite reductionist—it might rather be considered as providing a useful, enlightening template for interpreting mathematical notions—within their respective domains. This constitutes a strong shift in thinking about mathematical notions: what is most interesting about them, are their universal properties. A tensor product is important primarily because it is an object representing an important functor, not interesting per se. Even the Cartesian product of two sets is rather viewed as an object with appropriate arrows—and this is important, not the particular ordered pairs.

An important general insight is therefore perhaps: mathematics does not need foundation—but organization. As Mac Lane [24, p. 406] put it:

> Alternatively, set theory and category theory may be viewed as proposals for the organization of Mathematics. The canons of set theory provide guides to the formulation of new concepts and emphasize the extensional character of Mathematics: A "property" is completely determined by knowing all the elements which have that property. Similarly, the canons of category theory emphasize the importance of considering not just the objects but also their morphisms. They also emphasize the use of universal constructions and their associated adjoint functors.

For Mac Lane, none of these proposals are fully successful. Category theory works well in algebra and topology, but not so well in analysis. Set theory, for Mac Lane, contains many artificial constructions and, as Mac Lane [24, p. 407] has repeatedly said following Hermann Weyl: "it contains far too much sand".

CT serves as a method for organizing mathematics rather as a foundation. Nevertheless, CT still has ambitions to replace set theory's pride of place. In recent years, homotopy type theory and univalent foundations (HoTT/UF) have been intensively developed and has become a serious competitor to set theory. In principle, HoTT/UF can serve as a foundation for mathematics. It is a paradigm that marks a new way of thinking about the foundations of mathematics, which more faithfully than set theory, represents everyday mathematical practice (in this respect CT has always challenged set theory) on the one hand, and on the other HoTT/UF "is suited to computer systems and has been implemented in existing proof assistants" [42, p. 7].

In homotopy type theory the idea of a collection is realized by a type, just as in set theory the idea of a collection is realized by a set. The elements of types are points. A set is made up only of elements, but the type is made up of both points and ways of identifying points. Two points can be the same in many ways (let's take the sets $\{a, b\}$ and $\{c, d\}$ as the points and bijection as a relation of being the same, in this case there is more than one bijection; or: two topological spaces can be homeomorphic in many ways), so types also consist of ways in which elements are the same.

[6]This problem is discussed in the context of Topos Quantum Theory in [44].

Types are therefore certain spaces, not pure sets, more specifically they are ω-grupoids. In set theory the relation of being the same is given a priori, from the very beginning, as being in a sense ready. In homotopy type theory, if a collection is considered, reasons for "being the same element" must be considered simultaneously. For the foundations of mathematics it is important that the universe of sets is an appropriate part of the universe of types (through ETCS), therefore the model of ZFC can be reconstructed in HoTT/UF. Thus, if set theory reconstructs mathematical objects, then HoTT/UF does it all the more (for a more detailed explanation see a brief discussion of the idea in [38] and a full presentation of HoTT/UF in [42]). Undoubtedly HoTT/UF is much richer—ontologically speaking—than set theory [42, p. 1]:

> Homotopy type theory also brings new ideas into the very foundation of mathematics. On the one hand, there is Voevodsky's subtle and beautiful *univalence axiom*. The univalence axiom implies, in particular, that isomorphic structures can be identified, a principle that mathematicians have been happily using on workdays, despite its incompatibility with the "official" doctrines of conventional foundations. On the other hand, we have *higher inductive types*, which provide direct, logical descriptions of some of the basic spaces and constructions of homotopy theory: spheres, cylinders, truncations, localizations, etc. Both ideas are impossible to capture directly in classical set-theoretic foundations (...).

1.3.3 The Problem with Unifying Mathematics

> *Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level. [31]*

Notwithstanding the fact, that there are dozens of mathematical disciplines (and hundreds of subdisciplines), it is an empirical datum, that mathematics is a unity. Notions, methods, theorems are transferred in a very natural way from one mathematical discipline to another; examples are abundant. But in the history of mathematics, these theories often emerged as separate disciplines, and great contributions in the history of mathematics consisted often in transferring concepts from discipline D_1 to discipline D_2—applying algebra in geometry to solve classic questions (like the angle trisection) is the most obvious illustration. This mutual applicability of mathematical disciplines is almost a "raw data", a phenomenon which requires an explanation. The insights from CT might provide it.

From the point of view of reductive foundationalism (for instance—set theoretic), the answer is very natural: mathematics is a unity just because of the fact that all of Mathematics can be reduced to the Fundamental Theory (whatever it is). But this is not exactly what mathematical practice suggests: the mutual applications of diverse mathematical notions and results is not the result of translating everything into set-theoretic language, and then proceeding within set theory. The conceptual links are much more direct, that via a set-theoretic translation.

So, it is natural to ask, whether the phenomenon of unity could be explained not by postulating a reduction to set theory, but in a more direct way. This is the point of view of CT: the unity of mathematics stems from the fact that all mathematical theories and objects live within their natural environment(s) (which are the respective category/ies)—and the unity is explained by investigating the relationships between these categories. From this point of view, some notions become natural, and "environment-relative"—one of the simplest examples is the notion of isomorphism. In set theory there is an enormous amount of the "implementations" of this notion (isomorphism of groups is something different than the isomorphism of rings, even if the underlying set is the same[7]). In CT the isomorphism between objects in a category is defined via the properties of the morphisms. And the "transfer" of isomorphisms between different categories is explained via functorial notions (like the homeomorphism of topological spaces and algebraic isomorphisms). We might say, that the "essence" of the familiar notion of isomorphism (in all its variants) is captured by one category-theoretic definition.

In the development of mathematics, isolating proper, core notions for a discipline (e.g. by finding appropriate primitive notions for an axiomatization) was a natural problem. Identifying such notions can have great explanatory value, in a sense—metaphorically speaking—this is about identifying "the essence of the theory". And the proper identification can lead to a fertile conceptual recasting—being an explanatory presentation of a discipline.

This statement is perhaps difficult to grasp in a precise way: what does it mean, that a theory is presented in an explanatory or a non-explanatory way? An illuminating example is presented in [32], where Pringsheim's presentation of complex analysis is discussed.[8] According to Mancosu [32, p. 108]:

> The original approach to complex analysis defended by Pringsheim is based on the claim that only according to his method it is possible to "explain" a great number of results, which in previous approaches, in particular Cauchy's, remain mysterious and unexplained.

Of course, this is not a new theory, there are no new theorems—rather, this is an example of a shift in perspective. Mancosu's example does not concern CT in any way—but illustrates the phenomenon. Providing a reformulation of a theory can provide important insights. And the problem of "immersing" different subjects into one conceptual system is obviously connected with questions concerning the unity of a subject.

At this point it is worth mentioning that CT in the person of F. W. Lawvere led to a peculiar demythologisation of Gödel's famous theorems. It turns out that both Gödel's incompleteness theorem and Russell's paradox, as well as Cantor's theorem and Tarski's undefinability theorem—all these results are instances of a simple categorical construction. An interested reader should take a look at Lawvere's work in [21], and a reader who is not familiar with CT yet can easily follow Lawvere's result, thanks to N. Yanofsky's accessible introduction to the subject in [45] without the use of a CT-toolbox. This observation by Lawvere is a good example of how CT naturally

[7] Isomorphism of Lie groups *as* groups is something quite different than their isomorphism *as* topological spaces.

[8] Without going into details, it takes the notion of the mean value of a function as basic—and the other (standard) definitions are treated as derived facts.

finds surprising similarities between different mathematical phenomena. It is probably even more surprising that the topological operator of the interior is connected with the inclusion operator in the same way as the existential quantifier with the general quantifier—these are the examples of adjoint functors that were noticed in principle only thanks to CT. An overview of many examples of adjoint functors can be found in Mac Lane's classic work *Categories for the Working Mathematician* [27]. The issue of adjointness in the foundations of mathematics (understood not as "starting-point" or "justification" but as the study of what is universal in mathematics) is discussed by Lawvere in his work *Adjointness in Foundations* [20]. Lawvere shows how one can understand the game of *Formal* and *Conceptual* aspects of mathematics in the context of their adjointness.

CT is not an object-level theory, it is perhaps rather "a theory of theories"—and its contributions to our understanding are at a quite abstract level. As Spivak [41, p. 400] put it:

> Category theory is not a theory of everything. It is more like, as topologist Jack Morava put it, (...) "a theory of theories of anything". In other words, it is a model of models. It leaves each subject alone to solve its own problems, to sharpen and refine its toolset in the ways it sees fit. That is, CT does not micromanage in the affairs of any discipline. However, describing any discipline categorically tends to bring increased conceptual clarity, because conceptual clarity is CT's main concern, its domain of expertise. (...) Finally, category theory allows one to compare different models, thus carrying knowledge from one domain to another, as long as one can construct the appropriate "analogy", i.e., functor.

And this might be characteristic of CT: it takes on a new perspective. This new perspective is indeed an ontological shift that we mentioned earlier. It is a change of ontological form, whereby form, following Ingarden (in his *Controversy over the Existence of the World* [15]) we mean something radically non-qualitative. The form can be a parthood and a wholehood, a substance and its properties (the form here is the subject of properties) or exactly relationality, as in the case of CT. Existence in CT is only and exclusively being in relation. Hence, ontologically speaking, Eilenberg and Mac Lane made a formal-ontological shift towards pure relationality. This is not a technical (in the mathematical sense) shift, many mathematicians use both CT and set theory simultaneously. However, the difference between CT and set theory lies in the fundamentally different ontology behind them, hence discussions between fierce category theorists and set theorists resemble discussions between metaphysicists. CT is essentially a formal metaphysics of mathematics that defends a different vision of the world.

1.3.4 Category Theory in Metaphysics

Spivak [41, p. 382] pointed out that CT has served science as a modeling language for various studied phenomena:

> There is a good deal of work on using category theory to model high-level conceptual aspects of scientific subjects. For example, categories have been used by John Baez to model

1.4 Physics: Category Theory as an *Ontology Rescuing* Tool

Category theory entered physics seemingly only as a new language for old theories. But, in fact, one of the original purposes of the categorical (topos) approach to quantum mechanics, as presented in seminal papers of Christopher Isham, Andreas Döring, and Jeremy Butterfield has been to find an alternative *logical* foundation for the orthodox quantum theory [9, 16]. The most fundamental result of such a shift can be briefly explained by the following observations. The classical and quantum descriptions of the physical world differ considerably on the mathematical level. Classical systems are described in terms of a *phase space*, usually a differential manifold, its (measurable) subsets, coordinate systems, etc. Observables, i.e. physical quantities that we can measure, or, in general to which we can ascribe certain numerical values characterizing the observed system, are functions on the phase space. Observables, such like positions, momenta, energies, angular momenta etc. are some properties of systems like particles, ensembles of particles, rigid bodies, etc. They can change in time, but are properties that are possessed by systems alone and do not depend on whether or not they are actually measured at a particular moment. Moreover, at least in principle, we can measure them without disturbing them. Consequently, measurements can be performed in an arbitrary order, or even simultaneously, and provide the same results. We can thus pose questions about *exact* values of, say, the position and the momentum of a particle. Usually, however, due to e.g. inaccuracies of measurements we inquire into the probability that our particle is in a certain subset of the phase space. Such a probability is determined by the volumes of the relevant subsets.

Quantum mechanics offers a completely different picture. Here we do not have a phase space in the form of a manifold. Instead a system is described in terms of vectors and operators in a Hilbert space. We may ascribe to each system some properties that pretend to be the quantum analogues of classical ones like positions, momenta, angular momenta, energies, etc. (and some others that seem to be of a purely quantum mechanical nature, like spin, isospin, strangeness, hypercharge etc.). However, they are no longer *intrinsic* in the classical sense. They are not "carried" by a system during its evolution, rather they are "brought to life" by an act of measurement, which can be interpreted as an impossibility of a non-disturbing experiment. Each act of measurement disturbs the actual state of a system by bringing it to another state corresponding to a result of the measurement performed. Hence, the order in which measurements are taken does matter, and some measurements can not be taken simultaneously (the uncertainty principle). Moreover, although results of measurements depend on the actual state of the system prior to the act of a measurement, they do it only in a probabilistic manner. This is because for each observable (position, momentum, angular momentum, energy, spin, etc.) there is a corresponding selfadjoint operator, the eigenvalues of which are possible outcomes with probabilities depending on the state of a system before the measurement and the eigenvectors determine possible states after it.

It is thus clear that it is rather hard to find a unifying ontological basis for classical and quantum physics. The ontological status of such fundamental elements of physical reality, as positions, momenta, angular momenta, etc. have radically different ontological status in both theories. Whereas they are intrinsic and objective properties of a physical system, it is not so in quantum theory.

From a purely physical point of view this is not a danger. Ultimately, physics is an experimental science. It can and should answer experimental questions about outcomes of various measurements. Such an approach clearly puts more emphasis on the epistemology, moving apart, or even totally discarding ontological issues.

As an attempt to unify classical and quantum physics on common epistemological ground one can treat the quantum logic approach that goes back to Birkhoff and von Neumann [6]. The main idea is to analyze the structure of elementary experimental question/propositions about a system. In classical physics, elementary propositions can be reduced to statements that values of observed quantities (coordinates) belong to a certain subset of the phase space. The logical structure of the set of such propositions, determined by the rules concerning their negations, conjunctions and disjunctions isomorphically reflects the Boole algebra structure of the set of (measurable) subsets of the phase space. One of the characteristic features of a Boolean structure is the distributivity law, allowing for the distribution of conjunctions over disjunctions and vice versa.

In quantum mechanics elementary propositions concern positions of state vectors (characterizing a state of a system) with respect to eigenspaces of observables (self-adjoint operators in Hilbert space). As in the classical case we can ask composite questions corresponding to conjunctions and disjunctions. However, the ensuing logical structure is no longer distributive. The logic of a system described by a Hilbert space $\mathscr{H}$ is represented by the orthomodular lattice of closed subspaces in $\mathscr{H}$. The involution sending a subspace to its orthogonal complement represents logical negation, satisfying the law of an excluded middle: measuring the spin of an electron will yield either ‘up’ or ‘down’, *tertium non datur*. As said, the resulting lattice is non-distributive: x-spin up does not imply x-spin up and z-spin up or x-spin up and z-spin down (the incompatibility of the two measurements is reflected by the non-distributivity of the sub-lattice they ‘generate’, just as by the non-commutativity of the corresponding sub-algebra of operators). Having the lattice stand for the logic of the system, one derives its probability theory where states assign ‘probabilities’ to elements of the lattice, respecting the underlying structure (order and complementation). These states turn out to coincide with the usual density matrices by a celebrated theorem of Gleason (as long as $\dim \mathscr{H} \geq 3$, [11]).

Despite differences in the logical structures of both theories, such an approach definitely provides a unifying picture for the whole physics. The differences themselves reflect precisely the dissimilarities between the two theories. Does it really mean that we achieved the goal and we can look at classical quantum physics from the same point of view? For supporters of the purely epistemological approach briefly described above, most probably yes. But for those who pay more attention to the ontological basis of physical theories it might be disappointing. It seems that category theory could be called to come to the rescue.