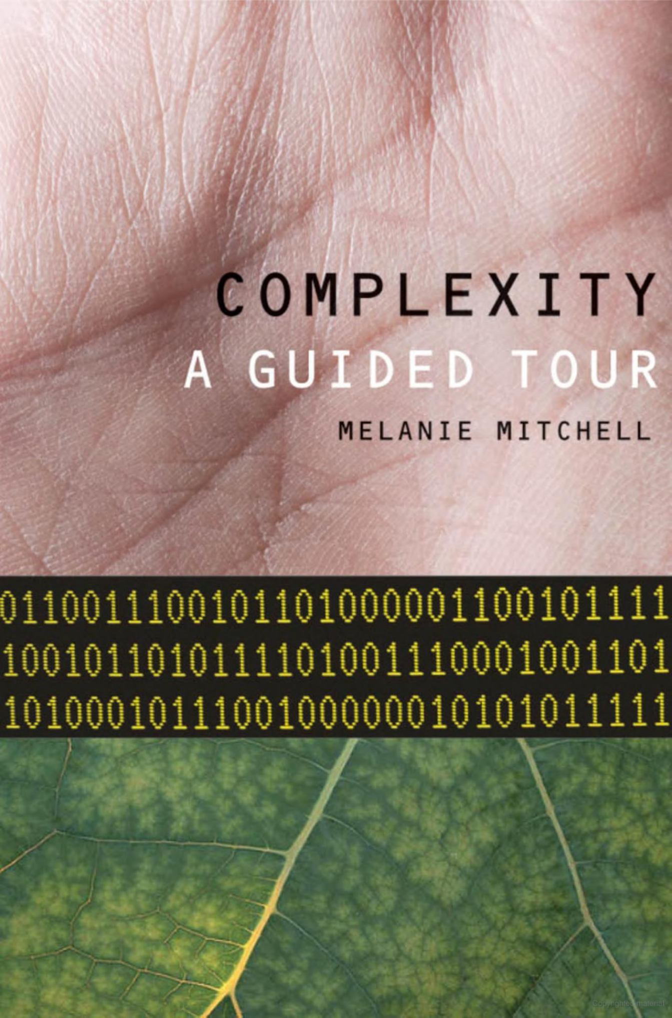
COMPLEXITY
A GUIDED TOUR
MELANIE MITCHELL
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MELANIE MITCHELL

Complexity

A Guided Tour

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PREFACE

REDUCTIONISM is the most natural thing in the world to grasp. It's simply the belief that "a whole can be understood completely if you understand its parts, and the nature of their 'sum.'" No one in her left brain could reject reductionism.

—Douglas Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid*

Reductionism has been the dominant approach to science since the 1600s. René Descartes, one of reductionism's earliest proponents, described his own scientific method thus: "to divide all the difficulties under examination into as many parts as possible, and as many as were required to solve them in the best way" and "to conduct my thoughts in a given order, beginning with the *simplest* and most easily understood objects, and gradually ascending, as it were step by step, to the knowledge of the most *complex*."[1]

Since the time of Descartes, Newton, and other founders of the modern scientific method until the beginning of the twentieth century, a chief goal of science has been a reductionist explanation of all phenomena in terms of fundamental physics. Many late nineteenth-century scientists agreed with the well-known words of physicist Albert Michelson, who proclaimed in 1894 that "it seems probable that most of the grand underlying principles have been firmly established and that further advances are to be sought chiefly in

1. Full references for all quotations are given in the notes.

the rigorous application of these principles to all phenomena which come under our notice."

Of course within the next thirty years, physics would be revolutionized by the discoveries of relativity and quantum mechanics. But twentieth-century science was also marked by the demise of the reductionist dream. In spite of its great successes explaining the very large and very small, fundamental physics, and more generally, scientific reductionism, have been notably mute in explaining the complex phenomena closest to our human-scale concerns.

Many phenomena have stymied the reductionist program: the seemingly irreducible unpredictability of weather and climate; the intricacies and adaptive nature of living organisms and the diseases that threaten them; the economic, political, and cultural behavior of societies; the growth and effects of modern technology and communications networks; and the nature of intelligence and the prospect for creating it in computers. The antireductionist catch-phrase, "the whole is more than the sum of its parts," takes on increasing significance as new sciences such as chaos, systems biology, evolutionary economics, and network theory move beyond reductionism to explain how complex behavior can arise from large collections of simpler components.

By the mid-twentieth century, many scientists realized that such phenomena cannot be pigeonholed into any single discipline but require an interdisciplinary understanding based on scientific foundations that have not yet been invented. Several attempts at building those foundations include (among others) the fields of cybernetics, synergetics, systems science, and, more recently, the science of complex systems.

In 1984, a diverse interdisciplinary group of twenty-four prominent scientists and mathematicians met in the high desert of Santa Fe, New Mexico, to discuss these "emerging syntheses in science." Their goal was to plot out the founding of a new research institute that would "pursue research on a large number of highly complex and interactive systems which can be properly studied only in an interdisciplinary environment" and "promote a unity of knowledge and a recognition of shared responsibility that will stand in sharp contrast to the present growing polarization of intellectual cultures." Thus the Santa Fe Institute was created as a center for the study of complex systems.

In 1984 I had not yet heard the term *complex systems*, though these kinds of ideas were already in my head. I was a first-year graduate student in Computer Science at the University of Michigan, where I had come to study *artificial intelligence*; that is, how to make computers think like people. One of my motivations was, in fact, to understand how *people* think—how abstract reasoning, emotions, creativity, and even consciousness emerge from trillions of tiny brain cells and their electrical and chemical communications. Having been deeply enamored of physics and reductionist goals, I was going through my own antireductionist epiphany, realizing that not only did current-day physics have little, if anything, to say on the subject of intelligence but that even neuroscience, which actually focused on those brain cells, had very little understanding of how thinking arises from brain activity. It was becoming clear that the reductionist approach to cognition was misguided—we just couldn't understand it at the level of individual neurons, synapses, and the like.

Therefore, although I didn't yet know what to call it, the program of complex systems resonated strongly with me. I also felt that my own field of study, computer science, had something unique to offer. Influenced by the early pioneers of computation, I felt that *computation* as an idea goes much deeper than operating systems, programming languages, databases, and the like; the deep ideas of computation are intimately related to the deep ideas of life and intelligence. At Michigan I was lucky enough to be in a department in which "computation in natural systems" was as much a part of the core curriculum as software engineering or compiler design.

In 1989, at the beginning of my last year of graduate school, my Ph.D. advisor, Douglas Hofstadter, was invited to a conference in Los Alamos, New Mexico, on the subject of "emergent computation." He was too busy to attend, so he sent me instead. I was both thrilled and terrified to present work at such a high-profile meeting. It was at that meeting that I first encountered a large group of people obsessed with the same ideas that I had been pondering. I found that they not only had a name for this collection of ideas—complex systems—but that their institute in nearby Santa Fe was exactly the place I wanted to be. I was determined to find a way to get a job there.

Persistence, and being in the right place at the right time, eventually won me an invitation to visit the Santa Fe Institute for an entire summer. The summer stretched into a year, and that stretched into additional years. I eventually became one of the institute's resident faculty. People from many different countries and academic disciplines were there, all exploring different sides of the same question. How do we move beyond the traditional paradigm of reductionism toward a new understanding of seemingly irreducibly complex systems?

The idea for this book came about when I was invited to give the Ulam Memorial Lectures in Santa Fe—an annual set of lectures on complex systems for a general audience, given in honor of the great mathematician Stanislaw Ulam. The title of my lecture series was "The Past and Future of the Sciences of Complexity." It was very challenging to figure out how to introduce the audience of nonspecialists to the vast territory of complexity, to give them a feel for what is already known and for the daunting amount that remains to be learned. My role was like that of a tour guide in a large, culturally rich foreign country. Our schedule permitted only a short time to hear about the historical background, to visit some important sites, and to get a feel for the landscape and culture of the place, with translations provided from the native language when necessary.

This book is meant to be a much expanded version of those lectures—indeed, a written version of such a tour. It is about the questions that fascinate me and others in the complex systems community, past and present: How is it that those systems in nature we call *complex* and *adaptive*—brains, insect colonies, the immune system, cells, the global economy, biological evolution—produce such complex and adaptive behavior from underlying, simple rules? How can interdependent yet self-interested organisms come together to cooperate on solving problems that affect their survival as a whole? And are there any general principles or laws that apply to such phenomena? Can life, intelligence, and adaptation be seen as mechanistic and computational? If so, could we build truly intelligent and *living* machines? And if we could, would we want to?

I have learned that as the lines between disciplines begin to blur, the content of scientific discourse also gets fuzzier. People in the field of complex systems talk about many vague and imprecise notions such as spontaneous order, self-organization, and emergence (as well as "complexity" itself). A central purpose of this book is to provide a clearer picture of what these people are talking about and to ask whether such interdisciplinary notions and methods are likely to lead to useful science and to new ideas for addressing the most difficult problems faced by humans, such as the spread of disease, the unequal distribution of the world's natural and economic resources, the proliferation of weapons and conflicts, and the effects of our society on the environment and climate.

The chapters that follow give a guided tour, flavored with my own perspectives, of some of the core ideas of the sciences of complexity—where they came from and where they are going. As in any nascent, expanding, and vital area of science, people's opinions will differ (to put it mildly) about what the core ideas are, what their significance is, and what they will lead to. Thus my perspective may differ from that of my colleagues. An important part of this book will be spelling out some of those differences, and I'll do my best to provide glimpses of areas in which we are all in the dark or just beginning to see some light. These are the things that make science of this kind so stimulating, fun, and worthwhile both to practice and to read about. Above all else, I hope to communicate the deep enchantment of the ideas and debates and the incomparable excitement of pursuing them.

This book has five parts. In part I I give some background on the history and content of four subject areas that are fundamental to the study of complex systems: information, computation, dynamics and chaos, and evolution. In parts II–IV I describe how these four areas are being woven together in the science of complexity. I describe how life and evolution can be mimicked in computers, and conversely how the notion of *computation* itself is being imported to explain the behavior of natural systems. I explore the new science of networks and how it is discovering deep commonalities among systems as disparate as social communities, the Internet, epidemics, and metabolic systems in organisms. I describe several examples of how complexity can be measured in nature, how it is changing our view of living systems, and how this new view might inform the design of intelligent machines. I look at prospects of computer modeling of complex systems, as well as the perils of such models. Finally, in the last part I take on the larger question of the search for general principles in the sciences of complexity.

No background in math or science is needed to grasp what follows, though I will guide you gently and carefully through explorations in both. I hope to offer value to scientists and nonscientists alike. Although the discussion is not technical, I have tried in all cases to make it substantial. The notes give references to quotations, additional information on the discussion, and pointers to the scientific literature for those who want even more in-depth reading.

Have you been curious about the sciences of complexity? Would you like to come on such a guided tour? Let's begin.

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I am grateful to the Santa Fe Institute (SFI) for inviting me to direct the Complex Systems Summer School and to give the Ulam Memorial Lectures, both of which spurred me to write this book. I am also grateful to SFI for providing me with a most stimulating and productive scientific home for many years. The various scientists who are part of the SFI family have been inspiring and generous in sharing their ideas, and I thank them all, too numerous to list here. I also thank the SFI staff for the ever-friendly and essential support they have given me during my association with the institute.

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This book is dedicated to Douglas Hofstadter and John Holland, who have done so much to inspire and encourage me in my work and life. I am very lucky to have had the benefit of their guidance and friendship.

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PART I

Background and History

Science has explored the microcosmos and the macrocosmos; we have a good sense of the lay of the land. The great unexplored frontier is complexity.

—Heinz Pagels, *The Dreams of Reason*

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CHAPTER 1 | What Is Complexity?

Ideas thus made up of several simple ones put together, I call Complex; such as are Beauty, Gratitude, a Man, an Army, the Universe.

—John Locke, *An Essay Concerning Human Understanding*

Brazil: The Amazon rain forest. Half a million army ants are on the march. No one is in charge of this army; it has no commander. Each individual ant is nearly blind and minimally intelligent, but the marching ants together create a coherent fan-shaped mass of movement that swarms over, kills, and efficiently devours all prey in its path. What cannot be devoured right away is carried with the swarm. After a day of raiding and destroying the edible life over a dense forest the size of a football field, the ants build their nighttime shelter—a chain-mail ball a yard across made up of the workers' linked bodies, sheltering the young larvae and mother queen at the center. When dawn arrives, the living ball melts away ant by ant as the colony members once again take their places for the day's march.

Nigel Franks, a biologist specializing in ant behavior, has written, "The solitary army ant is behaviorally one of the least sophisticated animals imaginable," and, "If 100 army ants are placed on a flat surface, they will walk around and around in never decreasing circles until they die of exhaustion." Yet put half a million of them together, and the group as a whole becomes what some have called a "superorganism" with "collective intelligence."

How does this come about? Although many things are known about ant colony behavior, scientists still do not fully understand all the mechanisms underlying a colony's collective intelligence. As Franks comments further, "I have studied *E. burchelli* [a common species of army ant] for many years, and for me the mysteries of its social organization still multiply faster than the rate at which its social structure can be explored."

The mysteries of army ants are a microcosm for the mysteries of many natural and social systems that we think of as "complex." No one knows exactly how any community of social organisms—ants, termites, humans—come together to collectively build the elaborate structures that increase the survival probability of the community as a whole. Similarly mysterious is how the intricate machinery of the immune system fights disease; how a group of cells organizes itself to be an eye or a brain; how independent members of an economy, each working chiefly for its own gain, produce complex but structured global markets; or, most mysteriously, how the phenomena we call "intelligence" and "consciousness" emerge from nonintelligent, nonconscious material substrates.

Such questions are the topics of *complex systems*, an interdisciplinary field of research that seeks to explain how large numbers of relatively simple entities organize themselves, without the benefit of any central controller, into a collective whole that creates patterns, uses information, and, in some cases, evolves and learns. The word *complex* comes from the Latin root *plectere*: to weave, entwine. In complex systems, many simple parts are irreducibly entwined, and the field of complexity is itself an entwining of many different fields.

Complex systems researchers assert that different complex systems in nature, such as insect colonies, immune systems, brains, and economies, have much in common. Let's look more closely.

Insect Colonies

Colonies of social insects provide some of the richest and most mysterious examples of complex systems in nature. An ant colony, for instance, can consist of hundreds to millions of individual ants, each one a rather simple creature that obeys its genetic imperatives to seek out food, respond in simple ways to the chemical signals of other ants in its colony, fight intruders, and so forth. However, as any casual observer of the outdoors can attest, the ants in a colony, each performing its own relatively simple actions, work together to build astoundingly complex structures that are clearly of great importance for the survival of the colony as a whole. Consider, for example, their use of soil, leaves, and twigs to construct huge nests of great strength and stability, with large networks of underground passages and dry, warm, brooding chambers whose temperatures are carefully controlled by decaying nest materials and the ants' own bodies. Consider also the long bridges certain species of ants build with their own bodies to allow emigration from one nest site to another via tree branches separated by great distances (to an ant, that is) (figure 1.1). Although much is now understood about ants and their social structures, scientists still can fully explain neither their individual nor group behavior: exactly how the individual actions of the ants produce large, complex structures, how the ants signal one another, and how the colony as a whole adapts to changing circumstances (e.g., changing weather or attacks on the colony). And how did biological evolution produce creatures with such an enormous contrast between their individual simplicity and their collective sophistication?

The Brain

The cognitive scientist Douglas Hofstadter, in his book *Gödel, Escher, Bach*, makes an extended analogy between ant colonies and brains, both being

FIGURE 1.1. Ants build a bridge with their bodies to allow the colony to take the shortest path across a gap. (Photograph courtesy of Carl Rettenmeyer.)

complex systems in which relatively simple components with only limited communication among themselves collectively give rise to complicated and sophisticated system-wide ("global") behavior. In the brain, the simple components are cells called *neurons*. The brain is made up of many different types of cells in addition to neurons, but most brain scientists believe that the actions of neurons and the patterns of connections among groups of neurons are what cause perception, thought, feelings, consciousness, and the other important large-scale brain activities.

Neurons are pictured in figure 1.2 (top). Neurons consists of three main parts: the cell body (*soma*), the branches that transmit the cell's input from other neurons (*dendrites*), and the single trunk transmitting the cell's output to other neurons (*axon*). Very roughly, a neuron can be either in an active state (*firing*) or an inactive state (*not firing*). A neuron fires when it receives enough signals from other neurons through its dendrites. *Firing* consists of sending an electric pulse through the axon, which is then converted into a chemical signal via chemicals called *neurotransmitters*. This chemical signal in turn activates other neurons through their dendrites. The firing frequency and the resulting chemical output signals of a neuron can vary over time according to both its input and how much it has been firing recently.

These actions recall those of ants in a colony: individuals (neurons or ants) perceive signals from other individuals, and a sufficient summed strength of these signals causes the individuals to act in certain ways that produce additional signals. The overall effects can be very complex. We saw that an explanation of ants and their social structures is still incomplete; similarly, scientists don't yet understand how the actions of individual or dense networks of neurons give rise to the large-scale behavior of the brain (figure 1.2, bottom). They don't understand what the neuronal signals mean, how large numbers of neurons work together to produce global cognitive behavior, or how exactly they cause the brain to think thoughts and learn new things. And again, perhaps most puzzling is how such an elaborate signaling system with such powerful collective abilities ever arose through evolution.

The Immune System

The immune system is another example of a system in which relatively simple components collectively give rise to very complex behavior involving signaling and control, and in which adaptation occurs over time. A photograph illustrating the immune system's complexity is given in figure 1.3.

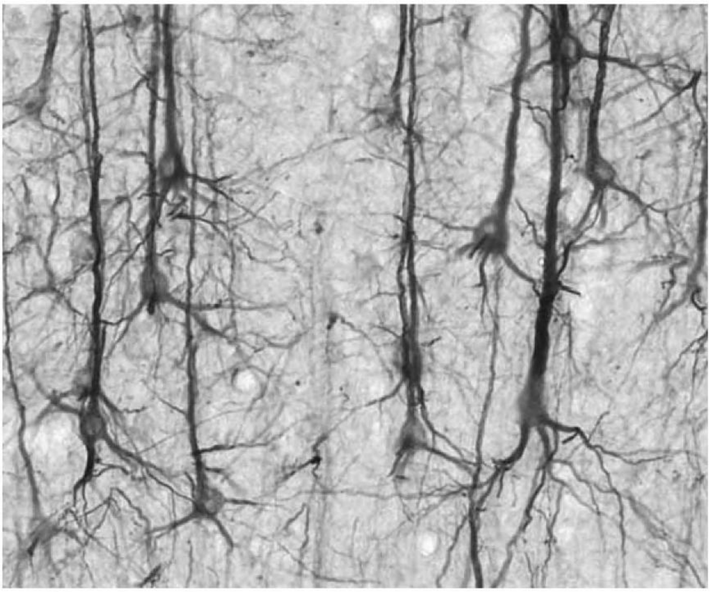


FIGURE 1.2. Top: microscopic view of neurons, visible via staining. Bottom: a human brain. How does the behavior at one level give rise to that of the next level? (Neuron photograph from brainmaps.org [http://brainmaps.org/smi32-pic.jpg], licensed under Creative Commons [http://creativecommons.org/licenses/by/3.0/]. Brain photograph courtesy of Christian R. Linder.)

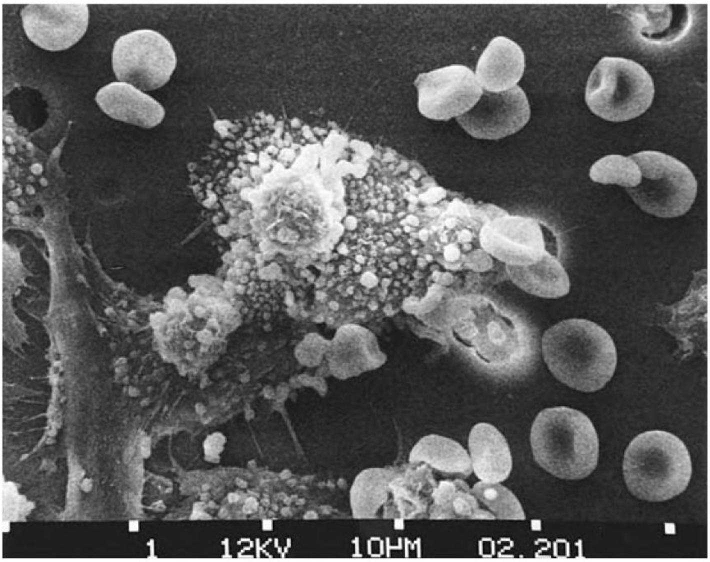


FIGURE 1.3. Immune system cells attacking a cancer cell. (Photograph by Susan Arnold, from National Cancer Institute Visuals Online [http://visualsonline.cancer.gov/details.cfm?imageid=2370].)

The immune system, like the brain, differs in sophistication in different animals, but the overall principles are the same across many species. The immune system consists of many different types of cells distributed over the entire body (in blood, bone marrow, lymph nodes, and other organs). This collection of cells works together in an effective and efficient way without any central control.

The star players of the immune system are white blood cells, otherwise known as *lymphocytes*. Each lymphocyte can recognize, via receptors on its cell body, molecules corresponding to certain possible invaders (e.g., bacteria). Some one trillion of these patrolling sentries circulate in the blood at a given time, each ready to sound the alarm if it is *activated*—that is, if its particular receptors encounter, by chance, a matching invader. When a lymphocyte is activated, it secretes large numbers of molecules—*antibodies*—that can identify similar invaders. These antibodies go out on a seek-and-destroy mission throughout the body. An activated lymphocyte also divides at an increased rate, creating daughter lymphocytes that will help hunt out invaders and secrete antibodies against them. It also creates daughter lymphocytes that will hang around and remember the particular invader that was seen, thus giving the body immunity to pathogens that have been previously encountered.

One class of lymphocytes are called *B cells* (the *B* indicates that they develop in the bone marrow) and have a remarkable property: the better the match between a B cell and an invader, the more antibody-secreting daughter cells the B cell creates. The daughter cells each differ slightly from the mother cell in random ways via mutations, and these daughter cells go on to create their own daughter cells in direct proportion to how well they match the invader. The result is a kind of Darwinian natural selection process, in which the match between B cells and invaders gradually gets better and better, until the antibodies being produced are extremely efficient at seeking and destroying the culprit microorganisms.

Many other types of cells participate in the orchestration of the immune response. *T cells* (which develop in the thymus) play a key role in regulating the response of B cells. *Macrophages* roam around looking for substances that have been tagged by antibodies, and they do the actual work of destroying the invaders. Other types of cells help effect longer-term immunity. Still other parts of the system guard against attacking the cells of one's own body.

Like that of the brain and ant colonies, the immune system's behavior arises from the independent actions of myriad simple players with no one actually in charge. The actions of the simple players—B cells, T cells, macrophages, and the like—can be viewed as a kind of chemical signal-processing network in which the recognition of an invader by one cell triggers a cascade of signals among cells that put into play the elaborate complex response. As yet many crucial aspects of this signal-processing system are not well understood. For example, it is still to be learned what, precisely, are the relevant signals, their specific functions, and how they work together to allow the system as a whole to "learn" what threats are present in the environment and to produce long-term immunity to those threats. We do not yet know precisely how the system avoids attacking the body; or what gives rise to flaws in the system, such as autoimmune diseases, in which the system does attack the body; or the detailed strategies of the human immunodeficiency virus (HIV), which is able to get by the defenses by attacking the immune system itself. Once again, a key question is how such an effective complex system arose in the first place in living creatures through biological evolution.

Economies

Economies are complex systems in which the "simple, microscopic" components consist of people (or companies) buying and selling goods, and the collective behavior is the complex, hard-to-predict behavior of markets as a whole, such as changes in the price of housing in different areas of the country or fluctuations in stock prices (figure 1.4). Economies are thought by some economists to be adaptive on both the microscopic and macroscopic level. At the microscopic level, individuals, companies, and markets try to increase their profitability by learning about the behavior of other individuals and companies. This microscopic self-interest has historically been thought to push markets as a whole—on the macroscopic level—toward an equilibrium state in which the prices of goods are set so there is no way to change production or consumption patterns to make everyone better off. In terms of profitability or consumer satisfaction, if someone is made better off, someone else will be made worse off. The process by which markets obtain this equilibrium is called *market efficiency*. The eighteenth-century economist Adam Smith called this self-organizing behavior of markets the "invisible hand": it arises from the myriad microscopic actions of individual buyers and sellers.

Economists are interested in how markets become efficient, and conversely, what makes efficiency fail, as it does in real-world markets. More recently, economists involved in the field of complex systems have tried to explain market behavior in terms similar to those used previously in the descriptions of other complex systems: dynamic hard-to-predict patterns in global behavior, such as patterns of market bubbles and crashes; processing of signals and information, such as the decision-making processes of individual buyers and sellers, and the resulting "information processing" ability of the market as a whole to "calculate" efficient prices; and adaptation and learning, such as individual sellers adjusting their production to adapt to changes in buyers' needs, and the market as a whole adjusting global prices.

The World Wide Web

The World Wide Web came on the world scene in the early 1990s and has experienced exponential growth ever since. Like the systems described above, the Web can be thought of as a self-organizing social system: individuals, with little or no central oversight, perform simple tasks: posting Web pages and linking to other Web pages. However, complex systems scientists have discovered that the network as a whole has many unexpected large-scale properties involving its overall structure, the way in which it grows, how information propagates over its links, and the coevolutionary relationships between the behavior of search engines and the Web's link structure, all of which lead to what could be called "adaptive" behavior for the system as a whole. The

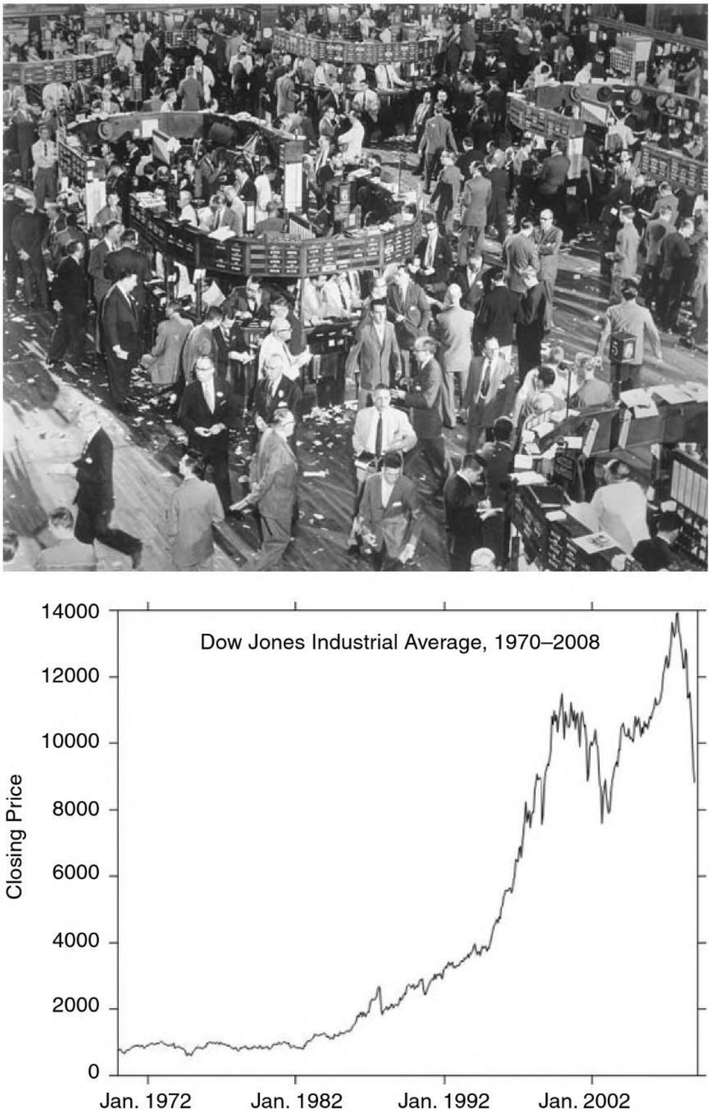


FIGURE 1.4. Individual actions on a trading floor give rise to the hard-to-predict large-scale behavior of financial markets. Top: New York Stock Exchange (photograph from Milstein Division of US History, Local History and Genealogy, The New York Public Library, Astor, Lenox, and Tilden Foundations, used by permission). Bottom: Dow Jones Industrial Average closing price, plotted monthly 1970–2008.

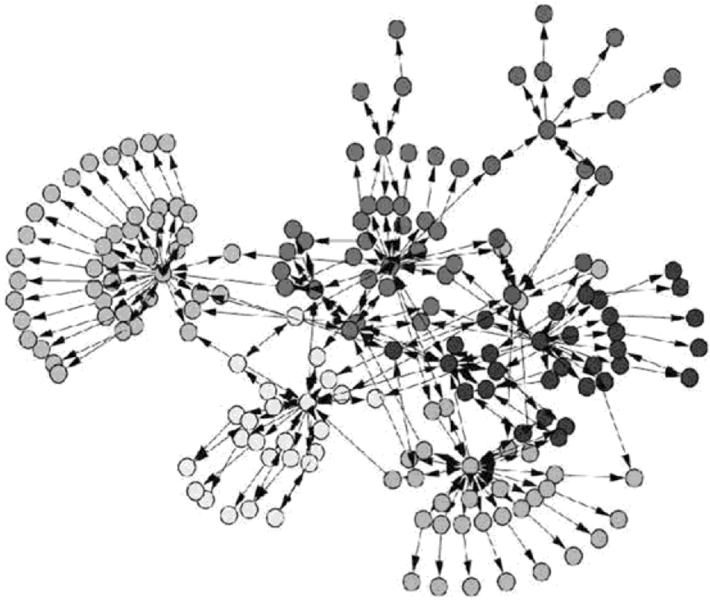


FIGURE 1.5. Network structure of a section of the World Wide Web. (Reprinted with permission from M.E.J. Newman and M. Girvin, *Physical Review Letters E*, 69,026113, 2004. Copyright 2004 by the American Physical Society.)

complex behavior emerging from simple rules in the World Wide Web is currently a hot area of study in complex systems. Figure 1.5 illustrates the structure of one collection of Web pages and their links. It seems that much of the Web looks very similar; the question is, *why*?

Common Properties of Complex Systems

When looked at in detail, these various systems are quite different, but viewed at an abstract level they have some intriguing properties in common:

1. **Complex collective behavior**: All the systems I described above consist of large networks of individual components (ants, B cells, neurons, stock-buyers, Web-site creators), each typically following relatively simple rules with no central control or leader. It is the collective actions of vast numbers of components that give rise to the complex, hard-to-predict, and changing patterns of behavior that fascinate us.

2. **Signaling and information processing**: All these systems produce and use information and signals from both their internal and external environments.
3. **Adaptation**: All these systems adapt—that is, change their behavior to improve their chances of survival or success—through learning or evolutionary processes.

Now I can propose a definition of the term *complex system*: **a system in which large networks of components with no central control and simple rules of operation give rise to complex collective behavior, sophisticated information processing, and adaptation via learning or evolution.** (Sometimes a differentiation is made between *complex adaptive systems*, in which adaptation plays a large role, and nonadaptive complex systems, such as a hurricane or a turbulent rushing river. In this book, as most of the systems I do discuss are adaptive, I do not make this distinction.)

Systems in which organized behavior arises without an internal or external controller or leader are sometimes called *self-organizing*. Since simple rules produce complex behavior in hard-to-predict ways, the macroscopic behavior of such systems is sometimes called *emergent*. Here is an alternative definition of a *complex system*: **a system that exhibits nontrivial emergent and self-organizing behaviors**. The central question of the sciences of complexity is how this emergent self-organized behavior comes about. In this book I try to make sense of these hard-to-pin-down notions in different contexts.

How Can Complexity Be Measured?

In the paragraphs above I have sketched some qualitative common properties of complex systems. But more quantitative questions remain: Just how *complex* is a particular complex system? That is, how do we measure *complexity*? Is there any way to say precisely how much more complex one system is than another?

These are key questions, but they have not yet been answered to anyone's satisfaction and remain the source of many scientific arguments in the field. As I describe in chapter 7, many different measures of complexity have been proposed; however, none has been universally accepted by scientists. Several of these measures and their usefulness are described in various chapters of this book.

But how can there be a science of complexity when there is no agreed-on quantitative definition of complexity?

I have two answers to this question. First, neither a single *science of complexity* nor a single *complexity theory* exists yet, in spite of the many articles and books that have used these terms. Second, as I describe in many parts of this book, an essential feature of forming a new science is a struggle to define its central terms. Examples can be seen in the struggles to define such core concepts as *information*, *computation*, *order*, and *life*. In this book I detail these struggles, both historical and current, and tie them in with our struggles to understand the many facets of complexity. This book is about cutting-edge science, but it is also about the history of core concepts underlying this cutting-edge science. The next four chapters provide this history and background on the concepts that are used throughout the book.

CHAPTER 2 | Dynamics, Chaos, and Prediction

It makes me so happy. To be at the beginning again, knowing almost nothing. . . . The ordinary-sized stuff which is our lives, the things people write poetry about—clouds—daffodils—waterfalls. . . .these things are full of mystery, as mysterious to us as the heavens were to the Greeks. . .It's the best possible time to be alive, when almost everything you thought you knew is wrong.

—Tom Stoppard, *Arcadia*

DYNAMICAL SYSTEMS THEORY (or *dynamics*) concerns the description and prediction of systems that exhibit complex *changing* behavior at the macroscopic level, emerging from the collective actions of many interacting components. The word *dynamic* means changing, and dynamical systems are systems that change over time in some way. Some examples of dynamical systems are

The solar system (the planets change position over time)

The heart of a living creature (it beats in a periodic fashion rather than standing still)

The brain of a living creature (neurons are continually firing, neurotransmitters are propelled from one neuron to another, synapse strengths are changing, and generally the whole system is in a continual state of flux)

The stock market

The world's population

The global climate

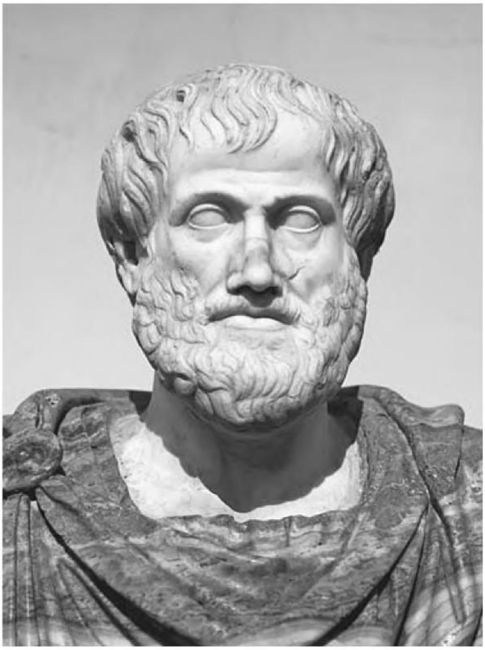
Dynamical systems include these and most other systems that you probably can think of. Even rocks change over geological time. Dynamical systems theory describes in general terms the ways in which systems can change, what types of macroscopic behavior are possible, and what kinds of predictions about that behavior can be made.

Dynamical systems theory has recently been in vogue in popular science because of the fascinating results coming from one of its intellectual offspring, the study of chaos. However, it has a long history, starting, as many sciences did, with the Greek philosopher Aristotle.

Early Roots of Dynamical Systems Theory

Aristotle was the author of one of the earliest recorded theories of motion, one that was accepted widely for over 1,500 years. His theory rested on two main principles, both of which turned out to be wrong. First, he believed that motion on Earth differs from motion in the heavens. He asserted that on

Aristotle, 384–322 B.C. (Ludovisi Collection)



Earth objects move in straight lines and only when something forces them to; when no forces are applied, an object comes to its natural resting state. In the heavens, however, planets and other celestial objects move continuously in perfect circles centered about the Earth. Second, Aristotle believed that earthly objects move in different ways depending on what they are made of. For example, he believed that a rock will fall to Earth because it is mainly composed of the element *earth*, whereas smoke will rise because it is mostly composed of the element *air*. Likewise, heavier objects, presumably containing more earth, will fall faster than lighter objects.

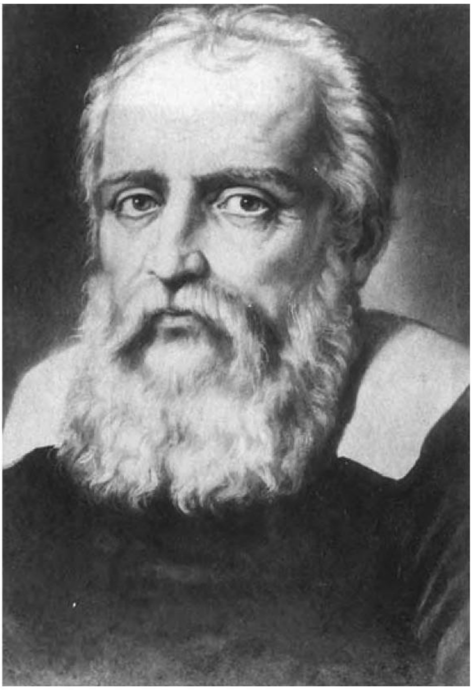
Clearly Aristotle (like many theorists since) was not one to let experimental results get in the way of his theorizing. His scientific method was to let logic and common sense direct theory; the importance of testing the resulting theories by experiments is a more modern notion. The influence of Aristotle's ideas was strong and continued to hold sway over most of Western science until the sixteenth century—the time of Galileo.

Galileo was a pioneer of experimental, empirical science, along with his predecessor Copernicus and his contemporary Kepler. Copernicus established that the motion of the planets is centered not about the Earth but about the sun. (Galileo got into big trouble with the Catholic Church for promoting this view and was eventually forced to publicly renounce it; only in 1992 did the Church officially admit that Galileo had been unfairly persecuted.) In the early 1600s, Kepler discovered that the motion of the planets is not circular but rather elliptical, and he discovered laws describing this elliptical motion.

Whereas Copernicus and Kepler focused their research on celestial motion, Galileo studied motion not only in the heavens but also here on Earth by experimenting with the objects one now finds in elementary physics courses: pendula, balls rolling down inclined planes, falling objects, light reflected by mirrors. Galileo did not have the sophisticated experimental devices we have today: he is said to have timed the swinging of a pendulum by counting his heartbeats and to have measured the effects of gravity by dropping objects off the leaning tower of Pisa. These now-classic experiments revolutionized ideas about motion. In particular, Galileo's studies directly contradicted Aristotle's long-held principles of motion. Against common sense, rest is *not* the natural state of objects; rather it takes *force* to stop a moving object. Heavy and light objects in a vacuum fall at the same rate. And perhaps most revolutionary of all, laws of motion on the Earth could explain some aspects of motions in the heavens. With Galileo, the scientific revolution, with experimental observations at its core, was definitively launched.

The most important person in the history of dynamics was Isaac Newton. Newton, who was born the year after Galileo died, can be said to have

Galileo, 1564–1642 (AIP Emilio Segre Visual Archives, E. Scott Barr Collection)



Isaac Newton, 1643–1727 (Original engraving by unknown artist, courtesy AIP Emilio Segre Visual Archives)

invented, on his own, the science of dynamics. Along the way he also had to invent calculus, the branch of mathematics that describes motion and change.

Physicists call the general study of motion *mechanics*. This is a historical term dating from ancient Greece, reflecting the classical view that all motion could be explained in terms of the combined actions of simple "machines" (e.g., lever, pulley, wheel and axle). Newton's work is known today as *classical mechanics*. Mechanics is divided into two areas: kinematics, which describes how things move, and dynamics, which explains why things obey the laws of kinematics. For example, Kepler's laws are kinematic laws—they describe *how* the planets move (in ellipses with the sun at one focus)—but not *why* they move in this particular way. Newton's laws are the foundations of dynamics: they explain the motion of the planets, and everything else, in terms of the basic notions of force and mass.

Newton's famous three laws are as follows:

1. Constant motion: Any object not subject to a force moves with unchanging speed.
2. Inertial mass: When an object is subject to a force, the resulting change in its motion is inversely proportional to its mass.
3. Equal and opposite forces: If object A exerts a force on object B, then object B must exert an equal and opposite force on object A.

One of Newton's greatest accomplishments was to realize that these laws applied not just to earthly objects but to those in the heavens as well. Galileo was the first to state the constant-motion law, but he believed it applied only to objects on Earth. Newton, however, understood that this law should apply to the planets as well, and realized that elliptical orbits, which exhibit a constantly *changing* direction of motion, require explanation in terms of a force, namely gravity. Newton's other major achievement was to state a universal law of gravity: the force of gravity between two objects is proportional to the product of their masses divided by the square of the distance between them. Newton's insight—now the backbone of modern science—was that this law applies everywhere in the universe, to falling apples as well as to planets. As he wrote: "nature is exceedingly simple and conformable to herself. Whatever reasoning holds for greater motions, should hold for lesser ones as well."

Newtonian mechanics produced a picture of a "clockwork universe," one that is wound up with the three laws and then runs its mechanical course. The mathematician Pierre Simon Laplace saw the implication of this clockwork view for prediction: in 1814 he asserted that, given Newton's laws and the current position and velocity of every particle in the universe, it was possible, in principle, to predict everything for all time. With the invention of electronic computers in the 1940s, the "in principle" might have seemed closer to "in practice."

Revised Views of Prediction

However, two major discoveries of the twentieth century showed that Laplace's dream of complete prediction is not possible, even in principle. One discovery was Werner Heisenberg's 1927 "uncertainty principle" in quantum mechanics, which states that one cannot measure the exact values of the position and the momentum (mass times velocity) of a particle at the same time. The more certain one is about where a particle is located at a given time, the less one can know about its momentum, and vice versa. However, effects of Heisenberg's principle exist only in the quantum world of tiny particles, and most people viewed it as an interesting curiosity, but not one that would have much implication for prediction at a larger scale—predicting the weather, say.

It was the understanding of *chaos* that eventually laid to rest the hope of perfect prediction of all complex systems, quantum or otherwise. The defining idea of chaos is that there are some systems—*chaotic* systems—in which even minuscule uncertainties in measurements of initial position and momentum can result in huge errors in long-term predictions of these quantities. This is known as "sensitive dependence on initial conditions."

In parts of the natural world such small uncertainties will not matter. If your initial measurements are fairly but not perfectly precise, your predictions will likewise be close to right if not exactly on target. For example, astronomers can predict eclipses almost perfectly in spite of even relatively large uncertainties in measuring the positions of planets. But sensitive dependence on initial conditions says that in chaotic systems, even the tiniest errors in your initial measurements will eventually produce huge errors in your prediction of the future motion of an object. In such systems (and hurricanes may well be an example) *any* error, no matter how small, will make long-term predictions vastly inaccurate.

This kind of behavior is counterintuitive; in fact, for a long time many scientists denied it was possible. However, chaos in this sense has been observed in cardiac disorders, turbulence in fluids, electronic circuits, dripping faucets, and many other seemingly unrelated phenomena. These days, the existence of chaotic systems is an accepted fact of science.

It is hard to pin down who first realized that such systems might exist. The possibility of sensitive dependence on initial conditions was proposed by a number of people long before quantum mechanics was invented. For example, the physicist James Clerk Maxwell hypothesized in 1873 that there are classes of phenomena affected by "influences whose physical magnitude is too small to be taken account of by a finite being, [but which] may produce results of the highest importance."

Possibly the first clear example of a chaotic system was given in the late nineteenth century by the French mathematician Henri Poincaré. Poincaré was the founder of and probably the most influential contributor to the modern field of dynamical systems theory, which is a major outgrowth of Newton's science of dynamics. Poincaré discovered sensitive dependence on initial conditions when attempting to solve a much simpler problem than predicting the motion of a hurricane. He more modestly tried to tackle the so-called three-body problem: to determine, using Newton's laws, the long-term motions of three masses exerting gravitational forces on one another. Newton solved the *two*-body problem, but the three-body problem turned out to be much harder. Poincaré tackled it in 1887 as part of a mathematics contest held in honor of the king of Sweden. The contest offered a prize of 2,500 Swedish crowns for a solution to the "many body" problem: predicting the future positions of arbitrarily many masses attracting one another under Newton's laws. This problem was inspired by the question of whether or not the solar system is stable: will the planets remain in their current orbits, or will they wander from them? Poincaré started off by seeing whether he could solve it for merely three bodies.

He did not completely succeed—the problem was too hard. But his attempt was so impressive that he was awarded the prize anyway. Like Newton with calculus, Poincaré had to invent a new branch of mathematics, *algebraic topology*, to even tackle the problem. Topology is an extended form of geometry, and it was in looking at the geometric consequences of the three-body problem that he discovered the possibility of sensitive dependence on initial conditions. He summed up his discovery as follows:

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon has been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomenon. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible. . . .

In other words, even if we know the laws of motion perfectly, two different sets of initial conditions (here, initial positions, masses, and velocities for

Henri Poincaré, 1854–1912 (AIP Emilio Segre Visual Archives)

objects), even if they differ in a minuscule way, can sometimes produce greatly different results in the subsequent motion of the system. Poincaré found an example of this in the three-body problem.

It was not until the invention of the electronic computer that the scientific world began to see this phenomenon as significant. Poincaré, way ahead of his time, had guessed that sensitive dependence on initial conditions would stymie attempts at long-term weather prediction. His early hunch gained some evidence when, in 1963, the meteorologist Edward Lorenz found that even simple computer models of weather phenomena were subject to sensitive dependence on initial conditions. Even with today's modern, highly complex meteorological computer models, weather predictions are at best reasonably accurate only to about one week in the future. It is not yet known whether this limit is due to fundamental chaos in the weather, or how much this limit can be extended by collecting more data and building even better models.

Linear versus Nonlinear Rabbits

Let's now look more closely at sensitive dependence on initial conditions. How, precisely, does the huge magnification of initial uncertainties come about in chaotic systems? The key property is *nonlinearity*. A linear system is one you can understand by understanding its parts individually and then putting them together. When my two sons and I cook together, they like to take turns adding ingredients. Jake puts in two cups of flour. Then Nicky puts in a cup of sugar. The result? Three cups of flour/sugar mix. The whole is equal to the sum of the parts.

A nonlinear system is one in which the whole is different from the sum of the parts. Jake puts in two cups of baking soda. Nicky puts in a cup of vinegar. The whole thing explodes. (You can try this at home.) The result? *More* than three cups of vinegar-and-baking-soda-and-carbon-dioxide fizz.

The difference between the two examples is that in the first, the flour and sugar don't really interact to create something new, whereas in the second, the vinegar and baking soda interact (rather violently) to create a lot of carbon dioxide.

Linearity is a reductionist's dream, and nonlinearity can sometimes be a reductionist's nightmare. Understanding the distinction between linearity and nonlinearity is very important and worthwhile. To get a better handle on this distinction, as well as on the phenomenon of chaos, let's do a bit of very simple mathematical exploration, using a classic illustration of linear and nonlinear systems from the field of biological population dynamics.

Suppose you have a population of breeding rabbits in which every year all the rabbits pair up to mate, and each pair of rabbit parents has exactly four offspring and then dies. The population growth, starting from two rabbits, is illustrated in figure 2.1.

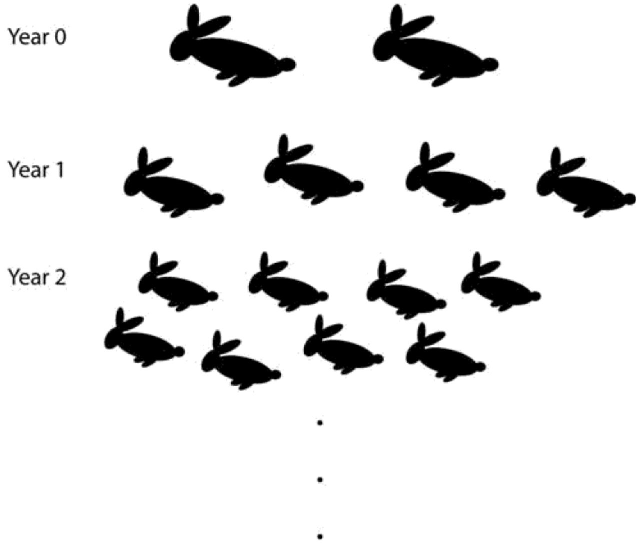


FIGURE 2.1. Rabbits with doubling population.

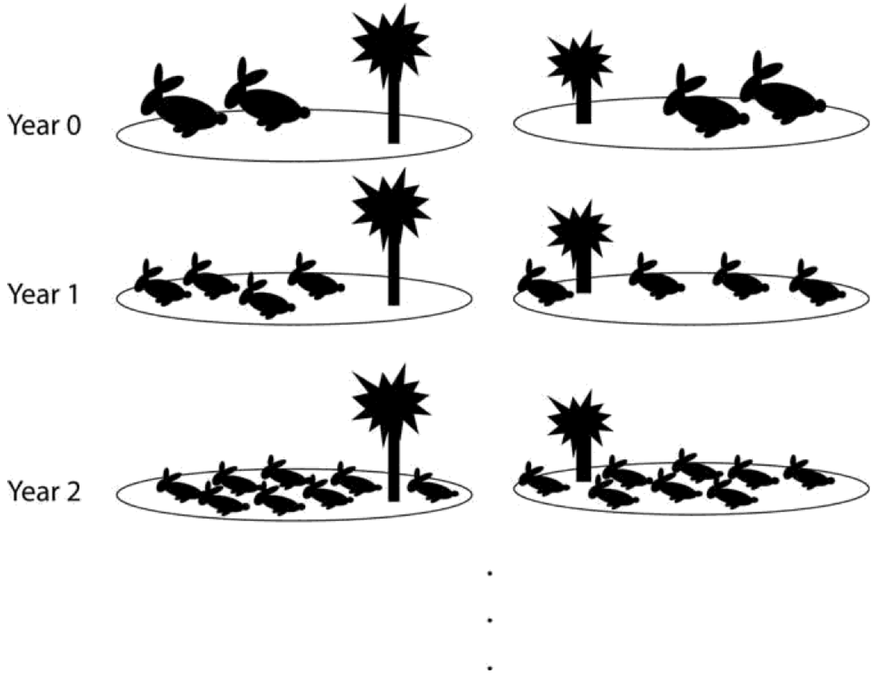


FIGURE 2.2. Rabbits with doubling population, split on two islands.

It is easy to see that the population doubles every year without limit (which means the rabbits would quickly take over the planet, solar system, and universe, but we won't worry about that for now).

This is a linear system: the whole is equal to the sum of the parts. What do I mean by this? Let's take a population of four rabbits and split them between two separate islands, two rabbits on each island. Then let the rabbits proceed with their reproduction. The population growth over two years is illustrated in figure 2.2.

Each of the two populations doubles each year. At each year, if you add the populations on the two islands together, you'll get the same number of rabbits that you would have gotten had there been no separation—that is, had they all lived on one island.

If you make a plot with the current year's population size on the horizontal axis and the next-year's population size on the vertical axis, you get a straight line (figure 2.3). This is where the term *linear system* comes from.

But what happens when, more realistically, we consider limits to population growth? This requires us to make the growth rule nonlinear. Suppose that, as before, each year every pair of rabbits has four offspring and then dies. But now suppose that some of the offspring die before they reproduce

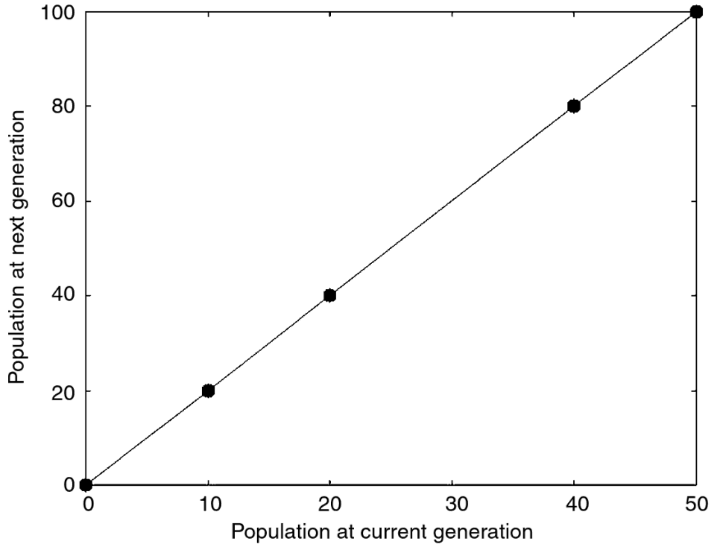


FIGURE 2.3. A plot of how the population size next year depends on the population size this year for the linear model.

because of overcrowding. Population biologists sometimes use an equation called the *logistic model* as a description of population growth in the presence of overcrowding. This sense of the word *model* means a mathematical formula that describes population growth in a simplified way.

In order to use the logistic model to calculate the size of the next generation's population, you need to input to the logistic model the current generation's population size, the *birth rate,* the *death rate* (the probability of an individual will die due to overcrowding), and the maximum *carrying capacity* (the strict upper limit of the population that the habitat will support.)

I won't give the actual equation for the logistic model here (it is given in the notes), but you can see its behavior in figure 2.4.

As a simple example, let's set *birth rate* = 2 and *death rate* = 0.4, assume the carrying capacity is thirty-two, and start with a population of twenty rabbits in the first generation. Using the logistic model, I calculate that the number of surviving offspring in the second generation is twelve. I then plug this new population size into the model, and find that there are still exactly twelve surviving rabbits in the third generation. The population will stay at twelve for all subsequent years.

If I reduce the death rate to 0.1 (keeping everything else the same), things get a little more interesting. From the model I calculate that the second generation has 14.25 rabbits and the third generation has 15.01816.

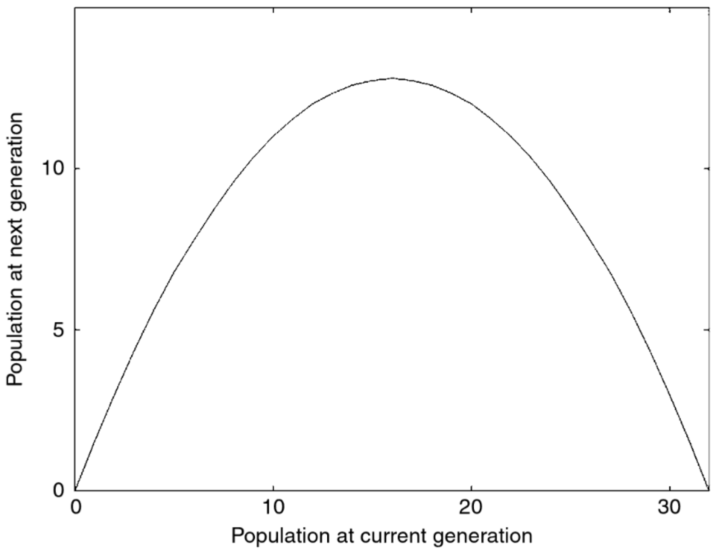


FIGURE 2.4. A plot of how the population size next year depends on the population size this year under the logistic model, with birth rate equal to 2, death rate equal to 0.4, and carrying capacity equal to 32. The plot will also be a parabola for other values of these parameters.

Wait a minute! How can we have 0.25 of a rabbit, much less 0.01816 of a rabbit? Obviously in real life we cannot, but this is a mathematical model, and it allows for fractional rabbits. This makes it easier to do the math, and can still give reasonable predictions of the actual rabbit population. So let's not worry about that for now.

This process of calculating the size of the next population again and again, starting each time with the immediately previous population, is called "iterating the model."

What happens if the death rate is set back to 0.4 and carrying capacity is doubled to sixty-four? The model tells me that, starting with twenty rabbits, by year nine the population reaches a value close to twenty-four and stays there.

You probably noticed from these examples that the behavior is more complicated than when we simply doubled the population each year. That's because the logistic model is nonlinear, due to its inclusion of death by overcrowding. Its plot is a parabola instead of a line (figure 2.4). The logistic population growth is not simply equal to the sum of its parts. To show this, let's see what happens if we take a population of twenty rabbits and segregate it

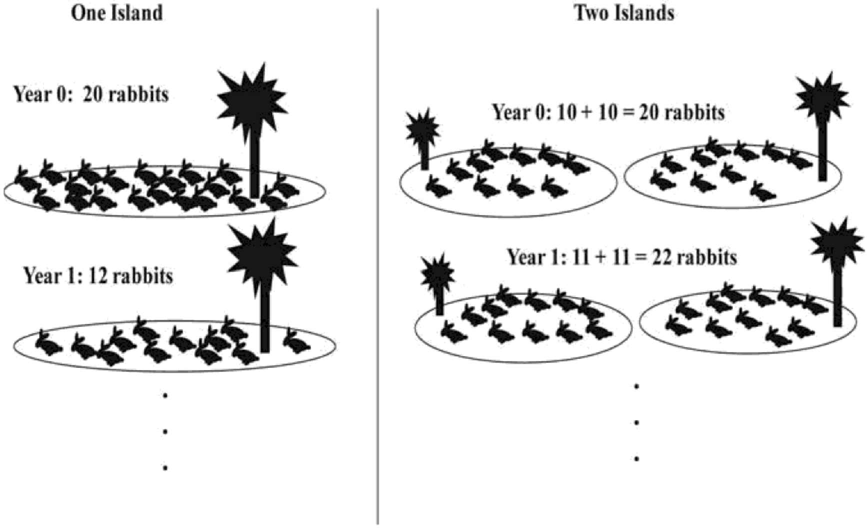


FIGURE 2.5. Rabbit population split on two islands, following the logistic model.

into populations of ten rabbits each, and iterate the model for each population (with *birth rate* = 2 and *death rate* = .4, as in the first example above). The result is illustrated in figure 2.5.

At year one, the original twenty-rabbit population has been cut down to twelve rabbits, but each of the original ten-rabbit populations now has eleven rabbits, for a total of twenty-two rabbits. The behavior of the whole is clearly not equal to the sum of the behavior of the parts.

The Logistic Map

Many scientists and mathematicians who study this sort of thing have used a simpler form of the logistic model called the *logistic map*, which is perhaps the most famous equation in the science of dynamical systems and chaos. The logistic model is simplified by combining the effects of birth rate and death rate into one number, called R. *Population size* is replaced by a related concept called "fraction of carrying capacity," called x. Given this simplified model, scientists and mathematicians promptly forget all about population growth, carrying capacity, and anything else connected to the real world, and simply get lost in the astounding behavior of the equation itself. We will do the same.

Here is the equation, where x_t is the current value of x and x_{t+1} is its value at the next time step:[1]

$$x_{t+1} = R\,x_t(1 - x_t).$$

I give the equation for the logistic map to show you how simple it is. In fact, it is one of the simplest systems to capture the essence of chaos: sensitive dependence on initial conditions. The logistic map was brought to the attention of population biologists in a 1971 article by the mathematical biologist Robert May in the prestigious journal *Nature*. It had been previously analyzed in detail by several mathematicians, including Stanislaw Ulam, John von Neumann, Nicholas Metropolis, Paul Stein, and Myron Stein. But it really achieved fame in the 1980s when the physicist Mitchell Feigenbaum used it to demonstrate *universal* properties common to a very large class of chaotic systems. Because of its apparent simplicity and rich history, it is a perfect vehicle to introduce some of the major concepts of dynamical systems theory and chaos.

The logistic map gets very interesting as we vary the value of R. Let's start with $R = 2$. We need to also start out with some value between 0 and 1 for x_0, say 0.5. If you plug those numbers into the logistic map, the answer for x_1 is 0.5. Likewise, $x_2 = 0.5$, and so on. Thus, if $R = 2$ and the population starts out at half the maximum size, it will stay there forever.

Now let's try $x_0 = 0.2$. You can use your calculator to compute this one. (I'm using one that reads off at most seven decimal places.) The results are more interesting:

$$\begin{aligned}
x_0 &= 0.2\\
x_1 &= 0.32\\
x_2 &= 0.4352\\
x_3 &= 0.4916019\\
x_4 &= 0.4998589\\
x_5 &= \mathbf{0.5}\\
x_6 &= \mathbf{0.5}\\
&\vdots
\end{aligned}$$

1. Authors of popular-audience science books are always warned of the following rule: every equation in your book will cut the readership by one-half. I'm no exception—my editor told me this fact very clearly. I'm going to give the logistic map equation here anyway, so the half of you who would throw the book out the window if you ever encountered an equation, please skip over the next line.

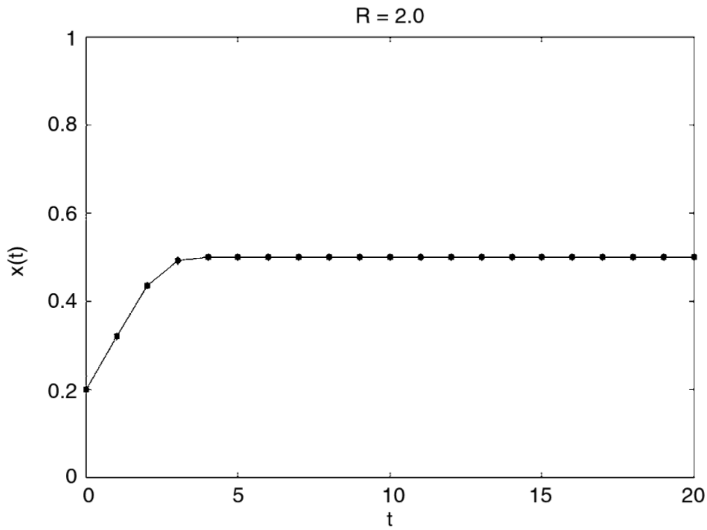


FIGURE 2.6. Behavior of the logistic map for $R = 2$ and $x_0 = 0.2$.

The same eventual result ($x_t = 0.5$ forever) occurs but here it takes five iterations to get there.

It helps to see these results visually. A plot of the value of x_t at each time t for 20 time steps is shown in figure 2.6. I've connected the points by lines to better show how as time increases, x quickly converges to 0.5.

What happens if x_0 is large, say, 0.99? Figure 2.7 shows a plot of the results.

Again the same ultimate result occurs, but with a longer and more dramatic path to get there.

You may have guessed it already: if $R = 2$ then x_t eventually always gets to 0.5 and stays there. The value 0.5 is called a *fixed point*: how long it takes to get there depends on where you start, but once you are there, you are fixed.

If you like, you can do a similar set of calculations for $R = 2.5$, and you will find that the system also always goes to a fixed point, but this time the fixed point is 0.6.

For even more fun, let $R = 3.1$. The behavior of the logistic map now gets more complicated. Let $x_0 = 0.2$. The plot is shown in figure 2.8.

In this case x never settles down to a fixed point; instead it eventually settles into an oscillation between two values, which happen to be 0.5580141 and 0.7645665. If the former is plugged into the formula the latter is produced, and vice versa, so this oscillation will continue forever. This oscillation will be reached eventually no matter what value is given for x_0. This kind of regular

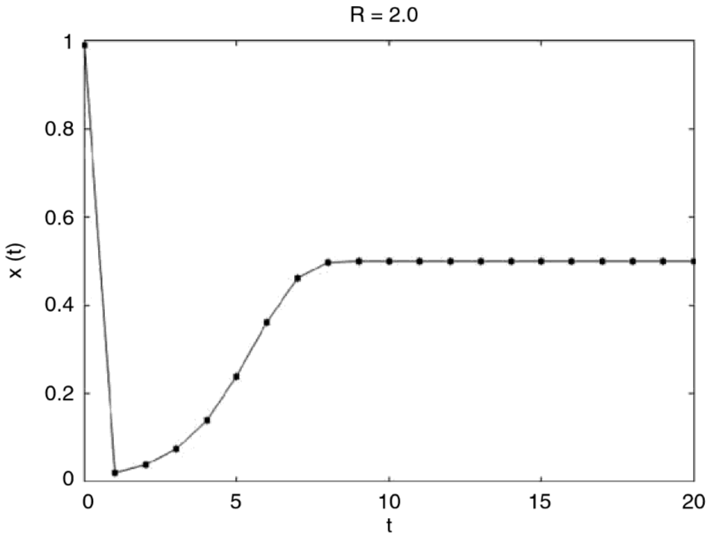


FIGURE 2.7. Behavior of the logistic map for $R = 2$ and $x_0 = 0.99$.

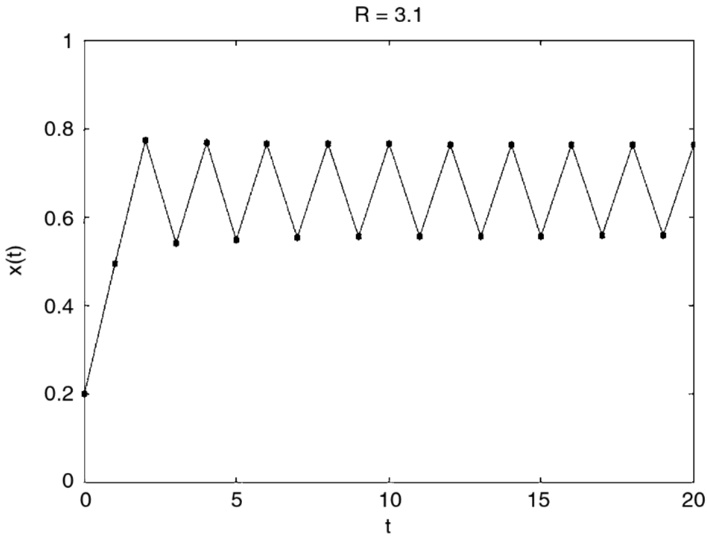


FIGURE 2.8. Behavior of the logistic map for $R = 3.1$ and $x_0 = 0.2$.

final behavior (either fixed point or oscillation) is called an "attractor," since, loosely speaking, any initial condition will eventually be "attracted to it."

For values of R up to around 3.4 the logistic map will have similar behavior: after a certain number of iterations, the system will oscillate between two different values. (The final pair of values will be different for each value of

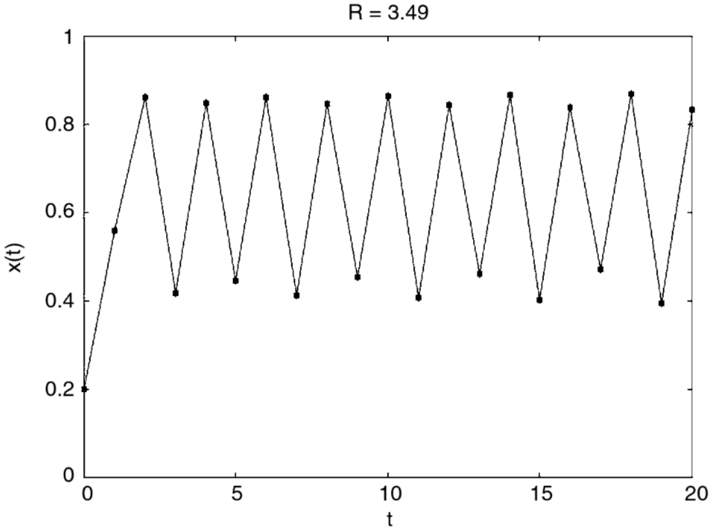


FIGURE 2.9. Behavior of the logistic map for $R = 3.49$ and $x_0 = 0.2$.

R.) Because it oscillates between two values, the system is said to have *period* equal to 2.

But at a value between $R = 3.4$ and $R = 3.5$ an abrupt change occurs. Given any value of x_0, the system will eventually reach an oscillation among *four* distinct values instead of two. For example, if we set $R = 3.49$, $x_0 = 0.2$, we see the results in figure 2.9.

Indeed, the values of x fairly quickly reach an oscillation among four different values (which happen to be approximately 0.872, 0.389, 0.829, and 0.494, if you're interested). That is, at some R between 3.4 and 3.5, the period of the final oscillation has abruptly doubled from 2 to 4.

Somewhere between $R = 3.54$ and $R = 3.55$ the period abruptly doubles again, jumping to 8. Somewhere between 3.564 and 3.565 the period jumps to 16. Somewhere between 3.5687 and 3.5688 the period jumps to 32. The period doubles again and again after smaller and smaller increases in R until, in short order, the period becomes effectively infinite, at an R value of approximately 3.569946. Before this point, the behavior of the logistic map was roughly predictable. If you gave me the value for R, I could tell you the ultimate long-term behavior from any starting point x_0: fixed points are reached when R is less than about 3.1, period-two oscillations are reached when R is between 3.1 and 3.4, and so on.

When R is approximately 3.569946, the values of x no longer settle into an oscillation; rather, they become chaotic. Here's what this means. Let's call the series of values x_0, x_1, x_2, and so on the *trajectory* of x. At values of

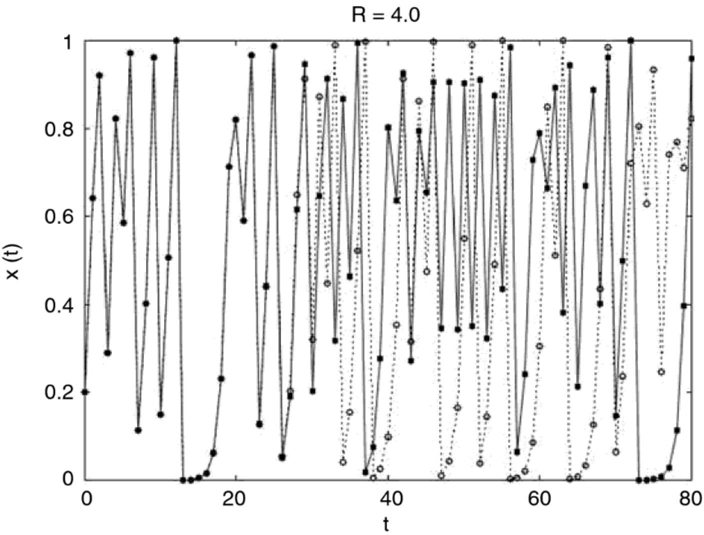


FIGURE 2.10. Two trajectories of the logistic map for $R = 4.0 : x_0 = 0.2$ and $x_0 = 0.2000000001$.

R that yield chaos, two trajectories starting from very similar values of x_0, rather than converging to the same fixed point or oscillation, will instead progressively diverge from each other. At $R = 3.569946$ this divergence occurs very slowly, but we can see a more dramatic sensitive dependence on x_0 if we set $R = 4.0$. First I set $x_0 = 0.2$ and iterate the logistic map to obtain a trajectory. Then I restarted with a new x_0, increased slightly by putting a 1 in the tenth decimal place, $x_0 = 0.2000000001$, and iterated the map again to obtain a second trajectory. In figure 2.10 the first trajectory is the dark curve with black circles, and the second trajectory is the light line with open circles.

The two trajectories start off very close to one another (so close that the first, solid-line trajectory blocks our view of the second, dashed-line trajectory), but after 30 or so iterations they start to diverge significantly, and soon after there is no correlation between them. This is what is meant by "sensitive dependence on initial conditions."

So far we have seen three different classes of final behavior (attractors): fixed-point, periodic, and chaotic. (Chaotic attractors are also sometimes called "strange attractors.") *Type of attractor* is one way in which dynamical systems theory characterizes the behavior of a system.

Let's pause a minute to consider how remarkable the chaotic behavior really is. The logistic map is an extremely simple equation and is completely deterministic: every x_t maps onto one and only one value of x_{t+1}. And yet the chaotic trajectories obtained from this map, at certain values of R, look very random—enough so that the logistic map has been used as a basis for generating pseudo-random numbers on a computer. Thus apparent randomness can arise from very simple deterministic systems.

Moreover, for the values of R that produce chaos, if there is any uncertainty in the initial condition x_0, there exists a time beyond which the future value cannot be predicted. This was demonstrated above with $R = 4$. If we don't know the value of the tenth and higher decimal places of x_0—a quite likely limitation for many experimental observations—then by $t = 30$ or so the value of x_t is unpredictable. For any value of R that yields chaos, uncertainty in any decimal place of x_0, however far out in the decimal expansion, will result in unpredictability at some value of t.

Robert May, the mathematical biologist, summed up these rather surprising properties, echoing Poincaré:

> The fact that the simple and deterministic equation (1) [i.e., the logistic map] can possess dynamical trajectories which look like some sort of random noise has disturbing practical implications. It means, for example, that apparently erratic fluctuations in the census data for an animal population need not necessarily betoken either the vagaries of an unpredictable environment or sampling errors: they may simply derive from a rigidly deterministic population growth relationship such as equation (1). . . . Alternatively, it may be observed that in the chaotic regime arbitrarily close initial conditions can lead to trajectories which, after a sufficiently long time, diverge widely. This means that, even if we have a simple model in which all the parameters are determined exactly, long-term prediction is nevertheless impossible.

In short, the presence of chaos in a system implies that perfect prediction *à la* Laplace is impossible not only in practice but also *in principle*, since we can never know x_0 to infinitely many decimal places. This is a profound negative result that, along with quantum mechanics, helped wipe out the optimistic nineteenth-century view of a clockwork Newtonian universe that ticked along its predictable path.

But is there a more positive lesson to be learned from studies of the logistic map? Can it help the goal of dynamical systems theory, which attempts to discover general principles concerning systems that change over time? In fact, deeper studies of the logistic map and related maps have resulted in an equally surprising and profound positive result—the discovery of universal characteristics of chaotic systems.

Universals in Chaos

The term *chaos*, as used to describe dynamical systems with sensitive dependence on initial conditions, was first coined by physicists T. Y. Li and James Yorke. The term seems apt: the colloquial sense of the word "chaos" implies randomness and unpredictability, qualities we have seen in the chaotic version of logistic map. However, unlike colloquial chaos, there turns out to be substantial order in mathematical chaos in the form of so-called *universal* features that are common to a wide range of chaotic systems.

THE FIRST UNIVERSAL FEATURE: THE PERIOD-DOUBLING ROUTE TO CHAOS

In the mathematical explorations we performed above, we saw that as R was increased from 2.0 to 4.0, iterating the logistic map for a given value of R first yielded a fixed point, then a period-two oscillation, then period four, then eight, and so on, until chaos was reached. In dynamical systems theory, each of these abrupt period doublings is called a *bifurcation*. This succession of bifurcations culminating in chaos has been called the "period doubling route to chaos."

These bifurcations are often summarized in a so-called bifurcation diagram that plots the attractor the system ends up in as a function of the value of a "control parameter" such as R. Figure 2.11 gives such a bifurcation diagram

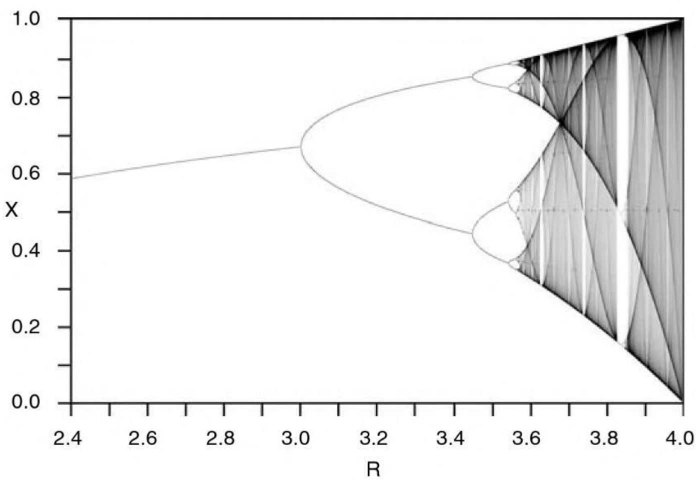


FIGURE 2.11. Bifurcation diagram for the logistic map, with attractor plotted as a function of R.

for the logistic map. The horizontal axis gives R. For each value of R, the final (attractor) values of x are plotted. For example, for $R = 2.9$, x reaches a fixed-point attractor of $x = 0.655$. At $R = 3.0$, x reaches a period-two attractor. This can be seen as the first branch point in the diagram, when the fixed-point attractors give way to the period-two attractors. For R somewhere between 3.4 and 3.5, the diagram shows a bifurcation to a period-four attractor, and so on, with further period doublings, until the onset of chaos at R approximately equal to 3.569946.

The period-doubling route to chaos has a rich history. Period doubling bifurcations had been observed in mathematical equations as early as the 1920s, and a similar cascade of bifurcations was described by P. J. Myrberg, a Finnish mathematician, in the 1950s. Nicholas Metropolis, Myron Stein, and Paul Stein, working at Los Alamos National Laboratory, showed that not just the logistic map but *any* map whose graph is parabola-shaped will follow a similar period-doubling route. Here, "parabola-shaped" means that plot of the map has just one hump—in mathematical terms, it is "unimodal."

THE SECOND UNIVERSAL FEATURE: FEIGENBAUM'S CONSTANT

The discovery that gave the period-doubling route its renowned place among mathematical universals was made in the 1970s by the physicist Mitchell Feigenbaum. Feigenbaum, using only a programmable desktop calculator, made a list of the R values at which the period-doubling bifurcations occur (where $\approx$ means "approximately equal to"):

$$
\begin{aligned}
R_1 &\approx 3.0\\
R_2 &\approx 3.44949\\
R_3 &\approx 3.54409\\
R_4 &\approx 3.564407\\
R_5 &\approx 3.568759\\
R_6 &\approx 3.569692\\
R_7 &\approx 3.569891\\
R_8 &\approx 3.569934\\
&\ \vdots\\
R_\infty &\approx 3.569946
\end{aligned}
$$

Here, R_1 corresponds to period $2^1(=\ 2), R_2$ corresponds to period $2^2(=\ 4)$, and in general, R_n corresponds to period 2^n. The symbol ∞

("infinity") is used to denote the onset of chaos—a trajectory with an infinite period.

Feigenbaum noticed that as the period increases, the *R* values get closer and closer together. This means that for each bifurcation, *R* has to be increased less than it had before to get to the next bifurcation. You can see this in the bifurcation diagram of Figure 2.11: as *R* increases, the bifurcations get closer and closer together. Using these numbers, Feigenbaum measured the *rate* at which the bifurcations get closer and closer; that is, the rate at which the *R* values *converge*. He discovered that the rate is (approximately) the constant value 4.6692016. What this means is that as *R* increases, each new period doubling occurs about 4.6692016 times faster than the previous one.

This fact was interesting but not earth-shaking. Things started to get a lot more interesting when Feigenbaum looked at some other maps—the logistic map is just one of many that have been studied. As I mentioned above, a few years before Feigenbaum made these calculations, his colleagues at Los Alamos, Metropolis, Stein, and Stein, had shown that any unimodal map will follow a similar period-doubling cascade. Feigenbaum's next step was to calculate the rate of convergence for some other unimodal maps. He started with the so-called sine map, an equation similar to the logistic map but which uses the trigonometric sine function.

Feigenbaum repeated the steps I sketched above: he calculated the values of *R* at the period-doubling bifurcations in the sine map, and then calculated the rate at which these values converged. He found that the rate of convergence was 4.6692016.

Feigenbaum was amazed. The rate was the same. He tried it for other unimodal maps. It was still the same. No one, including Feigenbaum, had expected this at all. But once the discovery had been made, Feigenbaum went on to develop a mathematical theory that explained why the common value of 4.6692016, now called *Feigenbaum's constant*, is universal—which here means the same for all unimodal maps. The theory used a sophisticated mathematical technique called *renormalization* that had been developed originally in the area of quantum field theory and later imported to another field of physics: the study of phase transitions and other "critical phenomena." Feigenbaum adapted it for dynamical systems theory, and it has become a cornerstone in the understanding of chaos.

It turned out that this is not just a mathematical curiosity. In the years since Feigenbaum's discovery, his theory has been verified in several laboratory experiments on physical dynamical systems, including fluid flow, electronic circuits, lasers, and chemical reactions. Period-doubling cascades

Mitchell Feigenbaum (AIP Emilio Segre Visual Archives, *Physics Today* Collection)

have been observed in these systems, and values of Feigenbaum's constant have been calculated in steps similar to those we saw above. It is often quite difficult to get accurate measurements of, say, what corresponds to R values in such experiments, but even so, the values of Feigenbaum's constant found by the experimenters agree well within the margin of error to Feigenbaum's value of approximately 4.6692016. This is impressive, since Feigenbaum's theory, which yields this number, involves only abstract math, no physics. As Feigenbaum's colleague Leo Kadanoff said, this is "the best thing that can happen to a scientist, realizing that something that's happened in his or her mind exactly corresponds to something that happens in nature."

Large-scale systems such as the weather are, as yet, too hard to experiment with directly, so no one has *directly* observed period doubling or chaos in their behavior. However, certain computer models of weather have displayed the period-doubling route to chaos, as have computer models of electrical power systems, the heart, solar variability, and many other systems.

There is one more remarkable fact to mention about this story. Similar to many important scientific discoveries, Feigenbaum's discoveries were also made, independently and at almost the same time, by another research team.