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MATHEMATICS ENTERS A NEW AGE

GILLES DOWEK

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Computation, Proof, Machine

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Introduction: In Which Mathematics Sets Out to Conquer New Territories

IT'S BEEN said again and again: the century that just ended was the true golden age of mathematics. Mathematics evolved more in the twentieth century than in all previous centuries put together. Yet the century just begun may well prove exceptional for mathematics, too: the signs seem to indicate that, in the coming decades, mathematics will undergo as many metamorphoses as in the twentieth century – if not more. The revolution has already begun. From the early seventies onward, the mathematical method has been transforming at its core: the notion of proof. The driving force of this transformation is the return of an old, yet somewhat underrated mathematical concept: that of computing.

The idea that computing might be the key to a revolution may seem paradoxical. Algorithms that allow us, among other things, to perform sums and products are already recognized as a basic part of mathematical knowledge; as for the actual calculations, they are seen as rather boring tasks of limited creative interest. Mathematicians themselves tend to be prejudiced against computing – René Thom said: "A great deal of my assertions are the product of sheer speculation; you may well call them reveries. I accept this qualification. . . . At a time when so many scientists around the world are computing, should we not encourage those of them who can to dream?" Making computing food for dreams does seem a bit of a challenge.

Unfortunately, this prejudice against computing is ingrained in the very definition of mathematical proof. Indeed, since Euclid, a proof has been defined as reasoning built on axioms and inference

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rules. But are mathematical problems always solved using a reasoning process? Hasn't the practice of mathematics shown, on the contrary, that solving a problem requires the subtle arrangement of reasoning stages and computing stages? By confining itself to reasoning, the axiomatic method may offer only a limited vision of mathematics. Indeed, the axiomatic method has reached a crisis, with recent mathematical advances, not all related to one another, gradually challenging the primacy of reasoning over computing and suggesting a more balanced approach in which these two activities play complementary roles.

This revolution, which invites us to rethink the relationship between reasoning and computing, also induces us to rethink the dialogue between mathematics and natural sciences such as physics and biology. It thus sheds new light on the age-old question of mathematics's puzzling effectiveness in those fields, as well as on the more recent debate about the logical form of natural theories. It prompts us to reconsider certain philosophical concepts such as analytic and synthetic judgement. It also makes us reflect upon the links between mathematics and computer science and upon the singularity of mathematics, which appears to be the only science where no tools are necessary.

Finally, and most interestingly, this revolution holds the promise of new ways of solving mathematical problems. These new methods will shake off the shackles imposed by past technologies that have placed arbitrary limits on the lengths of proofs. Mathematics may well be setting off to conquer new, as yet inaccessible territories.

Of course, the crisis of the axiomatic method did not come out of the blue. It had been heralded, from the first half of the twentieth century, by many signs, the most striking being two new theories that, without altogether questioning the axiomatic method, helped to reinstate computing in the mathematical edifice, namely the theory of computability and the theory of constructivity. We will therefore trace the history of these two ideas before delving into the crisis. However, let us first head for remote antiquity, where we will seek the roots of the very notion of computing and explore the "invention" of mathematics by the ancient Greeks.

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PART ONE

Ancient Origins

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In all these cases, the number $2 \times x^2$ is different from y^2. We could carry on searching, moving on to larger numbers. In all likelihood, Pythagoras's followers kept looking for the key to this problem for a long time, in vain, until they eventually became convinced that no such triangle existed. How did they manage to reach this conclusion, namely that the problem could not be solved? Not by trying out each and every pair of numbers one after the other, for there are infinitely many such pairs. Even if you tried out all possible pairs up to one thousand, or even up to one million, and found none that worked, you still could not state with any certainty that the problem has no solution – a solution might lie beyond one million.

Let's try to reconstruct the thought process that may have led the Pythagoreans to this conclusion.

First, when looking for a solution, we can restrict our attention to pairs in which at least one of the numbers x and y is odd. To see why, observe that if the pair $x = 202$ and $y = 214$, for example, were a solution, then, by dividing each number by two, we would find another solution, $x = 101$ and $y = 107$, where at least one of the numbers is odd. More generally, if you were to pick any solution and divide it by two, repeatedly if necessary, you would eventually come to another solution in which at least one of the numbers is odd. So, if the problem has any solution, there is necessarily a solution in which either x or y is an odd number.

Now, let's divide all pairs of numbers into four sets:

- pairs in which both numbers are odd;
- pairs in which the first number is even and the second number is odd;
- pairs in which the first number is odd and the second number is even;
- pairs in which both numbers are even.

We can now give four separate arguments to show that none of these sets holds a solution in which at least one of the numbers x and y is odd. As a result, the problem cannot be solved.

Begin with the first set: it cannot contain a solution in which one of the numbers x and y is odd, because if y is an odd number, then so is y^2; as a consequence, y^2 cannot equal $2 \times x^2$, which is necessarily an even number. This argument also rules out the second set,

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in which x is even and y odd. Obviously, the fourth set must also be ruled out, because by definition it cannot contain a pair where at least one number is odd. Which leaves us with the third set. In this case, x is odd and y is even, so that the number obtained by halving $2 \times x^2$ is odd, whereas half of y^2 is even – these two numbers cannot be equal.

The conclusion of this reasoning, namely that a square cannot equal twice another square, was reached by the Pythagoreans more than twenty-five centuries ago and still plays an important part in contemporary mathematics. It shows that, when you draw a right isosceles triangle whose short side is one meter long, the length of the hypotenuse measured in meters is a number (slightly greater than 1.414) that cannot be obtained by dividing x and y, two natural numbers, by each other. Geometry thus conjures up numbers that cannot be derived from integers using the four operations – addition, subtraction, multiplication, and division.

Many centuries later, this precedent inspired mathematicians to construct new numbers, called "real numbers." The Pythagoreans, however, did not go quite so far: they were not ready to give up what they regarded as the essential value of natural numbers. Their discovery felt to them more like a disaster than an opportunity.

Yet the Pythagorean problem was revolutionary not only because of its effects, but also because of how it is framed and how it was solved. To begin with, the Pythagorean problem is much more abstract than the question found on the Mesopotamian tablet, where 1,152,000 measures of grain were divided by 7 measures. Whereas the Mesopotamian question deals with measures of grain, the Pythagorean problem deals with numbers and nothing more. Similarly, the geometric form of the Pythagorean problem does not concern triangular fields but abstract triangles. Moving from a number of measures of grain to a number, from a triangular field to a triangle, may seem a trifle, but abstraction is actually a step of considerable importance. A field cannot measure more than a few kilometers. If the problem involved an actual triangular field, it would suffice, in order to solve it, to try every solution in which x and y are less than 10,000. But, unlike a triangular field, an abstract triangle can easily measure a million units, or a billion, or any magnitude.

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Clearly, a rift had opened between mathematical objects, which are abstract, and concrete, natural objects – and this exists even when the mathematical objects have been abstracted from the concrete ones. It is this rift that was the big breakthrough of the fifth century B.C.

The growing distance between mathematical objects and natural ones led some people to think that mathematics was not fit to describe natural objects. This idea dominated until the seventeenth century – Galileo's day – when it was refuted by advances in mathematical physics. Yet it persists today in those views that deny mathematics any relevance in the fields of social sciences – as when Marina Yaguello argues that the role of mathematics in linguistics is to "cover up its 'social' (hence fundamentally inexact) science with complex formulae."

This change in the nature of the objects under study – which, since the fifth century B.C., have been geometric figures and numbers not necessarily related to concrete objects – triggered a revolution in the method used to solve mathematical problems. Once again, let's compare the methods used by the Mesopotamians and those used by the Pythagoreans. The tablet shows that Mesopotamians solved problems by performing computations – to answer the question about grain, they did a simple division. When it comes to the Pythagoreans' problem, however, reasoning is necessary.

In order to do a division, all you have to do is apply an algorithm taught in primary school, of which the Mesopotamians knew equivalents. By contrast, when developing their thought process, the Pythagoreans could not lean on any algorithm – no algorithm recommends that you group the pairs into four sets. To come up with this idea, the Pythagoreans had to use their imaginations. Maybe one of Pythagoras's followers understood that the number y could not be odd and then, a few weeks or a few months later, another disciple helped make headway by discovering that x could not be an odd number either. Perhaps it was months or even years before another Pythagorean made the next big advance. When a Mesopotamian tackled a division, he knew he was going to achieve a result. He could even gauge beforehand how long the operation would take him. A Pythagorean tackling an arithmetic problem had no means of knowing how long it would be before he found the line

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of reasoning that would enable him to solve the problem – or even if he ever would.

Students often complain that mathematics is a tough subject, and they're right: it is a subject that requires imagination; there is no systematic method for solving problems. Mathematics is even more difficult for professional mathematicians – some problems have remained unsolved for decades, sometimes centuries. When trying to solve a math problem, there is nothing unusual about drawing a blank. Professional mathematicians often stay stumped too, sometimes for years, before they have a breakthrough. By contrast, no one dries up over a division problem – one simply commits the division algorithm to memory and applies it.

How did the change in the nature of mathematical objects bring about this methodological change? In other words, how did abstraction lead mathematicians to drop calculation in favor of the reasoning that so characterizes Ancient Greek mathematics? Why couldn't the Pythagorean problem be solved by simple calculation? Think back, once more, to the Mesopotamian question. It deals with a specific object (a grain-filled barn) of known size. In the Pythagorean problem, the size of the triangle is not known – indeed, that's the whole problem. So the Pythagorean problem does not involve a specific triangle but, potentially, all possible triangles. In fact, because there is no limit to the size a triangle might reach, the problem concerns an infinity of triangles simultaneously. The change in the nature of the objects being studied is thus accompanied by the irruption of the infinite into mathematics. It was this irruption that made a methodological change necessary and required reasoning to be substituted for computing. For, if the problem concerned a finite number of triangles – for example, all triangles whose sides measure less than 10,000 metres – we could still resort to calculation. Trying out every possible pair of whole numbers up to 10,000 would doubtless be tedious without the aid of a machine, but it is nonetheless systematic and would settle the finite problem. As we've observed, though, it would be futile against the infinite.

This is why the transition from computing to reasoning, in the fifth century B.C. in Greece, is regarded as the true advent of mathematics.

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THE FIRST REASONING RULES: PHILOSOPHERS AND MATHEMATICIANS

One crucial question remains: what is reasoning? Knowing that all squirrels are rodents, that all rodents are mammals, that all mammals are vertebrates, and that all vertebrates are animals, we can infer that all squirrels are animals.

One reasoning process – among others – enabling us to reach this conclusion consists in deducing, successively, that all squirrels are mammals, then that all squirrels are vertebrates, and finally that all squirrels are animals. Although this process is extremely simple, its structure is not fundamentally different from that of mathematical reasoning. In both cases, the thought process is made up of a series of propositions, each of which follows logically from the previous one through the application of an "inference rule." Here we used the same rule three times in a row: "if all *Y* are *X* and all *Z* are *Y*, then all *Z* are *X*."

The Greek philosophers were the first to compile a list of these inference rules that enable new propositions to be deduced from those already established and hence allow reasoning processes to make headway. For example, we have Aristotle to thank for the aforementioned rule. Indeed, Aristotle set up a list of rules that he called *syllogisms*. Syllogisms can take on various forms. Some follow the "all *Y* are *X*" pattern, others fall into the "some *Y* are *X*" category. Thus, knowing that all *Y* are *X*, and that some *Z* are *Y*, we can infer that some *Z* are *X*.

Aristotle was not the only ancient philosopher to take an interest in inference rules. In the third century B.C., the Stoics laid out other such rules. One rule allows the proposition *B* to be deduced from the propositions "if *A* then *B*" and *A*.

These two attempts to catalog inference rules occurred contemporaneously with the development of Greek arithmetic and geometry, after the revolutionary methodological switch from computing to reasoning. It would have made sense for Greek mathematicians to use the logic of Aristotle or that of the Stoics to support their reasoning. In order to prove that a square cannot be twice another square, for instance, they might have resorted to

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is the only way to solve mathematical problems. This position is in keeping with the importance the Ancient Greeks, both mathematicians and philosophers, attached to reasoning.

So Greek mathematicians discovered the axiomatic method and, with it, a whole new way of practicing mathematics. They might have tried to understand how this new sort of mathematics followed from Mesopotamian and Egyptian mathematics. If they had, this line of investigation would have led them to look for a way to combine computing and reasoning. But they did no such thing. Quite the contrary – they made a clean sweep of the past and abandoned computing altogether to replace it with reasoning.

For this reason, after the Greeks, computation held hardly a place in the rising edifice of mathematics.

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CHAPTER TWO

Two Thousand Years of Computation

ONCE THE AXIOMATIC METHOD had been adopted, reasoning was often spoken of as the one and only tool available for solving mathematical problems. In the discourse they developed about their science, mathematicians hardly ever mentioned computation. This doesn't mean that computing vanished from the practice of mathematics, however. Mathematicians would regularly put forward new algorithms to systematically solve certain types of problems. It seems that the history of mathematics has a bright side – that of conjectures, theorems, and proofs – and a hidden one – that of algorithms.

This chapter will focus on three important points in this history, each set in a different time period, and each raising important issues.

First we will tackle the apparent contradiction between mathematical discourse, which tends to overlook computation, and mathematical practice, which places great weight on it. We will also retrace the transition between the prehistory of mathematics and Ancient Greek mathematics.

Next we will examine the relative parts played in medieval mathematics by the Mesopotamian legacy and by the Greek legacy.

Finally we will explore why so many new geometric figures (the catenary curve, the roulette curve, etc.) appeared in the seventeenth century, whereas ancient geometry focused on only a small number of figures (the triangle, the circle, the parabola, etc.).

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EUCLID'S ALGORITHM: REASONING-BASED COMPUTATION

Euclid linked his name not only to geometry and the axiomatic method but also, ironically, to an algorithm that allows the calculation of the greatest common divisor of two integers. It is known as Euclid's algorithm.

The first method for calculating the greatest common divisor of two numbers consists of listing the divisors of each number – successively dividing the number by all smaller numbers and writing down all those for which there is no remainder – and identifying the largest number that appears on both lists. For instance, in order to calculate the greatest common divisor of 90 and 21, we start by listing the divisors of 90 (1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, and 90) and those of 21 (1, 3, 7, and 21). Then we observe that 3 is the largest number on both lists. Thus, to verify that 3 is the greatest common divisor of 90 and 21, or even to find out what the greatest common divisor of 90 and 21 is (according to how the problem is phrased), there is no need for reasoning. It suffices to apply this tiresome yet systematic algorithm (which boils down to a simple paraphrase of the definition of greatest common divisor).

Euclid's algorithm enables us to achieve the same result in a less tedious way. It rests on the following idea: in order to calculate the greatest common divisor of two numbers a and b – say, 90 and 21 – we start by dividing the greater number, a, by the smaller, b. If the division works out exactly and produces a quotient q, then $a = b \times q$. In that case, b is a divisor of a, therefore it is a common divisor of a and b, and it is bound to be the greatest one, because no divisor of b can be greater than b itself. As a conclusion, that number is the greatest common divisor of a and b. Now, if the division does not work out exactly but leaves a remainder r, then $a = b \times q + r$. In that case, the common divisors of a and b are also those of b and r. For that reason, we can replace the pair a and b by the pair b and r, which will have the same greatest common divisor. Euclid's algorithm consists in repeating that operation several times until we reach a pair of numbers for which the remainder is zero. The greatest common divisor is the smaller of those two numbers. Thus, when we calculate the greatest common divisor of 90 and 21 using Euclid's algorithm, we first replace the pair (90, 21) by the pair

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(21, 6), then by the pair (6, 3), and, finally, 6 being a multiple of 3, the result is 3.

For the numbers 90 and 21, Euclid's algorithm yields a result after three divisions. More generally, whatever numbers we start with, we will reach a result after a finite number of divisions. Because the number a is replaced with the number r, the numbers in the pair whose greatest common divisor we are looking for decrease, and a decreasing series of natural numbers is necessarily finite.

This example shows that, far from turning their backs on computation, the Greeks – among them, Euclid – participated in the devising of new algorithms. It also shows how intricately interwoven reasoning and computing are in mathematical practice. Whereas the first algorithm we discussed required no prior demonstration, in order to elaborate Euclid's algorithm, it was necessary to demonstrate several theorems: first, if the division of a by b works out exactly, then the greatest common divisor of a and b is b; second, if r is the remainder of the division of a by b, then the common divisors of a and b are the same as those of b and r; third, the remainder of a division is always less than the divisor; and last, a decreasing series of natural numbers is necessarily finite. Euclid established those results by reasoning processes similar to those used by the Pythagoreans to prove that a square cannot equal twice another square.

No significant reasoning was needed to build the first algorithm, but this is an exceptional case. More often than not, algorithms are like Euclid's and entail more than merely paraphrasing a definition: in order to elaborate the algorithm, we must conduct a reasoning process.

THALES AND THE PYRAMIDS: THE INVENTION OF MATHEMATICS

The fact that building an algorithm typically requires reasoning causes us to wonder, in retrospect, about Mesopotamian and Egyptian mathematics. How did the Mesopotamians, for instance, conceive a division algorithm without resorting to reasoning? The two peoples must have known an implicit form of reasoning. The fact that, unlike the Greeks, they did not make their reasoning processes explicit – by writing them down on tablets, for example – and that

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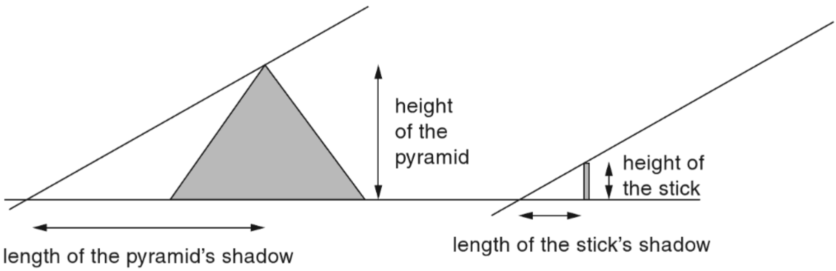

Figure 2.1 (pyramid, stick and shadow)

they probably were not aware of the importance of reasoning in the solving of abstract mathematical problems proves nothing: they may very well have reasoned the way Monsieur Jourdain, Molière's "Bourgeois Gentleman," spoke in prose – "unawares."

The necessity of mathematical reasoning in the building of algorithms has often been remarked upon. More rarely has it been noted that this necessity sheds light on the Greek miracle: the transition from computing to reasoning. Indeed, we can hypothesize that the importance of reasoning dawned on the Greeks precisely as they developed algorithms.

The "first" geometrical reasoning process is generally attributed to Thales. In order to calculate the size of a pyramid that was too high to be measured directly, Thales came up with an idea: he measured the length of the pyramid's shadow, the height of a stick, and the length of the stick's shadow, then proceeded to apply the rule of three (Figure 2.1).

It seems likely that Thales's aim was to devise a new algorithm to calculate the length of a segment; in doing so, he probably realized that he needed to prove that the ratio between the pyramid and its shadow was the same as that between the stick and its shadow. Thus, a theorem was born, the intrinsic value of which was later to be revealed.[1]

[1] This result, known today as the "intercept theorem," is referred to in many languages as Thales's theorem. It should not be confused with another result, more commonly known in English as Thales's theorem, which deals with points on a circle.

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makes simple addition and subtraction algorithms possible. More importantly, with this notation system, the multiplication algorithm is simplified: in order to multiply a number by ten, all you need to do is shift the number to the left and add a 0 at the end.

This positional notation for numbers has its origin in Mesopotamia, where a rough draft of this system was already in use by 2000 B.C. However, the Mesopotamian system was too complicated. The Indians were the first to simplify it. Then, in the ninth century, the Indian version of positional notation spread to the Arab world thanks to a book written by Muhammad ebne Mūsā al-Khwārizmī (from whose name the word "algorithm" is derived) called *Al-Jabr wa-al-Muqabilah* ("Book on Integration and Equation"). The system then reached Europe in the twelfth century. Mathematicians of the Middle Ages thus benefited from a double legacy: they inherited a lot from the Greeks, but also from the Mesopotamians, who handed down to them the all-important positional notation system. These mathematicians then spent many centuries developing and perfecting algorithms.

The discovery of the axiomatic method did not oust computation. On the contrary, computation thrived, through the Mesopotamian legacy, to become a key preoccupation in the eyes of medieval mathematicians.

CALCULUS

Having dealt with Euclid's algorithm and with algorithms designed to carry out arithmetic operations, we now move on to a third crucial event in the history of mathematics: the development of calculus. This branch of mathematics appeared in the seventeenth century with the works of Bonavantura Cavalieri, Isaac Newton, Gottfried Wilhelm Leibniz, and others. Its roots, however, go back much further: during antiquity, two discoveries of Archimedes's laid the groundwork for the invention of calculus. One of these discoveries concerns the area of the circle and the other, the area of the parabolic segment.

It is a well-known fact, today, that the area of a circle is obtained by multiplying the square of its radius by 3.1415926. . . . Archimedes did not get quite this far, but he did prove that, in order to

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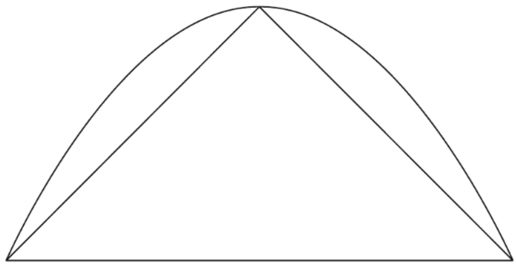

Figure 2.2a

calculate the area of a circle, you need to multiply a number bounded between $3 + 10/71 = 3.140\ldots$ and $3 + 10/70 = 3.142\ldots$ by the square of the circle's radius – in other words, he discovered the first two decimals of the number π. His work on the area of the parabolic segment was even more successful, as he reached an exact result: he correctly established that the area of a parabolic segment equals four-thirds the area of the triangle inscribed within that segment (Figure 2.2a).

In order to achieve that feat, Archimedes decomposed the parabolic segment into an infinity of successively smaller triangles, the areas of which he added up (Figure 2.2b).

If you take the area of the triangle inscribed within the parabolic segment as a unit, the area of first triangle is, by definition, 1. It can be proved that the two triangles on its sides have a total area of 1/4, then that the area of the next four triangles is 1/16, and so on. The total area of each set of triangles equals one fourth that of the

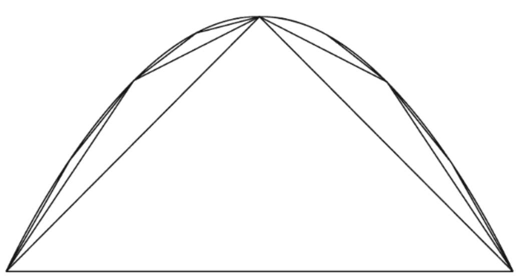

Figure 2.2b

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general in Frege's days, Russell's paradox proved it to be just another notion requiring its own axioms. So, in the 1920s, Russell's type theory was in turn simplified: David Hilbert stripped it of everything specific to the notion of set and created "predicate logic," which remains the framework of logic to this day. Since then axioms specific to the notion of set, as they were formulated by Ernst Zermelo in 1908, constituted just one theory among many others, namely set theory.

The separation between predicate logic and set theory undermines Russell's thesis of the universality – or lack of specificity – of mathematics. It is predicate logic that appears to be universal; within predicate logic, if you wish to practice mathematics, it is necessary to call upon axioms taken from set theory. It is therefore possible to conceive of a logical reasoning process that obeys the rules of predicate logic and yet rests on axioms other than those of set theory; the same cannot be said of mathematical reasoning.

Actually, these assertions call for some qualification. A theorem proved by Kurt Gödel in 1930 (but not Gödel's famous theorem) shows that any theory can be translated into set theory. Euclidian geometry, which a priori rests on a different set of axioms than set theory, can nevertheless be translated into set theory. This theorem revives Russell's thesis by conferring universality and ontological neutrality on set theory itself.

THE PROBLEM OF AXIOMS

Besides endangering Russell's thesis, the separation of predicate logic and set theory has another major drawback: it imperils Frege's whole philosophical project to define the notion of a natural number from purely logical notions, then to show that the proposition "$2 + 2 = 4$" follows from that definition. In predicate logic, without axioms it is impossible to define the integers so as to make this proposition provable. As soon as we introduce axioms – such as those of set theory – it becomes possible. Around the same time that Frege put forward his axioms, Peano devised some of his own, namely axioms of arithmetic. These also made it possible to prove the proposition "$2 + 2 = 4$" and, more generally, to demonstrate all known theorems concerning integers, only in a simpler way.

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