Praise for *Concept-Based Mathematics*

"I attended a Concepts Based Curriculum training course led by Jennifer Wathall and was really inspired by what I learned. Far too often, as teachers, we can become narrowly focused on the topics that we are covering; with concepts there is a whole new opportunity for students to understand the big ideas and the connections between different subjects. Jennifer skillfully guided us through how we can introduce students to a concept-based curriculum. I was really impressed with the method of writing generalizations which provide a framework for exploration. These generalizations can in fact make the focus of a lesson or series of lessons much more exciting, allowing students to break out of the constraints of a limited topic range. Ultimately, I left the course determined to try out a concepts-based model with a new unit we are developing on human rights. With the conceptual lens, this promises to be a much more thought-provoking unit for our students."

—John Edwards, Head of History Department

King George V School, Hong Kong

"Secondary teachers are constantly being encouraged to change their practice but few books have addressed the topics of secondary mathematics or given examples that secondary teachers can relate to. This book does that. Another strength is the connection of the content to the math processes and practices—the heart of good instruction. The figures provided to summarize big ideas are excellent. I love the potential of this book for using it as a text for middle and secondary teachers, a guide for professional development, and a place for individual reflection. I know for sure that I would use it for my student teacher seminar class and anytime that I was instructing upper level math teachers. I've been waiting for this!"

—Barbara Fox, Adjunct Professor, Student Teacher Supervisor

University of Massachusetts, Lesley University, Regis College

"Jennifer Wathall's *Concept-Based Mathematics* is one of the most forward-thinking mathematics resources on the market. While highlighting the essential tenets of Concept-Based Curriculum design, her accessible explanations and clear examples show how to move students to deeper conceptual understandings. This book

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Foreword

H. Lynn Erickson

How many times have you heard the lament, "I was so bad at mathematics in school?" Yet, those with an affinity for mathematics view it as a beautiful abstract language that cuts across fields of knowledge to solve problems, raise questions, explain mysteries, and create wondrous works of art. Jennifer Wathall is one of those people with this affinity. She desires to share her understanding and passion for mathematics with the world. How lucky we are!

I wish all of my prior mathematics teachers had been able to read Jennifer's book and learn from her. As I reflect on my years as a student and my mathematics education specifically, I remember feeling confused as we drilled on daily computations and struggled with word problems. I could *do* math but I did not *understand* math. In my own journey as a teacher, I came to realize the critical importance of conceptual understanding across all of the disciplines. Why had I not been trained to teach for deeper conceptual understanding?

Secondary mathematics teachers across the world will appreciate reading Jennifer's insights about the other half of the equation—the conceptual understanding of mathematics. In traditional mathematics education, we have "assumed" students understand the concepts of mathematics if they could perform the algorithms. It was a step forward as we required students to "explain their thinking" on mathematical problems, but this still did not ensure that students really understood the conceptual relationships inherent in the problem. Jennifer shows us that students need to demonstrate and verbalize their conceptual understanding of mathematics as well as apply it across multiple contexts.

Concept-Based Mathematics: Teaching for Deep Understanding in Secondary Classrooms is a clear explanation of the content and process structures of mathematics supported by salient examples. Jennifer provides practical, engaging, and meaningful learning experiences that draw students to the beauty and power of mathematical concepts and their relationships.

One of the strengths of concept-based curriculum and instruction models is that they are not "programs." They are the explicit expression of the previously implied design principles for quality curriculum design and pedagogy. They overlay *any* curriculum and instruction model and should not be a choice. They are the principles that ensure deeper conceptual thinking and the transfer of understandings through time, across cultures, and across situations. Whether school curricula are textbook based or teacher developed, they must reflect the three-dimensional design principles expressed in this book and other books on concept-based curriculum and instruction, or they will remain a lower level, two-dimensional design model—coverage of facts and skills.

This book is cutting edge. It is the next step to bring mathematics education into the 21st century. It needs to be in the hand of every secondary mathematics teacher and teacher educator. All students deserve to experience the wonder and passion for mathematics that Jennifer so obviously feels. It is time to change the age-old lament to empowering testimonials—"I love mathematics!" "I can use mathematics every day to think and create!" "Math is fun!" "I get math!"

Preface

Purpose of the Book

Traditional curriculum focuses on rules and procedures with little understanding of the conceptual relationships of mathematics—and mathematics is a *language* of conceptual relationships. Traditional curriculum *assumes* the deep understanding of concepts and fails to teach for transferability or to consider context. This book expands and develops the work of Lynn Erickson and Lois Lanning on concept-based curriculum into the realm of mathematics.

When learning math, students need to be given time and space to explore and discover the beauty and creativity in math without being fearful of mistakes. Math anxiety exists because of an overemphasis on the processes and skills of this discipline. This book addresses how to create concept-based and inquiry-led curriculum and instruction with a goal to make math enjoyable and accessible to all of our students.

Concept-based curriculum is a three-dimensional design model of curriculum and instruction that frames factual content and skills and processes with disciplinary concepts, generalizations, and principles. In concept-based curriculum and instruction, the development of intellect is achieved through higher order, synergistic thinking in which teachers use the facts, processes, and skills in concert with the concepts, generalizations, and principles. A traditional two-dimensional design model for curriculum and instruction focuses on factual content, processes, and skills and assumes conceptual understanding. The research and consensus on the benefits of developing conceptual understanding is undeniable. Concept-based curriculum produces deeper emotional and intellectual engagement in learning and therefore develops attributes such as critical thinking, reasoning, and creativity.

The intention of this book is to extend the work of Lynn Erickson and Lois Lanning on the Structure of Knowledge and the Structure of Process specifically to mathematics and to help math educators understand how to convey mathematical concepts and ideas using the vehicle of inquiry. All definitions used in the Structure of Knowledge and the Structure of Process derive from the work of Lynn Erickson and Lois Lanning. We need to help students understand that everyone is capable doing math and it is not a matter of whether you can or can't do math.

This book expands and develops the work of Lynn Erickson and Lois Lanning on concept-based curriculum into the realm of mathematics.

Special Features

Special features include sample lessons, samples of student work, vignettes from international educators, and discussion questions that may be used in a book study with fellow teachers or in a professional development setting. As an individual teacher or as part of a study group, read each chapter and use the discussion questions at the end of each chapter to reflect on your own practice. Metacognition logs are included at the end of the book, to help you to process, synthesize, and self-reflect on each chapter of the book. There is a chapter on integrating technology to enhance learning and conceptual understanding (Chapter 7) and a Glossary to guide you through the terms used in the book.

The main text is accompanied by a suite of free online resources, which include more sample instructional units and templates for worksheets that foster deeper conceptual understanding of particular math topics for secondary school.

After reading this book, you will be able to focus instruction on deeper conceptual understandings and equip students for future success. It will provide you with practical examples of concept-based lessons, unit webs, unit planners, and different assessment tools to enable you to develop a concept-based approach to your curriculum and instruction.

Concept-based mathematics is grounded in the philosophy that in order to develop intellect, instruction and curriculum needs to focus on the big ideas or conceptual understandings. This can be achieved through instructional practice and designing tasks that do not sacrifice the content or rigor of any prescribed syllabus. In fact, concept-based curriculum challenges students to employ higher order thinking skills. Concept-based curriculum can overlay any curriculum, such as the Common Core State Standards and Basal curriculum (United States), GCSE and A Levels (UK), as well as the International Baccalaureate Middle Years (MYP) and Diploma (DP) mathematics programs.

In this ever changing, dynamic and complex world, mathematics education must engage students intellectually and emotionally. The ability to think conceptually, transfer understandings across contexts and situations, and to enjoy learning and problem solving are major goals for mathematics education today so we can prepare our students for future success. Technological advancements of even the last decade have influenced instruction, and the key to utilizing technology effectively is not *what* tool is being used but *how* the technology is used to enhance learning.

I hope this book inspires you on your journey to develop conceptual understanding in your students and to eradicate math anxiety and fear by fostering a growth mindset. I hope you will join me on this journey for this much-needed math education reform.

In this ever changing, dynamic and complex world, mathematics education must engage students intellectually and emotionally.

How to Use the Companion Website

http://www.resources.corwin.com/WathallConceptBasedMathematics

The companion website offers the following resources to supplement this book:

- Straightforward activities designed to help teachers understand and apply concept-based curriculum and instruction;
- Examples that model each aspect of concept-based curricula;
- Blank templates for designing unit planners and writing quality generalizations;
- Guiding questions to help you and your book study group to reflect on the process of implementation and next steps;
- A metacognition log: a powerful tool for self-reflection that focuses on the end-of-chapter discussion questions in this book.

If you are working with a book study group or PLC, you might want to upload the activities, templates, companion website discussion questions, and the metacognition log to a cloud on an app such as Google Drive so that you can share your personal written reflections with your team as you write them.

Here are some suggestions for how an individual educator can use the website:

- Read *Concept-Based Mathematics: Teaching for Deep Understanding in Secondary Schools,* ensuring you address the discussion questions at the end of each chapter, and write a reflection on the metacognition log template. You can write your reflections on a piece of paper, on your personal computer, or you can upload them to a cloud to share with your book study group.
- Make notes on areas you would like to develop from the main book.
- Go through the website, using the templates to create your own examples, and answer the discussion questions.
- Think about a unit of work you would like to develop to ensure more conceptual understanding and use the templates to support your planning.
- Trial your ideas in the classroom and modify accordingly.

- Share with colleagues.

Here are some suggestions for how a group of educators can use the website as a book study. Read one chapter a week and meet with the book study group to discuss them.

- Read a chapter of *Concept-Based Mathematics: Teaching for Deep Understanding in Secondary Schools* and write a reflection in the metacognition log.
- Share your metacognition log notes, one chapter at a time, with your book study group.
- Use the discussion questions from each chapter to stimulate sharing of ideas during your meetings with your study group.
- Go through the website, using the templates to create your own examples.
- Think about a unit of work you would like to develop to ensure more conceptual understanding, and use the templates to support your planning. This could be a collaborative effort with three or four other teachers.
- Trial your ideas in the classroom.
- Share and review your unit planners with colleagues, either in person or in a cloud.

Audience

Drawn from my 24 years as an international educator and presenter, this book will uncover the secrets to help all students in middle and high school understand how to convey the conceptual language of mathematics. This book is intended for middle and high school teachers, trainee teachers in undergraduate education programs, and graduate education courses ranging from bachelor of education, diploma in education, to masters in arts specializing in education.

Chapter Overview

Part I (Chapters 1 and 2) of this book discusses what a concept-based curriculum looks like for mathematics and explains, in detail, Lynn Erickson's Structure of Knowledge and Lois Lanning's Structure of Process applied to the topic of functions. Examples of levels of inquiry (structured and guided) and inductive teaching are given. The key to inductive teaching is that students draw and form generalizations by working on specific examples initially.

Part II (Chapters 3 and 4) guides readers in the practice of applying concept-based curriculum and instruction to math. Chapter 3 deals with crafting generalizations, which are statements of conceptual understanding. Lynn Erickson provides a three-step guide to writing quality generalizations, which are statements of what we want our students to understand from their program of study. Chapter 4 includes models of unit webs and unit planners.

Part III (Chapters 5 through 8) looks at instructional strategies to intellectually and emotionally engage students to ensure deep conceptual understanding. Chapter 5 discusses eight strategies for lesson planning and captivating your students' hearts and minds. Chapter 6 looks into more detail about formative assessment strategies to track student learning. Chapter 7 discusses how to integrate technology effectively and gives practical activities and digital tools that support conceptual understanding. These tools include using mathematical graphing software, flipped classroom models, multimedia projects, collaborative digital tools, and various educational apps for the classroom. Chapter 8 looks at the elements of an ideal math classroom. It includes rubrics to support the developing concept-based teacher and for developing concept-based instruction. Chapter 8 also addresses common concerns and misconceptions about concept-based curriculum and instruction.

After reading this book,

- You will have a better understanding of the benefits of a concept-based instructional design model;
- You will be able to overlay a concept-based curriculum and instruction model onto any curriculum and implement it in your classroom; and
- You will have ideas and resources to engage your students and increase their conceptual understanding and enjoyment of mathematics.

Acknowledgments

There are very few people you meet who have such an impact on your life as H. Lynn Erickson has had on mine. I feel so fortunate that Lynn has chosen me to mentor and guide me through my journey into concept-based mathematics. I will forever be indebted for the care, time, and dedication she has shown me during the writing of this book. Lynn: You are a remarkable educator and you have been a wonderful role model for me, helping me believe that anything is possible. Thank you for showing me that I am only at the beginning of my journey in education.

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About the Author

Jennifer T. H. Wathall

has been a teacher of mathematics for more than 20 years. She graduated from the University of Sydney with a BSc majoring in mathematics and completed postgraduate studies at the University of Hong Kong.

She has worked in several international schools, including South Island School, Hong Kong; The United Nations International School (UNIS), New York; and she is currently working at Island School, Hong Kong as head of mathematics. In the international arena, she has presented workshops such as "How to Effectively Integrate Multimedia into the Classroom" at the 21st Century Conference in Hong Kong and Shanghai and the Asian Technology Conference in Mathematics, Bangkok and Beijing. She has also given talks around Asia about how to effectively integrate a 1:1 program into the mathematics classroom.

As a qualified International Baccalaureate workshop leader ("Mathematics, Concepts, and Inquiry in the Diploma Program and Approaches to Teaching and Learning"), Jennifer has delivered numerous workshops in the Asia Pacific region. Her role as a field representative for the IB Asia Pacific serves as part of the quality assurance framework. She has consulted for IB mathematics textbooks and has developed an IB Category 3 workshop on "The Use of the Casio GDC in IB Mathematics." Jennifer has delivered presentations at the IB Asia Pacific Conference ("Using Inquiry in the IB Mathematics Classroom") and at the IB Americas Annual Conference ("Concept-Based Mathematics"). Currently she is part of the external curriculum review group for IB diploma mathematics based in The Hague and Cardiff. As an expert in IB mathematics, Jennifer serves as an honorary faculty advisor and part-time instructor for the University of Hong Kong.

She is a certified trainer in the DISC™ behavior assessment tool, and she is a certified independent consultant in Concept-Based Curriculum Design by Dr. H. Lynn Erickson. Jennifer works as a consultant helping math departments and schools transition to concept-based curriculum and instruction. She utilizes her skills as a certified performance coach to facilitate transition and change.

Author's Note

I was born to be a teacher. I love being in the classroom and just spending time with my students with the goal of inspiring a love for learning. Nothing excites me more than seeing those light bulb moments during a lesson when students have a gleam in their eye because they get it. That gleam tells me my students understand on a deeper level than what a textbook or video can explain. I have been so lucky that all of my life I knew what my vocation would be.

My father fostered my love of learning and teaching, as he was a teacher himself. He taught English in the air force before joining the diplomatic corps. He read to me most nights: sometimes Jane Austen or Charles Dickens and sometimes famous Chinese fables to teach me about Chinese culture and history. He was patient, intelligent, and possessed a lifelong thirst for knowledge. His passing in July 2014 inspired me to complete my two years of research and to write this book. Math education needs to change. Too many students have been scarred for life because of their negative experiences in math classrooms. Everybody can do math in an environment focused on conceptual understanding and a growth mindset. I hope to start a revolution in math classrooms and help teachers to think and reflect about what they are teaching. What do we want our students to learn and understand, and what is mathematics? Is it a discipline of processes?

Mathematics comes from the Greek word *máthe-ma,* which means "that which is learnt." In Modern Greek, *máthe-ma* means "to learn." Math lessons need to focus on *learning* and not on *performing*. Many mathematicians have different interpretations of what mathematics is. Below are my favorite quotes from mathematicians.

> *Pure mathematics is, in its way, the poetry of logical ideas.*
>
> —Albert Einstein, German-born theoretical physicist and 1921 Nobel Prize winner, 1879–1955

> *Nature's great book is written in mathematics.*
>
> —Galileo Galilei; Italian physicist, mathematician, astronomer, and philosopher; 1564–1642

> *Mathematics is the queen of sciences and number theory is the queen of mathematics. She*

often condescends to render service to astronomy and other natural sciences, but in all relations she is entitled to the first rank.

—Carl Friedrich Gauss; German mathematician, physicist, and prodigy; 1777–1855

A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

—Godfrey H. Hardy, English mathematician known for his achievements in number theory and mathematical analysis, 1877–1947

Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency.

—René Descartes; French philosopher, mathematician, scientist, and writer; 1596–1650

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

—Stan Gudder, mathematics professor, University of Denver

Whenever I am in a social situation and tell someone I am a math teacher, I receive one of two reactions: anxiety alongside an alarming panic, with people expressing how much they hated math at school; and the less common response—how much they loved math—which begins a lively conversation about the usefulness of math. The first response saddens me. How can mathematics elicit such fear and negativity? English, art, and even science teachers do not elicit such strong emotions in people. A longstanding tradition sees mathematics as an elusive discipline that few could comprehend. Many people recall negative experiences when learning mathematics that have instilled fear of the discipline. Timed tasks, rote memorization of formulae with little conceptual understanding, and a focus on performance have created math fear and reinforce these negative experiences.

As a person who made mathematics education her career, I fortunately did not have those negative experiences as a child. I loved the challenge of puzzles and problems that were presented and possessed a passion for mathematics throughout my school life. When I was 12 years old, my mother took me to a fortune teller in Taiwan who looked into my eyes and said I would follow my passion to become a math teacher.

Who knows if the fortune teller could really tell, but from that day as a child, I felt I knew my destiny and have been fortunate enough to be able to share my joy for math education well into my third decade.

In loving memory of my father,

David Kuo Cheng Chang,

who inspired me to be a lifelong learner

1929–2014

Part I What Is Concept-Based Curriculum and Instruction in Mathematics? Research and Theory

Chapter 1 Why Is It Important for My Students to Learn Conceptually?

Around the world, mathematics is highly valued and great importance is placed on learning mathematics. Private tutors in non-Asian countries serve a remedial purpose, whereas in Asia, everyone has a tutor for providing an increased knowledge base and skill development practice. Many students in Asia enroll in programs like "Kumon," which focus on practicing skills (which has its place) and "doing" math rather than "doing and understanding" math. When you ask students who are well rehearsed in skills to problem solve and apply their understanding to different contexts, they struggle. The relationship between the facts, skills, and conceptual understandings is one that needs to be developed if we want our students to be able to apply their skills and knowledge to different contexts and to utilize higher order thinking.

Why Do We Need to Develop Curriculum and Instruction to Include the Conceptual Level?

According to Daniel Pink (2005), author of *A Whole New Mind,* we now live in the Conceptual Age. It is unlike the Agricultural Age, Information Age, or the Industrial Age because we no longer rely on the specialist content knowledge of any particular person. The Conceptual Age requires individuals to be able to critically think, problem solve, and adapt to new environments by utilizing transferability of ideas. "And now we're progressing yet again—to a society of creators and empathizers, of pattern recognizers and meaning makers" (Pink, 2005, p. 50).

Gao and Bao (2012) conducted a study of 256 college-level calculus students. Their findings show that students who were enrolled in concept-based learning environments scored higher than students enrolled in traditional learning environments. Students in the concept-based learning courses also liked the approaches more. A better grasp of concepts results in increased understanding and transferability.

With the exponential growth of information and the digital revolution, success in this modern age requires efficient processing of new information and a higher level of abstraction. Frey and Osborne (2013) report that in the next two decades, 47% of jobs in the United States will no longer exist due to automation and computerization. The conclusion is that we do not know what new jobs may be created in the next two decades. Did cloud service specialists, android developers, or even social marketing companies exist 10 years ago?

How will we prepare our students for the future? How will our students be able to stand out? What do employers want from their employees? It is no longer about having a wider knowledge base in any one area.

Hart Research Associates (2013) report the top skills that employers seek are the following:

- Critical thinking and problem solving,
- Collaboration (the ability to work in a team),
- Communication (oral and written), and
- The ability to adapt to a changing environment.

How do we develop curriculum and instruction to prepare our students for the future?

> We owe our students more than asking them to memorize hundreds of procedures. Allowing them the joy of discovering and using mathematics for themselves, at whichever level they are able, is surely a more engaging, interesting and mind-expanding way of learning. Those "A-ha" moments that you see on their faces; that's why we are teachers.
>
> David Sanda, Head of Mathematics Chinese International School, Hong Kong

The Structure of Knowledge and the Structure of Process

Knowledge has a structure like other systems in the natural and constructed world. Structures allow us to classify and organize information. In a report titled *Foundations for Success,* the U.S. National Mathematics Advisory Panel (2008) discussed three facets of mathematical learning: the factual, the procedural, and the conceptual. These facets are illustrated in the Structure of Knowledge and the Structure of Process, developed by Lynn Erickson (2008) and Lois Lanning (2013).

The **Structure of Knowledge** is a graphical representation of the relationship between the topics and facts, the concepts that are drawn from the content under study, and the generalization and principles that express conceptual relationships (transferable understandings). The top level in the structure is Theory.

Theory describes a system of conceptual ideas that explain a practice or phenomenon.

Examples include the Big Bang theory and Darwin's theory of evolution.

The **Structure of Process** is the complement to the Structure of Knowledge. It is a graphical representation of the relationship between the processes, strategies, skills and concepts, generalizations, and principles in process-driven disciplines like English language arts, the visual and performing arts, and world languages.

For all disciplines, there is interplay between the Structure of Knowledge and the Structure of Process, with particular disciplines tipping the balance beam toward one side or another, depending on the purpose of the instructional unit. The Structure of Knowledge and the Structure of Process are complementary models. Content-based disciplines such as science and history are more knowledge based, so the major topics are supported by facts. Process-driven disciplines such as visual and performing arts, music, and world languages rely on the skills and strategies of that discipline. For example, in language and literature, processes could include the writing process, reading process, or oral communication, which help to understand the author's craft, reader's craft, or the listener's craft. These process-driven understandings help us access and analyze text concepts or ideas.

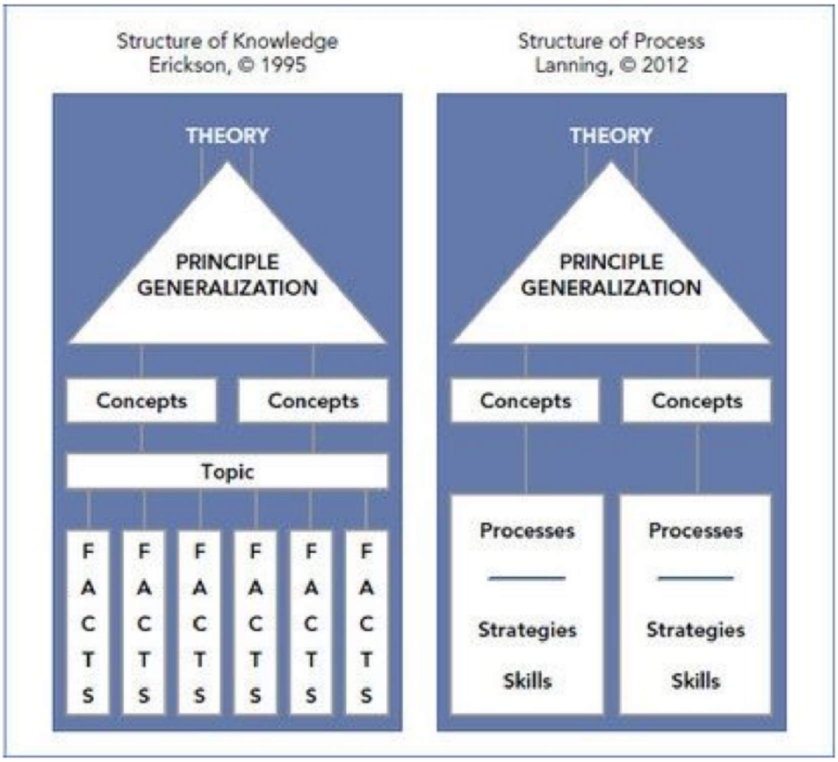
Both structures have concepts, principles, and generalizations, which are positioned above the facts, topics, or skills and strategies. Figure 1.1 illustrates both structures. Figure 1.1 can also be found on the companion website, to print out and use as a reference.

The Structure of Knowledge and the Structure of Process for Functions

Mathematics *can* be taught from a purely content-driven perspective. For example, functions can be taught just by looking at the facts and content; however, this does not support learners to have complete conceptual understanding. There are also processes in mathematics that need to be practiced and developed that could also reinforce the conceptual understandings. Ideally it is a marriage of the two, which promotes deeper conceptual understanding. Figure 1.2 illustrates the Structure of Knowledge for the topic of functions.

Topics organize a set of facts related to specific people, places, situations, or things. Unlike history, for example, mathematics is an inherently conceptual language, so "Topics" in the Structure of Knowledge are actually *broader concepts,* which break down into micro-concepts at the next level.

Figure 1.1: Side by Side: The Structure of Knowledge and the Structure of Process



Transitioning to Concept-Based Curriculum and Instruction, Corwin Press Publishers, Thousand Oaks, CA.

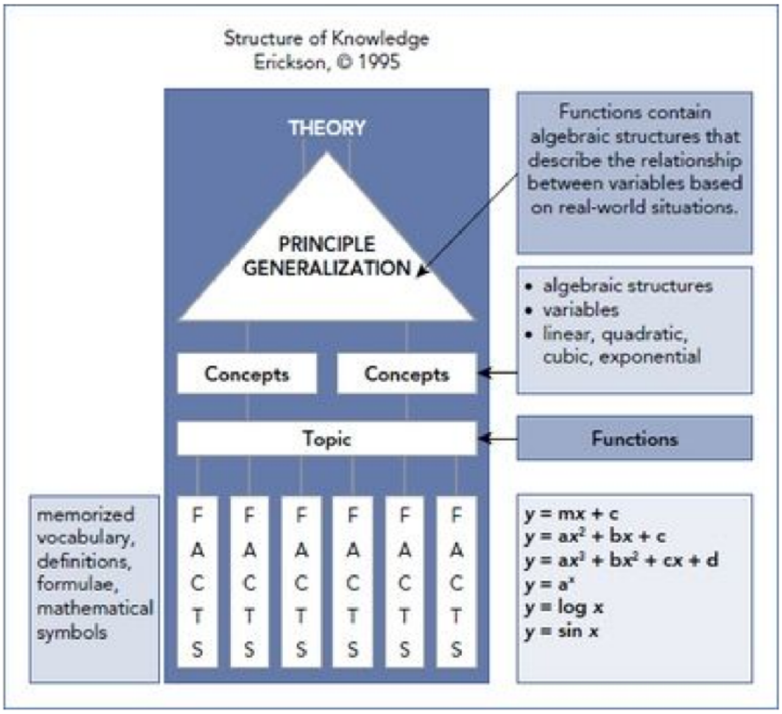
As explained by Lynn Erickson (2007), "The reason mathematics is structured differently from history is that mathematics is an inherently conceptual language of concepts, subconcepts, and their relationships. Number, pattern, measurement, statistics, and so on are the broadest conceptual organizers" (p. 30).

More about concepts in mathematics will be discussed in Chapter 2.

Facts are specific examples of people, places, situations, or things. Facts do not transfer and are locked in time, place, or a situation. In the functions example seen in Figure 1.2, the facts are $y = mx + c$, $y = ax^2 + bx + c$, and so on. The factual content in mathematics refers to the memorization of definitions, vocabulary, or formulae. When my student knows the *fact* that $y = mx + c$, this does not mean she understands the *concepts* of linear relationship, *y*-intercept, and gradient.

According to Daniel Willingham (2010), automatic factual retrieval is crucial when solving complex mathematical problems because they have simpler problems embedded in them. Facts are the critical content we wish our students to know, but they do not themselves provide evidence of deep conceptual understanding.

Figure 1.2: The Structure of Knowledge for Functions



Adapted from original Structure of Knowledge figure from *Transitioning to Concept-Based Curriculum and Instruction,* Corwin Press Publishers, Thousand Oaks, CA.

Formulae, in the form of symbolic mathematical facts, support the understanding of functions. This leads to a more focused understanding of the *concepts* of linear functions, quadratic functions, cubic functions, exponential functions, variables, and algebraic structures in Figure 1.2. The generalization "Functions contain algebraic structures that describe the relationship between two variables based on real-world situations" is our ultimate goal for conceptual understanding related to the broad concept of functions. Please take a look at the companion website for more examples of the Structure of Knowledge and the Structure of Process on the topic of linear functions. See Figures M1.1 and M1.2.

Concepts are mental constructs, which are timeless, universal, and transferable across time or situations. Concepts may be broad and abstract or more conceptually specific to a discipline. “Functions” is a broader concept, and the micro-concepts at the next level are algebraic structures, variables, linear, quadratic, cubic, and exponential. Above the concepts in Figure 1.2 are the principles and generalizations.

Principles and **generalizations** are transferable understandings that allow students to make connections between two or more concepts. In mathematics, the principles are the theorems, the cornerstone truths. Though generalizations and principles are both statements of conceptual relationship, the principles do not contain a qualifier such as *often, can,* or *may* because they are immutable “truths” as we know them. Because generalizations do not rise to the level of a law or theorem, they *may* require a qualifier if they do not hold true in all cases. Principles and generalizations are often exemplified in a real-life context for mathematics; however, they are not exclusively portrayed in this way. In Figure 1.2, another generalization could have been the following: “Algebraic tools allow highly complex problems to be solved and displayed in a way that provides a powerful image of change over time” (Fuson, Kalchman, & Bransford, 2005, p. 351).

Although the Structure of Knowledge provides the deep understanding of the content of mathematics, the processes, strategies, and skills also provide important conceptual understanding.

The Structure of Process represents the procedural facet of learning mathematics. Processes, skills, and strategies are included in the lowest levels in the Structure of Process. “Skills are smaller operations or actions that are embedded in strategies, and when appropriately applied ‘allow’ the strategies to work. Skills underpin a more complex strategy” (Lanning, 2013, p. 19).

Strategies are systematic plans that learners consciously adapt and monitor to improve learning performance. As explained by Erickson and Lanning (2014), “Strategies are complex because many skills are situated within a strategy. In order to effectively employ a strategy, one must have control over a variety of the skills that support the strategy.” (p. 46). An example of a strategy in math would be making predictions or drawing conclusions.

Processes are actions that produce results. A process is continuous and moves through stages during which inputs (materials, information, people’s advice, time, etc.) may transform or change the way a process flows. A process defines what is to be done—for example, the writing process, the reading process, the digestive process, the respiratory process, and so on.

Figure 1.3 illustrates an example of the mathematical process of creating representations and the generalizations associated with this mathematical process. Throughout this functions unit, students will learn different strategies and skills that support the process of creating representations. This could include using a table of values or an algebraic or geometric form of a function.

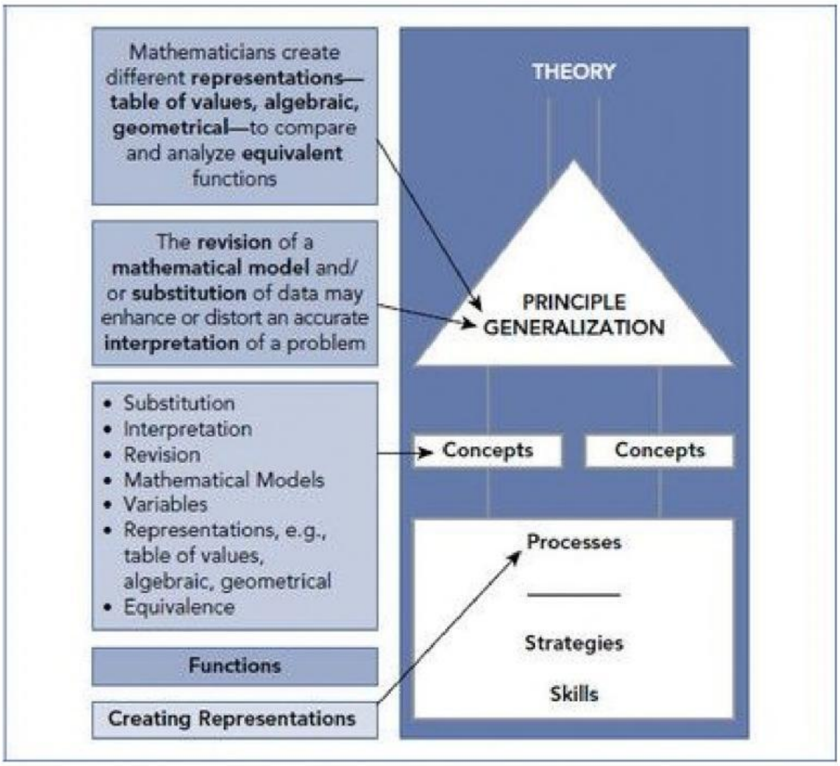
Concepts that can be drawn from this process include substitution, revision, interpretation, and models. Two or more of the concepts are used to write unit generalizations, which are also known as process generalizations. The process generalizations in Figure 1.3 are as follows:

> **M**athematicians create different representations—table of values, algebraic, geometrical—to compare and analyze equivalent functions.

> **T**he revision of a mathematical model or substitution of data may enhance or distort an accurate interpretation of a problem.

When students are guided to these generalizations, they demonstrate their understanding of the creating representations process.

Figure 1.3: Structure of Process Example for Functions



Adapted from original Structure of Process figure from *Transitioning to Concept-Based Curriculum and Instruction,* Corwin Press Publishers, Thousand Oaks, CA.

Other strategies and skills, such as graphing and analytical skills, support the process of creating representations. This process supports the concepts of mathematical models, substitution, interpretation, revision, variables, equivalence, and so on.

In Figure 1.4, we look at the dual part that the Structure of Knowledge and the Structure of Process each play in ensuring a deep understanding of content and process in mathematics. For the concept of functions, we include the content that needs to be learned as well as the skills and strategies that are employed fluently to aid the process of creating representations. The ability to employ strategies and skills fluently is referred to as *procedural fluency.* Visit the companion website to see additional summaries of the components of the Structures of Knowledge and Process. See Figures M1.3 and M1.4.

To help understand the generalization "Functions contain algebraic structures that describe the relationship between two variables based on real-world situations," we work to ensure the conceptual relationships are revealed. The concepts of algebraic structures, variables, linear, quadratic, and cubic help us connect the facts to give mathematical content more meaning and promote deeper understanding. The mathematical process involved is creating representations, and it supports the understanding of the concepts substitution, interpretation, revision, variables, mathematical models, and equivalence. Mathematical processes will be discussed in detail in the next chapter.

Generalizations are statements that connect two or more concepts.

> **T**he language of mathematics is different to languages like English and Chinese. There are things that are strictly allowed and there are things that are strictly not. It is the formal nature of the language that often causes confusion and errors in learners. However, overemphasis on the formality, and some teachers are only concerned with practicing formal exercises, prevents understanding of the beauty, creativity, and utility of mathematics.
>
> Chris Binge, Principal Island School, Hong Kong

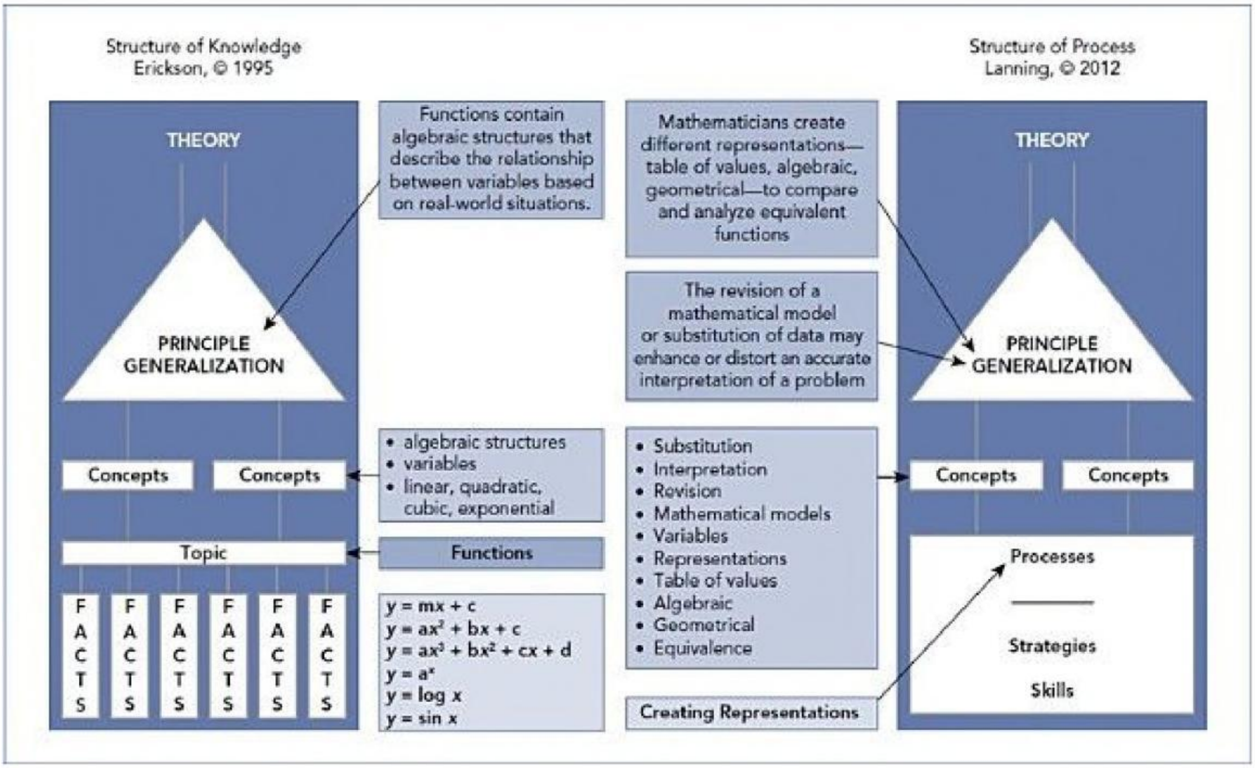
Applying the Structure of Knowledge and the Structure of Process

Inductive vs. Deductive Teaching

In my first years of teaching, it was common practice in the mathematics classroom to adopt the PPP model (presentation, practice, and production) of **deductive, teacher-led instruction**. The PPP approach typically looks like this:

> Step 1: Teacher introduces the formula, such as the Pythagorean theorem, and demonstrates three working examples.
> Step 2: Ask students to practice using the formula.
> Step 3: Ask students to produce their own examples.

Figure 1.4: The Structure of Knowledge and The Structure of Process for the Topic Functions, Side by Side



Adapted from original Structure of Knowledge and Structure of Process figures from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.

The **two-dimensional model of instruction**, which focuses on the facts and content of the subject and the rote memorization of procedures and topics, is intellectually shallow. A two-dimensional curriculum and instruction model focuses on the bottom levels of the Structure of Knowledge and the Structure of Process. This encourages students to work at a low-order level of thinking (such as memorization of facts or perfunctory performance of lower level skills) in a content/skill-based, coverage-centered curriculum. A two-dimensional model often presents the generalization or new concepts at the beginning of the learning cycle and follows a direct teaching methodology.

This is typical of a deductive approach in teaching. I have witnessed many, many lessons utilizing this approach, and to me, this is like telling our students what the present is before they open it! The concept-based model is generally an inductive teaching model that draws the understandings from the students as a result of structured or guided inquiry.

An **inductive approach**, like mathematical induction, allows learners to start with specific examples and form generalizations for themselves. In his research on how the brain learns mathematics, David Sousa (2015) states that the human brain is a powerful pattern seeker, and we have an innate number sense or what scientists call "numerosity." The inductive approach utilizes this innate quality for number sense and pattern finding. The teacher acts as a facilitator, helping students to discover relationships and seek patterns for themselves.

The **three-dimensional model of instruction** suggests a more sophisticated design with a third level: the conceptual level. In a three-dimensional curriculum and instruction model, the lower levels of the Structure of Knowledge and the Structure of Process are important components, but the third dimension of concepts, principles, and generalizations ensures that conceptual thinking and understanding are prominent.

A three-dimensional, inductive approach encourages students to construct generalizations at the end of the learning cycle through the use of inquiry. As stated by Erickson and Lanning (2014), "Deep understanding and the transfer of knowledge and skills require that teachers understand the relationship between the factual/skill level and the conceptual level, and use this relationship effectively in instruction" (p. 23).

Figure 1.5 illustrates the difference between inductive and deductive approaches.

Figure 1.5: Inductive vs. Deductive Approaches

Deductive Approach	Students are given the generalizations at the beginning of a lesson	Students then practice the generalizations through specific examples
Inductive Approach	Students are given specific examples at the beginning of the lesson	Students construct generalizations from

An inductive model is a student-centered approach, helping students to think logically and scientifically and allowing students to generalize by utilizing higher order thinking. Discovering inductive approaches changed my entire teaching practice and influences every student learning experience I plan for my students. The inductive approach provides a framework; it is a structure for all mathematical concepts to be conveyed to students in an analytical, coherent fashion. The key to inductive teaching is that students draw and form generalizations by working on specific examples initially.

Introducing the Pythagorean theorem utilizing an inductive approach would look like this:

1. Look at the following right-angled triangles and work out the squares of each of the sides. (Students work out specific numerical examples.)
2. What generalization can you make about the relationship between all three sides when they are squared? (Students now generalize by pattern seeking.)

Bransford, Brown, and Cocking (2000) offer a comprehensive survey of neurological and psychological research that provides strong support for constructivism and inductive methods. “All new learning involves transfer of information based on previous learning” (p. 53).

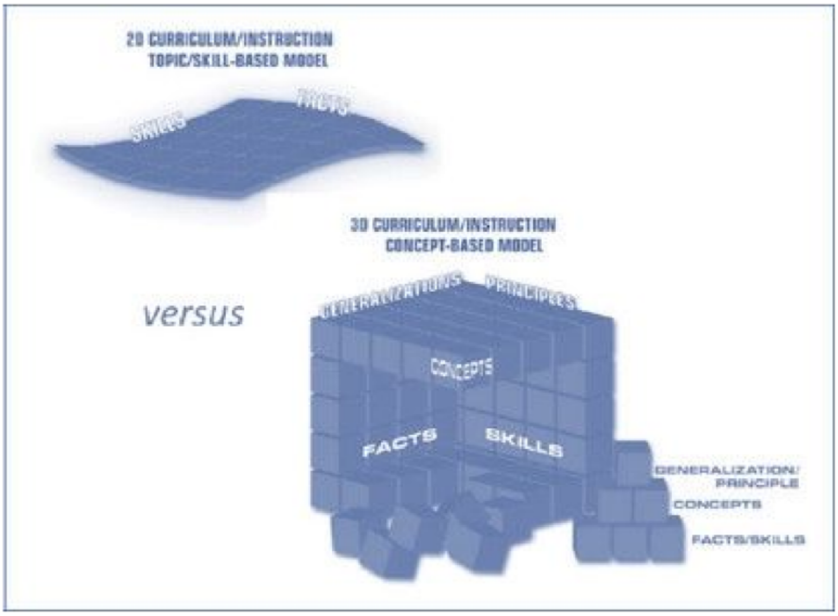
Inductive instruction presents new information in the context of situations, issues, and problems to which students can relate, so there is a much greater chance that the information can be linked to their existing cognitive structures. John D. Bransford et al. (2000) explain, “Motivation to learn affects the amount of time students are willing to devote to learning. Learners are more motivated when they can see the usefulness of what they are learning and when they can use it to do something that has an impact on others” (p. 61).

Inductive methods, such as problem-based learning, support techniques that use authentic situations and problems.

Generalizations and principles in the Structure of Knowledge and the Structure of Process are timeless, universal, transcend cultures, and are transferable ideas. They allow the learner to connect the facts and concepts for deeper meaning and understanding. The three-dimensional model of curriculum and instruction, according to Erickson and Lanning (2014), includes concepts, generalizations, and principles to ensure that curriculum and instruction focus on intellectual depth, the transfer of understanding, and the development of conceptual brain schemata. The three-dimensional model is contrasted with the traditional two-dimensional model of coverage and memorization.

Figure 1.6 illustrates the two-dimensional model, also known as the “inch deep, mile wide” approach to curriculum. In contrast, the three-dimensional model represents a more comprehensive, sophisticated design for curriculum and instruction.

Figure 1.6: Two-Dimensional vs. Three-Dimensional Curriculum/Instruction Models



Transitioning to Concept-Based Curriculum and Instruction, Corwin Press Publishers, Thousand Oaks, CA.

Inductive approaches lead to generalization formation.

Teaching for Inquiry

Inquiry is a vehicle and is about not telling students what the surprise is before opening the present. I have met many teachers in my travels, and often I hear the following about inquiry:

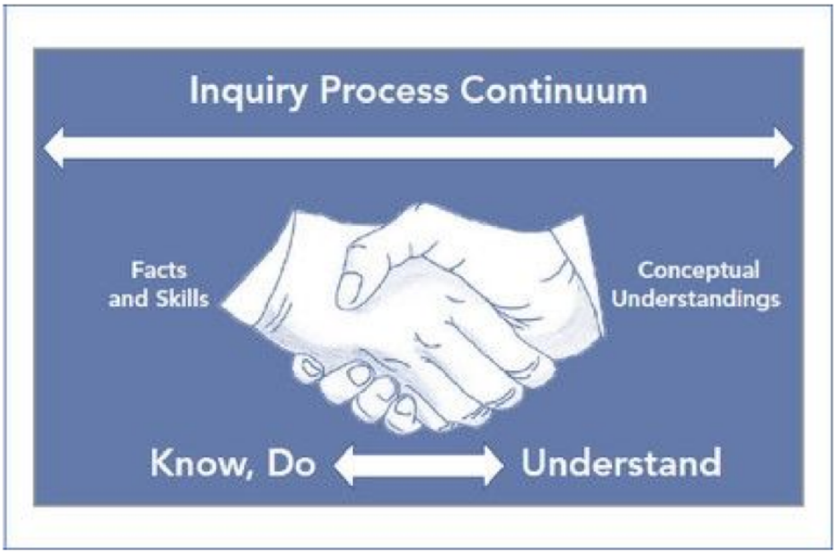
> "I don't have time for inquiry! I need to get through the content!"
> "I have inquiry lessons once per week!"
> "Inquiry just doesn't work with my students; they need to be spoon fed!"
> "Inquiry does not work for my students; they do not have the ability!"

Inquiry refers to posing questions, problems, or scenarios rather than providing established facts or knowledge. Inquiry means to seek truth, information, or knowledge, and individuals carry out the natural process of inquiry throughout their lives. Unfortunately, traditional curriculum discourages inquiry; students learn not to ask questions and to accept facts that are given. A study by Gelman, Gruber, and Ranganath (2014) found that learning is more effective when students are curious. Memory is also enhanced when students are in a state of curiosity. Inquiry encourages curiosity in students by posing questions to engage thought and interest.

Through inquiry and a variety of pedagogical approaches, such as cooperative and problem-based learning, students can develop skills for success while understanding the concepts involved (Barron & Darling-Hammond, 2008). Lynn Erickson encapsulates this idea as follows: "Information without intellect is meaningless." Figure 1.7 illustrates the synergistic relationship between the facts, skills, and concepts all being achieved through a continuum of inquiry.

In order to develop intellect in our students we need to establish synergistic thinking through the inquiry continuum.

Figure 1.7: Developing Intellect Through Inquiry Process Continuum Model



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Erickson and Lanning (2014) state that "Synergistic thinking requires the interaction of factual knowledge and concepts. Synergistic thinking requires a deeper level of mental processing and leads to an increased understanding of the facts related to concepts, supports personal meaning making, and increases motivation for learning" (p. 36).

The vehicle of inquiry is used to foster synergistic thinking. The design of guiding questions in the form of factual, conceptual, and debatable questions also supports synergistic thinking and allows students to bridge the gap between the facts and skills and conceptual understandings.

For additional resources, visit the companion website where you will find an example of a traditional activity as well as guidance on how to facilitate synergistic thinking and a template to plan a synergistic student activity of your own. See Figures M1.6 & M1.7.

As an example, in order to understand the concepts of linear functions, parameters, and variables, one must know facts, such as $y = mx + c$ or $Ax + By + C = 0$, and be able to plot points and create different representations. The inquiry process would ask students to investigate linear functions for different values for the parameters m and c. This supports the understanding of the concepts of linear, parameters, variables, and functions. Inquiry also stimulates student motivation and interest and leads to a deeper understanding of transferable concepts.

I have had the pleasure of working with Mike Ollerton, a pioneer in inquiry-based learning of mathematics from the United Kingdom. In his short piece on "Enquiry-Based Learning" (2013) he writes, "The underpinning pedagogy of enquiry-based learning (EBL) is for learners to gain and to use & apply knowledge in ways which places responsibility for the learning upon students. This is at the heart of supporting independent learning and requires the teacher become a facilitator of students' knowledge construction; as a key aspect of sense making."

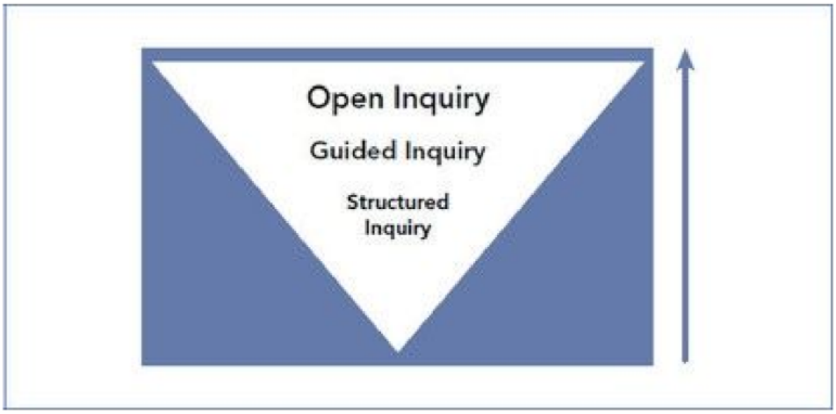
Different levels of inquiry are used as appropriate to the context and classroom situation. Figure 1.8 describes the levels of inquiry, adapted from the work of Andrew Blair (http://www.inquirymaths.com). Figure 1.9 shows the hierarchy of the levels of inquiry. The triangle represents the progression of inquiry levels, which can start off being quite narrow and structured, then move to a guided approach, and then ultimately to open inquiry, giving all students more opportunities to explore.

The three levels of inquiry—structured, guided, and open—originated in the learning approaches of science-based disciplines (Banchi & Bell, 2008). The important questions here are why and when do we use the different levels of inquiry?

Figure 1.8: Levels of Inquiry

Level	Description
Structured	• Heavily scaffolded • Predictable line of inquiry • Predictable outcomes
Guided	• Different lines of inquiry • Predictable outcomes
Open	• Different lines of inquiry • Unpredictable outcomes

Figure 1.9: Levels of Inquiry Hierarchy



Structured inquiry is heavily scaffolded and suitable perhaps for learners and teachers who are new to inquiry. Structured inquiry fosters confidence in learners while promoting autonomy and independence. Teachers who are not accustomed to using inquiry find it difficult to "let go" of control, and structured inquiry provides a happy medium. The outcomes are predictable and predetermined by the design of the task.

Guided inquiry presents learners with opportunities for different lines of inquiry, with predicable outcomes. For example, ask students for different methods to prove a particular theorem (e.g., the Pythagorean theorem). Guided inquiry has fewer prompts and gives the learner more freedom to choose his or her own pathways to the desired outcome.

Open inquiry promotes different lines of inquiry with unpredictable outcomes. Truly authentic, open inquiry engages the learner's interest and creativity. For example, the International Baccalaureate Mathematics Standard and Higher Levels include an internal assessment called a "personal exploration." Students are asked to choose an area of mathematics, conduct their own research, and draw their own conclusions. One of my past students, who was a ranked Hong Kong tennis champion, chose to write about tennis and binomial theorem. Another student with scoliosis looked at the curvature of her spine over the years using statistical analysis.

Open inquiry is not to be confused with pure "discovery" learning, when very little guidance is given to the learner. There is a misconception that inquiry is about giving students an open problem and letting them "run with it" with little guidance or input from the teacher. This is far from the intention of inquiry. Inquiry is student centered, inherently inductive, and peaks students' motivation and interest. Inquiry is not an excuse for passivity. The teacher's role is vital in facilitating and guiding the students during different stages of learning.

On the following pages there are three examples of student tasks on the same topic: proving the Pythagorean theorem. The topic is presented in three different ways to illustrate structured, guided, and open levels of inquiry.

Figure 1.10 summarizes the main features and the difference between the three levels of inquiry for the Pythagorean theorem task. Figures 1.11, 1.12, and 1.13 are the student tasks.

Figure 1.10: Levels of Inquiry for Proving the Pythagorean Theorem Task

Task: Proving the Pythagorean Theorem	Features
Structured approach	Step-by-step scaffolded questions and prompts The table allows students to calculate the areas of different shapes within the large square and prompts students to find a relationship.
Guided approach	Fewer scaffolded prompts Given the large square with the tilted square inside, students must work out that finding the areas of the shapes inside.
Open approach	Students are asked to research their own proof with hundreds to choose from. They need to explain and show understanding of their proof.

Through inductive inquiry, students are given opportunities to find generalizations and patterns they observe from specific examples. Studies have shown that a concept-based curriculum using an inductive approach results in a higher level of retention and conceptual understanding of the content.

According to Borovik and Gardiner (2007, pp. 3–4), the following are some of the top traits of mathematically able students:

- Ability to make and use generalizations—often quite quickly. One of the basic abilities, easily detectable even at the level of primary school: after solving a single example from a series, a child immediately knows how to solve all examples of the same kind.
- Ability to utilize analogies and make connections.
- Lack of fear of "being lost" and having to struggle to find one's way through the problem.

Notice these abilities are described as traits that are not genetic predispositions but qualities that can be nurtured and developed in students. Opportunities to fail or "get stuck" give students the ability to lack fear of being lost or "stuck." In her 2008 Harvard commencement address, J. K. Rowling, author of the *Harry Potter* books, said, “It is impossible to live without failing at something, unless you live so cautiously that you might as well not have lived at all—in which case, you fail by default.”

There are three principles outlined in the report *How Students Learn: Mathematics in the Classroom* (Bransford et al., 2005) that are consistent with the concept-based curriculum model:

> Principle 1: Teachers must engage students’ preconceptions. (p. 219)

This refers to recognition of students’ prior knowledge and prior strategies and the need to build on them to create new strategies and new learning.

> Principle 2: Understanding requires factual knowledge and conceptual frameworks. (p. 231)

This principle suggests the importance of the factual and conceptual and providing a framework for learners to connect the two in the form of generalizations. Learners need to have procedural fluency as well as know the conceptual relationships in order to develop mathematical proficiency.

> Principle 3: A metacognitive approach enables student self-monitoring. (p. 236)

Learners need to be given time and space to explore mathematical concepts—in other words, to self-monitor. More opportunities to reflect on their experiences will help learners to construct their ideas into larger categories and take control of their own learning.

With this overwhelming evidence, you may now ask, how do we develop curriculum and instruction using a concept-based and inquiry-led model? In Chapter 2, we will look at the facts, skills, and strategies in mathematics and how to use them to build conceptual understanding through the Structure of Knowledge and the Structure of Process. Subsequent chapters provide practical activities to guide your journey in developing a three-dimensional concept-based model for curriculum and instruction.

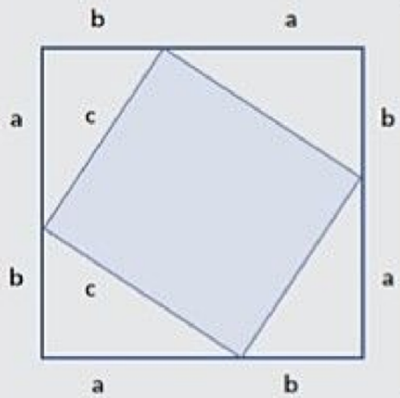
Open inquiry is student centered, with extensive input from the teacher.

Sample Student Learning Experience

Figure 1.11: A structured inquiry example

Proving Pythagorean Theorem

Find the area of the following shapes and complete this table.



A Area of the large square	B Area of the tilted square	C Area of the four triangles	B + C	Connecting A, B, and C

Explain in words the relationship you have discovered. Use a diagram to illustrate your explanation.

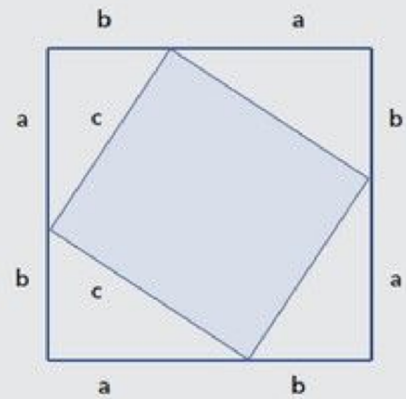
For a completed version of Figure 1.11, please visit the companion website.

Sample Student Learning Experience

Figure 1.12: A Guided Inquiry Example

Proving the Pythagorean Theorem

Investigate the relationship between a, b, and c using the following diagram.



Sample Student Learning Experience

Figure 1.13: An Open Inquiry Example

Proving the Pythagorean Theorem

There are hundreds of proofs for the Pythagorean theorem. Research one proof and explain the proof with diagrams. Use any medium to explain your proof. This could include a poster, movie, applet, or Google presentation.

To state a theorem and then to show examples of it is literally to teach backwards.

E. Kim Nebeuts From Inspirational Quotes, Word, Sayings (2015)

Deductive approaches are the norm in traditional math classrooms—we rote-learn processes in a mechanical way without understanding the true reasoning and meaning behind the problem itself. Inquiry-based learning requires us to think and analyze for ourselves, then come up with a conclusion or generalization, which is the fun and beauty behind learning mathematics. We are encouraged to challenge ourselves and step away from our comfort zones in order to expand our knowledge of mathematics. Both learning methods are effective in the short term for an exam. But I have found inductive, inquiry-based approaches allow new information and working methods to be stored in my long-term memory as I actually understand what I am doing.

Chun Yu Yiu, Grade 12 student Island School, Hong Kong

Northside ISD (San Antonio, TX) has been involved in concept-based curriculum for 10 years. It was important for this district that serves 103,000 students to have a K–12 curriculum in all major content areas that was developed using the tenets of concept-based curriculum. Our curriculum staff have been trained and certified by Lynn Erickson. Our teachers and administrators are clear about what our students are expected to know, understand, and do. Concept-based curriculum is without a doubt one of the main reasons Northside ISD continues to be a high performing district.

Linda Mora, Deputy Superintendent for Curriculum and Instruction Northside ISD, San Antonio, Texas

Chapter Summary

This chapter laid the foundation for why we need to move from a two-dimensional to a three-dimensional curriculum and instruction model to include the conceptual level. Evidence supports the effectiveness of a concept-based curriculum, which is grounded in an inductive and inquiry-led approach. Concept-based models lead to increased mathematical proficiency and understanding. The chapter discussed what a concept-based curriculum looks like for math and the benefits to students' learning. An overview of the symbiotic relationship between the Structure of Knowledge and the Structure of Process in the realm of mathematics was also provided. Developing intellect requires synergistic thinking, which, according to Lynn Erickson (2007), is an interplay between the factual and conceptual levels of thinking. Synergistic thinking is at the heart of a concept-based curriculum and instruction.

An inductive model is a student-centered approach, helping students to think logically and scientifically, allowing students to generalize by utilizing higher order thinking. The inductive approach provides a framework; it is a structure for all mathematical concepts to be conveyed to students in an analytical, coherent fashion. The key to inductive teaching is that students draw and form generalizations by working on specific examples initially.

Levels of inquiry provide teachers and learners with the opportunity to gain confidence when exploring mathematical concepts. Structured and guided inquiry facilitates differentiation and promotes student and teacher confidence.

Extensive studies in mathematics education indicate a need for curriculum and instruction to include the conceptual level for enduring, deeper understandings. If we are to prepare our students for an unknown future, due to vast technological advances, we must ensure we foster higher order thinking skills.

The next chapter will explain, in detail, the Structure of Knowledge and the Structure of Process as applied to the facts, skills, strategies, and processes of mathematics.

Discussion Questions

1. Does math education need to undergo a reform? Why or why not?
2. Why do educators need to include the conceptual understandings of a topic represented in a three-dimensional curriculum model?
3. How do the Structures of Knowledge and Process apply to the mathematics realm?
4. What are the features of inductive teaching and the benefits of an inductive approach when learning mathematics?
5. How does synergistic thinking develop intellect?
6. How would you use the different levels of inquiry in your classroom? Think of examples of when you might use each (structured, guided, open).

Chapter 2 What Are the Levels of the Structures of Knowledge and Process for Mathematics?

Quite a few years ago, in my first lesson on trigonometry with a new class, I asked my students whether they had learned about trigonometric ratios in right-angled triangles. They all replied, “No.” When I wrote *SOHCAHTOA* on the board, they said, “Oh, that’s what you mean.” They all knew the formula $\cos x = \frac{adjacent}{hypotenuse}$ but did not understand that SOHCAHTOA represents similarity in a set of right-angled triangles sharing a common acute angle. This was a mere memorized fact and algorithm for them, and they had little understanding of the concepts of ratio, similarity, and angles in right-angled triangles. They had been taught to focus on the facts and algorithms first, and there was an assumption of conceptual understanding. These lower levels of thinking are represented in the Structure of Knowledge and the Structure of Process.

As mentioned in Chapter 1, knowledge has an inherent structure, just as the animal and plant kingdoms have structures. With this structure, we are able to classify and recognize similarities, differences, and relationships. Concept-based curriculum requires an understanding of the different levels in the Structure of Knowledge and the Structure of Process and how they affect curriculum design and instruction. Concept-based models include the higher level of intellectual thinking: the conceptual level. An understanding of the Structure of Knowledge and the Structure of Process gives us the ability to plan curriculum and instruction for intellectual development. Let us recap the levels in the Structure of Knowledge.

The Levels of the Structure of Knowledge

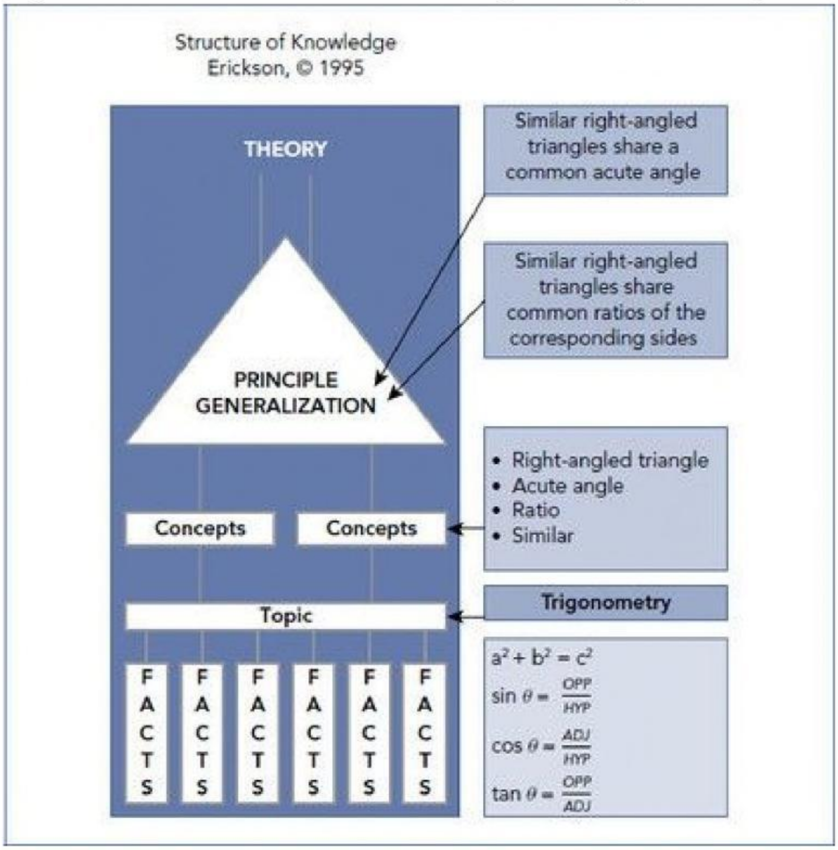
The Factual Level

The lowest level in the Structure of Knowledge is the factual level. Factual knowledge includes rote memorization and does not guarantee conceptual depth of understanding.

Facts are specific examples of people, places, situations, or things. They are locked in time, place, or situation. Facts are not transferable and include definitions, formulae in the form of symbols (e.g., $y = mx + c$), and the different names of polygons (e.g., pentagon, hexagon).

Figure 2.1 illustrates the Structure of Knowledge applied to the topic of trigonometry.

Figure 2.1: The Structure of Knowledge for Trigonometry



Adapted from original Structure of Knowledge figure from *Transitioning to Concept-Based Curriculum and Instruction,* Corwin Press Publishers, Thousand Oaks, CA.

The factual level includes knowing the shape of the graph of $y = \cos x$ without necessarily understanding how this is generated and memorized formula such as $\cos x = \frac{adjacent}{hypotenuse}$. These facts help students to support the understanding of the concepts of ratio, magnitude, angle, and direction when learning about right-angled trigonometry in a concept-based curriculum.

Knowing a definition or a formula does not imply understanding. Memorized formulae in mathematics are facts that support the broader concepts in mathematics. These facts include the vocabulary, definitions, and formulae in the form of mathematical symbols. For mathematical proficiency and understanding, learners need to know the facts to reinforce their understanding of the related concepts. To *know* means to memorize facts or definitions that are critical to understanding the generalizations (statements of conceptual relationships) for a particular unit.

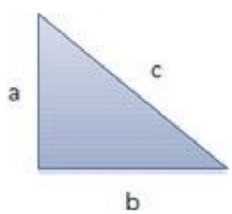
To continue with the theme of right-angled trigonometry, let us look at the example of the Pythagorean theorem to illustrate this point:

> For right-angled triangles, the area of the square drawn from the hypotenuse represents the sum of the areas of the squares drawn from the other sides.

This is a statement of conceptual understanding, which connects the concepts of hypotenuse, area, squares, and sum applied to right-angled triangles. There are numerous inquiry tasks that guide students to understand this principle, one being from http://nrich.maths.org/2293 called "Tilted Squares." In this task, students are asked to spot patterns, make generalizations, and even discover the Pythagorean theorem by finding the areas of tilted squares.

The formula for the Pythagorean theorem is $a^2 + b^2 = c^2$. This is a memorized fact, which does not reflect conceptual understanding. This fact only applies to a specific question, such as the following:

Find c when a = 3 and b = 4.



The recall of these facts is highly compressible in the brain and is crucial when problem solving and learning math. Once you understand the process or concept, the brain has an incredible capacity to file this information away for later use—in other words, compress it in the brain.

William Thurston (1990), a Fields Medal winner in mathematics, defined compression particularly well when he wrote, "Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics (p. 847).

The Difference Between Formulae and Theorems in the

Structure of Knowledge

A **formula** is an equation that uses mathematical symbols or variables to show a relationship and is represented by the *facts* in the Structure of Knowledge.

Theorems are statements that have been proven and connect explanations of conceptual understandings. Theorems are represented by *principles* in the Structure of Knowledge.

Let us look at the fundamental theorem of calculus. The first part of this theorem describes the relationship between differentiation and integration as inverse processes of each other. The second part of the fundamental theorem of calculus helps students to evaluate a definite integral without having to go back to the definition of taking the limit of a sum of rectangles.

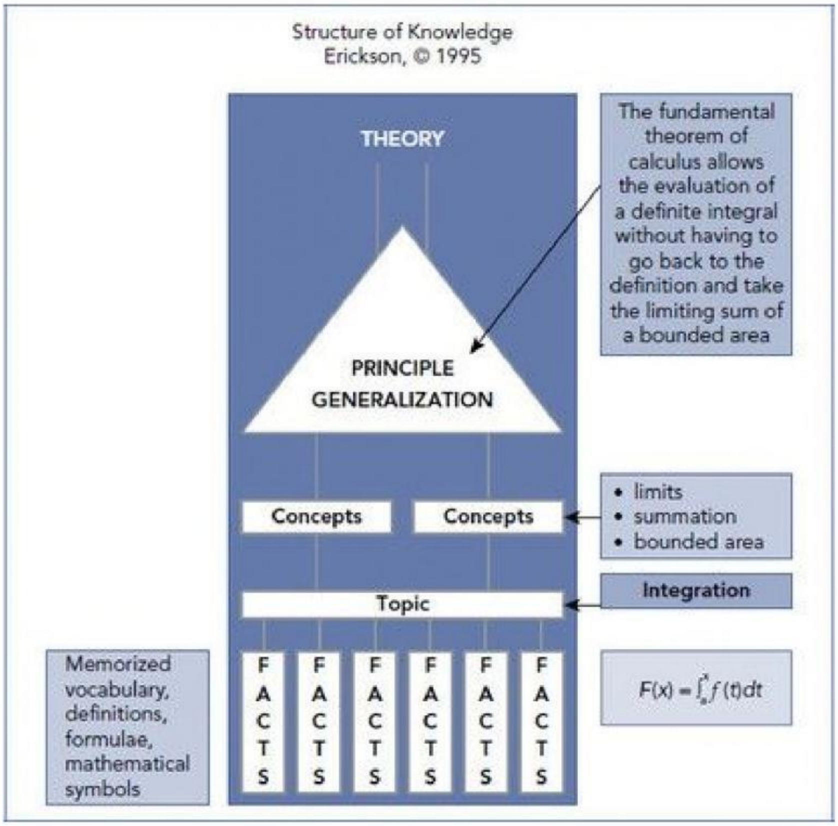
The fundamental theorem of calculus may also be expressed as a fact or formula in mathematical symbols:

Let $f(x)$ be continuous in the interval [a, b] and $F'(x) = f(x)$, then
$F(x) = \int_a^x f(t)dt$

If students know this fact or formula, do they have a deep understanding of the fundamental theorem of calculus? Do they understand the concept of integration as being an inverse process of differentiation or understand that calculus allows the evaluation of a definite integral without having to go back and take the limiting sum of a bounded area?

Figure 2.2 shows the fundamental theorem of calculus depicted in the Structure of Knowledge.

Figure 2.2: The Fundamental Theorem of Calculus in the Structure of Knowledge



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Facts in math include memorized formulae in math symbols and vocabulary and definitions.

The Topic and Concepts Levels

Topics in math are broader concepts that break down to specific micro concepts at the next level. In Figure 2.2, the topic "integration" is supported by the micro concepts "limits," "summation," and "bounded areas." More on the classification of math concepts will be discussed later in this chapter.

The Generalizations and Principles Level

Generalizations and principles are statements of conceptual understanding that allow students to make connections between two or more concepts. In mathematics, the principles are theorems—the cornerstone truths. In Figure 2.1, the generalizations are as follows:

> Similar right-angled triangles share a common acute angle

> Similar right-angled triangles share common ratios of corresponding sides

In Figure 2.2, the fundamental theorem of calculus represents a principle:

> Theorems in calculus allow the evaluation of a definite integral without having to go back to the definition and take the limiting sum of a bounded area.

Crafting quality generalizations requires an investment of time. How to craft generalizations will be discussed in the next chapter.

The Levels of the Structure of Process

Mathematical Processes, Algorithms, Strategies, and Skills in the Structure of Process

Students in my first trigonometry class were able to easily find lengths of sides and angles in right-angled triangles, but when I asked them to explain why this worked, no one could explain using the concepts of similarity and ratios of sides. To my students, SOHCAHTOA were buttons on a calculator and a memorized procedure or algorithm to get an answer. These students had been exposed to traditional methods that focused on memorizing algorithms. Mathematics classrooms worldwide have tended to focus on rote learning procedures or algorithms, and often too little attention is paid to why or how. Why do we multiply by the reciprocal of the divisor when we divide fractions? Why can we not divide by zero? Why do two negatives make a positive? These are examples of important questions that need to be addressed to support conceptual understanding.

Waterbury School System in Connecticut embarked on the concept-based curriculum model in 2012. Darren Schwartz, the Instructional Leadership Director of Waterbury Public Schools, explains the reasoning behind adopting the concept-based approach:

> The mission of the Waterbury School System is to establish itself as the leader in Connecticut for urban education reform in partnership with the State Department of Education and the entire Waterbury community. The school system will provide opportunities for all students to maximize their skills and talents in an

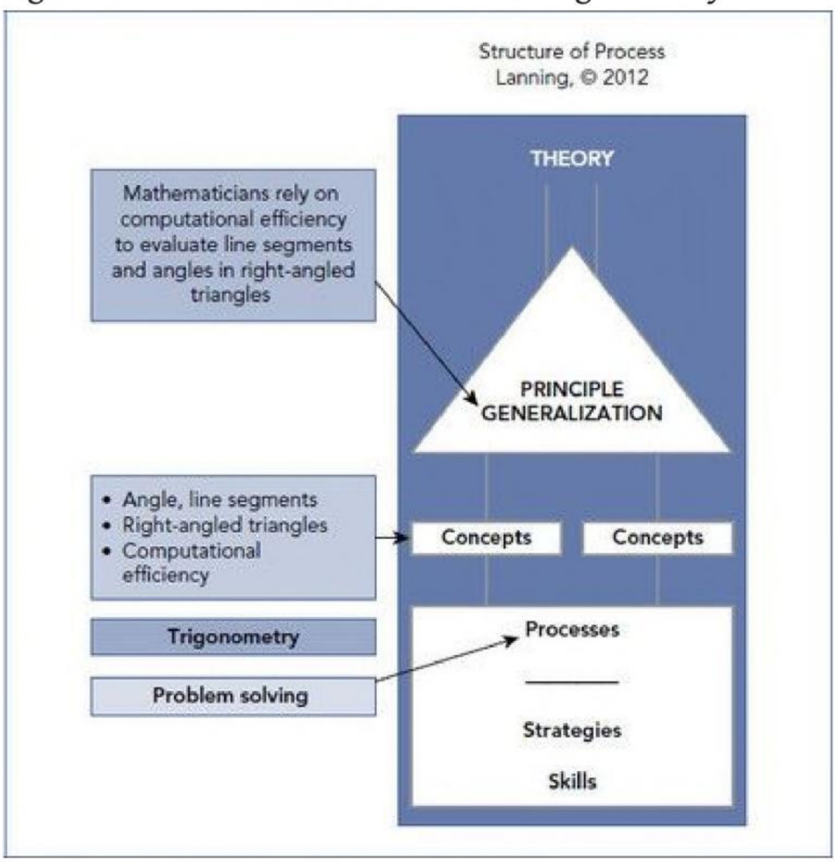
atmosphere where teaching and learning flourish under the never-wavering belief that all students can be exemplary students, while becoming respectful, responsible, productive citizens vital to our community.

There has been a long-standing tradition in math instruction to teach and model using algorithms first. In our district we focus firstly on the conceptual understanding of math and provide the opportunity for students to discover algorithms through an inquiry based learning process.

Darren Schwartz, Instructional Leadership Director Waterbury Public Schools, Connecticut

Strategies are a number of skills that learners use in a methodical and systematic way to support learning.

Figure 2.3: The Structure of Process for Trigonometry



Adapted from original Structure of Knowledge figure from *Transitioning to Concept-Based Curriculum and Instruction*, Corwin Press Publishers, Thousand Oaks, CA.