Gregory J. Chaitin

Conversations with a Mathematician

Math, Art, Science and The Limits of Reason

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A collection of his most wide-ranging and non-technical lectures and interviews

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Introduction

In 1970 when I was twenty-three years old and living in Buenos Aires, I visited a university in Rio de Janeiro, PUC.[1] This was the week before Carnival,[2] and I seem to recall hearing the news of Bertrand Russell's death while I was there. (I have an even more vivid memory as a small boy, of seeing a dramatic headline in red, "EINSTEIN DEAD".)

But my thoughts in Rio were not on death, they were on life! Inspired by the beautiful beaches, the beautiful women, and the tropical lushness of Rio, my mind was working well.

While in Rio I published a two-part PUC research report. The first part was my Rio breakthrough and is the subject of this book: I realized that using the ideas that I had been developing in order to define randomness or lack of structure, I could come up with an information-theoretic approach to the mysterious incompleteness phenomenon discovered by Gödel, that limits the power of formal axiomatic mathematical theories.[3]

The second half of my PUC research report was an English

[1] That's the Pontifícia Universidade Católica do Rio de Janeiro.

[2] My host at PUC who didn't want me to miss Carnival was Roberto Lins de Carvalho. Thank you, Roberto!

[3] I told this story in an interview, my third and latest TV interview, that was broadcast by Globo News TV in Brazil in June 2001 (see pp. 143–154). The Globo News TV channel is simultaneously webcast to the rest of the world, so I was able to see this interview on my PC in NY at the same time that it was on TV in Brazil! Thirty years ago at the Ipanema beach, how could I have imagined that this would be possible?

translation of a paper that I had presented at a meeting in Buenos Aires the year before. This paper was called "To a mathematical definition of 'life'," and it was my initial attempt to apply my program-size complexity ideas to biology in order to define what a living organism is and how to measure its complexity.[4]

However, the four-day weekend of Carnival all the math came to a full stop, or so I thought! I danced in the street all night to irresistible Brazilian and African rhythms, and watched the sensual Samba Parade. Now, thirty years later, I can see that this was information theory too. After all, from a biological point of view, the purpose of love-making is to exchange information, that's really what Carnival in Rio is all about, about information![5]

[4]I don't think that my work in this area was too successful, since it did not lead to a general, abstract mathematical theory of evolution, as I'll explain at the end of the first lecture (pp. 37–39) and in the last interview (pp. 152–153). In his forthcoming *A New Kind of Science,* Stephen Wolfram argues that there is no *essential* difference between us and any universal computer, and therefore no such general theory of evolution is needed. While his thesis is interesting, I feel that it is not the whole story.

[5]By the way, I fell in love in Rio. At PUC I bought a copy of the *LISP 1.5 Programmer's Manual,* which was not available in Buenos Aires. That was the beginning of my life-long love affair with LISP!

So to me, "information" is definitely a sexy subject, and it includes my algorithmic information, which is measured in bits of software, biological information, which is measured in kilobases of DNA, and psychological information and thought and the soul,[6] which we know very little about, but will hopefully someday understand. The ideas in this book on algorithmic information and the limits of formal reasoning may seem cold and inhuman, but I hope that they are the first step in the direction of a new, more sensual mathematics of life and creativity.

To the Future![7]

[6] As I have argued at the end of my book *The Unknowable* and in the last interview (p. 150), one can think of the soul as software that is moving from machine to machine. But then what about feelings?

[7] Regarding the reason for sex and future possibilities, see Mark Ridley, *Mendel's Demon* [UK title], *The Cooperative Gene* [US title]. For an account of the role of information theory in the early work on molecular biology and DNA, see Lily E. Kay, *Who Wrote the Book of Life?*

Lecture — A Century of Controversy over the Foundations of Mathematics[1]

[This 1999 talk at UMass-Lowell was my last major lecture of the previous century, and it summarizes that century's work on the foundations of mathematics, discusses connections with physics, and proposes a program of research for the next century. Not to be confused with another talk with the same title, my Distinguished Lecture given at Carnegie-Mellon University in 2000.]

Prof. Ray Gumb:

We're happy to have Gregory Chaitin from IBM's Thomas J. Watson Research Lab to speak with us today. He's a world-renowned figure, and the developer as a teenager of the theory of algorithmic information. And his newest book *The Unknowable,* which is accessible to undergraduates, and I hope will be of great appeal to our undergraduates in particular, is available on the Web and comes with LISP programs to run with it. It's kind of like a combination

[1]Originally published in C. S. Calude and G. Păun, *Finite versus Infinite,* Springer-Verlag, 2000, pp. 75–100.

of mathematics, computer science, and philosophy. Greg—

Greg Chaitin:

Thanks a lot! Okay, a great pleasure to be here! [Applause] Thank you very much! I'm awfully sorry to be late! You've got a beautiful town here! Those old brick buildings and the canals are really breathtaking! And thanks for being here for this talk! It's such a beautiful spring day—I think one has to be crazy to be indoors!

Okay, I'd like to talk about some crazy stuff. The general idea is that sometimes ideas are very powerful. I'd like to talk about theory, about the computer as a concept, a philosophical concept.

We all know that the computer is a very practical thing out there in the real world! It pays for a lot of our salaries, right? But what people don't remember as much is that really—I'm going to exaggerate, but I'll say it—the computer was invented in order to help to clarify a question about the foundations of mathematics, a philosophical question about the foundations of mathematics.

Now that sounds absurd, but there's some truth in it. There are actually lots of threads that led to the computer, to computer technology, which come from mathematical logic and from philosophical questions about the limits and the power of mathematics.

The computer pioneer Turing was inspired by these questions. Turing was trying to settle a question of Hilbert's having to do with the philosophy of mathematics, when he invented a thing called the Turing machine, which is a mathematical model of a toy computer. Turing did this before there were any real computers, and then he went on to actually build computers. The first computers in England were built by Turing.

And von Neumann, who was instrumental in encouraging the creation of computers as a technology in the United States, (unfortunately as part of a war effort, as part of the effort to build the atom bomb), he knew Turing's work very well. I learned of Turing by reading von Neumann talking about the importance of Turing's work.

So what I said about the origin of the computer isn't a complete lie, but it is a forgotten piece of intellectual history. In fact, let me start off with the final conclusion of this talk...In a way, a lot of this came from work of Hilbert. Hilbert, who was a very well-known German mathematician around the beginning of this century, had proposed formalizing completely all of mathematics, all of mathematical reasoning—deduction. And this proposal of his is a tremendous, glorious failure!

In a way, it's a spectacular failure. Because it turned out that you couldn't formalize mathematical reasoning. That's a famous result of Gödel's that I'll tell you about, done in 1931.

But in another way, Hilbert was really right, because formalism has been the biggest success of this century. Not for reasoning, not for deduction, but for programming, for calculating, for computing, that's where formalism has been a tremendous success. If you look at work by logicians at the beginning of this century, they were talking about formal languages for reasoning and deduction, for doing mathematics and symbolic logic, but they also invented some early versions of programming languages. And **these** are the formalisms that we all live with and work with now all the time! They're a tremendously important technology.

So formalism for reasoning did not work. Mathematicians don't reason in formal languages. But formalism for computing, programming languages, are, in a way, what was right in the formalistic vision that goes back to Hilbert at the beginning of this century, which was intended to clarify epistemological, philosophical questions about mathematics.

So I'm going to tell you this story, which has a very surprising outcome. I'm going to tell you this surprising piece of intellectual history.

The Crisis in Set Theory

So let me start roughly a hundred years ago, with Cantor...

Georg Cantor

The point is this. Normally you think that pure mathematics is static, unchanging, perfect, absolutely correct, absolute truth... Right? Physics may be tentative, but math, things are certain there! Well, it turns out that's not exactly the case.

In this century, in this past century there was a lot of controversy over the foundations of mathematics, and how you should do math, and what's right and what isn't right, and what's a valid proof. Blood was almost shed over this... People had terrible fights and ended up in insane asylums over this. It was a fairly serious controversy. This isn't well known, but I think it's an interesting piece of intellectual history.

More people are aware of the controversy over relativity theory. Einstein was very controversial at first. And then of the controversy over quantum mechanics... These were the two revolutions in the physics of this century. But what's less well known is that there were tremendous revolutions and controversies in pure mathematics too. I'd like to tell you about this. It really all starts in a way from Cantor.

Georg Cantor

What Cantor did was to invent a theory of infinite sets.

Infinite Sets

He did it about a hundred years ago; it's really a little more than a hundred years ago. And it was a tremendously revolutionary theory, it was **extremely** adventurous. Let me tell you why.

Cantor said, let's take 1, 2, 3, ...

$$1, 2, 3, \ldots$$

We've all seen these numbers, right?! And he said, well, let's add an infinite number after this.

$$1, 2, 3, \ldots \omega$$

He called it ω, lowercase Greek omega. And then he said, well, why stop here? Let's go on and keep extending the number series.

$$1, 2, 3, \ldots \omega, \omega + 1, \omega + 2, \ldots$$

Omega plus one, omega plus two, then you go on for an infinite amount of time. And what do you put afterwards? Well, two omega? (Actually, it's omega times two for technical reasons.)

$$1, 2, 3, \ldots \omega \ldots 2\omega$$

Then two omega plus one, two omega plus two, two omega plus three, two omega plus four...

$$1, 2, 3, \ldots 2\omega, 2\omega + 1, 2\omega + 2, 2\omega + 3, 2\omega + 4, \ldots$$

Then you have what? Three omega, four omega, five omega, six omega, ...

$$1, 2, 3, \ldots 3\omega \ldots 4\omega \ldots 5\omega \ldots 6\omega \ldots$$

Well, what will come after all of these? Omega squared! Then you keep going, omega squared plus one, omega squared plus six omega plus eight... Okay, you keep going for a long time, and the next interesting thing after omega squared will be? Omega cubed! And then you have omega to the fourth, omega to the fifth, and much later?

$$1, 2, 3, \ldots \omega \ldots \omega^2 \ldots \omega^3 \ldots \omega^4 \ldots \omega^5$$

Omega to the omega!

$$1, 2, 3, \ldots \omega \ldots \omega^2 \ldots \omega^\omega$$

And then much later it's omega to the omega to the omega an infinite number of times!

$$1, 2, 3, \ldots \omega \ldots \omega^2 \ldots \omega^\omega \ldots \omega^{\omega^{\omega^{\omega^{\cdots}}}}$$

I think this is usually called epsilon nought.

$$\varepsilon_0 = \omega^{\omega^{\omega^{\omega^{\cdots}}}}$$

It's a pretty mind-boggling number! After this point things get a little complicated...

And this was just one little thing that Cantor did as a warm-up exercise for his main stuff, which was measuring the size of infinite sets! It was spectacularly imaginative, and the reactions were extreme. Some people loved what Cantor was doing, and some people thought that he should be put in an insane asylum! In fact he had a nervous breakdown as a result of those criticisms. Cantor's work was very influential, leading to point-set topology and other abstract fields in the mathematics of the twentieth century. But it was also very controversial. Some people said, it's theology, it's not real, it's a fantasy world, it has nothing to do with serious math! And Cantor never got a good position and he spent his entire life at a second-rate institution.

Bertrand Russell's Logical Paradoxes

Then things got even worse, due mainly, I think, to Bertrand Russell, one of my childhood heroes.

Bertrand Russell

Bertrand Russell was a British philosopher who wrote beautiful essays, very individualistic essays, and I think he got the Nobel prize in literature for his wonderful essays. Bertrand Russell started off as a mathematician and then degenerated into a philosopher and finally into a humanist; he went downhill rapidly! [Laughter] Anyway, Bertrand Russell discovered a whole bunch of disturbing paradoxes, first in Cantor's theory, then in logic itself. He found cases where reasoning that seemed to be okay led to contradictions.

And I think that Bertrand Russell was tremendously influential in spreading the idea that there was a serious crisis and that these contradictions had to be resolved somehow. The paradoxes that Russell discovered attracted a great deal of attention, but strangely enough only one of them ended up with Russell's name on it! For example, one of these paradoxes is called the Burali-Forti paradox, because when Russell published it he stated in a footnote that it had been suggested to him by reading a paper by Burali-Forti. But if you look at the paper by Burali-Forti, you don't see the paradox!

But I think that the realization that something was seriously wrong, that something was rotten in the state of Denmark, that reasoning was bankrupt and something had to be done about it pronto, is due principally to Russell. Alejandro Garciadiego, a Mexican historian of math, has written a book which suggests that Bertrand Russell really played a much bigger role in this than is usually realized: Russell played a key role in formulating not only the Russell paradox, which bears his name, but also the Burali-Forti paradox and the Berry paradox, which don't. Russell was instrumental in discovering them and in realizing their significance. He told everyone that they were important, that they were not just childish word-play.

Anyway, the best known of these paradoxes is called the Russell paradox nowadays. You consider the set of all sets that are not members of themselves. And then you ask, "Is this set a member of itself or not?" If it is a member of itself, then it shouldn't be, and vice versa! It's like the barber in a small, remote town who shaves all the men in the town who don't shave themselves. That seems pretty reasonable, until you ask "Does the barber shave himself?" He shaves himself if and only if he doesn't shave himself, so he can't apply that rule to himself!

Now you may say, "Who cares about this barber!" It was a silly rule anyway, and there are always exceptions to the rule! But when you're dealing with a **set**, with a mathematical concept, it's not so easy to dismiss the problem. Then it's not so easy to shrug when reasoning that seems to be okay gets you into trouble!

By the way, the Russell paradox is a set-theoretic echo of an earlier paradox, one that was known to the ancient Greeks and is math language with a symbolic logic you should be able to achieve *perfect rigor.* You've heard the word "rigor", as in "rigor mortis", used in mathematics? [Laughter] It's not that rigor! But the idea is that an argument is either completely correct or else it's total nonsense, with nothing in between. And a proof that is formulated in a formal axiomatic system should be absolutely clear, it should be completely sharp!

In other words, Hilbert's idea was that we should be completely precise about what the rules of the game are, and about the definitions, the elementary concepts, and the grammar and the language—all the rules of the game—so that we can all agree on how mathematics should be done. In practice it would be too much work to use such a formal axiomatic system, but it would be philosophically significant because it would settle once and for all the question of whether a piece of mathematical reasoning is correct or incorrect.

Okay? So Hilbert's idea seemed fairly straightforward. He was just following the axiomatic and the formal traditions in mathematics. Formal as in formalism, as in using formulas, as in calculating! He wanted to go all the way, to the very end, and formalize all of mathematics, but it seemed like a fairly reasonable plan. Hilbert wasn't a revolutionary, he was a conservative... The amazing thing, as I said before, was that it turned out that Hilbert's rescue plan **could not work**, that it couldn't be done, that it was **impossible** to make it work!

Hilbert was just following the whole mathematics tradition up to that point: the axiomatic method, symbolic logic, formalism... He wanted to avoid the paradoxes by being absolutely precise, by creating a completely formal axiomatic system, an artificial language, that avoided the paradoxes, that made them impossible, that **outlawed** them! And most mathematicians probably thought that Hilbert was right, that *of course* you could do this—it's just the notion that in mathematics things are absolutely clear, black or white, true or false.

So Hilbert's idea was just an extreme, an exaggerated version of the normal notion of what mathematics is all about: the idea that we can decide and agree on the rules of the game, all of them, once and for all. The big surprise is that it turned out that this **could not** be done. Hilbert turned out to be wrong, but wrong in a tremendously fruitful way, because he had asked a very good question. In fact, by asking this question he actually created an entirely new field of mathematics called **meta**mathematics.

Metamathematics

Metamathematics is mathematics turned inward, it's an introspective field of math in which you study what mathematics can achieve or can't achieve.

What is Metamathematics?

That's my field—metamathematics! In it you look at mathematics from above, and you use mathematical reasoning to discuss what mathematical reasoning can or cannot achieve. The basic idea is this: Once you entomb mathematics in an artificial language *à la* Hilbert, once you set up a completely formal axiomatic system, then you can forget that it has any meaning and just look at it as a game that you play with marks on paper that enables you to deduce theorems from axioms. You can forget about the meaning of this game, the game of mathematical reasoning, it's just combinatorial play with symbols! There are certain rules, and you can study these rules and forget that they have any meaning!

What things do you look at when you study a formal axiomatic system from above, from the outside? What kind of questions do you ask?

Well, one question you can ask is if you can prove that "0 equals 1"?

$$0 = 1\,?$$

Hopefully you can't, but how can you be sure? It's hard to be sure!

And for any question A, for any affirmation A, you can ask if it's possible to settle the matter by either proving A or the opposite of A, not A.

$$A? \ \neg A?$$

That's called *completeness.*

Completeness

A formal axiomatic system is complete if you can settle any question A, either by proving it (A), or by proving that it's false ($\neg A$). That would be nice! Another interesting question is if you can prove an assertion (A) and you can also prove the contrary assertion ($\neg A$). That's called *inconsistency,* and if that happens it's very bad! *Consistency* is much better than *inconsistency*!

Consistency

So what Hilbert did was to have the remarkable idea of creating a new field of mathematics whose subject would be mathematics itself. But you can't do this until you have a completely formal axiomatic system. Because as long as any "meaning" is involved in mathematical reasoning, it's all subjective. Of course, the reason we do mathematics is because it has meaning, right? But if you want to be able to study mathematics, the power of mathematics, using mathematical methods, you have to "desiccate" it to "crystallize out" the meaning and just be left with an artificial language with completely precise rules, in fact, with one that has a **mechanical proof-checking algorithm**.

Proof-Checking Algorithm

The key idea that Hilbert had was to envision this perfectly desiccated or crystallized axiomatic system for all of mathematics, in which the rules would be so precise that if someone had a proof there would be a referee, there would be a mechanical procedure, which would either say, "This proof obeys the rules" or "This proof