Deep Thinking

What Mathematics Can Teach Us About the Mind

William Byers

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Chapter 1

What is Deep Thinking?

1.1 Introduction

The objective of this introductory chapter is to introduce "deep thinking." I cannot begin with a formal definition; deep thinking does not work by writing formal definitions and deducing inevitable consequences. I shall instead proceed by exploring deep thinking in a variety of situations and characterize it by isolating a number of its basic properties. I shall begin by looking at recent work in child development, more specifically, the development of the number concept in infants. This will lead me to examine various conceptual systems for number from two points of view—the manner that they arose historically and the way in which children learn them. These sources will be enough for me to list a series of general characteristics of deep thinking. They will give us a pretty good feeling for deep thinking—enough, at least to continue the exploration.

However before we can describe deep thinking let's spend a moment considering the more general question, "What is thinking?" Thinking is a mental process, a way of using the mind, which involves directing it towards something—normally a sequence of ideas, statements, or propositions. Thought, which is the content of thinking, usually occurs through the medium of words but it is also possible to think using non-verbal elements such as symbols, images, sounds, or even feelings. Nevertheless for most of us thinking is predominantly verbal.

Rational thinking consists of a sequence of propositions arranged in a logical order that is organized so that each step follows from the preceding one by means of the rules of logical inference. The first statement is usually called the hypothesis and the last, the conclusion. For some people rational thinking is the highest form of thinking and all other ways of thinking are deficient.

The fact remains, however, that there are other ways of using the mind, which are neither rational nor sequential. "Deep thinking" is the name that I am giving to a kind of non-sequential thinking. Most people are not normally aware of the existence of such thinking nor that they are capable of it. Non-rational thinking would appear to be amorphous and vague precisely because we are used to identifying thought with rationality. One way to get a feeling for what is being discussed would be to substitute the word "creative" for "deep" because deep thinking is the way the mind is used in creative work. Creative thinking is not strictly rational but neither is it irrational. The relationship between rational thinking and deep thinking is important and one of the themes of this book. For now let me just say that the discovery of rational thinking was a creative act that initially arose out of deep thinking. Logical thinking has its role in the various processes that I shall describe but it is a role that is secondary to the more basic process of deep thinking.

Deep thinking is something that is difficult to get a handle on, not because it is complex, but, on the contrary, because it is so basic. Deep thinking is the way the mind functions naturally, not something that the mind must be coerced into doing. The "default condition" of the mind is active, dynamic, and creative. It does not have to be "turned on." It arrives on the scene already turned on. We shall get evidence for the dynamic default condition of the mind by considering the way infants think as they make their initial attempts to get a conceptual understanding of the world. It is from recent work on the development of conceptual systems in infants that we shall begin to discern some of the essential ingredients that make up deep thinking. For this reason yet another term for deep thinking might be "developmental thinking."

Such thinking is universal—every child is capable of it. Since such thinking is essentially creative and every step in the conceptual development of children requires an act of creativity, it follows that children are naturally creative. This means that the potential for creativity also exists, latently at least, in all adults. It is not just reserved for the brilliant and talented. Why do most adults lose touch with it?

Why does our mind, the adult mind, so often feel torpid and dull? Why does it take so much effort to turn it on? Why is creativity so hard? These are all excellent questions but for now it is enough to make the observation that deep thinking is the natural condition of the mind even if this state is difficult for most people to actualize.

In other words the usual order of things is reversed. Instead of viewing creativity as a higher form of rational thought, rational thought is seen as a way of capturing and amplifying certain aspects of deep thinking. Deep thought is more basic and so it is futile to develop systems of artificial intelligence that are truly creative. The most such programs can possibly do is to *simulate* creativity and deep thinking.

The word "deep" in deep thinking refers to the property of being fundamental so it is natural to turn to the development of concepts in young children especially one of the most elementary of all concepts—the concept of number.

1.2 Deep Thinking in the Conceptual Development of Infants

Our guide in this foray into child development will be Susan Carey, Professor of Psychology at Harvard University and her recent seminal book, *The Origin of Concepts*. Carey's book is built around three major theses. The first is that human beings (and non-human animals as well) "have systems of core cognition, and that core cognition is the developmental foundation of human conceptual understanding."[i] The existence of such core systems—systems that are built-in to our biology—suggests that the conceptual development of infants builds on a foundation of an earlier biological development. There appears to be an arc of development that begins with the evolution of brain and mind structures, proceeds to the development of core conceptual systems in infants, then to the development of the more sophisticated conceptual structures that young people learn in schools and universities. This arc culminates in the creative research of science and mathematics. At every stage in this entire developmental arc, from the primitive to the most sophisticated, it is possible to discern the operation of a similar process of cognition—deep thinking.

"Carey's second major thesis is that new representational resources emerge in development—representational systems with more expressive power than those they are built from, as well as representational systems that are incommensurable with those they are built from. That is, conceptual development involves theoretically important discontinuities."[ii] Here Carey isolates a number of essential characteristics of deep thinking, which involves an indispensible element of discontinuity, that is, a jump from one point of view to another. Further the new point of view is incommensurable with the old. This implies that deep thinking is always difficult.

"Carey's third major thesis is that the bootstrapping processes that have been described in the literature on the history and philosophy of science underlie the construction of new representational resources in childhood as well."[iii] This is an attempt to explain the great conundrum of both development and creativity. If it is so difficult then how does it happen? How do development and learning come about? How do human beings manage to come up with ideas and theories that are fundamentally new and constitute a radical break with the past? Where do these new ideas and representations come from? It is all quite mysterious and even miraculous.

I find the argument around the existence of systems of core cognition fascinating but it is not my intention to go into it here. I will merely accept this thesis even though it may be controversial to some. It supports the main thrust of my argument but is not crucial to it. My interest in this chapter is in the second and third theses—the discontinuous emergence of new representational systems and the way in which they come about.

It is interesting that Carey in her third thesis uses the literature of the history and philosophy of science to support her work in child development. I also intend to use both of these sources, conceptual development as well as episodes from the history of science, to support my general thesis that there exists a common root to learning, creativity, and development, namely what I am calling deep thinking.

1.3 Core Cognitive Systems for Number

I now turn to Carey's description of a couple of systems of core cognition that deal with what has been called the "number sense[iv]." She says, "The number sense is a paradigm example of core cognition …" and lays out the evidence for two distinct systems of core cognition with numerical content.

The first of these core systems is the "analog magnitude representation of number. Number is represented by a physical magnitude" (say the length of a line segment) "that is roughly proportional to the number of individuals being enumerated." Thus the number seven would be represented by a line of length that is seven times as long as some unit length. This system is still operational in all of us. It is not precise since most people would have difficulty telling the difference between a line segment of length twelve and one of length thirteen. The fascinating thing about this core system is that it can discriminate two magnitudes as a function, not of their cardinality or their difference, but of their ratio[v]. So a six-month old infant can discriminate successfully between 24 objects and 12 objects (2 to 1) but not between 24 and 16 (3 to 2). Carey maintains that animals also represent number in this way. She produces data that supports "the existence of an evolutionary ancient representational system in which number is encoded by an analog magnitude proportional to the number of objects in the set." It is important to Carey's argument that such "number representations are conceptual; their content goes beyond spatio-temporal and sensory vocabulary."[vi]

Then Carey turns to a second core system with numerical content: "parallel individuation of small sets." This system stores in parallel, that is, simultaneously, the number of objects in small collections of 1,2, or 3 objects. This is a precise representation but it applies only to very small collections of objects.

"There is massive evidence for two distinct core cognition systems with numerical content, in one of which (parallel individuation) number is represented only implicitly and in the other, number is represented by a mental magnitude that is proportional to the cardinal value of the set of items under consideration."[vii] [This] "… raises several questions about the relations between them. In what senses are they distinct? Do they ever become integrated in a single system for representing number? If so, when and how?"[viii]

Carey emphasizes the radical difference between the two systems. These differences include the fact that the former system number represents magnitude and is geometric, continuous and approximate whereas the latter represents multitude[ix] and is discrete and precise. Why do we have more than one core system for number? Isn't number a singular concept so why don't we have a unique way to represent number and then develop this basic notion as we grow and are educated? The fact that evolution has provided us two (or more) systems for number is an important indication of how development and learning proceeds. There wouldn't be the same need for development if not for the situation of having two separate and essential systems that carry different perspectives on number. All of the historical extensions of the idea of number—including the fractions, negative numbers, real numbers, and so on—are resolutions of the tension that is implicit in our core systems. Without that tension there might well have been no development.

In my book, *The Blind Spot*[x], I discussed at length this tension that lies at the heart of the idea of number and the way in which this ambiguity has been resolved in many different ways in the history of mathematics. For example the Greeks discovered that the root of two was a fine geometric number since it arose as the length of a line segment in Pythagoras' Theorem but that it was difficult to incorporate this number into the world of multitude since it is irrational or incommensurable with the rational numbers. This incompatibility between the discrete and the continuous is only really resolved in a satisfactory way with the invention of the real numbers.

To repeat, the conflict between an infant's core conceptual systems provides the impetus for development, which is fueled by the need for some sort of resolution. The implicit conflict is resolved through the development of a new conceptual system—the counting numbers. However this resolution is not complete or definitive. It is a step on the long voyage that constitutes the development of our modern notion of number. On this voyage the primal differences between our core systems always remain active and so remain capable of generating more complex resolutions.

1.4 The Counting Numbers

It is surprising how many people still believe that numbers derive from counting—that counting is primitive. The developmental evidence that I have been summarizing tells us that the counting numbers are not at the origin of "number" but rather that counting numbers develop from more primitive core systems. Nevertheless the counting numbers (1, 2, 3, and so on) are a vital step in every person's cognitive development. Today, it is the first conceptual system for number that is learned. The system of counting numbers integrates major aspects of the core numerical systems while reconciling many of their conflicts. It supports new computations such as addition, subtraction, multiplication, and division. It is a sophisticated system, which is effectively infinite in terms of its potential for further development. Even today it is a major area of research in mathematics with very concrete and practical applications in areas such as encryption.

This system of counting numbers is the context within which many fundamental concepts are developed. These include linear order, cardinality, and infinity, to name just three. Yet young children become quite familiar with it—familiar enough to answer the question, "What is the largest number?" with the sophisticated answer, "There is none—they go on forever." Much of elementary arithmetic is concerned with grasping this conceptual system, making it one's own. The counting numbers are the most elementary example of the way human beings, even at a relatively early age, can access and work with cultural systems of incredible subtlety and depth.

Many people's understanding of number never goes much beyond this basic system. For them a number always remains a counting number. And so it was in the history of mathematics where, as we shall see, even the recognition that the ratio between two numbers, such as the ratio of 2 to 3 is (or can be represented by) a number, is a major development that takes us into a new conceptual system.

The mathematician Leopold Kronecker famously remarked that "The integers were created by God; all else is man-made." However it is much more likely that the positive integers are learned. Carey makes a detailed and convincing argument, that I do not have the space to replicate here, to show two things. The first is that the movement from one conceptual system to another (here from the core systems to the counting numbers) is discontinuous. The second proposes a mechanism for this movement, which she calls "Quinian bootstrapping," and I shall discuss later on in this chapter.

It is the movement from the core systems for number to the counting numbers that allows Carey to isolate crucial features of deep thinking. "… If CS_2 [the more developed conceptual system—here the counting numbers] transcends CS_1 in the sense of containing concepts not represented in CS_1, it must be the case that CS_2 is difficult for children to learn." The discontinuous learning that is involved in the mastery of a new conceptual system is inevitably difficult. When we come to look more closely at the nature of this difficulty we shall see that learning a new conceptual system is hard in precisely the same sense that being creative is hard. In fact learning a new conceptual system is *the* quintessential act of creativity. Continuous learning, which means acquiring the technical skill to make computations on the basis of a previously acquired conceptual system, does not carry with it the same sort of difficulty. For example, learning the multiplication tables and then the algorithm for multiplying two three digit numbers takes work but it a different kind of work that is required to move from living in the conceptual universe of the counting numbers to that of fractions.

Let me end this section with another cautionary note about the danger of identifying "number" with "counting number." Even Carey slips into this mistake at various places. However there is a fundamental difference—in fact a chasm—between them. We all know what the counting numbers are but "number", in its most general and basic sense, lives neither in the world of formal mathematical structures nor in the world of cognition. It is not, strictly speaking, either an object or a concept. It is an aspect of reality that is necessarily informal and so can be represented in multiple ways by means of different number systems.

Number is not so much a formal concept as it is what could be called a proto-concept, which generates multiple concepts.

Number is very subtle—you can't pin it down but you can't say it is nothing. One might say that number is a way of evoking the unity of the world as it manifests itself as order and pattern. We might identify it with "a tendency towards a complex order" that is present both in our mental processes and in the natural world. This is something the Greeks understood when they asserted that "the world is number;" it is the deeper meaning of the old saying, "God is a mathematician" and possibly what the psychoanalyst Carl Jung meant when he asserted that number is an archetype. Number is informal yet incredibly significant. It is no exaggeration to say that it is perhaps the most important foundational element of our entire scientific and technological civilization.

1.5 Core Systems for Geometry

Carey's colleague Elizabeth Spelke[xi] and her co-workers Sang Ah Lee and Véronique Izard have extended Carey's work on number to geometry. They claim that an analogous situation occurs in this domain, and hypothesize: "Like natural number, natural geometry is founded on two evolutionarily ancient, early developing, and cross-culturally universal cognitive systems that capture abstract information about the shape of the surrounding world: two *core systems of geometry*. … Children … construct a new system of geometric representation that is more complete and general."

It is fascinating that Spelke, like Carey, posits two core systems. Her work follows Carey's lead in viewing development more as an outcome of the integration of the two core systems. Nevertheless it would take me too far afield to discuss geometry in its own right at this time. I mention geometry here because of the danger that artificially separating "number" and "geometry" in this way will result in a misunderstanding of both, especially the development of number. The development of the concept of number does not only arise out of the activity of counting, as one might believe from reading these researchers, but also equally from the activity of measuring. For the Greeks one could argue that measuring was the more important source—numbers *were* measuring numbers, especially the length of line segments. In other words number has its origins in multiple core systems, which are antecedent to both counting and geometry.

The development of mathematical concepts does not proceed with a number concept that goes from "numerical" core concepts to counting whereas that of geometry goes from the core geometric concepts to a more sophisticated "Euclidean" system. Rather the same core systems give rise to different conceptual systems that involve different resolutions of the underlying discrepancies. Number is as much a measuring (geometric) concept as it is a counting object. Both counting numbers and measuring numbers are resolutions that succeed in resolving some but not all of the ambiguities inherent in the core systems. Both contribute to the development of more sophisticated notions of number.

1.6 The Rational Numbers as a Conceptual System

The fractions are the conceptual system that is learned after the counting numbers. Number systems in mathematics are basically situations in which you can do arithmetic, that is, you can add, subtract, multiply, and divide. However closure under these operations is sometimes incomplete. For example, adding or multiplying two counting numbers results in another one but subtracting or dividing may or may not do so. You can't subtract 8 from 3 in the counting numbers but trying to do so will ultimately lead to the creation of a new number systems—the integers—and its associated conceptual system, as we shall see in chapter three.

In this section I shall focus not on subtraction but on division. You can divide in the system of counting numbers. You can divide 12 beads into 3 collections and see that there are 4 beads in each collection. Alternately you can cut a pie into 12 regular pieces and consider that a sub-collection of 3 pieces bears a certain geometrical relationship to the whole pie. In other words a fraction is introduced as a relationship between two counting numbers, here 3 and 12. At this stage it is a relationship and not yet a number. You could also call it a computation over the counting numbers because the operation of division is occurring within the conceptual system of the counting numbers. You can divide whole numbers perfectly well without ever developing the concept of a fraction as we all did when we first learned that "three divides into seven two times and leaves a remainder of one."

But fractions are more than a representation system for ratios. To appreciate this extra dimension it is necessary to ask oneself in all seriousness, "Why is a ratio a number at all?" Don't answer, "Of course it is!" There is nothing obvious about it. Grasping that a ratio is a number and integrating that realization with the earlier system of counting numbers is a major intellectual accomplishment that we all made as children. It is a leap to a new world of numbers—the rational numbers.

We may appreciate this leap through an anecdote that the great mathematician, William Thurston, told of his own childhood[xii]. Thurston tells of the day when he ran excitedly up to his father and told him that he had just realized that 134 divided by 29 was a number and not just a problem in long division. His father's response was, "Of course it is." A mathematician would probably say, "That's obvious." But on the contrary it is the opposite of obvious. The father was speaking from the conceptual system of the fractions in which it is indeed obvious. But Thurston was describing the moment of insight when you transcend the system of the counting numbers and make the leap to the fractions. From this point of view it is a revelation! The two responses capture what is involved in going from one conceptual system to another and point out that what is involved in the process of deep thinking includes a discontinuous leap that gets you very excited because it gives you a vision of an entirely new world.

Let's think of this developmental problem in an entirely different way and look at it from the point of view of the ancient Greeks. For the Greeks numbers were not so much counting numbers as they were measuring numbers—lengths or areas. So asking whether something is a number comes down to asking if it can be constructed—in Euclidean geometry this construction can only use a straightedge and compass.

The geometric problem is: given line segments of lengths $\boldsymbol{a}$ and $\boldsymbol{b}$, can you construct segments of lengths $\boldsymbol{ab}$ and $\boldsymbol{a/b}$? For the Greek geometers the construction could only use a straight-edge and compass. The required construction for the product involved the theory of ratio and proportion, more specifically, that similar triangles (triangles of the same shape and having exactly the same angles) have sides that are proportional. The idea behind the construction is based on the geometric equation, $ab/a = b/1$, where we think of all of these numbers as lengths. Draw a triangle ***ABC*** such that the length of $AB = b$ and the length of $AC = 1$. Extend the side ***AC*** to ***D*** so that the length of ***AD*** is ***a***. Draw a line ***DE*** parallel to ***CB*** hitting the extended line ***AB*** at ***E***. Then the length of ***AD*** is ***ab*** because the triangles ***ABC*** and ***ADE*** are similar. (The quotient can be constructed in much the same way.)

Because this construction is possible every rational number can be represented concretely as a length, and so a ratio for the Greeks moved from being merely the relationship between two lengths to a length in its own right. But children are not experts in Euclidean geometry. It is not so obvious to them that the relationship between 5 pieces of a pie and the 12 pieces of the whole pie is a number that stands on the same footing as the numbers 5 and 12. In fact at first glance ratios and counting numbers are completely different things. For a child who defines number by the conceptual system of the counting numbers the ratio of 5 to 12 (or even of 3 to 12) is definitely *not* a number. If it is a number where does it fit into the counting numbers? Is it bigger than 1 or smaller than 1? Is it even comparable to 1? Such questions are not simple. They are hard, even "impossible" if they are looked at from the "wrong" point of view.

In what sense are such questions "hard?" The answer is that the two conceptual systems in question, the counting numbers and the fractions, are incompatible with one another. A person in the first system sees number in one way whereas a person in the second sees it in a completely different way. As Carey says, "The extension and conceptual role of the concept[xiii] of number are markedly different before and after the construction of the rationals." The child's answer to the question, "How many numbers are there between one and two?" will depend on which conceptual system the child is living in at the time. In other words the world of number changes and grows whenever you ascend the hierarchy of numerical conceptual systems. The change from one to the other comes suddenly; it has the nature of an insight. This discontinuous aspect comes directly from the strong incompatibility, or incommensurability, of the two number worlds.

Of course when you ascend to the world of fractions you learn that it contains what is essentially a copy of the counting numbers, now considered as the fractions **n/1**. From then on we successfully manage the ambiguity of considering an integer **n** as both a counting number and (simultaneously) as a fraction. The mathematician would say that we have isomorphically embedded the counting numbers into the fractions, which means that the copying process preserves all of the operations of arithmetic. The mathematician thinks of this as a formal process and so relatively easy, and does not always bear in mind that for the learner it involves the change of conceptual systems and that is always hard.

1.7 Ambiguity

The observation that the symbolic representation of number (**2** as counting number and fraction) suppresses the context (the conceptual system in question) that gives a precise meaning to the concept of number, demonstrates that even elementary mathematics contains an essential element of ambiguity.[xiv] This sort of ambiguity is not an error but an essential part of the conceptual structure of mathematics.

In recent books I highlighted the controversial notion that ambiguity is a key element in mathematics and science.[xv] Independently but almost simultaneously the philosopher of mathematics, Emily Grosholz, published an excellent book about "productive ambiguity" in mathematics and science that approached mathematics from this same perspective.[xvi] Ambiguity involves a singular situation or idea that can be represented or understood in more than one way. An ambiguous situation is one that has multiple frames of reference and these frames of reference often contain a mutual incompatibility that cannot be avoided. Thus an ambiguous situation has a problem at its core that requires resolution. This problem may be trivial or it may be profound. If it is interesting at all; if its resolution comes from a new way of thinking about the situation then we could say that the ambiguity is productive. In