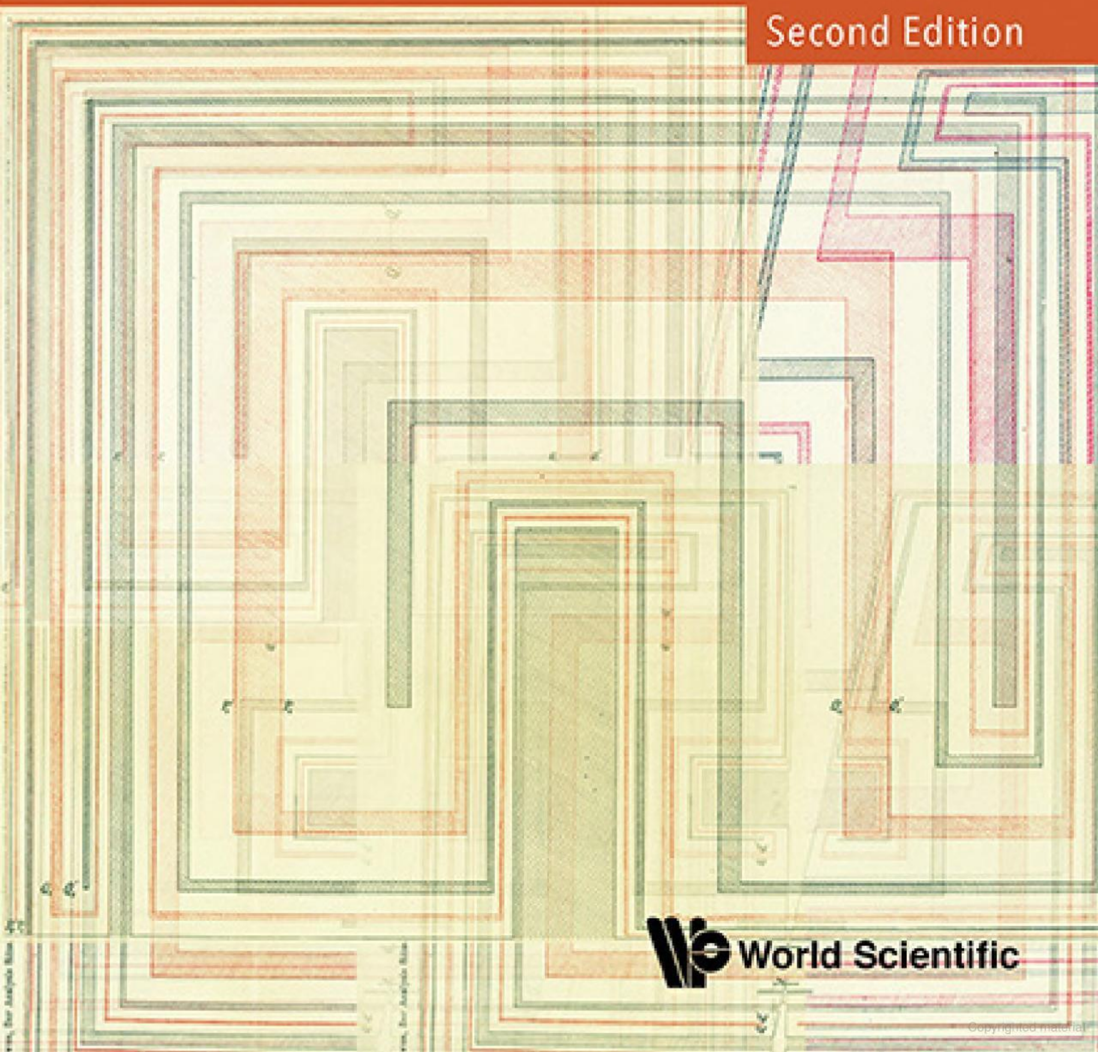
doing mathematics

Convention, Subject, Calculation, Analogy

Martin H Krieger

Second Edition



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Contents

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Preface

Doing Mathematics focuses on the work of mathematics and mathematicians, and the work of those who use mathematics in the physical sciences and the social sciences. Still, I have been assured by some lay persons that the book is readable with suitable skipping.

In this second edition, I have tried to deepen and clarify the text. I have come to understand more of some of the mathematics and the examples, and so I have been able to better discern my themes. In the last decade there have been remarkable advances, and some of them are relevant to the discussion. The Prolog epitomizes these themes.

In the case of the Ising model, there is a great deal of rigorous mathematical reformulations that may well be useful for understanding its analogy with other parts of mathematics.[1]

I shall be describing some ways of doing mathematical work and the subject matter that is being worked upon and created. I shall argue that the *conventions* mathematicians adopt, the *subject* areas they delimit, what they can *prove* and *calculate* about the (physical) world, the *analogies* that work for mathematicians, and the known *tools and techniques* they borrow from a Library of Mathematics—all depend on the mathematics, what will work out and what won't. And the mathematics, as it is done, is shaped and supported, or not, by convention, subject matter, calculation, analogy, and tools. These features correspond to chapter 2 on means and variances as conventional statistics, chapter 3 on the subject of topology, chapter 4 on strategy, structure, and tactics in long apparently "messy" proofs, chapter 5 on analogy in and between two programs for research, those of Robert Langlands (1936–), and go back to Richard Dedekind (1831–1916), in number theory, and of Lars Onsager (1903–1976) in statistical mechanics, and chapter 6 on some of the tools in that Library and how they are improved when they are loaned out. The examples I shall use are drawn from applied mathematics (and mathematical physics) as well as from pure mathematics.

Mathematics is done by mathematicians located in a particular culture (chapter 6), where mathematicians may choose to work on some problems and not others, in some ways and not others. But what they can in the end prove depends on the mathematics. And by "depends on" I mean that only some possible statements are both true and provable given our current imaginations and resources, that only some definitions are fruitful, and that only some models of the physical world are in accord with how it behaves, given what we know now.

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In effect, the Library of Mathematics has a wide variety of volumes, but not all we might wish for, and some of the time borrowed volumes are defaced in inventive ways and become even more useful. We shall notice that some of the time, what the mathematicians need is just what the physicist have been doing in their everyday work (albeit in a not so rigorous or general fashion). Or, what the physicists need has been developed on its own by the mathematicians and is available in the Library. Mathematicians also borrow from the Library, and so fields of mathematics deeply influence other fields. For example, curves representing algebraic expressions are then understood using deep algebraic methods (algebraic geometry).

Along the way, I am interested in saying something about what is really going on in the mathematics I describe. (And in some places, I have been rather more tentative and less successful, to be sure.) In saying what we might mean by "what is really going on," I shall argue that what is really going on is really going on only when it works in a proof or derivation and, ideally, we can show in detail just how and why it does the work. And "when it works" is itself a judgment; while, "showing in detail just how" is an interpretation of what is going on. Usually, we discover what is really going on from multiple perspectives on the same subject matter, different roads to exposition and proof, so that what at first seems miraculous and amazing (and it really is) is eventually shown to be manifest from the right points of view. We have "an identity in a manifold presentation of profiles," to use the term of art from philosophy.

Moreover, in this sort of work there is an intimate interaction of ideas and calculation, of insight and computation, of the conceptual and the algorithmic. Perhaps "proofs should be driven not by calculation but solely by ideas," as Hilbert averred in what he called Riemann's principle.[2] But, in fact, mathematical work is an interplay of both. So a combination of ingenuity, mathematical maturity, and a willingness to calculate and invent along the way is seen in Charles Fefferman's work, in the paper by C. N. Yang we shall discuss, in a series of papers by T. T. Wu and collaborators, and in Rodney Baxter's various exact solutions of lattice models in statistical mechanics. Moreover, mathematical rigor is in the end about ideas and the world; it is philosophical in that rigor often reveals aspects and counterexamples and cases we would not have otherwise been aware of. Rigor also allows mathematicians to be sure of their work, since it is error displaying, as Frank Quinn has argued.[3] And in order to implement ideas, one must calculate, theorize, and prove.

As for rigor and mathematical niceties, one often encounters comments such as,

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> The mathematical calculations that lead to exact results for quantities like the spontaneous magnetization have a complexity to them that many physicists writing on the subject feel obscures the essential physics.[4]

yet some of the essential physics is revealed by that complexity. But we also find,

> "The two-dimensional Ising model is a Free Fermion." Remarks to this effect are commonplace in the physics literature, although for mathematicians it sounds like a cross species identification.[5]

and

> The two-dimensional Ising model is nothing but the theory of elliptic curves.[6]

There are in fact many such indentifications and "nothing but"s, and it is in their variety that tells us more of what is going on.

The Topical of Table of Contents (next page) indicates where various particular examples appear in the text. It also indicates the main theme of each chapter, and a leitmotif or story or fairy tale that I use to motivate and organize my account. The reader is welcome to skip the fairy tale. My purpose in each case is to express poignantly the lesson of the chapter.

As for the leitmotifs or stories, some are perhaps part of the everyday discourse of mathematicians when they describe the work they do, often under the rubric of "the nature and purpose of mathematical proof." Sometimes, the mathematician is said to be climbing mountains, or exploring and discerning peaks in the distance that might be worthy of ascent. Or, perhaps the mathematician is an archaeologist. Having found some putative mathematical fact, the mathematician is trying to connect it with mathematics that is known already in the hopes of greater understanding.[7] Perhaps the mathematician tries to explore different aspects of some mathematical object, finding out what is crucial about it, getting at the facts that allow for its variety of aspects and appearances, in effect a phenomenological endeavor. In any case, here I have tried to describe rather more concretely some of mathematicians' work, without claiming anything generic about the nature of mathematics or mathematicians' work itself.

Again, my goal is to provide a description of some of the work of mathematics, a description that a mathematician would find recognizable, and which takes on some reasonably substantial mathematics. Yet I hope it is a description that lay readers

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FIGURE P.2: TOPICAL TABLE OF CONTENTS

Chapter	*Story*	*Theme*	*Examples**
2. Means and Variances	Anthropologist Studying a Culture	Conventions in Practice	Scaling
3. The Fields of Topology	Sibling Sub-Fields in Tension	A Subject or Field Defined in Practice, Algebra Everywhere	Ising Matter
4. Strategy, Structure, and Tactics in Proof	Interpreting Works of Art and Craft	Ideas and Calculation	Matter, Scaling
5. Syzygies of Research Programs	The Witch *is* the Stepmother	Analogy	Theta Functions, Ising, Scaling, Langlands, and Onsager
6. Mathematics *In Concreto*	People and Cities	Mutual Embeddedness of Mathematics and World	The City, the Embodied Prover, God as an Infinite Set

*Ising = The Two-Dimensional Ising Model; Langlands = Dedekind-Langlands Program in Number Theory; Matter = The Stability of Matter; Onsager = Onsager Program in Statistical Mechanics; Scaling = Ising in an Asymptotic Regime, or Automorphic Modular Functions, or the Central Limit Theorem.

might find familiar enough, as well, especially in terms of generic notions such as convention, subject matter, strategy and structure, and analogy. As a consequence of addressing these very different sorts of readers, the level of technical detail provided and the assumptions I make about what the reader knows already vary more than is usual. I hope that the simplifying images or the technical detail will not put off readers at each end of the spectrum.

I have been generous in repeating some details of several of the examples, at the various places where an example is employed. Each chapter may be read independently; and, sometimes, sections of chapters, of differing degrees of technical difficulty, may be read independently. It is noteworthy that some of the work I describe is better known by reputation than by having been read and assimilated, either because it is long and technical and hard to understand, or

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because it is deemed to be part of "what everybody knows," so few now feel compelled to take a close look at it.

Notionally, and somewhat technically, one recurrent and perhaps unifying substantive object is the central limit theorem of statistics: a sum of "nice" random variables ends up looking like a Gaussian or bell-shaped curve whose variance is the sum of the components' variances. The central limit theorem provides a model of scaling, the $\sqrt{N}$ growth of the Gaussian distribution; it is the large-N or large time (that is, asymptotic) story of the random walk on a grid or lattice; it turns out to be the foundation for a description of an ideal gas (the Maxwell-Boltzmann law of molecular velocities); and it provides a description of diffusion or heat flow, and so is governed by the heat equation (the laplacian or, in imaginary time, the Schrödinger equation), one of the partial differential equations of mathematical physics.[8]

If the Gaussian and the central limit theorem is the archetypal account of independent random variables, it would appear that matrices whose elements are random numbers or variables, and probability distributions of their basic symmetries (their eigenvalues), those distributions also determined by distant relations of the trigonometric functions, the Painlevé transcendents, are the archetype for measures of the connection between them (that is, correlation functions) and the extreme statistics (such as the maximum) of strongly dependent random variables.[9] These distributions appear as the natural asymptotic limit of correlation functions of nuclear energy levels or of zeros of zeta- and L-functions (in each case the separation between adjacent levels or zeros), in lengths of longest increasing sequences in random permutations, and in correlations of spins within a crystal lattice.

All of these themes will recurrently appear, as connected to each other, in our discussions.

Another unifying theme is set by the question, "Can you hear the shape of a drum?" That is, can you connect the sound spectrum of an object with its geometry, namely, its zeros with its symmetries?[10] (More concretely, can you connect the trace of a matrix with the determinant (a volume element) of another: the sum of the eigenvalues of one matrix with the product of the eigenvalues of another.) How is the local connected to the global, why is there regularity in both frequency and in scale size? Again, the connections will be multifold and wide-ranging.

And recurrently, we shall see that phenomena of geometry or topology or the calculus are mirrored in algebra, and so the alebraicization of much of mathematics, along the way transforming algebra itself.

More generally, there are *why* and *how* questions: Why is there *regularity* (in the combinatorial numbers, for example), seen as scaling and in fourier coefficients of nice functions? Why group *representations*, those matrices? What are the

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symmetries that underlie the relationship of counting to scaling? Why do we have *asymptotics* and these asymptotic forms?[11]

How does the algebra do the combinatorics (the motivation of Kac and Ward's famous paper)? And how does the algebra of counting lead to scaling or automorphy? More generally, how and why are objects that package combinatorial information exhibit nice scaling behavior. (And to show how does not really answer the why question.)

Descartes begins his *Rules for the Direction of the Mind* (1628),

> Whenever people notice some similarity between two things, they are in the habit of ascribing to the one what they find true of the other, even when the two are not in that respect similar. Thus they wrongly compare the sciences, which consist wholly in knowledge acquired by the mind, with the arts, which require some bodily aptitude and practice.[12]

Descartes is warning against what he takes to be incorrect analogy, and he is advocating a "universal wisdom," which allows the mind to form "true and sound judgments about whatever comes before it." I shall be arguing that whatever we might understand about universal wisdom, in actual practice our work is particular and concrete, that the temptation to analogy and to comparing the sciences and the arts is just the way we do our work.

After the introduction, *the chapters might be read out of order*. I have tried to make them self-contained for the most part. For those who prefer to start with the index, notes, and bibliography, I have tried to make them lead to the main points of the book.

As for studies of science, my analysis has been driven by the particular examples I am using, rather than by the demands of theory, sociology, philosophy, or historiography.[13] I imagine the analyses may prove useful for social and philosophical studies of science and mathematics. But other than my claims about convention, subject, calculation, and analogy, I have not tried to do such studies. Chapter 6, about the city, the body, and God, goes a bit further, but each of these topics demands a separate monograph, at least, if the chapter's suggestions were to be thoroughly worked out.

Rather than employing mathematics or physics to instantiate conventional philosophic (or sociological) problems, the cases we discuss suggest another set of problems: How are mathematics and physics made useful for each other? What is it about the physicist's world that makes it amenable to mathematical technologies, and how is that world so adjusted? What does it mean to have a

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precise mathematical description or definition of a physical phenomenon? How and why do the details in a mathematical analysis reveal physics that is otherwise not so appreciated? How is having many different solutions or proofs of the same problem useful and revealing? How do ideas and calculation support each other? How is mathematics' dynamic character fed by both applications and internal developments? How do ugly first proofs or derivations have an inner beauty? Just how does analogy actually work in these areas?

My particular choice of mathematics reflects some ongoing interests: elementary statistics (and what is not learned by students); statistical mechanics; the proof of Fermat's theorem by Ribet, Serre, Frey, and Wiles and Taylor in the 1990s, with the Langlands Program in the background; and, the recurrent appearance of scaling (or automorphy and renormalization), elliptic functions, and elliptic curves in many of these subjects. Some of this is mathematics in the service of physics. Although I cannot be sure, I doubt whether these particular choices affect my descriptions of how mathematicians do their work.[14]

As for technical knowledge of mathematics: Again, I have tried to write so that the professional scientist will find what I say to be correct mathematically and of practical interest. Yet the general reader or the philosopher should be able to follow the argument, if not always in detail then in its general flow. To that end, I have employed a number of signals:

Technically, I have sometimes flagged the more technical material by the device I used to begin this sentence. (In other cases, I have used brackets or parentheses to mark off technical material.)

***If three stars precede a paragraph, I have there provided a summary of what has gone before or a preview of a larger segment of the argument.

As for notation, of course the notation is consistent within any single example. But I have reused letters and symbols, with new meanings in different examples. The index points to the definitions of the letters and symbols.

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From the first edition: Abraham Polonsky and Gian-Carlo Rota encouraged my work and understood just what I was trying to do. Jay Caplan, Eric Livingston, Sam Schweber, and Robert Tragesser have been abiding readers and friends. I am grateful for my undergraduate education at Columbia College and my mathematics teachers there: J. Eels, Jr., A. Dold, and J. van Heijenoort. Colin McLarty and Craig Tracy helpfully answered my questions. The usual disclaimers apply, a fortiori.

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A sabbatical leave from the University of Southern California allowed me time to write. And my students, colleagues, and deans have provided a fertile environment for my work. Over the last decades, I have received external financial support from the National Humanities Center, the Russell Sage Foundation, the Exxon Education Foundation, the Zell-Lurie professorship at the University of Michigan, the Lilly Endowment, the Hewlett Foundation, and the Haynes Foundation, support that was for other books and projects. But I would not have been able to write this one without having done that work.

My son, David, always wants to know what is *really* going on. He continues to ask the best questions, and to demand the most thoughtful answers.

More recently, I have received support from the Haynes Foundation, the Kauffman Foundation, and the Price Charities, and again it was for other books and projects. I teach in a school of urban planning and public policy. I keep finding that the notions I develop in this book enrich my ability to teach our students.

I am grateful to my physicians at the University of Southern California's Keck School of Medicine and Hospital for their care for me and my son. My son David is now more than ten years older than when I wrote the first edition. He is still my toughest questioner.

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Prolog

It will be useful to preview our recurrent themes and examples: just what is *really* going on in the mathematics; how do ideas and calculation interact; how do the subjects of mathematics change as we learn more; ugly proofs have an inner beauty; analogy is a destiny we embrace; and, rigor and details are substantively informative. Substantively: we can hear the shape of a drum since sound spectrum and geometry are related; periodicity and self-similarity appear to accompany counting-up; local facts and global characteristics are systematically connected; and, you create mathematical objects so that those objects add up. In a bit greater detail:

1. When we ask *what is really going on in the mathematics*, or when we ask what is behind the various proofs of a theorem (or the various solutions of the Ising model), we are seeking what the phenomenological philosophers call *an identity in a manifold presentation of profiles*. Just what is the source of all the various seeming tricks and methods that make them work in these contexts? Put differently, how can the objects we are studying allow for such a variety of presentations. Most of the time, we have partial answers, and so fail to discern that identity as fully as we might hope.[1]

2. *Ideas and calculation* play against each other, for ideas without calculation are at best informed speculation, and calculation without ideas may or may not get anywhere or make sense. Moreover, devices and tricks employed along the way convey *meaning* and information, even if they appear jury-rigged or convenient. Again, what is really going on?

   Moreover, we might think of *mathematicians as master machinists,* using esoteric devices to make machinery that is sometimes adopted by engineers (physicists, for example) to fabricate what they want. The machinists are to some extent autonomous, tinkering and inventing, to some extent dependent on market demand.

3. *Fields of mathematics are dynamic*, changed by what we learn and can prove, what notions we invent or discover that turn out to be fruitful, and what happens in other fields of mathematics or physics (or computer

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science,…). Ideas and visions may transform a field, if they work out. And we may forget useful mathematics and examples along the way. Notions from one field may find use and meaning in another area, and most notions are present in more than one area or complex.

4. *Ugly proofs have an inner beauty*, revealed after subsequent proofs show us more of what is going on (in those proofs, and more generally). And often, initial proofs are ugly.[2]

5. *Analogy is destiny*, but that destiny and just what is analogized to what depend on what we do with the analogy. Analogy may be to other mathematics, to physics, or to everyday life.

6. *Details and fine points matter*. Rigor—just what demands those details and fine points—allows us to learn more of what is going on, and also to reveal stuff we had not noticed. The exceptional case we have to work around may well be quite revealing. In effect, we are philosophical analysts, seeking meaning through differential comparison of cases.

As for substantive themes:

A. Often, we can *hear the shape of a drum*.[3]

   So an object may be known by its audible sound (and so its frequency components) and/or by its everyday appearance. Spectrum and geometry, equation and curve, particle and field, the discrete and the smooth, algebraic object and topological space,…are intimately connected, albeit the connection may be of a different sort for each pair. So, an algebraic construction can stand in place for a topological or other sort of space; and we can often use a geometrical object or space to stand in for what appear to be collections of numbers or equations or an algebraic object

B. *Periodic Regularity and Self-Similarity or Scaling Symmetry* would seem to accompany various efforts at counting and enumeration.

   Brownian motion—a sum of random moves—looks the same at a very wide range of scales. Partition functions (or L-functions) that package information (say, about the number of solutions to an equation,

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modulo a prime, *p*), or their close relatives, exhibit a scaling symmetry (self-similarity, "automorphy"). Think of the central limit theorem and Gaussians growing as $\sqrt{N}$. The periodicity reflects a regularity in the packaged numbers, and we want to understand the source of that regularity.

C. Often, there is a connection between *local facts and global characteristics*.

There is a connection between locally-seen regularities (as in the number of mod *p* solutions to equations) and harmonic analysis (that is, a nice transform of something).

We may discover obstructions to those connections, and those obstructions are deeply informative. What we often have is a hierarchy, at each level fully adequate—as in the Standard Model of particle physics and its effective field theories.

D. We are looking for *individuals that add up*, one way or the other.

Usually they add up as in arithmetic, or linearly, as in a vector space. What are the right parts, the right variables, the right degrees of freedom? Complex interactions should be the sum of two-body interactions, invariant to the order of adding them up—again, as in arithmetic. And that addition would seem to lead to canonical asymptotic forms, such as the Gaussian.

A and *B* say that the sum of the spectrum is related to a volume ("Weyl asymptotics"). *B* says that partition functions that package combinatorial information have asymptotic forms that are self-similar. *C* says that partition functions, as global objects, are something like Fourier transforms of properties of local objects. And *D* is about the canonical form of those partition functions.[4]

I shall be *describing* how mathematics and mathematicians work, rather than theorizing or philosophizing—although these descriptions may help in theoretical and philosophical work.[5] Again, much of this is the legacy of Riemann and Maxwell. But I have no good answers to why group representations are so useful here, or why there are the periodic regularities people notice in counting-up, or why permutations are related to self-similarity. Kant would counsel that some things are beyond scientific knowledge. Husserl would suggest that we are discovering or uncovering various aspects of a phenomenon, that identity in a manifold

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presentation of profiles. I suspect that fifty years from now, or maybe just a few years from now, we'll have better answers and just as frustrating questions.

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1

Introduction

I

I want to provide a description of some of the work that mathematicians do, employing modern and sophisticated examples. I describe just how a *convention* is legitimated and established, just how a *subject* or a field of study comes to be defined, just how organization and *structure* provide meaning in the manipulations and calculations performed along the way in an extensive proof or derivation, and just how a profound *analogy* is employed in mathematical work. These just-hows are detailed and particular. So that, when we demand a particular feature in a rigorous proof, such as uniform continuity, there is lots to be learned from that.

Ideally, a mathematician who reads this description would say: "That's the way it is. Just about right." Thus, laypersons might learn something of how mathematics is actually done through this analytic description of the work. Students, too, may begin to appreciate more fully the technical moves made by their teachers.

To say that mathematicians prove theorems does not tell you much about what they actually do. It is more illuminating to say what it means in actual practice to do the work that is part of proving theorems: say, that mathematicians devise *structures* of argument that enable one to prove theorems, to check more readily the correctness of those proofs, and to see why what is being shown is almost surely the case.[1] Mathematicians might be said to construct notions and definitions that are fruitful, and theories that then enable them to prove things. The organization of those theories, or those structures of proof or demonstration or those notions, may often be shown to correspond to facts about the world. The sequencing and waypoints say something about physical or mathematical objects, or so we may convince ourselves. Whether those proofs look logical or narrative is a stylistic choice that varies among mathematicians and cultures.

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> In Burnside's classical style of writing [ca. 1897], Theorem *x* means Theorem *x* together with the discussion around it. In other words, although everything is proved, only some of the conclusions are called theorems. This "classical" style is quite different from the so-called "[E.] Landau Style" of Satz-Beweis-Bemerkung [Statement-Proof-Remark]. In the classical style, you first discuss things and then suddenly say that you have proved such and such; in other words, the proof precedes the statement.[2]

One might provide a description of writing fiction or poetry similar to the one I provide here for mathematics, emphasizing convention, genre, grammar and prosody, and analogy. My claim is simple: In the doing of mathematics one uses interpretive and rhetorical devices that are peculiar to mathematics, yet of the sort seen elsewhere. And, these devices are an essential part of the mathematics, for they are mathematics as such.[3]

Hence, one of my ancillary purposes is to better understand the mathematics as mathematics, to better read and understand difficult and complex work, to get at what is really going on in a proof or a derivation.

It is also true that mathematics is done by people, whose intuitions, examples, and ideas stem from and resonate with their own culture. To illustrate this fact, in chapter 6 I speculate on the resonances between nineteenth-century cities and some of Riemann's mathematical ideas, between the fact that we have bodies and the techniques of algebraic topology, and between religious accounts of God's infinitude and transcendence and mathematicians' notions of infinity. This speculation is intended to be suggestive. I make no claim that mathematics is reducible in any useful sense to something else, or the other way around. Rather, it is a reminder that even our most abstract endeavors are embedded in history and society. That truth should appear within history is no challenge to the facticity or truth of mathematics. It is important to understand not only the history of the mathematics and how what we now know came to be elucidated, but also to appreciate the actual process of achieving insight through new proofs, reformulations, and contrastive examples, and how we take mathematical objects as objects for us (that facticity). This was of enormous interest to mathematicians such as Hermann Weyl (1885–1955), influenced as he was by the philosopher and phenomenologist and mathematician Edmund Husserl (1859–1941) in the early part of the twentieth century.[4]

I am concerned with analyzing the particular features of particular fields of mathematical activity, rather than discussing what is conventionally claimed to be philosophy of mathematics. I focus on the concreteness of the examples, no matter

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how abstract the mathematics that is instantiated in those examples. Mathematicians usually report they are making discoveries; that proofs are a means of discovery and of checking the truth; that they are getting at the truth through various new proofs and formulations; and that even when they invent new objects there is a sense in which the objects are already-there in the material the mathematicians possessed already. I take these comments as a fiducial report, notwithstanding the fact that many philosophers of mathematics subject those reports to criticism.

I shall not at all address claims about mathematics being the language of science, or that having a language still does not provide a deeper explanation. In such instances, mathematics may be one of the main languages employed. But actual discourse is in a vernacular that mixes formal mathematics, physical and spatial intuitions, and everyday metaphorical speech. It is perhaps no more surprising that mathematics is suited to describing the physical realm than it is that ordinary everyday metaphorical speech is so suited. And the connections between mathematics and the physical sciences are as surprising as the connections between the fields of mathematics.[5]

Finally, I have focused on mathematics as published in journals and books, or codified in textbooks, and on the network of ideas in that mathematics. From our retrospective viewpoint, we might understand what is going on in the mathematics and what the mathematicians were doing. This is quite different from history of mathematics, for historians are concerned with what the scientist and their contemporaries understood, what they saw themselves as doing, and with the nature of influences and developments, and so are obliged to look more widely at additional documentary material.

We shall work out our ideas about mathematical practice through four examples of mathematical work: (i) means and variances in statistics, (ii) the subject of topology, (iii) classical analysis employed in the service of mathematical physics, and (iv) number theory's "learning to speak Riemannian." I take my warrant for these studies from Hermann Weyl, who writes about the practical need in mathematics for an historical-philosophical understanding.

> The problems of mathematics are not problems in a vacuum. There pulses in them the life of ideas which realize themselves in concreto through our human endeavors in our historical existence, but forming an indissoluble whole transcending any particular science.[6]

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That "life of ideas" is, for my purposes, the legacies of Riemann and of Maxwell. And what we discover is that the concrete, the particular, and the exceptional in mathematics are made to carry along the abstract, the arbitrary, and the norm, and vice versa.

I have chosen my case studies because of their power to illuminate actual practice. But, notionally, they might be linked by the question: "Can one hear the shape of a drum?" If you listen to a drum, and hear its resonant tones, can you infer something about the size and shape of that drumskin? This is a beautiful and archetypal problem that epitomizes much of our substantive mathematical discussion.[7] More generically, can you connect the sound spectrum to the geometry? The answer is a qualified Yes: For example, given our ordinary experience, we know that a bigger drum has deeper tones. More technically, you can hear its area, its perimeter, and whether its drumskin has punctures or holes. More generally, however, the answer is No, for there are inequivalent shapes with the same sound spectrum.

A drum's loudness is proportional to the number of resonant tones, essentially the story of the central limit theorem of statistics, the subject of chapter 2. That the holes in the drumskin have consequences for the resonant tones is just the connection between spatial facts and set-theoretic facts and the calculus that is the foundation for topology, and which is also a source of topology's internal tensions, the theme of chapter 3. The connection between the shape and the sound of the drum turns out to model the balance between the electrical forces among the electrons and nuclei and the electrons' angular momentum within ordinary matter, a balance that ensures that such matter does not implode, the story of chapter 4. And the connection between the shape, the sound, and an accounting or partition function which encodes the resonant frequencies and their loudness, is just the analogy explored in chapter 5.[8]

"Weyl's asymptotics," as all these facts are sometimes called, is a Christmas tree on which the various themes of this book might be arrayed. And it is Weyl (in 1913) who reworks Riemann's ideas and makes them the foundation for much of modern mathematics. Hence, this is a book about the legacy of Riemann (and of Maxwell), and of Weyl, too.

Another theme is the algebraicization of mathematics. Topology becomes algebraic, both point-set and combinatorial. And more generally, a logic of analogy, functorial relationships (a picture of *A* in *B*) that mirror both objects and the relationships between them, model the systematics of analogy.

In chapter 6, I describe several themes common to both nineteenth-century urban development and nineteenth-century mathematics. My thesis here is not about

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influence of one domain upon the other; rather, both domains appear to draw on the same sorts of ideas. I begin with a description of what I call the Library of Mathematics, and how mathematicians and physical scientists add volumes, borrow volumes, and sometimes usefully deface volumes they have borrowed and then return them in a transmogrified form. Then I describe how our actual bodies are employed in mathematical work (without at all suggesting that proverbial Martians would have different mathematics than ours). And, finally, I survey the theological and philosophical tradition concerning the relationship of God's infinitude to notions of the mathematical infinite. Again, the lesson in this chapter is that we need not be reductionist or relativist when we place mathematics within its larger culture. And, that historical-philosophic analysis can be helpful for understanding the mathematics as mathematics, and just how technical details matter.

II

The following survey of the next four chapters highlights their main themes, indicates relationship of the studies to each other, and provides an account of the argument absent of technical apparatus.

CONVENTION

Means and variances and Gaussians have come to be natural quantitative ways of taking hold of the world.[9] Not only has statistical thinking become pervasive, but these particular statistics and this distribution play a central role in thought and in actual practice: in the distribution of the actual data; the assumed distribution of that data; or, the distribution of the values of the measured statistic or estimator. Such conventions are entrenched by statistical *practice*, by deep mathematical *theorems* from probability, and by *theorizing* in the various natural and social sciences. But, entrenchment can be rational without its being as well categorical, that is, excluding all other alternatives—even if that entrenchment claims to provide for categoricity.

I describe the culture of everyday statistical thought and practice, as performed by lay users of statistics (say, most students and often their teachers in the natural and applied social sciences) rather than by sophisticated professional statisticians.[10] Professional statisticians can show how in their own work they are fully aware of the limitations I indicate, and that they are not unduly bound by means-variance thinking about distributions and data (as they have assured me); and, that much of

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their work involves inference and randomization, which I barely discuss. Still, many if not most ordinary users of statistics in the social sciences and the professions, and even in the natural sciences, are usually not so aware of the limitations in their practice.

A characteristic feature of this entrenchment of conventions by practice, theorems, and theorizing, is its highly technical form, the canonizing work enabled by apparently formal and esoteric mathematical means. So, it will prove necessary to attend to formal and technical issues.

We might account for the naturalness of means and variances in a history that shows how these conventions came to be dominant over others. Such a history shows just how means and variances as least-squares statistics—namely, the mean minimizes the sum of the squares of the deviations from the midpoint, more or less independent of the actual distribution—were discovered to be good measures of natural and social processes, as in errors in astronomical observation and as in demographic distributions. There is now just such a literature on the history of statistics and probability.[11] Here, however, I shall take a different tack, and examine the contemporary accepted ahistorical, abstract, technical *justifications* for these conventions, justifications which may be taken to replace historically and socially located *accounts*. More generally, these conventions come to be abstractly justified, so that least-squares thinking becomes something less of an historical artifact and rather more of an apparently necessary fact. Socially, such justifications, along with schematized histories, are then used to make current practice seem inevitable and necessary. Put differently: What might be taken as a matter of Occam's razor—for are not means and variances merely *first* and *second* moments, averages of x and x^2, and so are just good first and second order approximations, so to speak?—requires in fact a substantial armamentarium of justification so that it can then appear as straightforwardly obvious and necessary.

One might well have written a rather more revolutionary analysis, assuming a replacement theory had more or less arrived and that means and variances and their emblematic representative, the Gaussian, had seen their day. In a seminal 1972 paper by the mathematician Peter Huber, much of what I say here about the artificiality of means and Gaussians is said rather more condensedly, and a by-then well worked out alternative is reviewed.[12] Huber refers to the "dogma of normality" and discusses its historical origins. Means and Gaussians are just mutually supporting assumptions. He argues that since about 1960 it was well known that "one never had very accurate knowledge of the true underlying distribution." Moreover, the classical least-squares-based statistics such as means were sensitive to alterations or uncertainties in the underlying distribution, since that distribution

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is perhaps not so Gaussian, and hence their significance is not so clear, while other statistics (called "robust") were much less sensitive.

Now if one were to take, say, Fredrick Mosteller and John Tukey's 1977 text on statistics as the current gospel, then the alternatives hinted at here and by Huber are in fact fully in place. As they say, "Real distributions often straggle a lot compared to a normal distribution."[13] Means and variances and Gaussians are seen to be an idiosyncratic and parochial case. With modern computational and graphical capacities, and the technical developments of robust statistics, we are in a new era in which we can "play" with the data and become much more intimately acquainted with its qualities, employing sophisticated mathematics to justify our modes of play.[14] I should note that obituaries for John Tukey (2000) suggest much wider practical influence for his ideas than I credit here.[15] Moreover, probability distributions much broader than the Gaussian do play an important role in the natural and the economic sciences. And computation allows for empirical estimates of variation (bootstrap, resampling).

THE FIELDS OF TOPOLOGY

The third chapter describes some of the fundamental notions in topology as it is practiced by mathematicians, and the motivations for topological investigations. Topology comes in several subfields, and their recurrent and episodic interaction is the leitmotif of this chapter.[16] Such a philosophy of topology is about the mathematics itself.[17] What are the ideas in this field, and how do they do the work? (So in a first book on topology, the Urysohn Lemma, the Tietze Extension Theorem, and the Tychonoff Theorem are so described as embodying "ideas."[18])

Topology might be seen as the modern conception of continuity, in its infinitesimal and its global implications. The great discovery was that once we have a notion of a neighborhood of a point, the proverbial epsilons and deltas are no longer so needed.[19] A second discovery was that we might approximate a space by a tessellation or a space-frame or a skeleton, and from noting what is connected to what we can find out about the shape of that space. And the notion of space is generalized, well beyond our everyday notions of space as what is nearby to what. A third was that a space might be understood either in terms of neighborhoods *or* tessellations or, to use an electromagnetic analogy, in terms of localized charges *or* global field configurations, sources *or* fields. In general, we use both formulations, one approach more suitable for a particular problem than the other. And a fourth was that we might provide an algebraic account of continuity.

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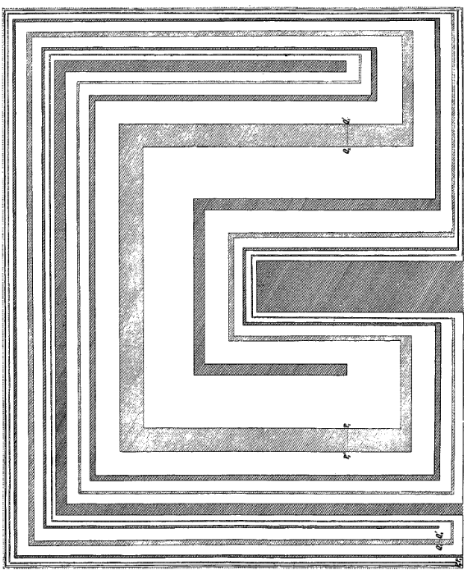
In the practice of topology, some recurrent themes are: (1) decomposing something into its putative parts and then showing how it might be put together again; and, how those parts and their relationships encode information about the object; (2) employing diagrams to make mathematical insights and proofs obvious and apparently necessary; and, (3), justifiably treating something in a variety of apparently conflicting ways, say that Δx is zero and also nonzero (as in computing the derivative), depending on the context or moment—themes more generally present in mathematics.[20] Another recurrent theme is the connection of local properties to global ones, when you can or cannot go from the local to the global, and the obstructions to doing so. A much larger theme is the algebraicization of mathematics, so that mathematical problems from analysis or geometry or topology come to be expressed in algebraic and formal terms. And this theme has profound implications for the practice of topology and, eventually and in return, for the practice of algebra.



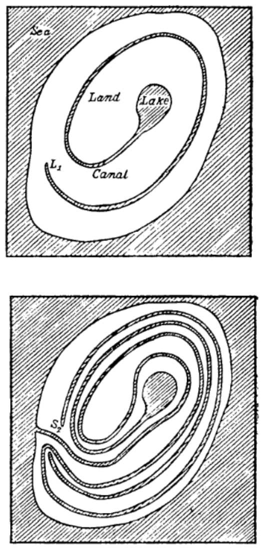


FIGURE 1.1: Brouwer's indecomposable plane, from "Zur Analysis Situs," *Mathematische Annalen* (1910).
FIGURE 1.2: Isles of Wada, from Yoneyama, "Theory of continuous sets of points," *Tôhoku Mathematics Journal* (1917)

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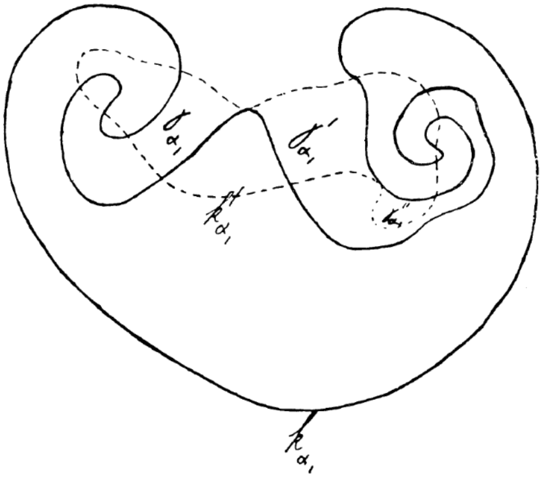

FIGURE 1.3: Proof diagram, Figure 5, from Brouwer, "Continuous one-one transformations of surfaces in themselves," *KNAW Proceedings* (1909).

We might better appreciate the tension between set theory and algebra in topology by reviewing some of the work of L.E.J. Brouwer (1881–1966). In 1909, Brouwer develops a beautiful and remarkable example, an "indecomposable continuum," accompanied by an extraordinarily handsome two-color diagram (as originally printed in the journal).[21] Here, a curve splits the plane into three parts, of which the curve is the common boundary; it is a continuum that is not the union of two sub-continua. A way of presenting this example (the "Isles of Wada"), begins:

> Suppose that there is a land surrounded by sea, and that in this land there is a fresh lake. Also, suppose that, from these lake and sea, canals are built to introduce the waters of them into the land according to the following scheme. . . . On the first day a canal is built such that it does not meet the sea-water [top map of Figure 1.2]. . . . The end point of this canal is denoted by L_1.
>
> On the second day, a canal is built from the sea, never meeting the fresh water of the lake and canal constructed the day before, and the work is continued . . .

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> On the third day, the work is begun from L_1, never cutting the canals already built, . . . [22]

Diagrams such as in Figure 1.3 appear in his 1910 and earlier "Cantor-Schönflies" style work.[23]

Brouwer then developed a notion of the degree of a continuous map or function, the n in z^n (a winding number, the number of times a map winds the circle around itself, z^n doing it n times, or the curl or circulation, or the index or charge). And, using "fantastically complicated constructions" he proves all of the then outstanding problems in topology, including "invariance of dimension," that dimension is preserved by continuous mappings (1911).[24] His notebook in which he begins to develop the notion of what we now call the Brouwer degree is entitled by him, "Potential Theory and Vector Analysis" (December 1909–January 1910), in effect, going back to Gauss and to Riemann and Maxwell.[25]

Brouwer's 1910 proof that a closed continuous non-selfintersecting line in the plane divides the plane into two disjoint parts (the Jordan curve theorem) employs set-theoretic topological devices (curves that are deformable into each other, "homotopy"), and combinatorial devices (those tessellations, a "linear simplicial approximation").

However, Brouwer never employs the algebraic technology developed earlier by Poincaré for revealing the invariant features of an object in various dimensions and their relationship (namely, homology and chain complexes). It took some time for these algebraic technologies to be incorporated systematically into topological research, so that eventually combinatorial device would be interpreted through algebraic machinery, Betti numbers becoming Betti (or homology) groups. (The crucial figure is Emmy Noether (1882–1935) and her (1920s) influence on P. Alexandroff and on H. Hopf, and independently in L. Vietoris's work.) For until 1930 or so "set theory lavished beautiful methods on ugly results while combinatorial topology boasted beautiful results obtained by ugly means."[26]

Here, we shall be following the Riemannian path of Felix Klein (1849–1925) and Hermann Weyl. Eventually, the various paths lead to a homological algebra—the wondrous fact that the homology groups project topology into algebra. Just how and why algebra and topology share similar apparatus, why there is a useful image of topology in algebraic terms and vice versa—defines a tension of the field, as well as one unifying theme: decomposition or resolution of objects into simpler parts. Another theme is how an algebraic account of a topological space

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> An analogous construction arose in invariant theory a century ago. . . . we seek "relations on relations" (called syzygies). . . . Hilbert's theorem on syzygies says that if this is done *n* times . . . The sequence of . . . free modules so obtained can be assembled into an exact [roughly, a nicely decomposing] sequence . . .[27]

As preparation for what is to come, at the end of the chapter there is an appendix on the Ising model of ferromagnetism (a permanent magnet). I should note once more, there is some deliberate repetition among the discussions of examples, in order to make each discussion reasonably self-contained.

IDEAS AND CALCULATION

Some calculations and proofs appear magical at first. Technical details that are apparently unavoidable in the first proofs seem to be pulled out of air. In part, you do not see the piles of paper on which unproductive paths are worked out; you only see what actually worked. You do not fully appreciate the personal toolkit of techniques possessed by a mathematician, tools that have worked for her in the past. Yet in retrospect those calculations turn out to build in the deepest ideas and objects, in effect what you must do (your jury-rigged devices) turns out to be done for a good reason, and the objects you uncover have a life of their own. But again, you only understand that after further proofs and calculations by others. Again, if some particular device is needed it is likely that that device will point to objects and properties you only discover latterly.

On the other hand, ideas need to be made concrete through calculations and the invention of technical devices. To make precise what you might mean by unique factorization of numbers will lead to ideal numbers and algebraic numbers. Again, we rarely hear about ideas that are not productive in proof, theory, and calculation.

In effect, we have the phenomenologist's "identity in a manifold presentation of profiles." An initial proof or calculation is then supplemented by many others, and in time one has a sense of what is really going on. An idea, realized in various contexts and calculations and examples becomes something real for us.

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> The paper was very difficult because it did not describe the strategy of the solution. It just described the detailed steps, one by one. Reading it I felt I was being led by the nose around and around until suddenly the result dropped out. It was a frustrating and discouraging process. However, in retrospect, the effort was not wasted.[30]

It helps enormously to have someone (perhaps even the author) show you what is really going on, or why the paper is organized just the way it is—say at the blackboard or in a seminar presentation—at least the author's best understanding, even if they do not really understand why it works. Writing up a paper for publication may make it all too neat, hiding the often still-inchoate insights and entrees that inform the effort.

Still, I believe that any of these proofs when studied at the right level of detail, usually reveals a strategy and a structure that makes very good sense, even if at the most detailed levels one is clearing a path through an uncharted jungle, and at the surface the various parts seem to hang together but they do not make sense except that they do achieve the proof. It may not be apparent just what is that right level of detail. Subsequent work by others often provides the appropriate point-of-view and suggests that right level of detail.

At the end of the most tortuous of paths, there is often a comparatively simple solution, although sometimes it is a mixed reward. Onsager ended up with expressions in terms of elliptic functions and hyperbolic geometry. Dyson and Lenard ended up with the simple inequality they were searching for, albeit with a very large constant of proportionality. (Lieb and Thirring's was about 10^{13} times smaller, of order 10.) Fefferman and Seco found simple numerical fractions for the coefficients of an asymptotic series. Yang's derivation of the spontaneous magnetization provided in the end a simple algebraic expression. And Wu, McCoy, Tracy, and Barouch found that the asymptotic correlation functions were expressible in terms of Painlevé transcendents, distant relations of the Bessel and elliptic and trigonometric functions. These are hard-earned saving graces. They are temptations, as well, that would appear to say that if we really understood what was going on we should be able to achieve these simple solutions rather more straightforwardly.

We might still ask, Why do elliptic functions (or Painlevé transcendents, or Fredholm determinants, or Toeplitz matrices, or Tracy-Widom asymptotic distributions) appear in these derivations. Subsequent work may suggest a very good reason, often about the symmetries of the problem that are embodied in these mathematical objects. Or, what was once seen as an idiosyncratic but effective technique, turns out to be generic for a class of problems of which this

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one is an exemplar (technically, here about nonlinear integrable systems and their associated linear differential equations). Still, someone had to do the initial solving or derivation, that *tour de force* or miracle, to discover that nice mathematical object *X* at the end, so that one might begin to ask the question, "Why does *X* appear?"

The applications of classical analysis may still demand lengthy and intricate calculations, endemic to the enterprise. But our assessment of those calculations, and their estimates and inequalities, can be rather more generous than Dyson's. Dyson and Lenard's work shows an enormous amount of physical understanding, whatever the size of the proportionality constant. This is evident in the organization of the proof. Moreover, the aesthetics are more complicated than terms such as clean or messy might suggest. For we might see in the inequalities some of the harmonies of nature:

> Since a priori estimates [of partial differential equations] lie at the heart of most of his [Jean Leray's] arguments, many of Leray's papers contain symphonies of inequalities; sometimes the orchestration is heavy, but the melody is always clearly audible.[31]

ANALOGY AND SYZYGY

> . . . members of any group of functions [*U*, *V*, *W*, . . .], more than two in number, whose nullity is implied in the relation of double contact [namely, $aU+bV+cW=0$, *a*, *b*, *c* integers] , . . . must be in syzygy. Thus *PQ*, *PQR*, and *QR*, must form a syzygy. (J.J. Sylvester, 1850[32])

P —————— Q —————— R

FIGURE 1.4: Three points in syzygy, as Sylvester might refer to them.

When three heavenly bodies, say *P*, *Q*, and *R*, are in line, they are said to be in syzygy—in effect, yoked together. In mathematics, a syzygy has come to mean a relation of relations. So, for example, the Hilbert Syzygy Theorem (1890), referred to earlier in the quote about homology theory, says that one can decompose a certain kind of object into a finite number of finite simply-composed parts, so that there are no residual relations or syzygies among them.

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More generally, a syzygy is an analogy of analogies. Much modern work in mathematics forms relations of relations through forming functors of functors (functors being simultaneous transformations of spaces and mappings), structure-preserving ("natural") transformations among functors, and more generally pursues a philosophy of functoriality and syzygy.

In chapter 5 I want to describe how one quite self-conscious analogy is employed in mathematics, what is nowadays called the Langlands Program in number theory and representation theory—and which in earlier forms has borne the names of Riemann (1826–1866), Dedekind (1831–1916) and Weber (1843–1912), Hilbert (1862–1943), Artin (1898–1962), and Weil (1906–1998), among many others. I suggest how that analogy, which I shall call the Dedekind-Langlands Program, is analogous to another analogy, which I shall call the Onsager Program in mathematical physics, with an equally long list of names. In one sense this analogy of analogies is not too surprising, given that in each case one has a partition function that packages some combinatorial information about an object such as the integers or a crystal. One then studies analytical properties of that function to discover general properties of the associated object. In order to do so, one studies an analogous mathematical object that also yields that function. However, the details of the syzygy are rather more precise and informative than we might at first suspect.

The role of analogy in mathematical discovery and proof is a recurrent theme in writing about mathematics.[33] Analogy may be intuitive or formal, at the level of a single fact or between substantial theoretical structures. I came to write about analogies, and analogies of analogies, from recurrently bumping into them in quite specific contexts. Earlier, in *Doing Physics* (1992, 2012), I called them tools in a toolkit, models as analogies, which then appear ubiquitously. In my subsequent work on how mathematics is employed in mathematical physics (*Constitutions of Matter*, 1996), I had come to see in rough outline further connections between the various kinds of mathematics and the varieties of physical models of matter.[34] What was then striking to me was my subsequent reading of André Weil's (1940) description, drawn from Richard Dedekind's work of the 1880s, of a triple of parallels or analogies among different fields of mathematics: geometry, function theory, and arithmetic—respectively: Riemann surfaces and analytic functions, algebraic functions, and algebraic numbers. Weil called these parallels "un texte trilingue," or a Rosetta stone (he called two of the languages Riemannian and Italian). Weil's triple deliberately echoed Riemannian themes, linking curves to the complex spaces in which they reside, and algebraic

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themes linking curves to algebraic function theory modeled on the abstract theory of algebraic numbers (due to Dedekind and Heinrich Weber, 1882).[35]

More to my point, Weil's triple paralleled the triple that was discernible in the mathematical physics describing a gridwork of interacting atoms, what is called lattice statistical mechanics, exemplified by the variety of solutions to Ising model.[36] Before me was a relation of relations, an analogy of analogies, a syzygy.

Roughly at the same time I was reading about elliptic curves—in part to try to understand a bit of the then recent proof of Fermat's Theorem by Andrew Wiles. By way of some bibliographic guidance provided by Anthony Knapp, in his book *Elliptic Curves* (1992), I then came upon the Langlands Program and saw how it was the contemporary version of Dedekind's analogy.[37] I cannot claim to understand much of the technical features of the Langlands Program. But I believe that the Dedekind-Langlands and the Langlands programs do provide for an analogy with the Onsager Program in mathematical statistical mechanics (and others have so indicated in the scientific literature).[38]

What the mathematicians find generically, the physicists would, for their particular cases, seem to have taken for granted. And what the physicists find, the mathematicians have come to expect. The syzygy illuminates its various elements, leading to a deeper understanding of what is going on in the mathematics and what is going on in the physics. Such relations and syzygies reflect both formal mathematical similarities and substantive ones. For a very long time, scientists (whom we now classify mostly as physicists) and mathematicians have borrowed models and techniques from each other. And, of course, many potential syzygies do not in fact work out.

We do not know the meaning of an analogy, just how something is analogous to something else, until we see how it is worked out in concrete cases. Still, there is excitement to be found in the prospect of discovery, in drawing these diagrams of relationship. Often, one is figuring them out on one's own, or hears about them orally. Rarely are these written down and published in the mainline scientific literature, for they appear so tenuous yet tempting, so unprovable for the moment even if it would seem that they must be true. A diagram I drew for myself about the Ising model, the Onsager program, and the Langlands program—before reading Weil—made it possible for me to be struck by the syzygy. (See note 34.) As Weil points out, this is only the beginning of the mathematics. It is that beginning I am describing here.

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III

In 1960, the distinguished theoretical physicist Eugene Wigner published an article, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," which concludes, "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."[39]

There is a substantial literature discussing and explaining Wigner's claim. His article is subtle and wide-ranging, pointing out the value of new concepts from mathematics, and concerned with the uniqueness of physical theories. But it is the title that has crystallized responses to it.

Now, there is another effectiveness that strikes me as at least as "unreasonable," namely, *the unreasonable effectiveness of physics in mathematics*, the appropriateness of the language of physics for the formulation and foundation of mathematical concepts and theories, and for suggesting new directions for the development of those theories. (I will use "physics" here implicitly referring to natural and social science as well.) Physicists invent mathematical devices to do their work, or discover them in the Library of Mathematics (as Wigner suggests) to do new work: whether they be Newton's calculus; Maxwell's coupled set of partial differential equations (which eventually become matters of differential geometry and topology); Heaviside (1850–1925) and Dirac's (1902–1984) delta function (and eventually Schwartz's distributions); Heisenberg's matrix mechanics for quantum theory (leading to developments in operator theory); Schroedinger's partial differential equation, or more recently, Witten and others' topological field theories.

What is remarkable is that the mathematicians can then take these practical devices and new uses of their concepts, and not only make them rigorous but some of the time make them the foundation for a rich field of mathematics. In part, as I indicate above, the physicists are actually borrowing some mathematics that is already extant. But their use of the mathematics extends it in unexpected directions, and those directions are then articulated by mathematicians into rich theories. Also, presumably, the physicists are in part driven by physical phenomena or analogies to known phenomena, so that it is the actual world as physicists understand it that then leads to these articulations.

Feynman's path integral approach to quantum mechanics (originally suggested by a remark by Dirac), the various groups (including nonabelian or

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noncommutative groups) used by Wigner and others to understand the realm of particles and symmetries, the scaling symmetry methods developed by Kenneth Wilson, the discovery that the otherwise unemployed Painlevé transcendents (descendants of the sines and cosines) play a natural role in some physical problems (by Wu, McCoy, Tracy, and Barouch),…—all demanded and fruitfully received further mathematical development and rigorization. The Standard Model of the elementary particles, and more generally, quantum field theories, have not only borrowed from the Library of Mathematics. They have, as well, presented problems to mathematicians that are proving fruitful and interesting within mathematics itself.

Two further examples, we discuss in later chapters: Charles Fefferman and collaborators' (pure mathematicians, all) derivation of the ground state energy of atoms in terms fractional powers of their number of electrons (Z^n), not only makes rigorous earlier derivations by physicists and extends those derivations to higher powers of Z, it develops as well innovative methods of classical analysis. Here a distinguished mathematician takes on the terrific rough-and-ready work of physicists, makes it rigorous, extends that work, and advances the mathematics itself. Second, in derivations of the properties of the Ising model of a phase transition, and related models, Rodney Baxter and collaborators (all physicists) have developed many ingenious methods of solution that demand deeper work by mathematicians (concerning analyticity and integrability) to make sense of their meaning and why they work.

Some mathematics is developed without any influence from the physicists, perhaps most of it. But the demands of the geneticists, physiologists, engineers, computer scientists, and others in the natural sciences, also provide impetus. The usual line derived from G. H. Hardy is that some of pure mathematics will never have connection to any application in the natural sciences is often denied by actual applications. But I am quite willing to believe that lots of mathematics, even if it is eventually applied, is autonomously developed by the mathematicians.

Also, I like to think that the physicists and other natural scientists, and social scientists as well, provide the mathematicians with rough-and-ready methods and theories, begging for rigorization and generalization. With luck, that effort launches interesting mathematics that is above and much beyond the original impetus. That may well eventually serve the physicist's or other scientists' needs, or not.

None of this is surprising when we think of other human activities. Some material or method is discovered by prospectors or inventors. Eventually, that material or method is used for practical manufacture of things we find useful and

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convenient. In effect that material or method becomes what economists call a resource. And scarcity and invention may then suggest the need for other materials or methods that until then did not seem of value.

I remain agnostic about the miracles referred to by Wigner. Rather, historical study will suggest how varied are the influences between mathematics and physics: how often mathematics is not of use to scientists, how often scientists search for mathematics and make do with what there is, and how often physicists' rough-and-ready inventions do not lead to any rich mathematical theories, where "how often" should be read as "how often or not."

I am also struck by the effectiveness of mathematics in the mathematical realm, developments in algebra proving useful in topology, for example. Again, some mathematics does not prove useful to other subfields.

IV

Wigner begins his article with the following:

> There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is pi." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

Now the population has nothing to do with the circumference of the circle. Rather, it is a statistical measure of the population that leads to the use of pi. As we shall see, again and again, when we try to characterize objects in terms of abstractions, we find pi-like and other mathematical objects.

Now let us move forward fifty years.

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Two friends from college meet after being out of touch for twenty years. Bernie Dedelands has become a mathematician doing number theory and representation theory. Larry Yangster has become a physicist doing statistical mechanics. When Bernie asks Larry to tell him about his current work, Larry mentions that he is working on a statistical counting problem that has three main methods of calculating its partition function, the physicists' way of packaging counting information about a system. The natural logarithm of the partition function is proportional to the thermodynamic free energy that the chemists so value. Bernie remarks that it is amazing that the packaging function could be so related to something a chemist would be interested in. Then, Larry tells Bernie how in order to solve the problem one might create a matrix that does the counting work, or one might look at the symmetries of the system, or one might focus on the analyticity of the solution of a functional equation and a Riemann surface that goes with an elliptic curve. In fact, although the proofs are not always there, Larry mentions that he has an intuition for when one might get away with the assumption of analyticity, and his work using functional equations to solve the problem are well-known among the physicists. It's how he got tenure.

Larry also mentions it would appear that the matrices might be seen as group representations, their trace or group character being the partition function and the group itself is often parametrized by the argument *u* of elliptic functions, sn(*u,k*), that group members commute if they have the same modulus, *k*, even if they have different arguments, and that there are connections between modulus *k* and modulus 1/*k* groups members' matrices (in effect low and high temperatures). And there is an equation that gives an account of the commutativity of the matrices, and it is intimately related to those elliptic functions.

(The elliptic functions are special in that $f(k)$ and say $f(1/k)$ are more generally related. Elliptic functions are said to be "modular," in the sense that, $f((az+b)/(cz+d)) = (cz + d)^m f(z)$, if ad-bc=1, so that $f(-1/z) = z^m f(z)$, and $f(z+1) = f(z)$ or perhaps $= q \times f(z)$.)

Larry goes on to mention that he is able to compute the partition function through counting or through modularity, whether it be through the relevant counting matrices or the group representations of the symmetries of the matrices or the functional equation.

Larry then mentions that there is something called universality, so that if two systems have similar symmetries, the crucial features of the partition functions turn out to be the same, even if the exact details of the systems differ substantially.

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Thematic Index

See also the list of sections that head each chapter. Names are indexed when they refer to papers (Dyson-Lenard), or styles of work (Baxter), or they are iconic (Onsager). Mentions are not indexed.

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