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Elements of ∞ – Category Theory

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Preface

Mathematical objects of a certain sophistication are frequently accompanied by higher homotopical structures: Maps between them might be connected by homotopies that witness the weak commutativity of diagrams, which might then be connected by higher homotopies expressing coherence conditions among these witnesses, which might then be connected by even higher homotopies ad infinitum. The natural habitat for such mathematical objects is not an ordinary 1-category but instead an ∞-*category* or, more precisely, an $(\infty, 1)$-*category*, with the index "1" referring to the fact that the morphisms above the lowest dimension – the homotopies just discussed – are weakly invertible.

Here the homotopies defining the higher morphisms of an ∞-category are to be regarded as *data* rather than as mere witnesses to an equivalence relation borne by the 1-dimensional morphisms. This shift in perspective is illustrated by the relationship between two algebraic invariants of a topological space: the fundamental groupoid, an ordinary 1-category, and the fundamental ∞-groupoid, an ∞-category in which all of the morphisms are weakly invertible. The objects in both cases are the points of the ambient topological space, but in the former, the 1-morphisms are homotopy classes of paths, while in the latter, the 1-morphisms are the paths themselves and the 2-morphisms are explicit endpoint-preserving homotopies. To encompass examples such as these, all of the categorical structures in an ∞-category are weak. Even at the base level of 1-morphisms, composition is not necessarily uniquely defined but is instead witnessed by a 2-morphism and associative up to a 3-morphism whose boundary data involves specified 2-morphism witnesses. Thus, diagrams valued in an ∞-category cannot be said to *commute* on the nose but are instead interpreted as *homotopy coherent*, with explicitly specified higher data.

A fundamental challenge in defining ∞-categories has to do with giving a precise mathematical meaning of this notion of a weak composition law, not just for the 1-morphisms but also for the morphisms in higher dimensions. Indeed, there is a sense in which our traditional set-based foundations for mathematics are not really suitable for reasoning about ∞-categories: Sets do not feature prominently in ∞-categorical data, especially when ∞-categories are considered in a morally correct fashion as objects that are only well-defined up to equivalence. When considered up to equivalence, ∞-categories, like ordinary categories, do not have a well-defined set of objects. In addition, the morphisms between a fixed pair of objects in an ∞-category assemble into an ∞-groupoid, which describes a well-defined homotopy type, though not a well-defined space.[1]

Precision is achieved through a variety of *models* of $(\infty, 1)$-categories, which are Bourbaki-style mathematical structures that represent infinite-dimensional categories with a weak composition law in which all morphisms above dimension 1 are weakly invertible. In order of appearance, these include *simplicial categories*, *quasi-categories* (née *weak Kan complexes*), *relative categories*, *Segal categories*, *complete Segal spaces*, and *1-complicial sets* (née *saturated 1-trivial weak complicial sets*), each of which comes with an associated array of naturally occurring examples. The proliferation of models of $(\infty, 1)$-categories begs the question of how they might be compared. In the first decades of the twenty-first century, Julia Bergner, André Joyal and Myles Tierney, Dominic Verity, Jacob Lurie, and Clark Barwick and Daniel Kan built various bridges that prove that each of the models listed above "has the same homotopy theory" in the sense of defining the fibrant objects in Quillen equivalent model categories.[2]

In parallel with the development of models of $(\infty, 1)$-categories and the construction of comparisons between them, Joyal pioneered and Lurie and many others expanded a wildly successful project to extend basic category theory from ordinary 1-categories to $(\infty, 1)$-categories modeled as quasi-categories in such a way that the new quasi-categorical notions restrict along the standard embedding $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ to the classical 1-categorical concepts. A natural question is then, does this work extend to other models of $(\infty, 1)$-categories? And to what extent are basic ∞-categorical notions invariant under change of model? For instance, $(\infty, 1)$-categories of manifolds are most naturally constructed as complete Segal spaces, so Kazhdan–Varshavsky [65], Boavida de Brito [34], and Rasekh [95, 96, 98] have recently endeavored to redevelop some of the category theory of quasi-categories using complete Segal spaces instead in order to have direct access to constructions and definitions that had previously been introduced only in the quasi-categorical model.

For practical, aesthetic, and moral reasons, the ultimate desire of practitioners

1 Grothendieck's homotopy hypothesis posits that ∞-groupoids up to equivalence correspond to homotopy types.

2 A recent book by Bergner surveys all but the last of these models and their interrelationships [15]. For a more whirlwind tour, see [3].

essential for the development of ∞-category theory. Similar proofs carry through to a weaker setting, at the cost of more time spent considering coherence of higher cells.

In Part I, we define and develop the notions of equivalence and adjunction between ∞-categories, limits and colimits in ∞-categories, and cartesian and cocartesian fibrations and their discrete variants, for which we prove a version of the Yoneda lemma. The majority of these results are developed from the comfort of the homotopy 2-category. In an interlude, we digress into abstract ∞-cosmology to give a more careful account of the full class of limit constructions present in any ∞-cosmos. This analysis is used to develop further examples of ∞-cosmoi, whose objects are pointed ∞-categories, or stable ∞-categories, or cartesian or cocartesian fibrations in a given ∞-cosmos.[8]

What is missing from this basic account of the category theory of ∞-categories is a satisfactory treatment of the "hom" bifunctor associated to an ∞-category, which is the prototypical example of what we call a *module*. An instructive exercise for a neophyte is the challenge of defining the ∞-groupoid-valued hom bifunctor in a preferred model. What is edifying is to learn that this construction, so fundamental to ordinary category theory, is prohibitively difficult.[9] In our axiomatization, any ∞-category in an ∞-cosmos has an associated ∞-category of arrows, equipped with domain and codomain projection functors that respectively define cartesian and cocartesian fibrations in a compatible manner. Such modules, which themselves assemble into an ∞-cosmos, provide a convenient vehicle for encoding universal properties as fibered equivalences. In Part II, we develop the calculus of modules between ∞-categories and apply this to define and study pointwise Kan extensions. This will give us an opportunity to repackage universal properties proven in Part I as part of the "formal category theory" of ∞-categories.

This work is all "model-agnostic" in the sense of being blind to details about the specifications of any particular ∞-cosmos. In Part III we prove that the category theory of ∞-categories is also "model independent" in a precise sense: all categorical notions are preserved, reflected, and created by any "change-of-model" functor that defines what we call a *cosmological biequivalence*. This model independence theorem is stronger than our axiomatic framework might initially suggest in that it also allows us to transfer theorems proven using analytic techniques to all biequivalent ∞-cosmoi. For instance, the four ∞-

8 The impatient reader could skip this interlude and take on faith that any ∞-cosmos begets various other ∞ without compromising their understanding of what follows – though they would miss out on some fun.

9 Experts in quasi-category theory know to use Lurie's straightening–unstraightening construction [78, 2.2.1.2] or Cisinski's universal left fibration [28, 5.2.8] and the twisted arrow quasi-category.

cosmoi whose objects model $(\infty, 1)$-categories are all biequivalent.[10] It follows that the analytically-proven theorems about quasi-categories from [78] hold for complete Segal spaces, and vice versa. We conclude with several applications of this transfer principle. For instance, in the ∞-cosmoi whose objects are $(\infty, 1)$-categories, we demonstrate that various universal properties are "pointwise-determined" by first proving these results for quasi-categories using analytical techniques and then appealing to model independence to extend these results to biequivalent ∞-cosmoi.

The question of the model invariance of statements about ∞-categories is more subtle than one might expect. When passing an ∞-category from one model to another and then back, the resulting object is typically equivalent but not identical to the original, and certain "evil" properties of ∞-categories fail to be invariant under equivalence: the assertion that an ∞-category has a single object is a famous example. A key advantage to our systematic approach to understanding the model independence of ∞-category theory is that it allows us to introduce a formal language and prove that statement about ∞-categories expressible in that language are model independent. This builds on work of Makkai that resolves a similar question about the invariance of properties of a 2-category under biequivalence [82].

Regrettably, space considerations have prevented us from exploring the homotopy coherent structures present in an ∞-cosmos. For instance, a companion paper [109] proves that any adjunction between ∞-categories in an ∞-cosmos extends homotopically uniquely to a homotopy coherent adjunction and presents a monadicity theorem for homotopy coherent monads as a mechanism for ∞-categorical universal algebra. The formal theory of homotopy coherent monads is extended further by Sulyma [124] who develops the corresponding theory of monadic and comonadic descent and Zaganidis [133] who defines and studies homotopy coherent monad maps. Another casualty of space limitations is an exploration of a "macrocosm principle" for cartesian fibrations, which proves that the codomain projection functor from the ∞-cosmos of cartesian fibrations to the base ∞-cosmos defines a "cartesian fibration of ∞-cosmoi" in a suitable sense [111]. We hope to return to these topics in a sequel.

The ideal reader might already have some acquaintance with enriched category theory, 2-category theory, and abstract homotopy theory so that the constructions and proofs with antecedents in these traditions will be familiar. Because ∞-categories are of interest to mathematicians with a wide variety of backgrounds,

[10] A closely related observation is that the Quillen equivalences between quasi-categories, complete Segal spaces, and Segal categories constructed by Joyal and Tierney in [64] can be understood as equivalences of $(\infty, 2)$-categories not just of $(\infty, 1)$-categories by making judicious choices of simplicial enrichments (see §E.2).

we review all of the material we need on each of these topics in Appendices A, B, and C, respectively. Some basic facts about quasi-categories first proven by Joyal are needed to establish the corresponding features of general ∞-cosmoi in Chapter 1. We state these results in §1.1 but defer the proofs that require lengthy combinatorial digressions to Appendix D, where we also introduce *n-complicial sets*, a model of (∞, n)-categories for any $0 \leq n \leq \infty$. The examples of ∞-cosmoi that appear "in the wild" can be found in Appendix E, where we also present general techniques that the reader might use to find ∞-cosmoi of their own. The final appendix addresses a crucial bit of unfinished business. Importantly, the synthetic theory developed in the ∞-cosmos of quasi-categories is fully compatible with the analytic theory developed by Joyal, Lurie, and many others. This is the subject of Appendix F.

We close with the obligatory disclaimer on sizes. To apply the theory developed here to the ∞-categories of greatest interest, one should consider three infinite inaccessible cardinals $\alpha < \beta < \gamma$, as is the common convention [5, 2]. Colloquially, α-small categories might be called "small," while β-small categories are the default size for ∞-categories. For example, the ∞-categories of (small) spaces, chain complexes of (small) abelian groups, or (small) homotopy coherent diagrams are all β-small. These normal-sized ∞-categories are then the objects of an ∞-cosmos that is γ-small – "large" in colloquial terms. Of course, if one is only interested in small simplicial sets, then the ∞-cosmos of small quasi-categories is β-small, rather than γ-small, and the theory developed here equally applies. For this reason, we set aside the Grothendieck universes and do not refer to these inaccessible cardinals elsewhere.

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It is difficult and time-consuming to learn a new language. The standard advice to "fake it til you make it" is disconcerting in mathematical contexts, where the validity of a proof hinges upon the correctness of the statements it cites. The aim in Part I of this text is to develop a substantial portion of the theory of ∞-categories from first principles, as rapidly and painlessly as possible – at least assuming that the reader finds classical abstract nonsense to be relatively innocuous.[11]

The axiomatic framework that justifies this is introduced in Chapter 1, but the impatient or particularly time-constrained reader might consider starting directly in Chapter 2 with the study of adjunctions, limits, and colimits. In adopting this approach, one must take for granted that there is a well-defined 2-category of ∞-categories, ∞-functors between them, and ∞-natural transformations between these. This 2-category is constructed in Chapter 1, where we see that any ∞-cosmos has a homotopy 2-category and that the familiar models of $(\infty, 1)$-categories define biequivalent ∞-cosmoi, with biequivalent homotopy 2-categories. To follow the proofs in Chapter 2, it is necessary to understand the general composition of natural transformations by *pasting diagrams*. This and other concepts from 2-category theory are reviewed in Appendix B, which should be consulted as needed.

The payoff for acquainting oneself with some standard 2-category theory is that numerous fundamental results concerning equivalences and adjunctions and limits and colimits can be proven quite expeditiously. We prove one such theorem, that right adjoint functors between ∞-categories preserve any limits found in those ∞-categories, via a formal argument that is arguably even simpler than the classical one.

The definitions of adjunctions, limits, and colimits given in Chapter 2 are optimized for ease of use in the homotopy 2-category of ∞-categories, ∞-functors, and ∞-natural transformations in an ∞-cosmos, but especially in the latter cases, these notions are not expressed in their most familiar forms. To encode a limit of a diagram valued in an ∞-category as a terminal cone, we introduce the powerful and versatile construction of the *comma ∞-category* built from a cospan of functors in Chapter 3. We then prove various "representability theorems" that characterize those comma ∞-categories that are equivalent to ones defined by a single functor. These general results specialize in Chapter 4 to the expected equivalent definitions of adjunctions, limits, and colimits. This theory is then applied to study limits and colimits of particular diagram

[11] Dan Freed defines the category number of a mathematician to be the largest integer n so that they may ponder n-categories for half an hour without developing a migraine. Here we require a category number of 2, which we note is much smaller than ∞!

shapes, which in turn is deployed to establish an equivalence between various presentations of the important notion of a *stable* ∞*-category*.

The basic theory of ∞-categories is extended in Chapter 5 to encompass *cocartesian* and *cartesian fibrations*, which can be understood as indexed families of ∞-categories acted upon covariantly or contravariantly by arrows in the base ∞-category. After developing the theory of the various classes of categorical fibrations, we conclude by proving a fibrational form of the Yoneda lemma that will be used to further develop the formal category theory of ∞-categories in Part II.

1

∞-Cosmoi and Their Homotopy 2-Categories

In this chapter, we introduce a framework to develop the formal category theory of ∞-categories, which goes by the name of an ∞-*cosmos*. Informally, an ∞-cosmos is an $(\infty, 2)$-category – a category enriched over $(\infty, 1)$-categories – that is equipped with $(\infty, 2)$-categorical limits. In the motivating examples of ∞-cosmoi, the objects are ∞-categories in some model. To focus this abstract theory on its intended interpretation, we recast "∞-category" as a technical term, reserved to mean an object of some ∞-cosmos.

Unexpectedly, the motivating examples permit us to use a quite strict interpretation of "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits": an ∞-cosmos is a particular type of simplicially enriched category and the $(\infty, 2)$-categorical limits are modeled by simplicially enriched limits. More precisely, an ∞-cosmos is a category enriched over quasi-categories, these being one of the models of $(\infty, 1)$-categories defined as certain simplicial sets. The $(\infty, 2)$-categorical limits are defined as limits of diagrams involving specified maps called *isofibrations*, which have no intrinsic homotopical meaning – since any functor between ∞-categories is equivalent to an isofibration – but allow us to consider strictly commuting diagrams.

In §1.1, we introduce quasi-categories, reviewing the classical results that are needed to show that quasi-categories themselves assemble into an ∞-cosmos – the prototypical example. General ∞-cosmoi are defined in §1.2, where several examples are given and their basic properties are established. In §1.3, we turn our attention to *cosmological functors* between ∞-cosmoi, which preserve all of the defining structure. Cosmological functors serve dual purposes, on the one hand providing technical simplifications in many proofs, and then later on serving as the "change of model" functors that establish the model independence of ∞-category theory.

Finally, in §1.4, we introduce a strict 2-category whose objects are ∞-categories, whose 1-cells are the ∞-functors between them, and whose 2-cells define

∞-natural transformations between these. Any ∞-cosmos has a 2-category of this sort, which we refer to as the *homotopy 2-category* of the ∞-cosmos. In fact, the reader who is eager to get on to the development of the category theory of ∞-categories can skip this chapter on first reading, taking the existence of the homotopy 2-category for granted, and start with Chapter 2.

1.1 Quasi-Categories

Before introducing an axiomatic framework that allows us to develop ∞-category theory in general, we first consider one model in particular: *quasi-categories*, which were introduced in 1973 by Boardman and Vogt [21] in their study of homotopy coherent diagrams. Ordinary 1-categories give examples of quasi-categories via the construction of Definition 1.1.4. Joyal first undertook the task of extending 1-category theory to quasi-category theory in [61] and [63] and in several unpublished draft book manuscripts. The majority of the results in this section are due to him.

Notation 1.1.1 (the simplex category). Let $\mathbf{\Delta}$ denote the **simplex category** of finite nonempty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps. These include in particular the

$$\text{elementary face operators} \qquad [n-1] \overset{\delta^i}{\rightarrowtail} [n] \qquad 0 \leq i \leq n$$

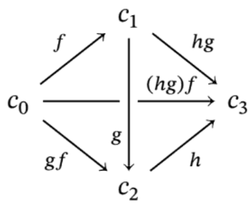
$$\text{elementary degeneracy operators} \qquad [n+1] \overset{\sigma^i}{\twoheadrightarrow} [n] \qquad 0 \leq i \leq n$$

whose images, respectively, omit and double up on the element $i \in [n]$. Every morphism in $\mathbf{\Delta}$ factors uniquely as an epimorphism followed by a monomorphism; these epimorphisms, the **degeneracy operators**, decompose as composites of elementary degeneracy operators, while the monomorphisms, the **face operators**, decompose as composites of elementary face operators.

The category of **simplicial sets** is the category $s\mathcal{S}et \coloneqq \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ of presheaves on the simplex category. We write $\Delta[n]$ for the **standard n-simplex** the simplicial set represented by $[n] \in \mathbf{\Delta}$, and $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$ for its k-**horn** and **boundary sphere**, respectively. The sphere $\partial\Delta[n]$ is the simplicial subset generated by the codimension-one faces of the n-simplex, while the horn $\Lambda^k[n]$ is the further simplicial subset that omits the face opposite the vertex k.

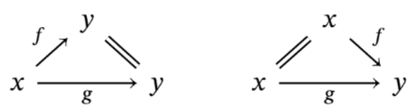
Given a simplicial set X, it is conventional to write X_n for the set of n-**simplices**, defined by evaluating at $[n] \in \mathbf{\Delta}$. By the Yoneda lemma, each n-simplex $x \in X_n$ corresponds to a map of simplicial sets $x \colon \Delta[n] \to X$. Accordingly, we write $x \cdot \delta^i$ for the ith face of the n-simplex, an $(n-1)$-simplex classified by to a 2-simplex exists precisely because any composable pair of arrows admits a (unique) composite.

An inner horn $\Lambda^1[3] \to C$ specifies the data of three composable arrows in C, as displayed in the following diagram, together with the composites gf, hg, and $(hg)f$.



Because composition is associative, the arrow $(hg)f$ is also the composite of gf followed by h, which proves that the 2-simplex opposite the vertex c_1 is present in C; by 2-coskeletality, the 3-simplex filling this boundary sphere is also present in C. The filler for a horn $\Lambda^2[3] \to C$ is constructed similarly. □

Definition 1.1.7 (homotopy relation on 1-simplices). A parallel pair of 1-simplices f, g in a simplicial set X are **homotopic** if there exists a 2-simplex whose boundary takes either of the following forms[2]



(1.1.8)

or if f and g are in the same equivalence class generated by this relation.

In a quasi-category, the relation witnessed by either of the types of 2-simplex on display in (1.1.8) is an equivalence relation and these equivalence relations coincide.

Lemma 1.1.9 (homotopic 1-simplices in a quasi-category). *Parallel 1-simplices f and g in a quasi-category are homotopic if and only if there exists a 2-simplex of any or equivalently all of the forms displayed in* (1.1.8).

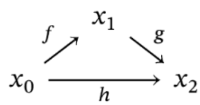
Proof Exercise 1.1.ii. □

Definition 1.1.10 (the homotopy category [44, §2.4]). By 1-truncating, any simplicial set X has an underlying reflexive directed graph with the 0-simplices of X defining the objects and the 1-simplices defining the arrows:

$$X_1 \;\overset{\cdot\delta^1}{\underset{\cdot\delta^0}{\underset{\longleftarrow\, \cdot\sigma^0 \,-}{\rightrightarrows}}}\; X_0,$$

[2] The symbol “ $=$ ” is used in diagrams to denote a degenerate simplex or an identity arrow.

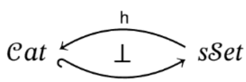
By convention, the source of an arrow $f \in X_1$ is its 0th face $f \cdot \delta^1$ (the face opposite 1) while the target is its 1st face $f \cdot \delta^0$ (the face opposite 0). The **free category** on this reflexive directed graph has X_0 as its object set, degenerate 1-simplices serving as identity morphisms, and nonidentity morphisms defined to be finite directed paths of nondegenerate 1-simplices. The **homotopy category** $\mathsf{h}X$ of X is the quotient of the free category on its underlying reflexive directed graph by the congruence[3] generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices



This relation implies in particular that homotopic 1-simplices represent the same arrow in the homotopy category.

The homotopy category of the nerve of a 1-category is isomorphic to the original category, as the 2-simplices in the nerve witness all of the composition relations satisfied by the arrows in the underlying reflexive directed graph. Indeed, the natural isomorphism $\mathsf{h}C \cong C$ forms the counit of an adjunction, embedding $\mathcal{C}at$ as a reflective subcategory of $s\mathcal{S}et$.

Proposition 1.1.11. *The nerve embedding admits a left adjoint, namely the functor which sends a simplicial set to its homotopy category:*



The adjunction of Proposition 1.1.11 exists for formal reasons (see Exercise 1.1.i), but nevertheless, a direct proof can be enlightening.

Proof For any simplicial set X, there is a natural map from X to the nerve of its homotopy category $\mathsf{h}X$; since nerves are 2-coskeletal, it suffices to define the map $\mathrm{sk}_2 X \to \mathsf{h}X$, and this is given immediately by the construction of Definition 1.1.10. Note that the quotient map $X \to \mathsf{h}X$ becomes an isomorphism upon applying the homotopy category functor and is already an isomorphism whenever X is the nerve of a category. Thus the adjointness follows from Lemma B.4.2 or by direct verification of the triangle equalities. □

The homotopy category of a quasi-category admits a simplified description.

Lemma 1.1.12 (the homotopy category of a quasi-category). *If A is a quasi-category then its* ***homotopy category*** *$\mathsf{h}A$ has*

[3] A binary relation $\sim$ on parallel arrows of a 1-category is a **congruence** if it is an equivalence relation that is closed under pre- and post-composition: if $f \sim g$ then $hfk \sim hgk$.

- *the set of 0-simplices A_0 as its objects*
- *the set of homotopy classes of 1-simplices A_1 as its arrows*
- *the identity arrow at $a \in A_0$ represented by the degenerate 1-simplex $a \cdot \sigma^0 \in A_1$*
- *a composition relation $h = g \circ f$ in $\mathsf{h}A$ between the homotopy classes of arrows represented by any given 1-simplices $f, g, h \in A_1$ if and only if there exists a 2-simplex with boundary*

$$\begin{array}{ccccc} & & a_1 & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ a_0 & & \xrightarrow[h]{} & & a_2 \end{array}$$

Proof Exercise 1.1.iii. □

Definition 1.1.13 (isomorphism in a quasi-category). A 1-simplex in a quasi-category is an **isomorphism**[4] just when it represents an isomorphism in the homotopy category. By Lemma 1.1.12 this means that $f : a \to b$ is an isomorphism if and only if there exists a 1-simplex $f^{-1} : b \to a$ together with a pair of 2-simplices

$$\begin{array}{ccccc} & & b & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle f^{-1}} & \\ a & & = & & a \end{array} \qquad \begin{array}{ccccc} & & a & & \\ & {\scriptstyle f^{-1}}\nearrow & & \searrow{\scriptstyle f} & \\ b & & = & & b \end{array}$$

The properties of the isomorphisms in a quasi-category are most easily proved by arguing in a closely related category where simplicial sets have the additional structure of a "marking" on a specified subset of the 1-simplices; maps of these so-called *marked simplicial sets* must then preserve the markings (see Definition D.1.1). For instance, each quasi-category has a *natural marking*, where the marked 1-simplices are exactly the isomorphisms (see Definition D.4.5). Since the property of being an isomorphism in a quasi-category is witnessed by the presence of 2-simplices with a particular boundary, every map between quasi-categories preserves isomorphisms, inducing a map of the corresponding naturally marked quasi-categories. Because marked simplicial sets seldom appear outside of the proofs of certain combinatorial lemmas about the isomorphisms in quasi-categories, we save the details for Appendix D.

Let us now motivate the first of several results proven using marked techniques. A quasi-category A is defined to have extensions along all *inner horns*. But when the initial or final edges, respectively, of an outer horn $\Lambda^0[2] \to A$ or

[4] Joyal refers to these maps as "isomorphisms" while Lurie refers to them as "equivalences." We prefer, wherever possible, to use the same term for ∞-categorical concepts as for the analogous 1-categorical ones.

$\Lambda^2[2] \to A$ map to isomorphisms in A, then a filler

$$\begin{array}{ccccc} & & a_1 & & \\ & \overset{f}{\underset{\sim}{\nearrow}} & & \overset{hf^{-1}}{\searrow} & \\ a_0 & & \xrightarrow[h]{} & & a_2 \end{array} \qquad \begin{array}{ccccc} & & a_1 & & \\ & \overset{g^{-1}h}{\nearrow} & & \overset{g}{\underset{\sim}{\searrow}} & \\ a_0 & & \xrightarrow[h]{} & & a_2 \end{array}$$

should intuitively exist. The higher-dimensional "special outer horns" behave similarly:

Proposition 1.1.14 (special outer horn filling). *Any quasi-category A admits fillers for those outer horns*

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{g} & A \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array} \qquad \begin{array}{ccc} \Lambda^n[n] & \xrightarrow{h} & A \\ \big\downarrow & \nearrow & \\ \Delta[n] & & \end{array} \qquad \text{for } n \geq 1$$

in which the edges $g|_{\{0,1\}}$ *and* $h|_{\{n-1,n\}}$ *are isomorphisms.*[5]

The proof of Proposition 1.1.14 requires clever combinatorics, due to Joyal, and is deferred to Proposition D.4.6. Here, we enjoy its myriad consequences. Immediately:

Corollary 1.1.15. *A quasi-category is a Kan complex if and only if its homotopy category is a groupoid.*

Proof If the homotopy category of a quasi-category is a groupoid, then all of its 1-simplices are isomorphisms, and Proposition 1.1.14 then implies that all inner and outer horns have fillers. Thus, the quasi-category is a Kan complex. Conversely, in a Kan complex, all outer horns can be filled and in particular fillers for the horns displayed in Definition 1.1.13 can be used to construct left and right inverses for any 1-simplex, which can be rectified to a single two-sided inverse by Lemma 1.1.12. □

A quasi-category contains A a canonical **maximal sub Kan complex** $A^{\simeq}$, the simplicial subset spanned by those 1-simplices that are isomorphisms. Just as the arrows in a quasi-category A are represented by simplicial maps $\mathbb{2} \to A$ whose domain is the nerve of the free-living arrow, the isomorphisms in a quasi-category can be represented by diagrams $\mathbb{I} \to A$ whose domain, called the **homotopy coherent isomorphism**, is the nerve of the free-living isomorphism:

[5] In the case $n = 1$, no condition is needed on the horns; degenerate 1-simplices define the required lifts.

Corollary 1.1.16. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a homotopy coherent isomorphism*

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & A \\ \big\downarrow & \nearrow & \\ \mathbb{I} & & \end{array}$$

Proof If f is an isomorphism, the map $f\colon \mathbb{2} \to A$ lands in the maximal sub Kan complex contained in A:

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & A^{\simeq} \subset A \\ \big\downarrow & \nearrow & \\ \mathbb{I} & & \end{array}$$

By Exercise 1.1.v, the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ can be expressed as a sequential composite of pushouts of outer horn inclusions. Since $A^{\simeq}$ is a Kan complex, this shows that the required extension exists and in fact lands in $A^{\simeq} \subset A$. □

The category of simplicial sets, like any category of presheaves, is cartesian closed. By the Yoneda lemma and the defining adjunction, an n-simplex in the exponential Y^X corresponds to a simplicial map $X \times \Delta[n] \to Y$, and its faces and degeneracies are computed by precomposing in the simplex variable. Our next aim is to show that the quasi-categories define an exponential ideal in the simplicially enriched category of simplicial sets: if X is a simplicial set and A is a quasi-category, then A^X is a quasi-category. We deduce this as a corollary of the "relative" version of this result involving certain maps called isofibrations that we now introduce.

Definition 1.1.17 (isofibration). A simplicial map $f\colon A \to B$ between quasi-categories is an **isofibration** if it lifts against the inner horn inclusions, as displayed below-left, and also against the inclusion of either vertex into the free-living isomorphism $\mathbb{I}$.

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & A \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta[n] & \longrightarrow & B \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{1} & \longrightarrow & A \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathbb{I} & \longrightarrow & B \end{array}$$

To notationally distinguish the isofibrations, we depict them as arrows "$\twoheadrightarrow$" with two heads.

Proposition 1.1.14 is subsumed by its relative analogue, proven as Theorem D.5.1:

encoded by the composite[8] functors

$$\mathsf{h}A \xrightarrow{\mathsf{h}\alpha} \mathsf{h}(A^{\mathbb{I}}) \longrightarrow (\mathsf{h}A)^{\mathbb{I}} \qquad \mathsf{h}B \xrightarrow{\mathsf{h}\beta} \mathsf{h}(B^{\mathbb{I}}) \longrightarrow (\mathsf{h}B)^{\mathbb{I}}$$

Definition 1.1.25. A map $f\colon X \to Y$ between simplicial sets is a **trivial fibration** if it admits lifts against the boundary inclusions for all simplices

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow{\wr}\, f \\ \Delta[n] & \longrightarrow & Y \end{array} \qquad \text{for } n \geq 0 \tag{1.1.26}$$

We write "$\xrightarrow{\sim}\!\!\!\!\rightarrow$" to decorate trivial fibrations.

Remark 1.1.27. The simplex boundary inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ "cellularly generate" the monomorphisms of simplicial sets (see Definition C.2.4 and Lemma C.5.9). Hence the dual of Lemma C.2.3 implies that trivial fibrations lift against any monomorphism between simplicial sets. In particular, it follows that any trivial fibration $X \xrightarrow{\sim}\!\!\!\!\rightarrow Y$ is a split epimorphism.

The notation "$\xrightarrow{\sim}\!\!\!\!\rightarrow$" is suggestive: the trivial fibrations between quasi-categories are exactly those maps that are both isofibrations and equivalences. This can be proven by a relatively standard although rather technical argument in simplicial homotopy theory, appearing as Proposition D.5.6.

Proposition 1.1.28. *For a map $f\colon A \to B$ between quasi-categories the following are equivalent:*

(i) *f is a trivial fibration*
(ii) *f is both an isofibration and an equivalence*
(iii) *f is a* ***split fiber homotopy equivalence****: an isofibration admitting a section s that is also an equivalence inverse via a homotopy α from id_A to sf that composes with f to the constant homotopy from f to f.*

$$\begin{array}{ccccc} A + A & & \xrightarrow{(\mathrm{id}_A, sf)} & & A \\ \big\downarrow & & \overset{\alpha}{\nearrow} & & \big\downarrow{\wr}\, f \\ A \times \mathbb{I} & \xrightarrow[\pi]{} & A & \underset{\sim}{\overset{f}{\twoheadrightarrow}} & B \end{array}$$

As a class characterized by a right lifting property, the trivial fibrations are also closed under composition, product, pullback, limits of towers, and contain

[8] Note that $\mathsf{h}(A^{\mathbb{I}}) \not\cong (\mathsf{h}A)^{\mathbb{I}}$ in general. Objects in the latter are homotopy classes of isomorphisms in A, while objects in the former are homotopy coherent isomorphisms, given by a specified 1-simplex in A, a specified inverse 1-simplex, together with an infinite tower of coherence data indexed by the nondegenerate simplices in $\mathbb{I}$.

the isomorphisms. The stability of these maps under Leibniz exponentiation is proven along with Proposition 1.1.20 in Proposition D.5.2.

Proposition 1.1.29. *If $i : X \to Y$ is a monomorphism and $f : A \to B$ is an isofibration, then if either f is a trivial fibration or if i is in the class cellularly generated by the inner horn inclusions and the map $\mathbb{1} \hookrightarrow \mathbb{I}$ then the induced Leibniz exponential map*

$$A^Y \xrightarrow{\widehat{i \pitchfork f}} B^Y \times_{B^X} A^X$$

a trivial fibration.

To illustrate the utility of these Leibniz stability results, we give an "internal" or "synthetic" characterization of the Kan complexes.

Lemma 1.1.30. *A quasi-category A is a Kan complex if and only if the map $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ induced by the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is a trivial fibration.*

Note that Proposition 1.1.20 implies that $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is an isofibration.

Proof The lifting property that characterizes trivial fibrations transposes to another lifting property, displayed below-right

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & A^{\mathbb{I}} \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta[n] & \longrightarrow & A^{\mathbb{2}} \end{array} \qquad \leftrightsquigarrow \qquad \begin{array}{ccc} \partial\Delta[n] \times \mathbb{I} \underset{\partial\Delta[n]\times\mathbb{2}}{\cup} \Delta[n] \times \mathbb{2} & \longrightarrow & A \\ \big\downarrow & \nearrow & \\ \Delta[n] \times \mathbb{I} & & \end{array}$$

that asserts that A admits extensions along maps formed by taking the *Leibniz product* – also known as the pushout product – of a simplex boundary inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$ with the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$. By Exercise 1.1.v(ii) the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ is a sequential composite of pushouts of left outer horn inclusions. By Corollary D.3.11, a key step along the way to the proofs of Propositions 1.1.20 and 1.1.29, it follows that the Leibniz product is also a sequential composite of pushouts of left and inner horn inclusions. If A is a Kan complex, then the extensions displayed above right exist, and, by transposing, the map $A^{\mathbb{I}} \twoheadrightarrow A^{\mathbb{2}}$ is a trivial fibration.

Conversely, if $A^{\mathbb{I}} \overset{\sim}{\twoheadrightarrow} A^{\mathbb{2}}$ is a trivial fibration then in particular it is surjective on vertices. Thus every arrow in A is an isomorphism, and Corollary 1.1.15 tells us that A must be a Kan complex. □

Digression 1.1.31 (the Joyal model structure). The category of simplicial sets bears a Quillen model structure, in the sense of Definition C.3.1, whose fibrant objects are exactly the quasi-categories and in which all objects are cofibrant.

Between fibrant objects, the fibrations, weak equivalences, and trivial fibrations are precisely the isofibrations, equivalences, and trivial fibrations just introduced. Proposition 1.1.28 proves that the trivial fibrations are the intersection of the fibrations and the weak equivalences. Propositions 1.1.20 and 1.1.29 reflect the fact that the Joyal model structure is a *cartesian closed model category*, satisfying the additional axioms of Definition C.3.10.

We decline to elaborate further on the Joyal model structure for quasi-categories since we have highlighted all of the features that we need. The results enumerated here suffice to show that the category of quasi-categories defines an ∞-*cosmos*, a concept to which we now turn.

Exercises

Exercise 1.1.i ([103, §1.5]). Given any cosimplicial object $C\colon \mathbf{\Delta} \to \mathcal{E}$ valued in any category $\mathcal{E}$, there is an associated **nerve functor** N_C defined by:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{N_C} & s\mathcal{S}et \\ E & \longmapsto & \hom(C^-, E) \end{array} \qquad \begin{array}{c} \mathbf{\Delta} \\ C \swarrow \quad \searrow よ \\ \mathcal{E} \underset{N_C}{\overset{\operatorname{lan}_よ C}{\underset{\bot}{\leftrightarrows}}} s\mathcal{S}et \end{array}$$

By construction n-simplices in $N_C E$ correspond to maps $C^n \to E$ in $\mathcal{E}$. Show that if $\mathcal{E}$ is cocomplete, then N_C has a left adjoint defined as the left Kan extension of the functor C along the Yoneda embedding $よ\colon \mathbf{\Delta} \hookrightarrow s\mathcal{S}et$. This gives a second proof of Proposition 1.1.11.

Exercise 1.1.ii (Boardman–Vogt [21]). Consider the set of 1-simplices in a quasi-category with initial vertex a and final vertex b.

(i) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary $\begin{array}{ccc} & b & \\ {}^{f}\nearrow & & \searrow\!\!\!\!= \\ a & \xrightarrow[g]{} & b \end{array}$ is an equivalence relation.

(ii) Prove that the relation defined by $f \sim g$ if and only if there exists a 2-simplex with boundary $\begin{array}{ccc} & a & \\ =\!\!\!\!\nearrow & & \searrow^{f} \\ a & \xrightarrow[g]{} & b \end{array}$ is an equivalence relation.

(iii) Prove that the equivalence relations defined by (i) and (ii) are the same.

This proves Lemma 1.1.9.

Exercise 1.1.iii. Consider the free category on the reflexive directed graph

$$A_1 \underset{\cdot\delta^0}{\overset{\cdot\delta^1}{\rightrightarrows}} A_0, \qquad A_1 \xleftarrow{\ \cdot\sigma^0\ } A_0$$

underlying a quasi-category A.

(i) Consider the binary relation that identifies sequences of composable 1-simplices with common source and common target whenever there exists a simplex of A in which the sequences of 1-simplices define two paths from its initial vertex to its final vertex. Prove that this relation is stable under pre- and post-composition with 1-simplices and conclude that its transitive closure is a **congruence**: an equivalence relation that is closed under pre- and post-composition.[9]

(ii) Consider the congruence relation generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{array}{ccc} & a_1 & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ a_0 & \xrightarrow[h]{} & a_2 \end{array}$$

and prove that this coincides with the relation considered in (i).

(iii) In the congruence relations of (i) and (ii), prove that every sequence of composable 1-simplices in A is equivalent to a single 1-simplex. Conclude that every morphism in the quotient of the free category by this congruence relation is represented by a 1-simplex in A.

(iv) Prove that for any triple of 1-simplices f, g, h in A, $h = g \circ f$ in the homotopy category $\mathsf{h}A$ of Definition 1.1.10 if and only if there exists a 2-simplex with boundary

$$\begin{array}{ccc} & a_1 & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ a_0 & \xrightarrow[h]{} & a_2 \end{array}$$

This proves Lemma 1.1.12.

Exercise 1.1.iv. Show that any quasi-category in which inner horns admit unique fillers is isomorphic to the nerve of its homotopy category.

Exercise 1.1.v. Let $\mathbb{I}$ be the nerve of the free-living isomorphism.

[9] Given a congruence relation on the hom-sets of a 1-category, the quotient category can be formed by quotienting each hom-set (see [81, §II.8]).

(i) Prove that $\mathbb{I}$ contains exactly two nondegenerate simplices in each dimension.

(ii) Inductively build $\mathbb{I}$ from $\mathbb{2}$ by expressing the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ as a sequential composite of pushouts of left outer horn inclusions[10] $\Lambda^0[n] \hookrightarrow \Delta[n]$, one in each dimension starting with $n = 2$.[11]

Exercise 1.1.vi. Prove the relative version of Corollary 1.1.16: for any isofibration $p\colon A \twoheadrightarrow B$ between quasi-categories and any $f\colon \mathbb{2} \to A$ that defines an isomorphism in A any homotopy coherent isomorphism in B extending pf lifts to a homotopy coherent isomorphism in A extending f.

$$\begin{array}{ccc} \mathbb{2} & \xrightarrow{f} & A \\ \downarrow & \nearrow & \downarrow p \\ \mathbb{I} & \longrightarrow & B \end{array}$$

Exercise 1.1.vii. Specialize Proposition 1.1.20 to prove the following:

(i) If A is a quasi-category and X is a simplicial set then A^X is a quasi-category.

(ii) If A is a quasi-category and $X \hookrightarrow Y$ is a monomorphism then $A^Y \twoheadrightarrow A^X$ is an isofibration.

(iii) If $A \twoheadrightarrow B$ is an isofibration and X is a simplicial set then $A^X \twoheadrightarrow B^X$ is an isofibration.

Exercise 1.1.viii. Anticipating Lemma 1.2.17:

(i) Prove that the equivalences defined in Definition 1.1.23 are closed under retracts.

(ii) Prove that the equivalences defined in Definition 1.1.23 satisfy the 2-of-3 property.

Exercise 1.1.ix. Prove that if $f\colon X \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow Y$ is a trivial fibration between quasi-categories then the functor $\mathsf{h}f\colon \mathsf{h}X \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow \mathsf{h}Y$ is a surjective equivalence of categories.

1.2 ∞-Cosmoi

In §1.1, we presented "analytic" proofs of a few of the basic facts about quasi-categories. The category theory of quasi-categories can be developed in a similar

[10] By the duality described in Definition 1.2.25, the right outer horn inclusions $\Lambda^n[n] \hookrightarrow \Delta[n]$ can be used instead.

[11] This decomposition of the inclusion $\mathbb{2} \hookrightarrow \mathbb{I}$ reveals which data extends homotopically uniquely to a homotopy coherent isomorphism. For instance, the 1- and 2-simplices of Definition 1.1.13 together with a single 3-simplex that has these as its outer faces with its inner faces degenerate. Homotopy type theorists refer to this data as a **half adjoint equivalence** [125, §4.2].

The data of a simplicial category can equivalently be encoded by a **simplicially enriched category** with a set of objects and a simplicial set $\mathcal{A}(x, y)$ of morphisms between each ordered pair of objects: an n-arrow in $\mathcal{A}_n$ from x to y corresponds to an n-simplex in $\mathcal{A}(x, y)$ (see Exercise 1.2.i). Each endo-hom space contains a distinguished identity 0-arrow (the degenerate images of which define the corresponding identity n-arrows) and composition is required to define a simplicial map

$$\mathcal{A}(y, z) \times \mathcal{A}(x, y) \xrightarrow{\circ} \mathcal{A}(x, z)$$

the single map encoding the compositions in each of the categories $\mathcal{A}_n$ and also the functoriality of the diagram (1.2.5). The composition is required to be associative and unital, in a sense expressed by the commutative diagrams of simplicial sets

$$\begin{array}{ccc} \mathcal{A}(y,z) \times \mathcal{A}(x,y) \times \mathcal{A}(w,x) & \xrightarrow{\circ \times \mathrm{id}} & \mathcal{A}(x,z) \times \mathcal{A}(w,x) \\ {\scriptstyle \mathrm{id} \times \circ} \downarrow & & \downarrow {\scriptstyle \circ} \\ \mathcal{A}(y,z) \times \mathcal{A}(w,y) & \xrightarrow[\circ]{} & \mathcal{A}(w,z) \end{array}$$

$$\begin{array}{ccc} \mathcal{A}(x,y) & \xrightarrow{\mathrm{id}_y \times \mathrm{id}} & \mathcal{A}(y,y) \times \mathcal{A}(x,y) \\ {\scriptstyle \mathrm{id} \times \mathrm{id}_x} \downarrow & \overset{\mathrm{id}}{\searrow} & \downarrow {\scriptstyle \circ} \\ \mathcal{A}(x,y) \times \mathcal{A}(x,x) & \xrightarrow[\circ]{} & \mathcal{A}(x,y) \end{array}$$

On account of the equivalence between these two presentations, the terms "simplicial category" and "simplicially enriched category" are generally taken to be synonyms.[15]

In particular, the underlying category $\mathcal{K}_0$ of an ∞-cosmos $\mathcal{K}$ is the category whose objects are the ∞-categories in $\mathcal{K}$ and whose morphisms are the 0-arrows, i.e., the vertices in the functor spaces. In all of the examples to appear in what follows, this recovers the expected category of ∞-categories in a particular model and functors between them.

Digression 1.2.6 (simplicially enriched limits, §A.4-A.5). Let $\mathcal{A}$ be a simplicial category. The **cotensor** of an object $A \in \mathcal{A}$ by a simplicial set U is characterized by a natural isomorphism of simplicial sets

$$\mathcal{A}(X, A^U) \cong \mathcal{A}(X, A)^U \tag{1.2.7}$$

[15] The phrase "simplicial object in $\mathcal{C}at$" is reserved for the more general yet less common notion of a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}at$ that is not necessarily comprised of identity-on-objects functors.

Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\begin{aligned} s\mathcal{S}et^{\mathrm{op}} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (U,A) &\longmapsto A^U \end{aligned}$$

in a unique way making the isomorphism (1.2.7) natural in U and A as well.

The other simplicial limit notions postulated by axiom 1.2.1(i) are **conical**, which is the term used for ordinary 1-categorical limit shapes that satisfy an enriched analog of the usual universal property (see Definition A.5.2). Such limits also define limits in the underlying category, but the usual universal property is strengthened. By applying the covariant representable functor $\mathcal{A}(X,-)\colon \mathcal{A}_0 \to s\mathcal{S}et$ to a limit cone $(\lim_{j\in J} A_j \to A_j)_{j\in J}$ in $\mathcal{A}_0$, we obtain a natural comparison map

$$\mathcal{A}(X, \lim_{j\in J} A_j) \longrightarrow \lim_{j\in J} \mathcal{A}(X, A_j). \tag{1.2.8}$$

We say that $\lim_{j\in J} A_j$ defines a **simplicially enriched limit** if and only if (1.2.8) is an isomorphism of simplicial sets for all $X \in \mathcal{A}$.

The general theory of enriched categories is reviewed in Appendix A.

Preview 1.2.9 (flexible weighted limits in ∞-cosmoi). The axiom 1.2.1(i) implies that any ∞-cosmos $\mathcal{K}$ admits all *flexible limits*, a much larger class of simplicially enriched "weighted" limits (see Definition 6.2.1 and Proposition 6.2.8).

We quickly introduce the three examples of ∞-cosmoi that are most easily absorbed, deferring more sophisticated examples to the end of this section. The first of these is the prototypical ∞-cosmos.

Proposition 1.2.10 (the ∞-cosmos of quasi-categories). *The full subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ of quasi-categories defines an ∞-cosmos in which the isofibrations, equivalences, and trivial fibrations coincide with the classes already bearing these names.*

Proof The subcategory $\mathcal{QC}at \subset s\mathcal{S}et$ inherits its simplicial enrichment from the cartesian closed category of simplicial sets: by Proposition 1.1.20, whenever A and B are quasi-categories, $\mathsf{Fun}(A,B) \coloneqq B^A$ is again a quasi-category.

The cosmological limits postulated in 1.2.1(i) exist in the ambient category of simplicial sets.[16] For instance, the defining universal property of the simplicial cotensor (1.2.7) is satisfied by the exponentials of simplicial sets. Moreover,

[16] Any category of presheaves is cartesian closed, complete, and cocomplete – a "cosmos" in the sense of Bénabou.

since the category of simplicial sets is cartesian closed, each of the conical limits is simplicially enriched in the sense discussed in Digression 1.2.6 (see Exercise 1.2.ii and Proposition A.5.4).

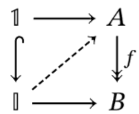
We now argue that the full subcategory of quasi-categories inherits all these limit notions and at the same time establish the stability of the isofibrations under the formation of these limits. In fact, this latter property helps to prove the former. To see this, note that a simplicial set is a quasi-category if and only if the map from it to the point is an isofibration. More generally, if the codomain of any isofibration is a quasi-category then its domain must be as well. So if any of the maps in a limit cone over a diagram of quasi-categories are isofibrations, then it follows that the limit is itself a quasi-category.

Since the isofibrations are characterized by a right lifting property, Lemma C.2.3 implies that the isofibrations contains all isomorphism and are closed under composition, product, pullback, and forming inverse limits of towers. In particular, the full subcategory of quasi-categories possesses these limits. This verifies all of the axioms of 1.2.1(i) and 1.2.1(ii) except for the last two: Leibniz closure and closure under exponentiation $(-)^X$. These last closure properties are established in Proposition 1.1.20, and in fact by Exercise 1.1.vii, the former subsumes the latter . This completes the verification of the ∞-cosmos axioms.

It remains to check that the equivalences and trivial fibrations coincide with those maps defined by 1.1.23 and 1.1.25. By Proposition 1.1.28 the latter coincidence follows from the former, so it remains only to show that the equivalences of 1.1.23 coincide with the **representably defined equivalences**: those maps of quasi-categories $f : A \to B$ for which $A^X \to B^X$ is an equivalence of quasi-categories in the sense of Definition 1.1.23. Taking $X = \Delta[0]$, we see immediately that representably defined equivalences are equivalences, and the converse holds since the exponential $(-)^X$ preserves the data defining a simplicial homotopy. □

Two further examples fit into a common paradigm: both arise as full subcategories of the ∞-cosmos of quasi-categories and inherit their ∞-cosmos structures from this inclusion (see Lemma 6.1.4). But it is also instructive, and ultimately takes less work, to describe the resulting ∞-cosmos structures directly.

PROPOSITION 1.2.11 (the ∞-cosmos of categories). *The category $\mathcal{C}at$ of 1-categories defines an ∞-cosmos whose isofibrations are the* ***isofibrations****: functors* *satisfying the displayed right lifting property:*

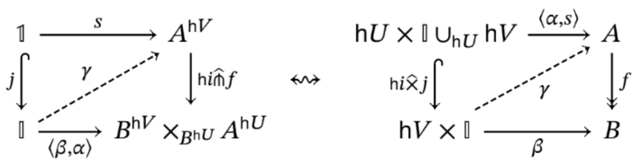


The equivalences are the equivalences of categories and the trivial fibrations are ***surjective equivalences****: equivalences of categories that are also surjective on objects.*

Proof It is well-known that the 2-category of categories is complete (and in fact also cocomplete) as a $\mathcal{C}at$-enriched category (see Definition A.6.17 or [67]). The categorically enriched category of categories becomes a quasi-categorically enriched category by applying the nerve functor to the hom-categories (see §A.7). Since the nerve functor is a right adjoint, it follows formally that these 2-categorical limits become simplicially enriched limits. In particular, as proscribed in Proposition A.7.8, the cotensor of a category A by a simplicial set U is defined to be the functor category $A^{\mathsf{h}U}$. This completes the verification of axiom (i).

Since the class of isofibrations is characterized by a right lifting property, Lemma C.2.3 implies that the isofibrations are closed under all of the limit constructions of 1.2.1(ii) except for the last two, and by Exercise 1.1.vii, the Leibniz closure subsumes the closure under exponentiation.

To verify that isofibrations of categories $f: A \twoheadrightarrow B$ are stable under forming Leibniz cotensors with monomorphisms of simplicial sets $i: U \hookrightarrow V$, we must solve the lifting problem below-left



which transposes to the lifting problem above-right, which we can solve by hand. Here the map β defines a natural isomorphism between $fs: \mathsf{h}V \to B$ and a second functor. Our task is to lift this to a natural isomorphism γ from s to another functor that extends the natural isomorphism α along $\mathsf{h}i: \mathsf{h}U \to \mathsf{h}V$. Note this functor $\mathsf{h}i$ need not be an inclusion, but it is injective on objects, which is enough.

We define the components of γ by cases. If an object $v \in \mathsf{h}V$ is equal to $i(u)$ for some $u \in \mathsf{h}U$ define $\gamma_{i(u)} := \alpha_u$; otherwise, use the fact that f is an isofibration to define γ_v to be any lift of the isomorphism β_v to an isomorphism in A with domain $s(v)$. The data of the map $\gamma\colon \mathsf{h}V \times \mathbb{I} \to A$ also entails the specification of the functor $\mathsf{h}V \to A$ that is the codomain of the natural isomorphism γ. On objects, this functor is given by $v \mapsto \mathrm{cod}(\gamma_v)$. On morphisms, this functor defined in the unique way that makes γ into a natural transformation:

$$(k\colon v \to v') \mapsto \gamma_{v'} \circ s(k) \circ \gamma_v^{-1}.$$

This completes the proof that $\mathcal{C}at$ defines an ∞-cosmos. Since the nerve of a functor category, such as $A^{\mathbb{I}}$, is isomorphic to the exponential between their nerves, the equivalences of categories coincide with the equivalences of Definition 1.1.23. It follows that the equivalences in the ∞-cosmos of categories coincide with equivalences of categories, and since the surjective equivalences are the intersection of the equivalences and the isofibrations, this completes the proof. □

Proposition 1.2.12 (the ∞-cosmos of Kan complexes). *The category $\mathcal{K}an$ of Kan complexes defines an ∞-cosmos whose isofibrations are the* ***Kan fibrations****: maps that lift against all horn inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$.*

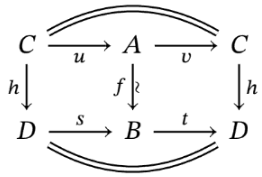
The proof proceeds along the lines of Lemma 6.1.4. We show that the subcategory of Kan complexes inherits an ∞-cosmos structure by restricting structure from the ∞-cosmos of quasi-categories.

Proof By Proposition 1.1.18, an isofibration between Kan complexes is a Kan fibration. Conversely, since the homotopy coherent isomorphism $\mathbb{I}$ can be built from the point $\mathbb{1}$ by attaching fillers to a sequence of outer horns, all Kan fibrations define isofibrations. This shows that between Kan complexes, isofibrations and Kan fibrations coincide. So to show that the category of Kan complexes inherits an ∞-cosmos structure by restriction from the ∞-cosmos of quasi-categories, we need only verify that the full subcategory $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ is closed under all of the limit constructions of axiom 1.2.1(i). For the conical limits, the argument mirrors the one given in the proof of Proposition 1.2.10, while the closure under cotensors is a consequence of Corollary D.3.11, which implies that the Kan complexes also define an exponential ideal in the category of simplicial sets. The remaining axiom 1.2.1(ii) is inherited from the analogous properties established for quasi-categories in Proposition 1.2.10. □

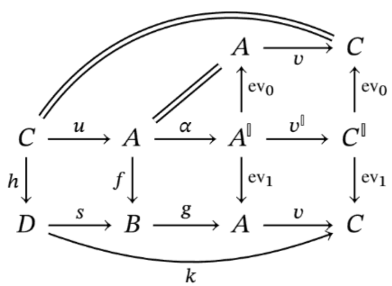
We mention a common source of ∞-cosmoi found in nature to build intuition for readers familiar with Quillen's model categories, a popular framework for abstract homotopy theory, but reassure newcomers that model categories are not needed outside of Appendix E where these results are proven.

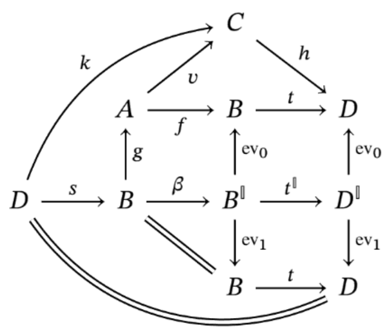
quasi-categories (see Exercise 1.1.viii). But for sake of completeness, we prove the general cosmological result without relying on this base case, subsuming Exercise 1.1.viii.

Proof Let $f : A \xrightarrow{\sim} B$ be an equivalence equipped with the data of (1.2.16) and consider a retract diagram



By Lemma 1.2.15, to prove that $h : C \to D$ is an equivalence, it suffices to construct the data of an inverse homotopy equivalence. To that end define $k : D \to C$ to be the composite vgs and then observe from the commutative diagrams





that $v^{\mathbb{I}}\alpha u : C \to C^{\mathbb{I}}$ and $t^{\mathbb{I}}\beta s : D \to D^{\mathbb{I}}$ define the required homotopy coherent isomorphisms.

Via Lemma 1.2.15, the 2-of-3 property for equivalences follows from the fact that the set of isomorphisms in a quasi-category is closed under composition. Homotopy coherent isomorphisms in a quasi-category represent isomorphisms in the homotopy category, whose composite in the homotopy category is then an isomorphism, which can be lifted to a representing homotopy coherent isomorphism by Corollary 1.1.16.[17] We now apply this to the homotopy coherent isomorphisms in the functor spaces of an ∞-cosmos that form part of the data of an equivalence of ∞-categories.

[17] In fact, by Example D.5.5, homotopy coherent isomorphisms can be composed directly, but we do not need this here.

To prove that equivalences are closed under composition, consider a composable pair of equivalences with their inverse equivalences

$$A \underset{\underset{k}{\overset{\sim}{\longleftarrow}}}{\overset{f}{\underset{\sim}{\longrightarrow}}} B \underset{\underset{h}{\overset{\sim}{\longleftarrow}}}{\overset{g}{\underset{\sim}{\longrightarrow}}} C$$

The equivalence data of Lemma 1.2.15 defines isomorphisms $\alpha\colon \mathrm{id}_A \cong kf \in \mathsf{Fun}(A,A)$ and $\gamma\colon \mathrm{id}_B \cong hg \in \mathsf{Fun}(B,B)$, the latter of which whiskers to define $k\gamma f\colon kf \cong khgf \in \mathsf{Fun}(B,B)$. Composing these, we obtain an isomorphism $\mathrm{id}_A \cong khgf \in \mathsf{Fun}(A,A)$, witnessing that kh defines a left equivalence inverse of gf. The other isomorphism is constructed similarly.

To prove that the equivalences are closed under right cancelation, consider a diagram

$$A \underset{\underset{k}{\overset{\sim}{\longleftarrow}}}{\overset{f}{\underset{\sim}{\longrightarrow}}} B \xrightarrow{g} C, \qquad C \xrightarrow[\sim]{\ell} A$$

with k an inverse equivalence to f and ℓ and inverse equivalence to gf. We claim that $f\ell$ defines an inverse equivalence to g. One of the required isomorphisms $\mathrm{id}_C \cong gf\ell$ is given already. The other is obtained by composing three isomorphisms in $\mathsf{Fun}(B,B)$

$$\mathrm{id}_B \xrightarrow[\approx]{\beta^{-1}} fk \xrightarrow[\approx]{f\delta k} f\ell gfk \xrightarrow[\approx]{f\ell g\beta} f\ell g.$$

The proof of stability of equivalence under left cancelation is dual. □

The trivial fibrations admit a similar characterization as split fiber homotopy equivalences.

Lemma 1.2.18 (trivial fibrations split). *Every trivial fibration admits a section*

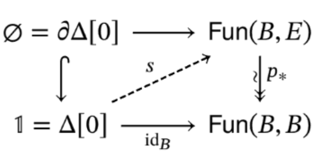
$$\begin{array}{ccc} & & E \\ & {\overset{s}{\nearrow}} & \big\downarrow{\scriptstyle\wr}\, p \\ B & = & B \end{array}$$

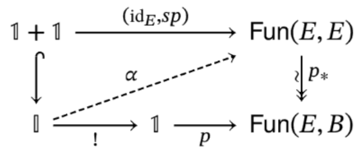
that defines a split fiber homotopy equivalence

$$\begin{array}{ccccc} E & \xrightarrow{\alpha} & E^{\mathbb{I}} & \xrightarrow[(\mathrm{ev}_0,\mathrm{ev}_1)]{} & E \times E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p^{\mathbb{I}}} & & \\ B & \xrightarrow[\Delta]{} & B^{\mathbb{I}} & & \end{array} \qquad (\mathrm{id}_E, sp)\colon E \to E\times E$$

and conversely any isofibration that defines a split fiber homotopy equivalence is a trivial fibration.

Proof If $p \colon E \twoheadrightarrow B$ is a trivial fibration, then by the final stability property of Lemma 1.2.14, so is $p_* \colon \mathsf{Fun}(X, E) \twoheadrightarrow Fun(X, B)$ for any ∞-category X. By Definition 1.1.25, we may solve the lifting problem below-left

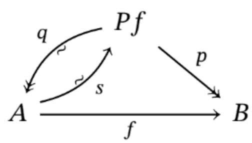




to find a map $s \colon B \to E$ so that $ps = \mathrm{id}_B$, and then solve the lifting problem above-right to construct the desired fibered homotopy. The converse is immediate from Lemma 1.2.15. □

A classical construction in abstract homotopy theory proves the following:

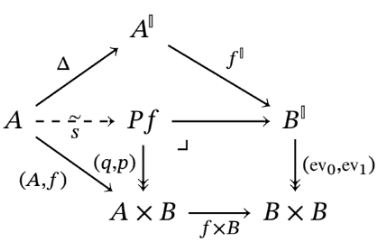
Lemma 1.2.19 (Brown factorization lemma). *Any functor $f \colon A \to B$ in an ∞-cosmos may be factored as an equivalence followed by an isofibration, where this equivalence is constructed as a section of a trivial fibration.*



(1.2.20)

Moreover, f is an equivalence if and only if the isofibration p is a trivial fibration.

Proof The displayed factorization is constructed by the pullback of an isofibration formed by the simplicial cotensor of the inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{I}$ into the ∞-category B.



Note the map q is a pullback of the trivial fibration $\mathrm{ev}_0 \colon B^{\mathbb{I}} \twoheadrightarrow B$ and is hence a trivial fibration. Its section s, constructed by applying the universal property of the pullback to the displayed cone with summit A, is thus an equivalence by the

2-of-3 property. Again by 2-of-3, it follows that f is an equivalence if and only if p is. □

Remark 1.2.21 (equivalences satisfy the 2-of-6 property). In fact the equivalences in any ∞-cosmos satisfy the stronger **2-of-6 property**: for any composable triple of functors

$$\begin{array}{ccccc} & & B & & \\ & {}^{f}\nearrow & & \searrow^{hg}_{\sim} & \\ A & & \xrightarrow{\quad hgf \quad} & & D \\ & {}_{gf}\searrow^{\sim} & \downarrow{\scriptstyle g} & \nearrow_{h} & \\ & & C & & \end{array}$$

if gf and hg are equivalences then f, g, h, and hgf are too. An argument of Blumberg and Mandell [20, 6.4] reproduced in Proposition C.1.8 uses Lemmas 1.2.17, 1.2.18, and 1.2.19 to prove that the equivalences have the 2-of-6 property (see Corollary C.1.9).

One of the key advantages of the ∞-cosmological approach to abstract category theory is that there are a myriad varieties of "fibered" ∞-cosmoi that can be built from a given ∞-cosmos, which means that any theorem proven in this axiomatic framework specializes and generalizes to those contexts. The most basic of these derived ∞-cosmoi is the ∞-cosmos of isofibrations over a fixed base, which we introduce now. Other examples of ∞-cosmoi are developed in Chapter 6, once we have a deeper understanding of the cosmological limits of axiom 1.2.1(i).

Proposition 1.2.22 (sliced ∞-cosmoi). *For any ∞-cosmos $\mathcal{K}$ and any ∞-category $B \in \mathcal{K}$ there is an* ***∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over*** *B whose*

(i) objects are isofibrations $p\colon E \twoheadrightarrow B$ with codomain B
(ii) functor spaces, say from $p\colon E \twoheadrightarrow B$ to $q\colon F \twoheadrightarrow B$, are defined by pullback

$$\begin{array}{ccc} \mathsf{Fun}_B(p\colon E \twoheadrightarrow B, q\colon F \twoheadrightarrow B) & \longrightarrow & \mathsf{Fun}(E,F) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle q_*} \\ \mathbb{1} & \xrightarrow{\quad p \quad} & \mathsf{Fun}(E,B) \end{array}$$

and abbreviated to $\mathsf{Fun}_B(E,F)$ when the specified isofibrations are clear from context

(iii) *isofibrations are commutative triangles of isofibrations over B*

$$\begin{array}{ccc} E & \xrightarrow{r} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

(iv) *terminal object is* $\mathrm{id}\colon B \twoheadrightarrow B$ *and products are defined by the pullback along the diagonal*

$$\begin{array}{ccc} \times_i^B E_i & \longrightarrow & \prod_i E_i \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \prod_i p_i} \\ B & \xrightarrow{\Delta} & \prod_i B \end{array}$$

(v) *pullbacks and limits of towers of isofibrations are created by the forgetful functor* $\mathcal{K}_{/B} \to \mathcal{K}$

(vi) *simplicial cotensor of* $p\colon E \twoheadrightarrow B$ *with* $U \in s\mathcal{S}et$ *is constructed by the pullback*

$$\begin{array}{ccc} U \pitchfork_B p & \longrightarrow & E^U \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle p^U} \\ B & \xrightarrow{\Delta} & B^U \end{array}$$

(vii) *and in which a map over B*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

is an equivalence in the ∞-cosmos $\mathcal{K}_{/B}$ *if and only if* f *is an equivalence in* $\mathcal{K}$.

Proof The functor spaces are quasi-categories since axiom 1.2.1(ii) asserts that for any isofibration $q\colon F \twoheadrightarrow B$ in $\mathcal{K}$ the map $q_*\colon \mathsf{Fun}(E, F) \twoheadrightarrow \mathsf{Fun}(E, B)$ is an isofibration of quasi-categories. Other parts of this axiom imply that each of the limit constructions – such as the products and cotensors constructed in (iv) and (vi) – define isofibrations over B. The closure properties of the isofibrations in $\mathcal{K}_{/B}$ follow from the corresponding ones in $\mathcal{K}$. The most complicated of these is the Leibniz cotensor stability of the isofibrations in $\mathcal{K}_{/B}$, which follows from the corresponding property in $\mathcal{K}$, since for a monomorphism of simplicial sets $i\colon X \hookrightarrow Y$ and an isofibration r over B as in (iii) above, the map $i \mathbin{\widehat{\pitchfork}_B} r$ is constructed by pulling back $i \mathbin{\widehat{\pitchfork}} r$ along $\Delta\colon B \to B^Y$.

The fact that the above constructions define simplicially enriched limits in a simplicially enriched slice category are standard from enriched category theory.

For any ∞-cosmos $\mathcal{K}$, there is a **dual ∞-cosmos** $\mathcal{K}^{\mathrm{co}}$ with the same objects but with functor spaces defined by:

$$\mathsf{Fun}_{\mathcal{K}^{\mathrm{co}}}(A,B) \coloneqq \mathsf{Fun}_{\mathcal{K}}(A,B)^{\mathrm{op}}.$$

The isofibrations, equivalences, and trivial fibrations in $\mathcal{K}^{\mathrm{co}}$ coincide with those of $\mathcal{K}$.

Conical limits in $\mathcal{K}^{\mathrm{co}}$ coincide with those in $\mathcal{K}$, while the cotensor of $A \in \mathcal{K}$ with $U \in s\mathcal{S}et$ is defined to be $A^{U^{\mathrm{op}}}$.

A 2-categorical justification for this notation is given in Exercise 1.4.ii.

Definition 1.2.26 (discrete ∞-categories). An ∞-category E in an ∞-cosmos $\mathcal{K}$ is **discrete** just when for all $X \in \mathcal{K}$ the functor space $\mathsf{Fun}(X,E)$ is a Kan complex.

In the ∞-cosmos of quasi-categories, the discrete ∞-categories are exactly the Kan complexes. Similarly, in the ∞-cosmoi of Example 1.2.24 whose ∞-categories are $(\infty,1)$-categories in some model, the discrete ∞-categories are the ∞-groupoids. Importantly for what follows, the discrete ∞-categories can be characterized "internally" to the ∞-cosmos as follows:

Lemma 1.2.27. *An ∞-category E is discrete if and only if $E^{\mathbb{I}} \xrightarrow{\sim} E^{\mathbb{2}}$ is a trivial fibration.*

Proof By Definition 1.2.2, the isofibration $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$ is a trivial fibration if and only if for all ∞-categories X the induced map on functor spaces

$$\begin{array}{ccc} \mathsf{Fun}(X,E^{\mathbb{I}}) & \twoheadrightarrow & \mathsf{Fun}(X,E^{\mathbb{2}}) \\ \wr\| & & \wr\| \\ \mathsf{Fun}(X,E)^{\mathbb{I}} & \twoheadrightarrow & \mathsf{Fun}(X,E)^{\mathbb{2}} \end{array}$$

is a trivial fibration of quasi-categories. Via the universal property of the simplicial cotensor, Lemma 1.1.30 tells us that this map is a trivial fibration if and only if $\mathsf{Fun}(X,E)$ is a Kan complex. □

The reader may check that the discrete ∞-categories in any ∞-cosmos assemble into an ∞-cosmos $\mathcal{K}^{\simeq}$. A proof appears in Proposition 6.1.6 where general techniques for producing new ∞-cosmoi from given ones are developed.

Exercises

Exercise 1.2.i. Define an equivalence between the categories of:

(i) simplicial categories, as in (1.2.5), and

(ii) categories enriched over simplicial sets.

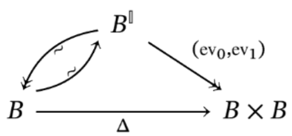
Exercise 1.2.ii. Elaborate on the proof of Proposition 1.2.10 by proving that the simplicially enriched category $\mathcal{QC}at$ admits conical products satisfying the universal property of Digression 1.2.6. That is:

(i) Define the cartesian product $A \times B$ and the projection maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ for a pair of quasi-categories A and B and prove that this data satisfies the usual (unenriched) universal property.
(ii) Given another quasi-category X, use (i) and the Yoneda lemma to show that the projection maps induce an isomorphism of quasi-categories

$$(A \times B)^X \xrightarrow{\cong} A^X \times B^X.$$

(iii) Explain how this relates to the universal property of Digression 1.2.6.
(iv) Express the usual 1-categorical universal property of (i) as the "0-dimensional aspect" of the universal property of (ii).

Exercise 1.2.iii. Prove that any object in an ∞-cosmos has a **path object**



constructed by cotensoring with the free-living isomorphism.

Exercise 1.2.iv. Show that if $\mathcal{K}$ is a cartesian closed ∞-cosmos then $\mathcal{K}^{\mathrm{co}}$ is as well.

Exercise 1.2.v (6.1.6). Use Proposition 1.2.12 to show that the discrete ∞-categories in any ∞-cosmos define an ∞-cosmos whose functor spaces are all Kan complexes.

1.3 Cosmological Functors

Certain "right adjoint type" constructions define maps between ∞-cosmoi that preserve all of the structures axiomatized in Definition 1.2.1. The simple observation that such constructions define *cosmological functors* between ∞-cosmoi streamlines many proofs.

Definition 1.3.1 (cosmological functor). A **cosmological functor** is a simplicial functor (see Definition A.2.6) between ∞-cosmoi that preserves the specified isofibrations and all of the cosmological limits.

In general, cosmological functors preserve any ∞-categorical notion that can be characterized *internally* to the ∞-cosmos – for instance, as a map equipped with additional structure – as opposed to *externally* – for instance, by a statement that involves a universal or existential quantifier. For example, the equivalences in an ∞-cosmos are characterized externally in Definition 1.2.2, which might lead one to suspect that a nonsurjective cosmological functor could fail to preserve them. However, Lemma 1.2.15 characterizes equivalences in terms of the presence of structures defined internally to an ∞-cosmos, so as a result:

Lemma 1.3.2. *Any cosmological functor also preserves equivalences and trivial fibrations.*

Proof By Lemma 1.2.15 the equivalences in an ∞-cosmos coincide with the "homotopy equivalences" defined by cotensoring with the free-living isomorphism. Since a cosmological functor preserves simplicial cotensors, it preserves the data displayed in (1.2.16) and hence carries equivalences to equivalences. The preservation of trivial fibrations follows. □

Remark 1.3.3. Similarly, arguing from Definition 1.2.26 it would not be clear whether cosmological functors preserve discrete ∞-categories, but using the internal characterization of Lemma 1.2.27 – an ∞-category A is discrete if and only if $A^{\mathbb{I}} \xrightarrow{\sim} A^{\mathbb{2}}$ is a trivial fibration – this follows from the fact that cosmological functors preserve simplicial cotensors and trivial fibrations.

We now demonstrate that cosmological functors are abundant:

Proposition 1.3.4. *The following constructions define cosmological functors for any ∞-cosmos $\mathcal{K}$:*

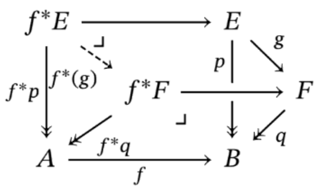
(i) *The functor space* $\mathsf{Fun}(X, -) : \mathcal{K} \to \mathcal{QC}at$*, for any ∞-category X.*
(ii) *The* ***underlying quasi-category*** *functor*

$$(-)_0 := \mathsf{Fun}(1, -) : \mathcal{K} \to \mathcal{QC}at,$$

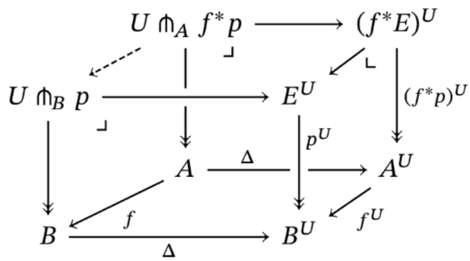
specializing (i) to the terminal ∞-category 1.
(iii) *The simplicial cotensor* $(-)^U : \mathcal{K} \to \mathcal{K}$*, for any simplicial set U.*
(iv) *The exponential* $(-)^A : \mathcal{K} \to \mathcal{K}$*, for any ∞-category A in a cartesian closed ∞-cosmos $\mathcal{K}$.*
(v) *Pullback of isofibrations* $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ *along any functor* $f : A \to B$ *in an ∞-cosmos $\mathcal{K}$.*
(vi) *Moreover, for any cosmological functor* $F : \mathcal{K} \to \mathcal{L}$ *and any ∞-category $A \in \mathcal{K}$, the induced map on slices* $F : \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ *defines a cosmological functor.*

Proof The first four of these statements are nearly immediate, the preservation of isofibrations being asserted explicitly as a hypothesis in each case and the preservation of limits following from familiar arguments.

For (v), pullback in an ∞-cosmos $\mathcal{K}$ is a simplicially enriched limit construction; one consequence of this is that $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ defines a simplicial functor. The action of the functor f^* on a 0-arrow g in $\mathcal{K}_{/B}$ is also defined by a pullback square: since the front and back squares in the displayed diagram are pullbacks the top square is as well



Since isofibrations are stable under pullback, it follows that $f^* : \mathcal{K}_{/B} \to \mathcal{K}_{/A}$ preserves isofibrations. It remains to prove that this functor preserves the simplicial limits constructed in Proposition 1.2.22, which is fundamentally a consequence of the commutativity of limit constructions. In each case, this can be verified explicitly. We illustrate this computation for simplicial cotensors by constructing the commutative cube:



Since the front, back, and right faces are pullbacks, the left is as well.

The final statement (vi) is left as Exercise 1.3.i. □

Example 1.3.5. By Propositions 1.2.11 and 1.2.12, the full subcategory inclusions $\mathcal{C}at \hookrightarrow \mathcal{QC}at$ and $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ both define cosmological functors (see also Lemma 6.1.4). These cosmological embeddings explicate the intuition that the formal category theory of 1-categories or of ∞-groupoids can be recovered as a special case of the formal category theory of $(\infty, 1)$-categories.

Non-examples of cosmological functors are also instructive:

Non-Example 1.3.6. The forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ is simplicial and preserves isofibrations but does *not* define a cosmological functor, failing to preserve cotensors and products. However, by Proposition 1.3.4(v), its right adjoint $- \times B \colon \mathcal{K} \to \mathcal{K}_{/B}$ does define a cosmological functor.

Non-Example 1.3.7. The cosmological embedding $\mathcal{K}an \hookrightarrow \mathcal{QC}at$ has a right adjoint $(-)^{\simeq} \colon \mathcal{QC}at \to \mathcal{K}an$ that carries each quasi-category to its "∞-groupoid core" or maximal sub Kan complex, the simplicial subset containing those n-simplices whose edges are all isomorphisms. This core functor preserves isofibrations and 1-categorical limits but is *not* cosmological since it is not simplicially enriched: any functor $K \to Q$ whose domain is a Kan complex and whose codomain is a quasi-category factors through the inclusion $Q^{\simeq} \hookrightarrow Q$ via a unique map $K \to Q^{\simeq}$ but in general $\mathsf{Fun}(K, Q) \not\cong \mathsf{Fun}(K, Q^{\simeq})$, since a natural transformation $K \times \Delta[1] \to Q$ only factors through $Q^{\simeq} \hookrightarrow Q$ in the case where its components are invertible (see Lemma 12.1.12 however).

Certain cosmological functors are especially well-behaved:

Definition 1.3.8 (cosmological biequivalence). A cosmological functor defines a **cosmological biequivalence** $F \colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ if it additionally

(i) is **essentially surjective on objects up to equivalence**: for all $C \in \mathcal{L}$ there exists $A \in \mathcal{K}$ so that $FA \simeq C$ and
(ii) defines a **local equivalence**: for all $A, B \in \mathcal{K}$, the action of F on functor spaces defines an equivalence of quasi-categories

$$\mathsf{Fun}(A, B) \xrightarrow{\sim} \mathsf{Fun}(FA, FB).$$

Cosmological biequivalences are studied more systematically in Chapter 10, where we think of them as "change-of-model" functors. Crucially for our proof of the "model independence" of $(\infty, 1)$-category theory in Chapter 11, there are a variety of cosmological biequivalences between the ∞-cosmoi of $(\infty, 1)$-categories:

Example 1.3.9 (§E.2).

(i) The underlying quasi-category functors defined on the ∞-cosmoi of complete Segal spaces, Segal categories, and 1-complicial sets

$$\mathcal{CSS} \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \quad \mathcal{S}egal \xrightarrow[\sim]{(-)_0} \mathcal{QC}at \quad 1\text{-}\mathcal{C}omp \xrightarrow[\sim]{(-)_0} \mathcal{QC}at$$

are all biequivalences. In the first two cases these are defined by "evaluating at the 0th row" and in the last case this is defined by "forgetting the markings."

2-category of an ∞-cosmos. The reader is then invited to revisit the creation of equivalences in Exercise 1.4.vi.

Exercises

Exercise 1.3.i. Prove that for any cosmological functor $F\colon \mathcal{K} \to \mathcal{L}$ and any $A \in \mathcal{K}$, the induced map $F\colon \mathcal{K}_{/A} \to \mathcal{L}_{/FA}$ defines a cosmological functor.

Exercise 1.3.ii. Sketch a proof that cosmological biequivalences create equivalences between ∞-categor ies without passing to homotopy categories, by lifting and composing the homotopy coherent isomorphisms given as part of the data of the hypothesized equivalences.

Exercise 1.3.iii. Suppose $F\colon \mathcal{K} \to \mathcal{L}$, $G\colon \mathcal{L} \to \mathcal{M}$, and $H\colon \mathcal{M} \to \mathcal{N}$ are cosmological functors, and assume that GF and HG are cosmological biequivalences. Show that F, G, H, and HGF are cosmological biequivalences.

1.4 The Homotopy 2-Category

Small 1-categories define the objects of a strict 2-category[22] $\mathcal{C}at$ of categories, functors, and natural transformations. Many basic categorical notions – those defined in terms of categories, functors, and natural transformations – can be defined internally to the 2-category $\mathcal{C}at$. This suggests a natural avenue for generalization: reinterpreting these same definitions in a generic 2-category using its objects in place of small categories, its 1-cells in place of functors, and its 2-cells in place of natural transformations.

In Chapter 2, we develop a significant portion of the theory of ∞-categories in any fixed ∞-cosmos following exactly this outline, working internally to a 2-category that we refer to as the *homotopy 2-category* that we associate to any ∞-cosmos. The homotopy 2-category of an ∞-cosmos is a quotient of the full ∞-cosmos, replacing each quasi-categorical functor space by its homotopy category. Surprisingly, this rather destructive quotienting operation preserves quite a lot of information. Indeed, essentially all of the development of the

[22] Appendix B introduces 2-categories and 2-functors, reviewing the 2-category theory needed here. Succinctly, in parallel with Digression 1.2.4, 2-categories (see Definition B.1.1) can be understood equally as:

- "two-dimensional" categories, with objects; **1-cells**, whose boundary are given by a pair of objects; and **2-cells**, whose boundary are given by a parallel pair of 1-cells between a pair of objects – together with partially defined composition operations governed by this boundary data
- or as categories enriched over $\mathcal{C}at$.

theory of ∞-categories in Part I takes place in the homotopy 2-category of an ∞-cosmos. This said, we caution the reader against becoming overly seduced by homotopy 2-categories, which are more of a technical convenience for reducing the complexity of our arguments than a fundamental notion of ∞-category theory.

The homotopy 2-category for the ∞-cosmos of quasi-categories was first introduced by Joyal in his work on the foundations of quasi-category theory [63].

Definition 1.4.1 (homotopy 2-category). Let $\mathcal{K}$ be an ∞-cosmos. Its **homotopy 2-category** is the 2-category $\mathfrak{h}\mathcal{K}$ whose

- objects are the the objects A, B of $\mathcal{K}$, i.e., the ∞-categories;
- 1-cells $f : A \to B$ are the 0-arrows in the functor space $\mathsf{Fun}(A, B)$, i.e., the ∞-functors; and
- 2-cells $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$ are homotopy classes of 1-simplices in $\mathsf{Fun}(A, B)$, which we call **∞-natural transformations**.

Put another way $\mathfrak{h}\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$ and with hom-categories defined by

$$\mathsf{hFun}(A, B) \coloneqq \mathsf{h}(\mathsf{Fun}(A, B)),$$

that is, $\mathsf{hFun}(A, B)$ is the homotopy category of the quasi-category $\mathsf{Fun}(A, B)$.

The **underlying category** of a 2-category is defined by simply forgetting its 2-cells. Note that an ∞-cosmos $\mathcal{K}$ and its homotopy 2-category $\mathfrak{h}\mathcal{K}$ share the same underlying category $\mathcal{K}_0$ of ∞-categories and ∞-functors in $\mathcal{K}$.

Digression 1.4.2 (change of base, §A.7). The homotopy category functor preserves finite products, as of course does its right adjoint. It follows that the adjunction of Proposition 1.1.11 induces a change-of-base adjunction

$$2\text{-}\mathcal{C}at \underset{\hookrightarrow}{\overset{\mathsf{h}_*}{\underset{\perp}{\leftarrow}}} s\mathcal{S}et\text{-}\mathcal{C}at$$

whose left and right adjoints change the enrichment by applying the homotopy category functor or the nerve functor to the hom objects of the enriched category. Here $2\text{-}\mathcal{C}at$ and $s\mathcal{S}et\text{-}\mathcal{C}at$ can each be understood as 2-categories – of enriched categories, enriched functors, and enriched natural transformations – and both change of base constructions define 2-functors (see Propositions A.7.3 and A.7.5). Since the nerve embedding is fully faithful, 2-categories can be identified as a full subcategory comprised of those simplicial categories whose hom spaces are nerves of categories.

The proof of Lemma 1.3.12 uses an observation worth highlighting:

Lemma 1.4.3.

(i) *Every 2-cell* $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$ *in the homotopy 2-category of an* ∞*-cosmos is represented by a map of quasi-categories as below-left or equivalently by a functor as below-right*

$$\begin{array}{ccc} & \mathbb{1}+\mathbb{1} & \\ \swarrow & & \searrow^{(f,g)} \\ \mathbb{2} & \xrightarrow[\alpha]{} & \mathsf{Fun}(A,B) \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} A & \xrightarrow{\ulcorner\alpha\urcorner} & B^{\mathbb{2}} \\ {}_{(g,f)}\searrow & & \swarrow_{(p_1,p_0)} \\ & B\times B & \end{array}$$

and two such maps represent the same 2-cell if and only if they are homotopic as 1-simplices in $\mathsf{Fun}(A,B)$.

(ii) *Every invertible 2-cell* $A \underset{g}{\overset{f}{\cong\Downarrow\alpha}} B$ *in the homotopy 2-category of an* ∞*-cosmos is represented by a map of quasi-categories as below-left or equivalently by a functor as below-right*

$$\begin{array}{ccc} & \mathbb{1}+\mathbb{1} & \\ \swarrow & & \searrow^{(f,g)} \\ \mathbb{I} & \xrightarrow[\alpha]{} & \mathsf{Fun}(A,B) \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} A & \xrightarrow{\ulcorner\alpha\urcorner} & B^{\mathbb{I}} \\ {}_{(g,f)}\searrow & & \swarrow_{(p_1,p_0)} \\ & B\times B & \end{array}$$

and two such maps represent the same invertible 2-cell if and only if their common restrictions along $\mathbb{2} \hookrightarrow \mathbb{I}$ *are homotopic as 1-simplices in* $\mathsf{Fun}(A,B)$.

The notion of homotopic 1-simplices referenced here is defined in Lemma 1.1.9. Since the 2-cells in the homotopy 2-category are referred to as ∞-natural transformations, we refer to the invertible 2-cells in the homotopy 2-category as **∞-natural isomorphisms**.

Proof The statement (i) records the definition of the 2-cells in the homotopy 2-category and the universal property (1.2.7) of the simplicial cotensor. For (ii), a 2-cell in the homotopy 2-category is **invertible** if and only if it defines an isomorphism in the appropriate hom-category $\mathsf{hFun}(A,B)$. By Corollary 1.1.16 it follows that each invertible 2-cell α is represented by a homotopy coherent isomorphism $\alpha\colon \mathbb{I} \to \mathsf{Fun}(A,B)$, which similarly internalizes to define a functor $\ulcorner\alpha\urcorner\colon A \to B^{\mathbb{I}}$. □

An upshot of Digression 1.4.2 is that change of base is an operation that applies to enriched functors as well as enriched categories, as can be directly verified in the case of greatest interest.

Lemma 1.4.4. *Any simplicial functor $F\colon \mathcal{K} \to \mathcal{L}$ between ∞-cosmoi induces a 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ between their homotopy 2-categories.*

Proof The action of the induced 2-functor $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ on objects and 1-cells is given by the corresponding action of $F\colon \mathcal{K} \to \mathcal{L}$; recall an ∞-cosmos and its homotopy 2-category have the same underlying 1-category. Each 2-cell in $\mathfrak{h}\mathcal{K}$ is represented by a 1-simplex in $\mathsf{Fun}(A,B)$ which is mapped via

$$\mathsf{Fun}(A,B) \xrightarrow{\quad F \quad} \mathsf{Fun}(FA,FB)$$

$$A \underset{g}{\overset{f}{\Downarrow\alpha}} B \longmapsto FA \underset{Fg}{\overset{Ff}{\Downarrow F\alpha}} FB$$

to a 1-simplex representing a 2-cell in $\mathfrak{h}\mathcal{L}$. Since the action $F\colon \mathsf{Fun}(A,B) \to \mathsf{Fun}(FA,FB)$ on functor spaces defines a morphism of simplicial sets, it preserves faces and degeneracies. In particular, homotopic 1-simplices in $\mathsf{Fun}(A,B)$ are carried to homotopic 1-simplices in $\mathsf{Fun}(FA,FB)$ so the action on 2-cells just described is well-defined. The 2-functoriality of these mappings follows from the simplicial functoriality of the original mapping. □

We now begin to relate the simplicially enriched structures of an ∞-cosmos to the 2-categorical structures in its homotopy 2-category by proving that homotopy 2-categories inherit products from their ∞-cosmoi that satisfy a 2-categorical universal property. To illustrate, recall that the terminal ∞-category $1 \in \mathcal{K}$ has the universal property $\mathsf{Fun}(X,1) \cong \mathbb{1}$ for all $X \in \mathcal{K}$. Applying the homotopy category functor we see that $1 \in \mathfrak{h}\mathcal{K}$ has the universal property $\mathsf{hFun}(X,1) \cong \mathbb{1}$ for all $X \in \mathfrak{h}\mathcal{K}$, which is expressed by saying that the ∞-category 1 defines a **2-terminal object** in the homotopy 2-category. This 2-categorical universal property has both a 1-dimensional and a 2-dimensional aspect. Since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ is a category with a single object, there exists a unique morphism $X \to 1$ in $\mathcal{K}$, and since $\mathsf{hFun}(X,1) \cong \mathbb{1}$ has only a single morphism, the only 2-cells in $\mathfrak{h}\mathcal{K}$ with codomain 1 are identities.

Proposition 1.4.5 (cartesian (closure)).

(i) The homotopy 2-category of any ∞-cosmos has 2-categorical products.
(ii) The homotopy 2-category of a cartesian closed ∞-cosmos is cartesian closed as a 2-category.

Proof While the functor $\mathsf{h}\colon s\mathcal{S}et \to \mathcal{C}at$ only preserves finite products, the restricted functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ preserves *all* products on account of the simplified description of the homotopy category of a quasi-category given in Lemma 1.1.12. Thus for any set I and family of ∞-categories $(A_i)_{i\in I}$ in $\mathcal{K}$, the homotopy category functor carries the isomorphism of functor spaces to an isomorphism of hom-categories

$$\begin{array}{ccc} \mathsf{Fun}(X, \prod_{i\in I} A_i) & \xrightarrow{\cong} & \prod_{i\in I} \mathsf{Fun}(X, A_i) \\ & \downarrow{\scriptstyle \mathsf{h}} & \\ \mathsf{hFun}(X, \prod_{i\in I} A_i) & \xrightarrow{\cong} & \prod_{i\in I} \mathsf{hFun}(X, A_i). \end{array}$$

This proves that the homotopy 2-category $\mathfrak{h}\mathcal{K}$ has products whose universal properties have both a 1- and 2-dimensional component, as described in the empty case for terminal objects above.

If $\mathcal{K}$ is a cartesian closed ∞-cosmos, then for any triple of ∞-categories $A, B, C \in \mathcal{K}$ there exist exponential objects $C^A, C^B \in \mathcal{K}$ characterized by natural isomorphisms

$$\mathsf{Fun}(A \times B, C) \cong \mathsf{Fun}(A, C^B) \cong \mathsf{Fun}(B, C^A).$$

Passing to homotopy categories we have natural isomorphisms

$$\mathsf{hFun}(A \times B, C) \cong \mathsf{hFun}(A, C^B) \cong \mathsf{hFun}(B, C^A),$$

which demonstrates that $\mathfrak{h}\mathcal{K}$ is cartesian closed as a 2-category: functors $A \times B \to C$ transpose to define functors $A \to C^B$ and $B \to C^A$, and natural transformations transpose similarly. □

There is a standard definition of *isomorphism* between two objects in any 1-category, preserved by any functor. Similarly, there is a standard definition of *equivalence* between two objects in any 2-category, preserved by any 2-functor:

Definition 1.4.6 (equivalence). An **equivalence** in a 2-category is given by

- a pair of objects A and B;
- a pair of 1-cells $f\colon A \to B$ and $g\colon B \to A$; and
- a pair of invertible 2-cells

$$A \underset{gf}{\overset{=}{\Downarrow}}\!{\scriptstyle \cong\alpha}\; A \qquad \text{and} \qquad B \underset{=}{\overset{fg}{\Downarrow}}\!{\scriptstyle \cong\beta}\; B$$

When A and B are **equivalent**, we write $A \simeq B$ and refer to the 1-cells f and g as **equivalences**, denoted by "$\xrightarrow{\sim}$."

is an isofibration of categories in the sense defined in Proposition 1.2.11. By axiom 1.2.1(ii), since $p\colon E \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, the induced map $p_*\colon \mathsf{Fun}(X,E) \twoheadrightarrow \mathsf{Fun}(X,B)$ is an isofibration of quasi-categories. So it suffices to show that the functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ carries isofibrations of quasi-categories to isofibrations of categories.

So let us now consider an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories. By Corollary 1.1.16, every isomorphism β in the homotopy category $\mathsf{h}B$ of the quasi-category B is represented by a simplicial map $\beta\colon \mathbb{I} \to B$. By Definition 1.1.17, the lifting problem

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\ e\ } & E \\ \big\downarrow & \overset{\gamma}{\nearrow} & \big\downarrow p \\ \mathbb{I} & \xrightarrow[\ \beta\ ]{} & B \end{array}$$

can be solved, and the map $\gamma\colon \mathbb{I} \to E$ so produced represents a lift of the isomorphism from $\mathsf{h}B$ to an isomorphism in $\mathsf{h}E$ with domain e. □

Convention 1.4.10 (on isofibrations in homotopy 2-categories). Since the converse to Proposition 1.4.9 does not hold, there is a potential ambiguity when using the term "isofibration" to refer to a map in the homotopy 2-category of an ∞-cosmos. We adopt the convention that when we declare a map in $\mathfrak{h}\mathcal{K}$ to be an isofibration we always mean this is the stronger sense of defining an isofibration in $\mathcal{K}$. This stronger condition gives us access to the 2-categorical lifting property of Proposition 1.4.9 and also to homotopical properties axiomatized in Definition 1.2.1, which ensure that the strictly defined limits of 1.2.1(i) are automatically equivalence invariant constructions (see §C.1 and Proposition 6.2.8).

We conclude this chapter with a final definition that can be extracted from the homotopy 2-category of an ∞-cosmos. The 1- and 2-cells in the homotopy 2-category from the terminal ∞-category $1 \in \mathcal{K}$ to a generic ∞-category $A \in \mathcal{K}$ define the objects and morphisms in the homotopy category of the ∞-category A.

Definition 1.4.11 (homotopy category of an ∞-category). The **homotopy category** of an ∞-category A in an ∞-cosmos $\mathcal{K}$ is defined to be the homotopy category of its underlying quasi-category, that is:

$$\mathsf{h}A \coloneqq \mathsf{hFun}(1,A) \coloneqq \mathsf{h}(\mathsf{Fun}(1,A)).$$

As we shall discover, homotopy categories generally inherit "derived" analogues of structures present at the level of ∞-categories. An early example of this appears in Proposition 2.1.7(ii).

Exercises

Exercise 1.4.i.

(i) What is the homotopy 2-category of the ∞-cosmos $\mathcal{C}at$ of 1-categories?
(ii) Prove that the nerve defines a 2-functor $\mathcal{C}at \hookrightarrow \mathfrak{h}\mathcal{QC}at$ that is locally fully faithful.

Exercise 1.4.ii. Demonstrate that the homotopy 2-category of the dual cosmos $\mathcal{K}^{\mathrm{co}}$ of an ∞-cosmos $\mathcal{K}$ is the co-dual of the homotopy 2-category $\mathfrak{h}\mathcal{K}$ – in symbols $\mathfrak{h}(\mathcal{K}^{\mathrm{co}}) \cong (\mathfrak{h}\mathcal{K})^{\mathrm{co}}$ – with the domains and codomains of 2-cells but not 1-cells reversed (see Definition B.1.6).

Exercise 1.4.iii. Consider a natural isomorphism $A \underset{g}{\overset{f}{\rightrightarrows}} B$ (with $\cong\Downarrow\alpha$) between a parallel pair of functors in an ∞-cosmos. Give two proofs that if either f or g is an equivalence then both functors are, either by arguing entirely in the homotopy 2-category or by appealing to Lemma 1.4.3.

Exercise 1.4.iv. Extend Lemma 1.2.27 to show that the following four conditions are equivalent, characterizing the discrete objects E in an ∞-cosmos $\mathcal{K}$:

(i) E is a discrete object in the homotopy 2-category $\mathfrak{h}\mathcal{K}$, that is, every 2-cell with codomain E is invertible.
(ii) For each $X \in \mathcal{K}$, the hom-category $\mathsf{hFun}(X, E)$ is a groupoid.
(iii) For each $X \in \mathcal{K}$, the mapping quasi-category $\mathsf{Fun}(X, E)$ is a Kan complex.
(iv) The isofibration $E^{\mathbb{I}} \twoheadrightarrow E^{\mathbb{2}}$, induced by the inclusion of simplicial sets $\mathbb{2} \hookrightarrow \mathbb{I}$, is a trivial fibration.

Exercise 1.4.v (10.3.1). Extend Lemma 1.4.4 to show that if $F\colon \mathcal{K} \to \mathcal{L}$ is a cosmological biequivalence then $F\colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{L}$ is a 2-categorical **biequivalence**, a 2-functor that is essentially surjective on objects up to equivalence that locally defines an equivalence of hom-categories.

Exercise 1.4.vi. Let $F\colon \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ be a cosmological biequivalence and let $A, B \in \mathcal{K}$. Re-prove part of the statement of Lemma 1.3.12: that if $FA \simeq FB$ in $\mathcal{L}$ then $A \simeq B$ in $\mathcal{K}$.

Exercise 1.4.vii (3.6.2). Let B be an ∞-category in the ∞-cosmos $\mathcal{K}$ and let $\mathfrak{h}\mathcal{K}_{/B}$ denote the 2-category whose

- objects are isofibrations $E \twoheadrightarrow B$ in $\mathcal{K}$ with codomain B;

- 1-cells are 1-cells in $\mathfrak{h}\mathcal{K}$ over B; and

$$\begin{array}{ccc} E & \longrightarrow & F \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

- 2-cells are 2-cells α in $\mathfrak{h}\mathcal{K}$

$$\begin{array}{ccc} E & \overset{f}{\underset{g}{\Downarrow\alpha}} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

  that lie over B in the sense that $q\alpha = \mathrm{id}_p$.

Argue that the homotopy 2-category $\mathfrak{h}(\mathcal{K}_{/B})$ of the sliced ∞-cosmos has the same underlying 1-category but different 2-cells. How do these compare with the 2-cells of $\mathfrak{h}\mathcal{K}_{/B}$?

2

Adjunctions, Limits, and Colimits I

Heuristically, ∞-categories generalize ordinary 1-categories by adding in higher dimensional morphisms and weakening the composition law. One could imagine "∞-tizing" other types of categorical structure similarly, by adding in higher dimension and weakening properties. The naïve hope is that proofs establishing the theory of 1-categories might similarly generalize to give proofs for ∞-categories, just by adding a prefix "∞-" everywhere. In this chapter, we make this dream a reality – at least for a library of basic propositions concerning equivalences, adjunctions, limits, and colimits and the interrelationships between these notions.

Recall that categories, functors, and natural transformations assemble into a 2-category $\mathcal{C}at$. Similarly, the ∞-categories, ∞-functors, and ∞-natural transformations in any ∞-cosmos assemble into a 2-category, namely the *homotopy 2-category* of the ∞-cosmos, introduced in §1.4. In fact, $\mathcal{C}at$ can be regarded as a special case of a homotopy 2-category (by Exercise 1.4.i). In this chapter, we use 2-categorical techniques to define *adjunctions* between ∞-categories and *limits* and *colimits* of diagrams valued in an ∞-category and prove that these notions interact in the expected ways. In the homotopy 2-category of categories, this recovers classical results from 1-category theory, and in some cases even specializes to the standard proofs. As these arguments are equally valid in any homotopy 2-category, our proofs also establish the desired generalizations by simply appending the prefix "∞-."

In §2.1, we define an adjunction between ∞-categories to be an adjunction in the homotopy 2-category of ∞-categories, ∞-functors, and ∞-natural transformations. While it takes some work to justify the moral correctness of this simple definition, it has the great advantage that proofs of a number of results concerning the calculus of adjunctions and equivalences can be taken "off the shelf" in the sense that anyone who is sufficiently well-acquainted with 2-categories might know them already. In §2.2, we specialize the theory of adjunctions between ∞-categories to define and study initial and terminal elements inside an ∞-category. This section also serves as a warmup for the more subtle general theory of limits and colimits of diagrams valued in an ∞-category, which is the subject of §2.3. Finally, in §2.4, we study the interactions between these notions, proving that right adjoints preserve limits and left adjoints preserve colimits.

Missing from this discussion is an account of the universal properties associated to the unit of an adjunction or to a limit cone. These will be incorporated when we return to these topics in Chapter 4 after introducing an appropriate "hom ∞-category" with which to state them.

2.1 Adjunctions and Equivalences

In §1.4, we encounter the definition of an *equivalence* between a pair of objects in a 2-category. In the case where the ambient 2-category is the homotopy 2-category of an ∞-cosmos, Theorem 1.4.7 observes that the 2-categorical notion of equivalence precisely recaptures the notion of equivalence between ∞-categories in the full ∞-cosmos. In each of the examples of ∞-cosmoi we have considered, the representably defined equivalences in the ∞-cosmos coincide with the standard notion of equivalences between ∞-categories as presented in that particular model.[1] Thus, the 2-categorical notion of equivalence is the "correct" notion of equivalence between ∞-categories.

Similarly, there is a standard definition of an *adjunction* between a pair of objects in a 2-category, which, when interpreted in the homotopy 2-category of ∞-categories, functors, and natural transformations in an ∞-cosmos, will define the correct notion of adjunction between ∞-categories.

Definition 2.1.1 (adjunction). An **adjunction** between ∞-categories is comprised of:

- a pair of ∞-categories A and B;
- a pair of ∞-functors $u\colon A \to B$ and $f\colon B \to A$; and
- a pair of ∞-natural transformations $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, called the **unit** and **counit** respectively,

[1] For instance, as outlined in Digression 1.2.13, the equivalences in the ∞-cosmoi of Example 1.2.24 recapture the weak equivalences between fibrant–cofibrant objects in the usual model structure.

(ii) for any ∞-category X,

$$\mathsf{hFun}(X,A) \underset{u_*}{\overset{f_*}{\underset{\bot}{\rightleftarrows}}} \mathsf{hFun}(X,B)$$

defines an adjunction between categories;

(iii) for any simplicial set U,

$$A^U \underset{u^U}{\overset{f^U}{\underset{\bot}{\rightleftarrows}}} B^U$$

defines an adjunction between ∞-categories; and

(iv) if the ambient ∞-cosmos is cartesian closed, then for any ∞-category C,

$$A^C \underset{u^C}{\overset{f^C}{\underset{\bot}{\rightleftarrows}}} B^C$$

defines an adjunction between ∞-categories.

For instance, taking $X = 1$ in (ii) yields a "derived" adjunction between the homotopy categories of the ∞-categories A and B (see Definition 1.4.11):

$$\mathsf{h}A \underset{u_*}{\overset{f_*}{\underset{\bot}{\rightleftarrows}}} \mathsf{h}B$$

Proof Any adjunction $f \dashv u$ in the homotopy 2-category $\mathfrak{h}\mathcal{K}$ is preserved by each of the 2-functors $\mathsf{Fun}(X,-) : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$, $\mathsf{hFun}(X,-) : \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, $(-)^U : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$, and $(-)^C : \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$. □

Remark 2.1.8. There are contravariant versions of each of the adjunction preservation results of Proposition 2.1.7, the first of which we explain in detail (see Exercise 2.1.i for further discussion). Fixing the codomain variable of the functor space at any ∞-category $C \in \mathcal{K}$ defines a 2-functor

$$\mathsf{Fun}(-,C) : \mathfrak{h}\mathcal{K}^{\mathrm{op}} \longrightarrow \mathfrak{h}\mathcal{QC}at$$

that is contravariant on 1-cells and covariant on 2-cells.[4] Such 2-functors preserve adjunctions, but exchange left and right adjoints: for instance, given $f \dashv u$

[4] On a 2-category, the superscript "op" is used to signal that the 1-cells should be reversed but not the 2-cells, the superscript "co" is used to signal that the 2-cells should be reversed but not the 1-cells, and the superscript "coop" is used to signal that both the 1- and 2-cells should be reversed (see Definition B.1.6).

in $\mathcal{K}$, we obtain an adjunction

$$\mathsf{Fun}(A,C) \underset{f^*}{\overset{u^*}{\underset{\bot}{\leftrightarrows}}} \mathsf{Fun}(B,C)$$

between the functor spaces.

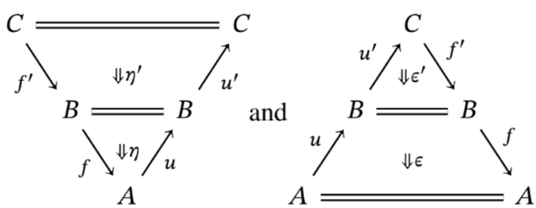
The next five results have standard proofs that can be taken "off the shelf" by querying any 2-category theorist who may happen to be standing nearby. The only novelty is the observation that these standard arguments can be applied to the theory of adjunctions between ∞-categories.

Proposition 2.1.9. *Adjunctions compose: given adjoint functors*

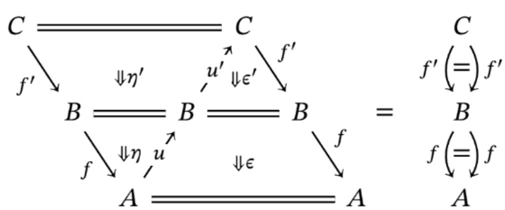
$$C \underset{u'}{\overset{f'}{\underset{\bot}{\rightleftarrows}}} B \underset{u}{\overset{f}{\underset{\bot}{\rightleftarrows}}} A \qquad \rightsquigarrow \qquad C \underset{u'u}{\overset{ff'}{\underset{\bot}{\rightleftarrows}}} A$$

the composite functors are adjoint.

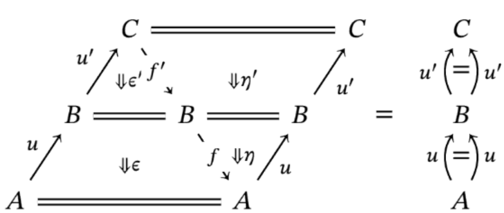
Proof Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_C \Rightarrow u'f'$, and $\epsilon'\colon f'u' \Rightarrow \mathrm{id}_B$ for the respective units and counits, the pasting diagrams



define the unit and counit of $ff' \dashv u'u$ so that the triangle equalities hold:



□

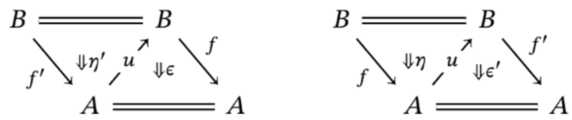


An adjoint to a given functor is unique up to natural isomorphism:

Proposition 2.1.10 (uniqueness of adjoints).

(i) If $f \dashv u$ *and* $f' \dashv u$*, then* $f \cong f'$*.*
(ii) Conversely, if $f \dashv u$ *and* $f \cong f'$*, then* $f' \dashv u$*.*

Proof Writing $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$, $\eta'\colon \mathrm{id}_B \Rightarrow uf'$, and $\epsilon'\colon f'u \Rightarrow \mathrm{id}_A$ for the respective units and counits, the pasting diagrams



define 2-cells $f \Rightarrow f'$ and $f' \Rightarrow f$. The composites $f \Rightarrow f' \Rightarrow f$ and $f' \Rightarrow f \Rightarrow f'$ are computed by pasting these diagrams together horizontally on one side or on the other. Applying the triangle equalities for the adjunctions $f \dashv u$ and $f' \dashv u$ both composites are easily seen to be identities. Hence $f \cong f'$ as functors from B to A.

Part (ii) is left as Exercise 2.1.ii. □

The following result weakens the hypotheses of Definition 2.1.1.

Lemma 2.1.11 (minimal adjunction data). *A pair of functors* $f\colon B \to A$ *and* $u\colon A \to B$ *form an adjoint pair* $f \dashv u$ *if and only if there exist natural transformations* $\mathrm{id}_B \Rightarrow uf$ *and* $fu \Rightarrow \mathrm{id}_A$ *so that the triangle equality composites* $f \Rightarrow fuf \Rightarrow f$ *and* $u \Rightarrow ufu \Rightarrow u$ *are both invertible.*

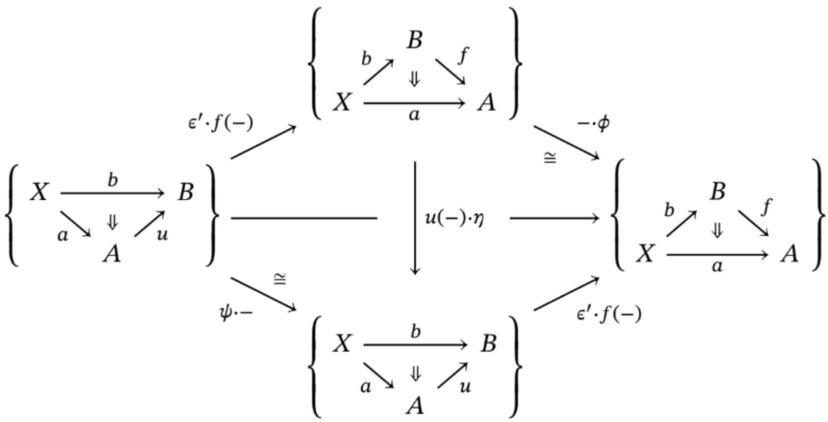
Proof The unit and counit of an adjunction certainly satisfy these hypotheses. For the converse, consider natural transformations $\eta\colon \mathrm{id}_B \Rightarrow uf$ and $\epsilon'\colon fu \Rightarrow \mathrm{id}_A$ so that the triangle equality composites

$$\phi := f \xRightarrow{f\eta} fuf \xRightarrow{\epsilon' f} f \qquad \psi := u \xRightarrow{\eta u} ufu \xRightarrow{u\epsilon'} u$$

are isomorphisms. We construct an adjunction $f \dashv u$ with unit η by modifying ϵ' to form the counit ϵ.[5] To explain the idea of the construction, note that for a fixed pair of generalized elements $b\colon X \to B$ and $a\colon X \to A$, pasting with η and

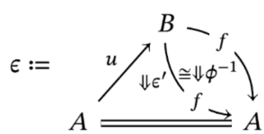
[5] By the co-dual of this construction, we could alternatively take ϵ' to be the counit at the cost of modifying η to form the unit (see Exercise 2.1.iii).

with ϵ' defines functions between the displayed sets of natural transformations:

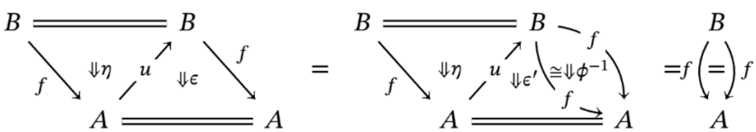


From the hypothesis that the triangle equality composites are isomorphisms, two of these functions are invertible, and then by the 2-of-6 property for isomorphisms all six maps are bijections.

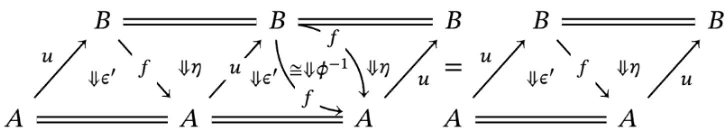
Define the "corrected" counit to be the composite:



so that one of the triangle equality composites reduces to the identity:



Now from the pasting equality

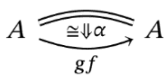


we see that $(u\epsilon \cdot \eta u) \cdot \psi = \psi$. Since ψ is invertible, we may cancel to conclude that $u\epsilon \cdot \eta u = \mathrm{id}_u$. □

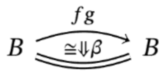
A standard 2-categorical result is that any equivalence in a 2-category can be promoted to an equivalence that also defines an adjunction:

Proposition 2.1.12 (adjoint equivalences). *Any equivalence can be promoted to an* ***adjoint equivalence*** *by modifying one of the 2-cells. That is, the invertible 2-cells in an equivalence can be chosen so as to satisfy the triangle equalities. Hence, if f and g are inverse equivalences then $f \dashv g$ and $g \dashv f$.*

Proof Consider an equivalence comprised of functors $f : A \to B$ and $g : B \to A$ and invertible 2-cells



and



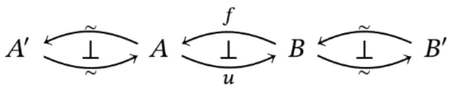
Since α and β are both invertible, the triangle equality composites are as well, and the construction of Lemma 2.1.11 applies. □

One use of Proposition 2.1.12 is to show that adjunctions are equivalence invariant:

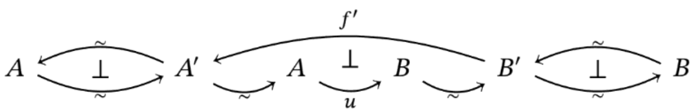
Proposition 2.1.13. *A functor $u : A \to B$ between ∞-categories admits a left adjoint if and only if, for any pair of equivalent ∞-categories $A' \simeq A$ and $B' \simeq B$, the equivalent functor $u' : A' \to B'$ admits a left adjoint.*

As we shall discover, all of ∞-category theory is equivalence invariant in this way.

Proof If $u : A \to B$ admits a left adjoint then by composing $f \dashv u$ with the adjoint equivalences $A' \simeq A$ and $B \simeq B'$ we obtain an equivalent adjunction:



Conversely, if the equivalent functor $u' : A' \xrightarrow{\sim} A \xrightarrow{u} B \xrightarrow{\sim} B'$ admits a left adjoint f' then again we obtain a composite adjunction:



whose right adjoint is naturally isomorphic to the original functor u. By Proposition 2.1.10 the displayed left adjoint is then a left adjoint to u. □

For later use, we close with an example of an abstractly defined adjunction that can be constructed for any ∞-category in any ∞-cosmos via the results proven in this section.

live inside ∞-categories A, which are the objects of ∞-cosmoi $\mathcal{K}$ – which themselves define "infinite-dimensional categories," albeit of a different sort.

Definition 2.2.1 (initial/terminal element). An **initial element** in an ∞-category A is a left adjoint to the unique functor $! \colon A \to 1$, as displayed below-left, while a **terminal element** in an ∞-category A is a right adjoint, as displayed below-right.

$$1 \underset{!}{\overset{i}{\underset{\bot}{\rightleftarrows}}} A \qquad\qquad 1 \underset{t}{\overset{!}{\underset{\bot}{\leftrightarrows}}} A$$

Let us unpack the definition of an initial element; dual remarks apply to terminal elements.

Lemma 2.2.2 (minimal data). *To define an initial element in an ∞-category A, it suffices to specify*

- *an element $i \colon 1 \to A$ and*
- *a natural transformation* $\begin{array}{ccc} & 1 & \\ {}^{!}\nearrow & \Downarrow\epsilon & \searrow^{i} \\ A & = & A \end{array}$ *from the constant functor at i to the identity functor*

so that the component $\epsilon i \colon i \Rightarrow i$, an arrow from i to i in $\mathfrak{h}A$, is invertible.

Proof Proposition 1.4.5, whose proof starts in the paragraph before its statement, demonstrates that the ∞-category $1 \in \mathcal{K}$ is 2-terminal in the homotopy 2-category $\mathfrak{h}\mathcal{K}$. The 1-dimensional aspect of this universal property implies that any element $i \colon 1 \to A$ defines a section of the unique map $! \colon A \to 1$, while the 2-dimensional aspect asserts that there exist no nonidentity 2-cells with codomain 1. In particular, the unit of the adjunction $i \dashv !$ is necessarily an identity and one of the triangle equalities comes for free. What remains of Definition 2.1.1 in this setting is the data of a counit natural transformation $\epsilon \colon i! \Rightarrow \mathrm{id}_A$ together with the condition that its component $\epsilon i = \mathrm{id}_i$. But in fact we can prove that this natural transformation must be the identity from the weaker and more natural assumption that $\epsilon i \colon i \cong i$ is invertible.

To see this consider, the horizontal composite

$$1 \xrightarrow{i} \begin{array}{ccccc} & 1 & & 1 & \\ {}^{!}\nearrow & \Downarrow\epsilon & \searrow^{i} \;\; {}^{!}\nearrow & \Downarrow\epsilon & \searrow^{i} \\ A & = & A & = & A \end{array} \quad \leftrightsquigarrow \quad \begin{array}{ccc} i!i!i & \xRightarrow{\epsilon i!i} & i!i \\ {\scriptstyle i!\epsilon i}\Downarrow & & \Downarrow{\scriptstyle \epsilon i} \\ i!i & \xRightarrow[\epsilon i]{} & i \end{array}$$

By naturality of whiskering,[7] we can evaluate this composite as a vertical composite in two ways. Since 1 is 2-terminal, the whiskered cell $!\epsilon = \mathrm{id}_!$, so the composition relation reduces to $\epsilon i \cdot \epsilon i = \epsilon i$. Thus ϵi is an idempotent isomorphism, and hence, by cancelation, an identity. ☐

Put more concisely, an initial element defines a *left adjoint right inverse* to the functor $! \colon A \twoheadrightarrow 1$, while a terminal element defines a *right adjoint right inverse* (see §B.4).

Lemma 2.2.3 (uniqueness). *Any two initial elements in an ∞-category A are isomorphic in $\mathsf{h}A$ and any element of $\mathsf{h}A$ that is isomorphic to an initial element is initial.*

Proof By Proposition 2.1.10, any two left adjoints i and i' to the functor $! \colon A \to 1$ are naturally isomorphic, and any $a \colon 1 \to A$ that is isomorphic to a left adjoint to $! \colon A \to 1$ is itself a left adjoint. A natural isomorphism between a pair of functors $i, i' \colon 1 \to A$ gives exactly the data of an isomorphism $i \cong i'$ between the corresponding elements of the homotopy category $\mathsf{h}A$. ☐

Remark 2.2.4. Applying the 2-functor $\mathsf{Fun}(X, -) \colon \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{QC}at$ to an initial element $i \colon 1 \to A$ of an ∞-category $A \in \mathcal{K}$ yields an adjunction

$$\mathbb{1} \cong \mathsf{Fun}(X,1) \underset{!}{\overset{i_*}{\underset{\bot}{\rightleftarrows}}} \mathsf{Fun}(X,A)$$

Via the isomorphism $\mathsf{Fun}(X,1) \cong \mathbb{1}$ that expresses the universal property of the terminal ∞-category 1, the constant functor at an initial element

$$X \xrightarrow{\ !\ } 1 \xrightarrow{\ i\ } A$$

defines an initial element of the functor space $\mathsf{Fun}(X, A)$. This observation can be summarized by saying that initial elements are representably initial at the level of the ∞-cosmos.

Conversely, if $i \colon 1 \to A$ is representability initial, then i defines an initial element of A. This is most easily seen by passing to the homotopy 2-category, where we can show that an initial element $i \colon 1 \to A$ is initial among all *generalized elements* $f \colon X \to A$ in the following precise sense.

[7] "Naturality of whiskering" refers to the observation of Lemma B.1.3 that any horizontal-composite of 2-cells in a 2-category can be expressed as a vertical composite of whiskerings of those cells in two different ways, in this case giving rise to the commutative diagram in $\mathsf{h}A \coloneqq \mathsf{hFun}(1, A)$ displayed above-right.

Lemma 2.2.5. *An element* $i\colon 1 \to A$ *is initial if and only if for all* $f\colon X \to A$ *there exists a unique 2-cell with boundary*

$$\begin{array}{ccc} & 1 & \\ {}^{!}\nearrow & \Downarrow\exists! & \searrow^{i} \\ X & \xrightarrow[f]{} & A \end{array}$$

Proof If $i\colon 1 \to A$ is initial, then the adjunction of Definition 2.2.1 is preserved by the 2-functor $\mathsf{hFun}(X,-)\colon \mathfrak{h}\mathcal{K} \to \mathcal{C}at$, defining an adjunction

$$\mathbb{1} \cong \mathsf{hFun}(X,1) \underset{!}{\overset{i_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{hFun}(X,A)$$

Via the isomorphism $\mathsf{hFun}(X,1) \cong \mathbb{1}$, this adjunction proves that the constant functor $i!\colon X \to A$ is initial in the category $\mathsf{hFun}(X,A)$ and thus has the universal property of the statement.

Conversely, if $i\colon 1 \to A$ satisfies the universal property of the statement, applying this to the generic element of A (the identity map $\mathrm{id}_A\colon A \to A$) produces the data of Lemma 2.2.2. □

Lemma 2.2.5 says that initial elements are representably initial in the homotopy 2-category. Specializing the generalized elements to ordinary elements, we see that initial and terminal elements in A respectively define initial and terminal elements in its homotopy category:

$$\mathbb{1} \underset{t}{\overset{i}{\rightleftarrows}} \mathsf{hA} \qquad \text{with } i \dashv ! \dashv t \tag{2.2.6}$$

In general the property of being "homotopy initial," i.e., initial in the homotopy category, is weaker than being initial in the ∞-category. However Nguyen, Raptis, and Schrade observe that a homotopy initial element in a complete $(\infty,1)$-category necessarily defines an initial element [88, 2.2.2].

Continuing the theme of the equivalence invariance of ∞-categorical notions:

Lemma 2.2.7. *If* A *has an initial element and* $A \simeq A'$ *then* A' *has an initial element and these elements are preserved up to isomorphism by the equivalences.*

Proof By Proposition 2.1.12, the equivalence $A \simeq A'$ can be promoted to an adjoint equivalence, which can immediately be composed with the adjunction characterizing an initial element i of A:

$$1 \underset{!}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} A \underset{\sim}{\overset{\sim}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} A'$$

The composite adjunction provided by Proposition 2.1.9 proves that the image of i defines an initial element of A', which by construction is preserved by the equivalence $A \xrightarrow{\sim} A'$. By the uniqueness of initial elements established in Lemma 2.2.3, this argument also shows that the equivalence $A' \xrightarrow{\sim} A$ preserves initial elements. □

We now turn to the general theory of limits and colimits of diagrams valued in an ∞-category. The theory of initial elements previews this material well since in fact an initial element can be understood as an example of both notions: an initial element is the colimit of the empty diagram and also the limit of the diagram encoded by the identity functor, as we explain in Example 2.3.11.

Exercises

Exercise 2.2.i. Use Lemma 2.2.5 to show that a representably initial element, as described in Remark 2.2.4, necessarily defines an initial element in A.

Exercise 2.2.ii. Prove that initial elements are preserved by left adjoints and terminal elements are preserved by right adjoints.

2.3 Limits and Colimits

We now introduce limits and colimits of diagram valued *inside* an ∞-category A in some ∞-cosmos. We consider two varieties of diagrams:

- diagrams indexed by a simplicial set J and valued in an ∞-category A in a generic ∞-cosmos and
- diagrams indexed by an ∞-category J and valued in an ∞-category A in a cartesian closed ∞-cosmos.[8]

Definition 2.3.1 (diagram ∞-category). For an ∞-category A and a simplicial set J – or possibly, in the case of a cartesian closed ∞-cosmos, an ∞-category J – we refer to A^J as the **∞-category of J-shaped diagrams in A**. A **diagram** of shape J in A is an element $d\colon 1 \to A^J$.[9]

[8] For the ∞-cosmoi of $(\infty,1)$-categories of Example 1.2.24, there is no essential difference between these notions: in $\mathcal{QC}at$ they are tautologically the same, and in all biequivalent ∞-cosmoi the ∞-category of diagrams indexed by an ∞-category J is equivalent to the ∞-category of diagrams indexed by its underlying quasi-category, regarded as a simplicial set (see Proposition 10.3.5).

[9] When A^J is the exponential of a cartesian closed ∞-cosmos, diagrams stand in bijection with functors $d\colon J \to A$.

Both constructions of the ∞-category of diagrams in an ∞-cosmos $\mathcal{K}$ define simplicial bifunctors

$$\begin{array}{ccc} s\mathcal{S}et^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} & \qquad & \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K} \\ (J, A) \longmapsto A^J & \qquad & (J, A) \longmapsto A^J \end{array}$$

In either indexing context, there is a terminal object 1 with the property that $A^1 \cong A$ for any ∞-category A. Restriction along the unique map $! \colon J \to 1$ induces the **constant diagram functor** $\Delta \colon A \to A^J$.

We deliberately conflate the notation for ∞-categories of diagrams indexed by a simplicial set or by another ∞-category because all of the results we prove in Part I about the former case also apply to the latter. For economy of language, we refer only to simplicial set indexed diagrams for the remainder of this section.

Definition 2.3.2 (limit and colimit functor). An ∞-category A **admits all colimits** of shape J if the constant diagram functor $\Delta \colon A \to A^J$ admits a left adjoint, while A **admits all limits** of shape J if the constant diagram functor admits a right adjoint:

$$A^J \;\underset{\lim}{\overset{\mathrm{colim}}{\underset{\perp}{\overset{\perp}{\longleftarrow \Delta \longrightarrow}}}}\; A$$

In the ∞-cosmos of categories, Definition 2.3.2 reduces to the classically defined limit and colimit functors, but in a general ∞-category limits and colimits should be thought of as analogous to the classical notions of "homotopy limits" and "homotopy colimits." In certain cases, this correspondence can be made precise. Every quasi-category is equivalent to the homotopy coherent nerve of a Kan complex enriched category [111, 7.2.2], and homotopy limit or homotopy colimit cones in the Kan complex enriched category correspond exactly to limit or colimit cones in the homotopy coherent nerve (see Lurie's [78, 4.2.4.1] or [113, 6.1.4, 6.2.7]). In the ∞-categorical context, no stricter notion of limit or colimit is available, so the "homotopy" qualifier is typically dropped.

Limits or colimits of set-indexed diagrams – the case where the indexing shape is a coproduct of the terminal object 1 indexed by a set J – are called **products** or **coproducts**, respectively.

Lemma 2.3.3. *Products or coproducts in an ∞-category A also define products or coproducts in its homotopy category $\mathrm{h}A$.*

Proof When J is a set, the ∞-category of diagrams itself decomposes as a

Proof The universal property of the absolute right lifting diagram

$$\begin{array}{ccc} X & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle f} \\ A & = & A \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & {\scriptstyle \exists!\Downarrow\beta}\ \nearrow^{u}\ {\scriptstyle \Downarrow\epsilon} & \downarrow{\scriptstyle f} \\ A & = & A \end{array}$$

asserts that every natural transformation $\alpha\colon fb \Rightarrow a$ has a unique transpose $\beta\colon b \Rightarrow ua$ across the adjunction between the hom-categories of the homotopy 2-category:

$$\mathsf{hFun}(X,B) \underset{u_*}{\overset{f_*}{\underset{\bot}{\rightleftarrows}}} \mathsf{hFun}(X,A)$$

Thus if $f \dashv u$ with counit ϵ, Proposition 2.1.7(ii) supplies this induced adjunction and (u,ϵ) defines an absolute right lifting of id_A through f.

Conversely, the unit and triangle equalities of an adjunction can extracted from the universal property of the absolute right lifting diagram. The details are left as Exercise 2.3.iii. □

In particular, the unit and counit of the adjunctions colim $\dashv \Delta \dashv$ lim of Definition 2.3.2 define absolute left and right lifting diagrams:

$$\begin{array}{ccc} & & A \\ & {}^{\mathrm{colim}}\nearrow \ \Uparrow\eta & \downarrow{\scriptstyle \Delta} \\ A^J & = & A^J \end{array} \qquad \begin{array}{ccc} & & A \\ & {}^{\lim}\nearrow \ \Downarrow\epsilon & \downarrow{\scriptstyle \Delta} \\ A^J & = & A^J \end{array}$$

By Lemma 2.3.6, these universal properties are retained upon restricting to any subobject of the ∞-category of diagrams. This motivates the following definition:

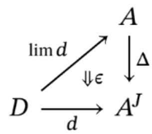
Definition 2.3.8 (limit and colimit). A **colimit** of a family of diagrams $d\colon D \to A^J$ of shape J in an ∞-category A is given by an absolute left lifting diagram

$$\begin{array}{ccc} & & A \\ & {}^{\mathrm{colim}\,d}\nearrow \ \Uparrow\eta & \downarrow{\scriptstyle \Delta} \\ D & \xrightarrow[d]{} & A^J \end{array}$$

comprised of a generalized element $\mathrm{colim}\,d\colon D \to A$ and a **colimit cone** $\eta\colon d \Rightarrow \Delta\,\mathrm{colim}\,d$.

Dually, a **limit** of a family of diagrams $d\colon D \to A^J$ of shape J in an ∞-

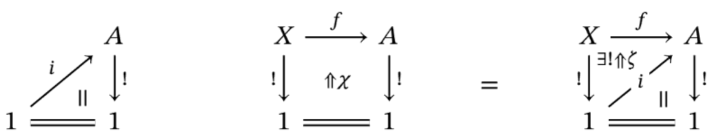
category A is given by an absolute right lifting diagram



comprised of a generalized element $\lim d \colon D \to A$ and a **limit cone** $\epsilon \colon \Delta \lim d \Rightarrow d$.

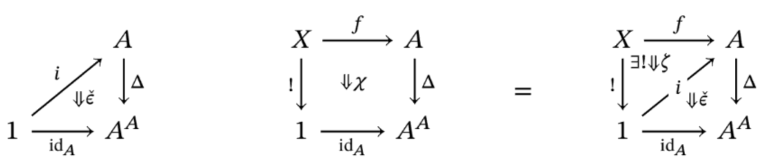
Remark 2.3.9. If A has all limits of shape J, then Lemma 2.3.6 implies that any family of diagrams $d \colon D \to A^J$ has a limit, defined by composing the limit functor $\lim \colon A^J \to A$ with d. In an ∞-cosmos of $(\infty, 1)$-categories, if every diagram $d \colon 1 \to A^J$ has a limit, then A admits all limits of shape J (see Corollary 12.2.10), but in general families of diagrams cannot be reduced to single diagrams.

Example 2.3.10. An initial element $i \colon 1 \to A$ can be regarded as a colimit of the empty diagram. The ∞-category $A^{\varnothing} \simeq 1$ of empty diagrams in A is terminal, so the constant diagram functor reduces to $! \colon A \to 1$. To show that initial elements are colimits in the sense of Definition 2.3.8, we must verify that an initial element defines an absolute left lifting diagram whose 2-cell is the identity:



Since the ∞-category 1 is 2-terminal, there is a unique 2-cell χ inhabiting the central square above, namely the identity. Thus, the universal property of the absolute left lifting diagram asserts the existence of a unique 2-cell $\zeta \colon i! \Rightarrow f$ for any $f \colon X \to A$, exactly as provided by Lemma 2.2.5.

Example 2.3.11. In a cartesian closed ∞-cosmos, an initial element $i \colon 1 \to A$ can also be regarded as a limit of the identity functor $\mathrm{id}_A \colon A \to A$.[11] The counit $\epsilon \colon i! \Rightarrow \mathrm{id}_A$ of the adjunction $i \dashv !$ transposes across the 2-adjunction $A \times - \dashv (-)^A$ of Proposition 1.4.5 to define the limit cone displayed below-left:



11 This result is extended to ∞-cosmoi that are not cartesian closed in Proposition 9.4.10.

The universal property displayed above-right is easiest to verify by transposing across the 2-adjunction $A \times - \dashv (-)^A$ again, where we must establish the pasting equality

$$
\begin{array}{ccc}
 & X & \\
{}^{\pi_X}\nearrow & & \\
X \times A & \Downarrow\hat{\chi} & \downarrow f \\
{}_{\pi_A}\searrow & & \\
 & A &
\end{array}
\;=\;
\begin{array}{ccccc}
 & X & \xrightarrow{\quad f \quad} & & \\
{}^{\pi_X}\nearrow & & \searrow ! & \exists!\Downarrow\zeta & \\
X \times A & & 1 & \xrightarrow{\;i\;} & A \\
{}_{\pi_A}\searrow & & \nearrow ! & \Downarrow\epsilon & \\
 & A & = & = & A
\end{array}
\tag{2.3.12}
$$

Observe that when we restrict the right-hand side of (2.3.12) along the functor $\mathrm{id}_X \times i \colon X \cong X \times 1 \to X \times A$ we recover the 2-cell ζ, since $\epsilon i = \mathrm{id}_i$. This tells us that given χ, we must necessarily define the 2-cell $\zeta \colon f \Rightarrow i!$ to be the restriction of $\hat{\chi}$ along the functor $\mathrm{id}_X \times i \colon X \to X \times A$.

From this definition of ζ and the 2-functoriality of the cartesian product – which tells us that $\epsilon\pi_A = \pi_A(X \times \epsilon)$ – we have

$$
\begin{array}{ccccc}
 & X & \xrightarrow{\quad f \quad} & & \\
{}^{\pi_X}\nearrow & & \searrow ! & \Downarrow\zeta & \\
X \times A & & 1 & \xrightarrow{\;i\;} & A \\
{}_{\pi_A}\searrow & & \nearrow ! & \Downarrow\epsilon & \\
 & A & = & = & A
\end{array}
\;=\;
X \times A \;
\begin{array}{c}
\xrightarrow{\mathrm{id}_X \times !} X \times 1 \xrightarrow{\mathrm{id}_X \times i} \\
\Downarrow X \times \epsilon \\
=\!=\!=\!=\!=\!=
\end{array}
\;
\begin{array}{ccc}
 & X & \\
{}^{\pi_X}\nearrow & & \\
X \times A & \Downarrow\hat{\chi} & \downarrow f \\
{}_{\pi_A}\searrow & & \\
 & A &
\end{array}
$$

By "naturality of whiskering" (see Lemma B.1.3), the right-hand pasted composite can be computed as the vertical composite of $\pi_X(X \times \epsilon)$ followed by $\hat{\chi}$, but $\pi_X(X \times \epsilon)$ is the identity 2-cell, so this composite is just $\hat{\chi}$. This verifies the desired pasting equality (2.3.12).

Certain limits and colimits in ∞-categories exist for formal reasons. For example, an abstract 2-categorical lemma enables a formal proof of a classical result from homotopy theory that computes the colimits, typically called *geometric realizations*, of "split" simplicial objects. Before proving this, we introduce the indexing shapes involved.

Definition 2.3.13 (split augmented (co)simplicial object). The simplex category $\mathbf{\Delta}$ of finite nonempty ordinals and order-preserving maps introduced in 1.1.1 defines a full subcategory of the category $\mathbf{\Delta}_+$ of finite ordinals and order-preserving maps, which freely appends the empty ordinal "$[-1]$" as an initial object. The category $\mathbf{\Delta}_+$ in turn defines a wide subcategory of a category $\mathbf{\Delta}_\perp$, which adds an "extra" degeneracy $\sigma^{-1} \colon [n+1] \twoheadrightarrow [n]$ between each pair of consecutive ordinals, including $\sigma^{-1} \colon [0] \twoheadrightarrow [-1]$. The category $\mathbf{\Delta}_+$ also defines a wide subcategory of a category $\mathbf{\Delta}_\top$, which adds an "extra" degeneracy

$\sigma^{n+1} : [n+1] \twoheadrightarrow [n]$ on the other side between each pair of consecutive ordinals, including $\sigma^0 : [0] \twoheadrightarrow [-1]$. The categories $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ can be described in another way: there are faithful embeddings of these categories into $\mathbf{\Delta}$ that act on objects by $[n] \mapsto [n+1]$ and identify $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ with the subcategories of finite nonempty ordinals and order-preserving maps that preserve the bottom and top elements respectively.

Covariant diagrams indexed by $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp, \mathbf{\Delta}_\top$ are, respectively, called **cosimplicial objects**, **coaugmented cosimplicial objects**, and **split coaugmented cosimplicial objects** (in the case of either $\mathbf{\Delta}_\perp$ or $\mathbf{\Delta}_\top$), while contravariant diagrams are respectively called **simplicial objects**, **augmented simplicial objects**, and **split augmented simplicial objects**. When it is useful to disambiguate between $\mathbf{\Delta}_\perp$ and $\mathbf{\Delta}_\top$ we refer to the former category as a "bottom splitting" and the latter category as a "top splitting," but this terminology is not standard.

A cosimplicial object $d : 1 \to A^{\mathbf{\Delta}}$ in an ∞-category A **admits a coaugmentation** or **admits a splitting** if it lifts along the restriction functors

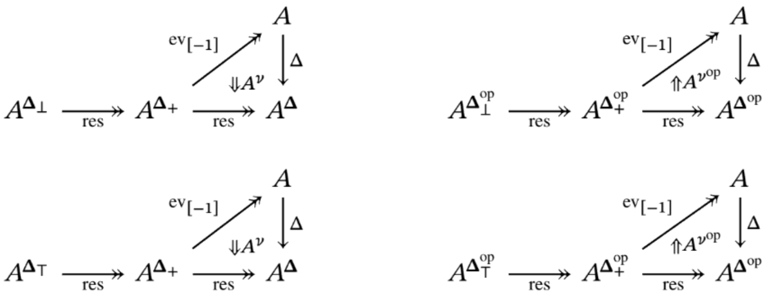
$$\begin{array}{ccc} & & A^{\mathbf{\Delta}_\perp} \\ & \nearrow & \downarrow \text{res} \\ & & A^{\mathbf{\Delta}_+} \\ & \nearrow & \downarrow \text{res} \\ 1 & \xrightarrow[d]{} & A^{\mathbf{\Delta}} \end{array}$$

where in the case of a top splitting, $\mathbf{\Delta}_\perp$ is replaced by $\mathbf{\Delta}_\top$. The family of cosimplicial objects admitting a coaugmentation and splitting is represented by the generalized element $\text{res} : A^{\mathbf{\Delta}_\perp} \twoheadrightarrow A^{\mathbf{\Delta}}$. In any augmented cosimplicial object, there is a cone over the underlying cosimplicial object whose summit is obtained by evaluating at $[-1] \in \mathbf{\Delta}_+$. This cone is defined by cotensoring with the unique natural transformation

$$\begin{array}{ccc} \mathbf{\Delta} & \hookrightarrow & \mathbf{\Delta}_+ \\ & {\scriptstyle !}\searrow \;\; \exists! \Uparrow \nu \;\; \nearrow{\scriptstyle [-1]} & \\ & \mathbb{1} & \end{array} \tag{2.3.14}$$

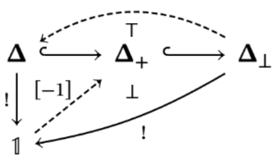
that exists because $[-1] : \mathbb{1} \to \mathbf{\Delta}_+$ is initial (see Lemma 2.2.5).

Proposition 2.3.15 (totalization/geometric realization). *Let A be any ∞-category. Every cosimplicial object in A that admits a coaugmentation and a splitting has a limit, whose limit cone is defined by the coaugmentation. Dually, every simplicial object in A that admits an augmentation and a splitting has a colimit, whose colimit cone is defined by the augmentation. That is, there exist* *absolute right and left lifting diagrams*

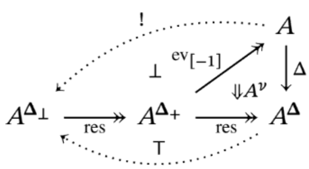


in which the 2-cells are obtained as restrictions of the cotensor of the 2-cell (2.3.14) *into A. Moreover, such limits and colimits are* ***absolute****, preserved by any functor* $f\colon A \to B$ *of* ∞*-categories.*

Proof By Example B.5.2, the inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_\perp$ admits a right adjoint, which can automatically be regarded as an adjunction "over $\mathbb{1}$" since $\mathbb{1}$ is 2-terminal in $\mathcal{C}at$. The initial element $[-1] \in \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp$ defines a left adjoint to the constant functor:



and the counit of this adjunction restricts along the inclusions $\mathbf{\Delta} \subset \mathbf{\Delta}_+ \subset \mathbf{\Delta}_\perp$ to the 2-cell (2.3.14). For any ∞-category A in an ∞-cosmos $\mathcal{K}$, these adjunctions are preserved by the 2-functor $A^{(-)}\colon \mathcal{C}at^{\mathrm{op}} \to \mathfrak{h}\mathcal{K}$, yielding a diagram



By Lemma B.5.1 these adjunctions witness the fact that evaluation at $[-1]$ and the 2-cell from (2.3.14) define an absolute right lifting of the canonical restriction functor $A^{\mathbf{\Delta}_\perp} \twoheadrightarrow A^{\mathbf{\Delta}}$ through the constant diagram functor, as claimed. The colimit case is proven similarly by applying the composite 2-functor

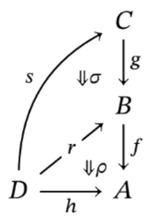
$$\mathcal{C}at^{\mathrm{coop}} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}at^{\mathrm{op}} \xrightarrow{A^{(-)}} \mathfrak{h}\mathcal{K}$$

A similar argument, starting from Example B.5.3, constructs the absolute lifting diagrams from the top splitting.

case where the ∞-categories in question admit *all* limits of a given shape: under these hypotheses, the limit functor is right adjoint to the constant diagram functor, which commutes with all functors between the base ∞-categories. Since the left adjoints commute, the uniqueness of adjoints (Proposition 2.1.10) implies that the right adjoints commute up to isomorphism. This outline gives a hint for Exercise 2.4.i.

A more delicate argument is needed in the general case, involving, say, the preservation of a single limit diagram without a priori assuming that any other limits exist. We appeal to a general lemma about composition and cancelation of absolute lifting diagrams:

Lemma 2.4.1 (composition and cancelation of absolute lifting diagrams). *Suppose (r, ρ) defines an absolute right lifting of h through f:*



Then (s, σ) defines an absolute right lifting of r through g if and only if $(s, \rho \cdot f\sigma)$ defines an absolute right lifting of h through fg.

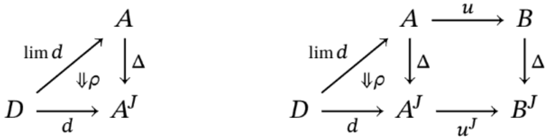
Proof Exercise 2.4.ii. □

Theorem 2.4.2. *Right adjoints preserve limits and left adjoints preserve colimits.*

The usual argument that right adjoints preserve limits is this: a cone over a J-shaped diagram in the image of a right adjoint u transposes across the adjunction $f^J \dashv u^J$ to a cone over the original diagram, which factors uniquely through the designated limit cone. This factorization transposes across the adjunction $f \dashv u$ to define the sought-for unique factorization through the image of the limit cone. An ∞-categorical proof along these lines can be given as well (see Exercise 2.4.iii), but instead we present a slicker packaging of the standard argument. We use absolute lifting diagrams to express the universal properties of limits and colimits (Definition 2.3.8) and adjoint transposition (Lemma 2.3.7), allowing us to suppress consideration of a generic test cone that must be shown to uniquely factor through the limit cone.

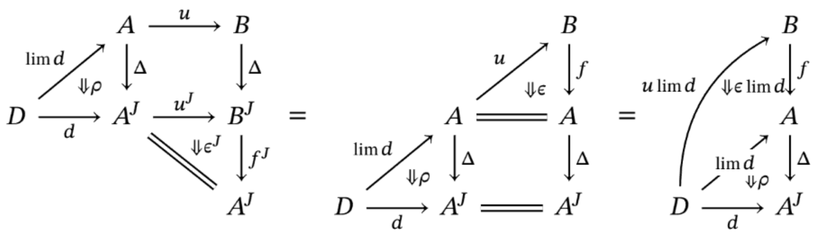
Proof We prove that right adjoints preserve limits. By taking co-duals the same argument demonstrates that left adjoints preserve colimits.

Suppose a functor $u: A \to B$ in an ∞-cosmos $\mathcal{K}$ admits a left adjoint $f: B \to A$ with counit $\epsilon: fu \Rightarrow \mathrm{id}_A$. Our aim is to show that any absolute right lifting diagram as displayed below-left is carried to an absolute right lifting diagram as displayed below-right:



(2.4.3)

By Proposition 2.1.7, the cotensor $(-)^J: \mathfrak{h}\mathcal{K} \to \mathfrak{h}\mathcal{K}$ carries the adjunction $f \dashv u$ to an adjunction $f^J \dashv u^J$ with counit ϵ^J. In particular, by Lemma 2.3.7, (u^J, ϵ^J) defines an absolute right lifting of the identity through f^J, which is then preserved by restriction along the functor d. Thus, by Lemma 2.4.1, the diagram on the right of (2.4.3) is an absolute right lifting diagram if and only if the pasted composite displayed below-left defines an absolute right lifting diagram:



As noted in the proof of Lemma 2.3.7, pasting the 2-cell on the right of (2.4.3) with the counit in this way amounts to transposing the cone $u^J\rho$ across the adjunction $f^J \dashv u^J$.

We now argue that this transposed cone above-left factors through the limit cone $(\lim d, \rho)$ in a canonical way. From the 2-functoriality of the simplicial cotensor in its exponent variable, $f^J\Delta = \Delta f$ and $\epsilon^J\Delta = \Delta\epsilon$. Hence, the pasting diagram displayed above-left equals the one displayed above-center, which equals the diagram above-right. This latter diagram is a pasted composite of two absolute right lifting diagrams, and is then an absolute right lifting diagram in its own right by Lemma 2.4.1; this universal property says that any cone over d whose summit factors through f factors uniquely through the limit cone $(\lim d, \rho)$ through a map that then transposes along the adjunction $f \dashv u$. Hence the diagram on the right-hand side of (2.4.3) is an absolute right lifting diagram as claimed. □

Proposition 2.4.4. *An equivalence $f: A \xrightarrow{\sim} B$ preserves, reflects, and creates limits and colimits.*

Proof By Proposition 2.1.12, equivalences define adjoint functors, so Theorem 2.4.2 implies that equivalences preserve limits. To see that limits are reflected, consider a J-shaped cone ρ in A whose image $f^J\rho$ is a limit cone in B. The inverse equivalence $g\colon B \xrightarrow{\sim} A$ carries this to a limit cone $g^J f^J \rho$ in A, which is naturally isomorphic to the original cone ρ. By Exercise 2.3.vi, ρ must also define a limit cone. Finally to see that limits are created, consider a diagram $d\colon D \to A^J$ so that fd has a limit cone ν in B. Then $g^J\nu$ defines a limit cone for the diagram gfd in A, and by Exercise 2.3.vi, a limit cone for d may be defined by composing with the isomorphism $gfd \cong d$. □

We turn now to a limit-preservation result of another sort, which can be used to simplify the calculation of limits or colimits of diagrams with particular shapes. This simplification comes about by reindexing the diagrams, by restricting along a functor $k\colon I \to J$. For certain functors, called "initial" or "final," this reindexing preserves and reflects limits or colimits, respectively.

At present, we give a teleological, rather than an intrinsic, description of these functors. The following definition makes sense for an arbitrary functor in a cartesian closed ∞-cosmos or for a map between simplicial sets serving as indexing shapes in an arbitrary ∞-cosmos. In Definition 9.4.11 we extend the adjectives "initial" and "final" to functors between ∞-categories in an arbitrary ∞-cosmos and prove that the functors characterized there satisfy the property described here.

Definition 2.4.5 (initial and final functor). A functor $k\colon I \to J$ is **final** if a J-shaped cone defines a colimit cone if and only if the restricted I-shaped cone is a colimit cone and **initial** if any J-shaped cone defines a limit cone if and only if the restricted I-shaped cone is a limit cone. That is, $k\colon I \to J$ is final if and only if for any ∞-category A, the square

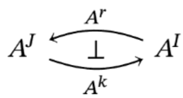
$$\begin{array}{ccc} A & = & A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ A^J & \xrightarrow[A^k]{} & A^I \end{array}$$

preserves and reflects all absolute left lifting diagrams, and initial if and only if this squares preserves and reflects all absolute right lifting diagrams.

Historically, final functors were called "cofinal" with no obvious name for the dual notion. Our preferred terminology hinges on the following mnemonic: the inclusion of an initial element defines an initial functor, while the inclusion of a terminal (aka final) element defines a final functor. These facts are special cases of a more general result we now establish, using exactly the same tactics as deployed to prove Theorem 2.4.2.

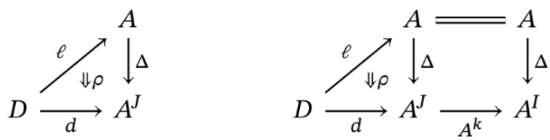
Proposition 2.4.6. *Left adjoints define initial functors and right adjoints define final functors.*

Proof If $k \dashv r$ with counit $\epsilon\colon kr \Rightarrow \mathrm{id}_J$, then cotensoring into A yields an adjunction



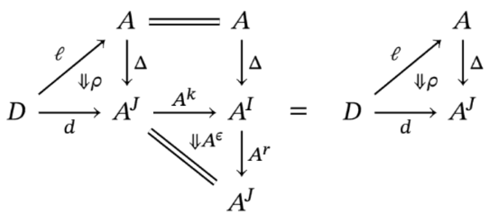
with counit $A^\epsilon\colon A^r A^k \Rightarrow \mathrm{id}_{A^J}$.

To prove that k is initial we must show that for any cone $\rho\colon \Delta\ell \Rightarrow d$ as displayed below-left,



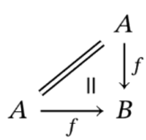
the left-hand diagram is an absolute right lifting diagram if and only if the right-hand diagram is an absolute right lifting diagram.

By Lemmas 2.3.7 and 2.4.1, the right-hand diagram is an absolute right lifting diagram if and only if the pasted composite displayed below-left is also an absolute right lifting diagram.



Since $A^r\Delta = \Delta$ and $A^\epsilon\Delta = \mathrm{id}_\Delta$, the left-hand side reduces to the right-hand side, which proves the claim. □

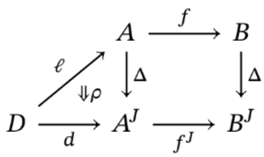
Exercise 2.3.v defines a functor $f\colon A \to B$ between ∞-categories to be **fully faithful** just when



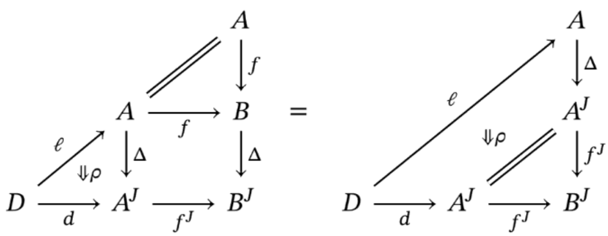
defines absolute right lifting diagram or equivalently an absolute left lifting diagram. Modulo a result we borrow from Chapter 3, we show:

Proposition 2.4.7. *A fully faithful functor $f : A \to B$ reflects any limits or colimits that exist in B.*

Proof The statement for limits asserts that for any family of diagrams $d : D \to A^J$ of shape J in A, any functor $\ell : D \to A$, and any cone $\rho : \Delta\ell \Rightarrow d$ so that the whiskered composite with $f^J : A^J \to B^J$ is an absolute right lifting diagram



then (ℓ, ρ) defines an absolute right lifting of $d : D \to A^J$ through $\Delta : A \to A^J$. By Exercise 2.3.v, to say that f is fully faithful is to say that $\mathrm{id}_A : A \to A$ defines an absolute right lifting of f through itself. So by Lemma 2.4.1, the composite diagram below-left is an absolute right lifting diagram, and by 2-functoriality of the simplicial cotensor with J, the diagram below-left coincides with the diagram below-right:



Now if we knew that $\mathrm{id}_{A^J} : A^J \to A^J$ defines an absolute right lifting of f^J through itself – that is, if we know that $f^J : A^J \to B^J$ is also fully faithful – then we could apply Lemma 2.4.1 again to conclude that (ℓ, ρ) is an absolute right lifting of d through Δ as required. And indeed this is the case: by Corollary 3.5.7, any cosmological functor, such as $(-)^J$, preserves absolute lifting diagrams. □

It is worth asking why we have not already proven that cosmological functors preserve absolute lifting diagrams, since after all, by Lemma 1.4.4, cosmological functors induce 2-functors between homotopy 2-categories, which is where absolute lifting diagrams are defined. But unlike adjunctions, which are defined by pasting equations in a 2-category, absolute lifting diagrams are defined using universal quantifiers and hence are not preserved by all 2-functors. However, the 2-functors that underlie cosmological functors *do* preserve absolute lifting diagrams, even when the cosmological functor is "forgetful" or fails to be essentially surjective. This is because the universal property of absolute lifting

Our aim in this chapter is to develop the general theory of comma constructions from the point of view of the homotopy 2-category of an ∞-cosmos. Our first payoff for this work appears in Chapter 4 where we study the universal properties of adjunctions, limits, and colimits along these lines. The comma construction also provides the essential vehicle in Part III for establishing the model independence of the categorical notions we introduce throughout this text.

There is a standard definition of a "comma object" that can be stated in any 2-category, defined as a particular weighted limit (see Example A.6.14). Comma ∞-categories do *not* satisfy this universal property in the homotopy 2-category, however. Instead, they satisfy a somewhat peculiar "weak" variant of the usual 2-categorical universal property that to our knowledge has not appeared elsewhere in the literature. The weak universal property is encoded by something we call a *smothering functor*, which relates homotopy coherent and homotopy commutative diagrams of suitable shapes. To introduce these universal properties in a concrete rather than abstract framework, we start in §3.1 by considering smothering functors involving homotopy categories of quasi-categories.

In §3.2, we use a smothering functor to encode the weak universal property of the ∞-category of arrows A^2 associated to an ∞-category A, considered as an object in the homotopy 2-category. In §3.3, we briefly study the analogous weak universal properties associated to the pullback of an isofibration, which we exploit to prove that the pullback of an equivalence along an isofibration is an equivalence.

Comma ∞-categories are introduced in §3.4 where we describe both their strict universal properties as simplicially enriched limits as well as their weak universal properties in the homotopy 2-category. Each have their uses, for instance in describing the induced actions on comma ∞-categories of various types of morphisms between their generating cospans. The weak 2-categorical universal property is deployed in §3.5 to prove a general representability theorem that characterizes those comma ∞-categories that are right or left represented by a functor. In Chapter 4, we reap the payoff for this work, achieving the desired representable characterizations of adjunctions, limits, and colimits as special cases of these general results.

In §3.6, we tighten the main theorem of §3.5 to say that a comma ∞-category is right represented by a functor if and only if its codomain-projection functor admits a terminal element, when considered as an object in the sliced ∞-cosmos. This result requires a careful analysis of the subtle difference between the homotopy 2-category of a sliced ∞-cosmos and the sliced 2-category of the homotopy 2-category of an ∞-cosmos. Those readers who would rather stay out of the weeds are invited to take note of Definition 3.6.5 and Corollary 3.6.10 but otherwise skip this section.

3.1 Smothering Functors

Let Q be a quasi-category. Recall from Lemma 1.1.12 that its homotopy category $\mathsf{h}Q$ has

- elements of Q as its objects;
- homotopy classes of 1-simplices of Q as its arrows, where parallel 1-simplices are homotopic just when they bound a 2-simplex whose remaining outer edge is degenerate; and
- a composition relation if and only if any chosen 1-simplices representing the three arrows bound a 2-simplex.

For a 1-category J, it is well-known in classical homotopy theory that the homotopy category of diagrams $\mathsf{h}(Q^J)$ is not equivalent to the category $(\mathsf{h}Q)^J$ of diagrams in the homotopy category – except in very special cases, such as when J is a set (see Lemma 2.3.3). The objects of $\mathsf{h}(Q^J)$ are *homotopy coherent* diagrams of shape J in Q, while the objects of $(\mathsf{h}Q)^J$ are mere *homotopy commutative* diagrams. There is, however, a canonical comparison functor

$$\mathsf{h}(Q^J) \longrightarrow (\mathsf{h}Q)^J$$

defined by applying $\mathsf{h} \colon \mathcal{QC}at \to \mathcal{C}at$ to the evaluation functor $Q^J \times J \to Q$ and then transposing; a homotopy coherent diagram is in particular homotopy commutative.

Our first aim in this section is to better understand the relationship between the arrows in the homotopy category $\mathsf{h}Q$ and the arrows of Q, meaning the 1-simplices in the quasi-category. To study this, we consider the quasi-category $Q^{\mathbb{2}}$ in which the arrows of Q live as elements, where $\mathbb{2} = \Delta[1]$ is the nerve of the walking arrow. Our notation deliberately imitates the notation commonly used for the **category of arrows**: if C is a 1-category, then $C^{\mathbb{2}}$ is the category whose objects are arrows in C and whose morphisms are commutative squares, regarded as a morphism from the arrow displayed vertically on the left-hand side to the arrow displayed vertically on the right-hand side. This notational conflation suggests our first question: how does the homotopy category of $Q^{\mathbb{2}}$ relate to the category of arrows in the homotopy category $\mathsf{h}Q$?

LEMMA 3.1.1. *The canonical functor* $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ *is*

(i) surjective on objects,

(ii) *full, and*

(iii) ***conservative**, i.e., reflects invertibility of morphisms,*

but not necessarily injective on objects nor faithful.

Proof Surjectivity on objects asserts that every arrow in the homotopy category $\mathsf{h}Q$ is represented by a 1-simplex in Q. This is the conclusion of Exercise 1.1.iii(iii) which outlines the proof of Lemma 1.1.12.

To prove fullness, consider a pair of arrows f and g in Q that form the source and target of a commutative square in $\mathsf{h}Q$. By (i), we may choose arbitrary 1-simplices representing each morphism in $\mathsf{h}Q$ and their common composite:

$$\begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ {\scriptstyle f}\downarrow & \overset{\ell}{\searrow} & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[k]{} & \bullet \end{array}$$

By Lemma 1.1.12, every composition relation in $\mathsf{h}Q$ is witnessed by a 2-simplex in Q; choosing a pair of such 2-simplices defines a diagram $\mathbb{2} \times \mathbb{2} \to Q$, which represents a morphism from f to g in $\mathsf{h}(Q^{\mathbb{2}})$, proving fullness.

Surjectivity on objects and fullness of the functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ are special properties having to do with the diagram shape $\mathbb{2}$, while conservativity holds for generic diagram shapes by Corollary 1.1.22. The construction of counterexamples illustrating the general failure of injectivity on objects and faithfulness is left to Exercise 3.1.i, with a hint. □

The properties of the canonical functor $\mathsf{h}(Q^{\mathbb{2}}) \to (\mathsf{h}Q)^{\mathbb{2}}$ frequently reappear, so we bestow them with a suggestive name:

Definition 3.1.2 (smothering functor). A functor $f : A \to B$ between 1-categories is **smothering** if it is surjective on objects, full, and conservative. That is, a functor is smothering if and only if it has the right lifting property with respect to the set of functors:

$$\left\{ \begin{array}{c} \emptyset \\ \big\downarrow \\ \mathbb{1} \end{array}, \quad \begin{array}{c} \mathbb{1}+\mathbb{1} \\ \big\downarrow \\ \mathbb{2} \end{array}, \quad \begin{array}{c} \mathbb{2} \\ \big\downarrow \\ \mathbb{I} \end{array} \right\}$$

Various elementary properties of smothering functors are established in Exercise 3.1.ii; here we highlight one worthy of particular attention:

Lemma 3.1.3 (smothering fibers). *Each fiber of a smothering functor is a nonempty connected groupoid.*

Proof Suppose $f\colon A \to B$ is smothering and consider the fiber

$$\begin{array}{ccc} A_b & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ \mathbb{1} & \xrightarrow{b} & B \end{array}$$

over an object b of B. By surjectivity on objects, the fiber is nonempty. Its morphisms are defined to be arrows between objects in the fiber of b that map to the identity on b. By fullness, any two objects in the fiber are connected by a morphism, indeed, by morphisms pointing in both directions. By conservativity, all the morphisms in the fiber are necessarily invertible. □

The argument used to prove Lemma 3.1.1 generalizes to:

Lemma 3.1.4. *If J is a 1-category that is free on a reflexive directed graph and Q is a quasi-category, then the canonical functor $\mathsf{h}(Q^J) \to (\mathsf{h}Q)^J$ is smothering.*

Proof Exercise 3.1.iii. □

Cotensors are one of the cosmological limits axiomatized in Definition 1.2.1. Other limit constructions listed there also give rise to smothering functors.

Lemma 3.1.5. *For any pullback diagram of quasi-categories in which p is an isofibration*

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow p \\ A & \xrightarrow[f]{} & B \end{array}$$

the canonical functor $\mathsf{h}(A \underset{B}{\times} E) \to \mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is smothering.

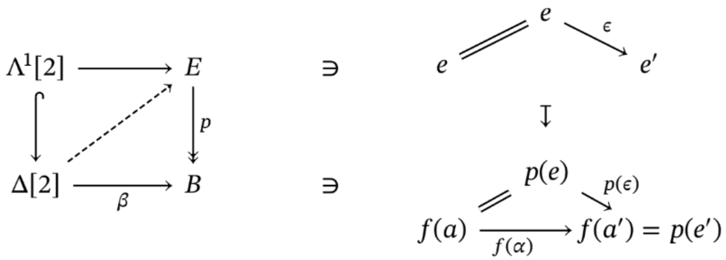
Proof As $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ does not preserve pullbacks, the canonical comparison functor of the statement is not an isomorphism. It is however bijective on objects since the composite functor

$$\mathcal{QC}at \xrightarrow{\mathsf{h}} \mathcal{C}at \xrightarrow{\mathrm{obj}} \mathcal{S}et$$

passes to the underlying set of vertices of each quasi-category, and this functor *does* preserve pullbacks.

For fullness, note that a morphism in $\mathsf{h}A \times_{\mathsf{h}B} \mathsf{h}E$ is represented by a pair of 1-simplices $\alpha\colon a \to a'$ in A and $\epsilon\colon e \to e'$ in E whose images are homotopic in B, a condition that implies in particular that $f(a) = p(e)$ and $f(a') = p(e')$.

By Lemma 1.1.9, we can configure this homotopy however we like, and thus we choose a 2-simplex witness β so as to define a lifting problem



Since p is an isofibration, a solution exists, defining an arrow $\tilde{\epsilon} \colon e \to e'$ in E in the same homotopy class as ϵ so that $p(\tilde{\epsilon}) = f(\alpha)$. The pair $(\alpha, \tilde{\epsilon})$ now defines the lifted arrow in $\mathsf{h}(E \times_B A)$.

Finally, consider an arrow $\mathbb{2} \to A \underset{B}{\times} E$ whose image in $\mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is an isomorphism, which is the case just when the projections to E and A define isomorphisms. By Corollary 1.1.16, we may choose a homotopy coherent isomorphism $\mathbb{I} \to A$ extending the given isomorphism $\mathbb{2} \to A$. This data presents us with a lifting problem

$$\begin{array}{ccccc} \mathbb{2} & \longrightarrow & A \underset{B}{\times} E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ \mathbb{I} & \longrightarrow & A & \xrightarrow{f} & B \end{array}$$

which Exercise 1.1.vi tells us we can solve. This proves that $\mathsf{h}(A \underset{B}{\times} E) \to \mathsf{h}A \underset{\mathsf{h}B}{\times} \mathsf{h}E$ is conservative and hence also smothering. □

A similar argument proves:

Lemma 3.1.6. *For any tower of isofibrations between quasi-categories*

$$\cdots \twoheadrightarrow E_n \twoheadrightarrow E_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow E_2 \twoheadrightarrow E_1 \twoheadrightarrow E_0$$

the canonical functor $\mathsf{h}(\lim_n E_n) \to \lim_n \mathsf{h}E_n$ *is smothering.*

Proof Exercise 3.1.iv. □

Lemma 3.1.7. *For any cospan between quasi-categories* $C \xrightarrow{g} A \xleftarrow{f} B$ *consider the quasi-category defined by the pullback*

$$\begin{array}{ccc} \mathsf{Hom}_A(f, g) & \longrightarrow & A^{\mathbb{2}} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (\mathrm{cod},\mathrm{dom})} \\ C \times B & \xrightarrow{g \times f} & A \times A \end{array}$$

the representing object. Here the identity functor $\mathrm{id}\colon A^{\mathbb{2}} \to A^{\mathbb{2}}$ is mapped to an element of $\mathsf{Fun}(A^{\mathbb{2}}, A)^{\mathbb{2}}$, a 1-simplex in $\mathsf{Fun}(A^{\mathbb{2}}, A)$, which by Lemma 1.4.3 represents a 2-cell κ in the homotopy 2-category.

To see that the source and target of κ must be the domain evaluation and codomain evaluation functors, defined by cotensoring with the endpoint inclusion $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{2}$, we use the naturality of the isomorphism (3.2.4) in the cotensor variable:

$$\begin{array}{ccc} \mathsf{Fun}(X, A^{\mathbb{2}}) & \cong & \mathsf{Fun}(X, A)^{\mathbb{2}} \\ {\scriptstyle (p_1, p_0)_*}\downarrow & & \downarrow{\scriptstyle (\mathrm{cod}, \mathrm{dom})} \\ \mathsf{Fun}(X, A \times A) & \cong & \mathsf{Fun}(X, A) \times \mathsf{Fun}(X, A) \end{array}$$

The identity functor maps around the top-right composite to the pair of functors $(\mathrm{cod}\,\kappa, \mathrm{dom}\,\kappa)$ and around the left-bottom composite to the pair (p_1, p_0). □

There is a 2-categorical limit notion that is analogous to Definition 3.2.1, which constructs, for any object A, the universal 2-cell with codomain A: namely the (categorical) cotensor with the 1-category $\mathbb{2}$. Its universal property is analogous to (3.2.4) but with the hom-categories of the 2-category in place of the functor spaces (see Definition A.4.1). In the 2-category of categories, the $\mathbb{2}$-cotensor defines the arrow category.

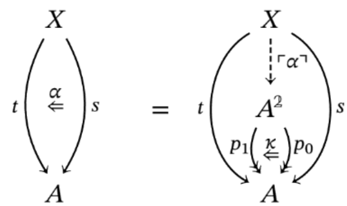
In the homotopy 2-category, by the Yoneda lemma again, the data (3.2.3) encodes a natural transformation

$$\mathsf{hFun}(X, A^{\mathbb{2}}) \to \mathsf{hFun}(X, A)^{\mathbb{2}}$$

of categories but this is *not* a natural isomorphism, nor even a natural equivalence of categories. However, it does furnish the ∞-category of arrows with a "weak" universal property of the following form:

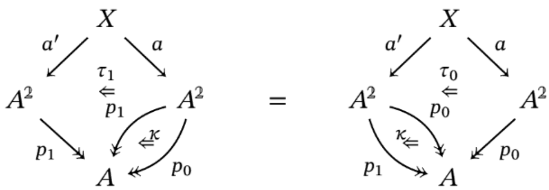
Proposition 3.2.5 (the weak universal property of the arrow ∞-category). *The generic arrow* (3.2.3) *with codomain A has a weak universal property in the homotopy 2-category given by three operations:*

(i) ***1-cell induction****: Given a natural transformation over A as below-left*



there exists a functor $\ulcorner\alpha\urcorner\colon X \to A^{2}$ *so that* $s = p_0\ulcorner\alpha\urcorner$, $t = p_1\ulcorner\alpha\urcorner$, *and* $\alpha = \kappa\ulcorner\alpha\urcorner$.

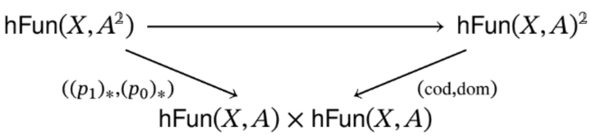
(ii) ***2-cell induction***: *Given functors* $a, a'\colon X \to A^{2}$ *and natural transformations* τ_1 *and* τ_0 *so that*



there exists a natural transformation $\tau\colon a \Rightarrow a'$ *so that* $p_1\tau = \tau_1$ *and* $p_0\tau = \tau_0$.

(iii) ***2-cell conservativity***: *For any natural transformation* $X \underset{a'}{\overset{a}{\underset{\Downarrow\tau}{\rightrightarrows}}} A^{2}$ *if both* $p_1\tau$ *and* $p_0\tau$ *are isomorphisms then* τ *is an isomorphism.*

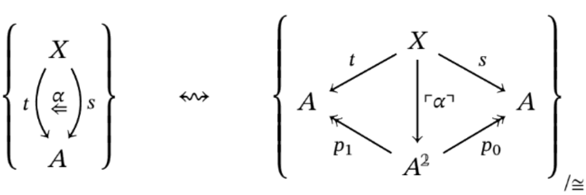
Proof Let $Q = \mathsf{Fun}(X, A)$ and apply Lemma 3.1.1 to observe that the natural map of hom-categories



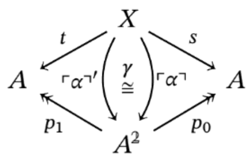
over $\mathsf{hFun}(X, A \times A) \cong \mathsf{hFun}(X, A) \times \mathsf{hFun}(X, A)$ is a smothering functor. Surjectivity on objects is expressed by 1-cell induction, fullness by 2-cell induction, and conservativity by 2-cell conservativity. □

Note that the functors $\ulcorner\alpha\urcorner\colon X \to A^{2}$ that represent a given natural transformation α with domain X and codomain A are not unique. However, they are unique up to "fibered" isomorphisms that whisker with $(p_1, p_0)\colon A^{2} \twoheadrightarrow A \times A$ to identities:

Proposition 3.2.6. *Whiskering with* (3.2.3) *induces a bijection between natural transformations with domain X and codomain A as displayed below-left*



and fibered isomorphism classes of functors $X \to A^{\mathbb{2}}$ *as displayed above-right, where the fibered isomorphisms are given by invertible 2-cells*



so that $p_1\gamma = \mathrm{id}_t$ *and* $p_0\gamma = \mathrm{id}_s$.

Proof Lemma 3.1.3 proves that the fibers of the smothering functor of Proposition 3.2.5 are connected groupoids. The objects of the fiber over α are functors $X \to A^{\mathbb{2}}$ that whisker with the generic arrow κ to α, and the morphisms are invertible 2-cells that whisker with $(p_1, p_0) \colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ to the identity 2-cell $(\mathrm{id}_t, \mathrm{id}_s)$. The action of the smothering functor defines a bijection between the objects of its codomain and their corresponding fibers. □

Our final task is to observe that the universal property of Proposition 3.2.5 is also enjoyed by any object $(e_1, e_0) \colon E \twoheadrightarrow A \times A$ that is equivalent to the ∞-category of arrows $(p_1, p_0) \colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ in the slice ∞-cosmos over $A \times A$. We have special terminology to allow us to concisely express the type of equivalence we have in mind.

Definition 3.2.7 (fibered equivalence). A **fibered equivalence** over an ∞-category B in an ∞-cosmos $\mathcal{K}$ is an equivalence

$$\begin{array}{ccc} E & \xrightarrow{\ \sim\ } & F \\ & \searrow \!\!\!\!\rightarrow \quad \swarrow\!\!\!\!\leftarrow & \\ & B & \end{array} \tag{3.2.8}$$

in the sliced ∞-cosmos $\mathcal{K}_{/B}$. We write $E \simeq_B F$ to indicate that the specified isofibrations with these domains are equivalent **over** B.

By Proposition 1.2.22(vii), a fibered equivalence is just a map between a pair of isofibrations over a common base that defines an equivalence in the underlying ∞-cosmos: the forgetful functor $\mathcal{K}_{/B} \to \mathcal{K}$ preserves and reflects equivalences. Note, however, that it does not create them: It is possible for two ∞-categories E and F to be equivalent without there existing any equivalence compatible with a pair of specified isofibrations $E \twoheadrightarrow B$ and $F \twoheadrightarrow B$.

Warning 3.2.9. At this point, there is some ambiguity about the 2-categorical data that presents a fibered equivalence in an ∞-cosmos $\mathcal{K}_{/B}$ related to the question posed in Exercise 1.4.vii about the relationship between the 2-categories

$\mathfrak{h}(\mathcal{K}_{/B})$ and $(\mathfrak{h}\mathcal{K})_{/B}$. But since Proposition 1.2.22(vii) tells us that a mere equivalence in $\mathfrak{h}\mathcal{K}$ involving a functor of the form (3.2.8) is sufficient to guarantee that this as-yet-unspecified 2-categorical data exists, we defer a careful analysis of this issue to Proposition 3.6.4.

Proposition 3.2.10 (uniqueness of arrow ∞-categories). *For any isofibration $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ that is fibered equivalent to $(p_1, p_0)\colon A^{\mathbb{2}} \twoheadrightarrow A \times A$ the 2-cell*

$$E \underset{e_1}{\overset{e_0}{\underset{\Downarrow \epsilon}{\rightrightarrows}}} A$$

encoded by the equivalence $e\colon E \xrightarrow{\sim} A^{\mathbb{2}}$ satisfies the weak universal property of Proposition 3.2.5. Conversely, if the isofibrations $(d_1, d_0)\colon D \twoheadrightarrow A \times A$ and $(e_1, e_0)\colon E \twoheadrightarrow A \times A$ are equipped with 2-cells

$$D \underset{d_1}{\overset{d_0}{\underset{\Downarrow \delta}{\rightrightarrows}}} A \qquad \textit{and} \qquad E \underset{e_1}{\overset{e_0}{\underset{\Downarrow \epsilon}{\rightrightarrows}}} A$$

satisfying the weak universal property of Proposition 3.2.5, then $D \simeq_{A \times A} E$.

Proof We prove the first statement. By the defining equation of 1-cell induction $\epsilon = \kappa e$, where κ is the generic arrow (3.2.3). Hence, the functor induced by pasting with ϵ factors as a composite

$$\begin{array}{ccccc} \mathsf{hFun}(X,E) & \xrightarrow[\sim]{e_*} & \mathsf{hFun}(X,A^{\mathbb{2}}) & \longrightarrow & \mathsf{hFun}(X,A)^{\mathbb{2}} \\ & {\scriptstyle ((p_1)_*,(p_0)_*)} \searrow & & \swarrow {\scriptstyle (\mathrm{cod},\mathrm{dom})} & \\ & & \mathsf{hFun}(X,A) \times \mathsf{hFun}(X,A) & & \end{array}$$

and our task is to prove that this composite functor is smothering. The first functor, defined by postcomposing with the equivalence $e\colon E \xrightarrow{\sim} A^{\mathbb{2}}$, is an equivalence of categories, and the second functor is smothering. Thus, the composite is clearly full and conservative. To see that it is also surjective on objects, note first that by 1-cell induction any 2-cell

$$X \underset{t}{\overset{s}{\underset{\Downarrow \alpha}{\rightrightarrows}}} A$$

is represented by a functor $\ulcorner \alpha \urcorner\colon X \to A^{\mathbb{2}}$ over $A \times A$. Composing with any