FINDING FIBONACCI

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The Quest to Rediscover the Forgotten Mathematical Genius Who Changed the World

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PRELUDE

Sputnik and Calculus

If my name is familiar to you, chances are it's because you have read one of my mathematics books (I've written around 35, many of them for the general reader), or perhaps you've read an article I have published in a magazine or newspaper, read one of my blogposts (I maintain four blogs), taken part in one of the sessions of my massively open online course (MOOC) on mathematics, or heard me on National Public Radio, where I am known as "the Math Guy." Yes, I am *that* Keith Devlin. And yes, I love math.

By most people's standards, I'm also pretty good at math. But it wasn't always so. In the first few years of elementary school, I was one of the poorest performing children in mathematics. In fact, I was the last kid in the class to memorize my multiplication tables, a rite of passage that loomed large in the mathematics education mandated in 1950s England. Mastering the multiplication tables (today's math educators call them "multiplicative number bonds," presumably to give them a more modern ring) caused me so much anxiety that my parents had to go to the teacher to explain that my abysmal performance was not due to lack of effort on my part. Rather, I simply did not get it. I was not, they said, a math type.

Things improved by the time I was aged 9, and as a result of precocious classroom performances in all disciplines, including math, I skipped year 10 and went straight into year 11, the final year of elementary school, taking the national "11 Plus" selection exam for secondary school placement at age 10. I was the only kid at that school to "pass" that exam, meaning I was allocated a place in an academically focused secondary school (a Grammar School), rather than the more common Secondary Modern School, designed to prepare the next generation of British factory workers, the kind of school to which all (and I mean all) of my classmates went on to attend.

Looking back, my early difficulty with math was likely because I got stuck trying to *understand* multiplication, whereas everyone else simply memorized the tables like a meaningless passage from Jabberwocky. But back then, no one else in my world thought of mathematics as something you understand; rather, it was a meaningless memorization hurdle you had to master to progress through the system, justified with the mantra "You will need it in later life." (In my case, it turned out I actually did make use of all that school math, including the multiplication tables, but that was because I became a mathematician. By the time I was at university, electronic calculators had arrived on the scene, rendering obsolete the one demonstrably practical advantage of mastery of the multiplication tables.)

I entered grammar school in 1958 (by then having reached the official age of 11), the year Russia launched the first satellite (Sputnik) into space. For this science-fiction-mad young male, that event was dramatic. Humankind was about to enter the Space Age, and I wanted to be part of it, the more so when the TV news showed US president John F. Kennedy announce that they (the

CHAPTER 1

The Flood Plain

Tuscany, Italy, September 2002. Like many present-day travelers to Pisa, I took the train from Florence—a small commuter train of four carriages pulled by a noisy diesel locomotive, quite different from the sleek Intercity Express that had whisked me southward from Trento. Even late in the season, the train was crammed with tourists, many of them young people carrying backpacks. Everyone was talking loudly to make themselves heard over the noise from the engine. In my carriage I heard Americans, British, Australians, Germans, French, Scandinavians, and Japanese. A port in the Roman era and a major Mediterranean trading hub in medieval times, Pisa clearly is still an international destination, though these days the main cargo seems to be foreign tourists.

Once the train had left Florence behind, the journey became spectacular, winding its way through the beautiful rolling hills of the Chianti wine region. On both sides of the railroad tracks, the steeply rising slopes were covered with an irregular checkerboard of bright green vineyards, each one laid out with geometric precision. Occasionally, a field would stretch right down to the side of the tracks, giving the passengers a closer view. Now, in late summer, the vines were heavy with the ripening purple grapes that would soon be harvested to make the wines the region is so famous for.

Eventually, the hills gave way to a large flat plain, stretching all the way to Pisa and beyond to the sea. There had been heavy rains just prior to my visit to Italy, and as the train left the vineyards it began to rain once again. As the engine slowed down to arrive at our destination, I saw that the land on both sides of the tracks was still under a foot or more of water. The land here floods regularly, a lasting reminder of why Pisa had become a port in the first place: In Roman times, and earlier, this is where Pisa's harbor used to be.

By the time the train pulled up in Pisa, the rain had turned into a sustained, heavy downpour. The small, quaint, inexpensive hotel I had booked via the Internet was perfectly located for sightseeing, right in the center of the old medieval city, close to the river. Unfortunately, the railway station was not—it is a "Central Station" in name only. As I had experienced many times in New York City, when it rains in Pisa, everyone travels by taxi. As a result, the station taxi stand before me stood empty. I waited in line for an hour, with only my umbrella to keep me dry, before I was finally able to secure a ride. I soon began to wish I too had my belongings in a backpack, so I could have walked to my destination, as many of my fellow passengers did. It was a damp end to my journey, both literally and figuratively. Still, I was in Pisa at last, about to take the first step in what would turn out to be a seven-year quest to piece together the story of one of the most influential figures in human history, a medieval mathematician who, over the years, had become something of an obsession with me.

My visit had come about quite by chance. I had been invited to Italy to give an address at an international conference in Rome on the newly emerging field of mathematical cognition. I was asked to give lectures at several other universities as well—the industrial powerhouse of Torino in the northwest, the vacation destination Trento in the mountainous wine region in the northeast, the ancient university town of Bologna partway from Trento to Florence, and the spectacular Siena where, more than 20 years earlier, I had been a visiting professor for several weeks.

I had decided to take a two-day detour to Pisa in between my lecturing commitments in Bologna and Siena, in an effort to find out something about Leonardo Fibonacci, a mysterious thirteenth-century mathematician who apparently played a key role in the making of the modern world, and in whose mathematical footsteps I had, in one important respect, been treading for the past 20 years.

Was there enough information to write a book about him? No one else had written one, so I suspected there was not. On the other

FIGURE 1. This Leonardo woodcut provides one of only two images we have of Leonardo. There is no evidence either is more than an artist's conception.

hand, that yawning gap in the written history of science meant that Fibonacci was the most famous and accomplished scientist never to have been the subject of a biography. I wanted to give it a try.

My interest was certainly not that of the historian, for such I am not. I am a mathematician. What intrigued me about Leonardo was that significant similarity between our mathematical careers. I sensed a kindred spirit.

As I sheltered under my umbrella, waiting for a taxi, I reflected briefly on how different my mathematical career had been from the future I had envisaged back in 1968, when I completed my bachelor's degree at the University of London and headed off to the University of Bristol to begin work on my doctorate.

Back then, when I was starting out, the only thing I knew about Fibonacci was that he was the mathematician who discovered the famous Fibonacci sequence (he didn't—I was wrong), which I knew had deep connections to human aesthetics (it doesn't—I was wrong). It was much later that I discovered he was one of the most influential men of all time. And that his greatness lay not in his mathematical discoveries—though he was without doubt the strongest mathematician of his time—but rather in his expository power. He had the ability to take what were at the time novel and difficult mathematical ideas and make them accessible to a wide range of people. Moreover, he had the instinct to do it in a way that in present-day terminology would be described as a "good marketing strategy."

As a young graduate student, my role models were not the likes of Leonardo Fibonacci, but the mathematicians who had made major mathematical discoveries—more recent mathematical giants such as Leonard Euler, Karl Friedrich Gauss, Pierre De Fermat, and Kurt Gödel. Like many young people embarking on a mathematical career, I dreamed of joining the ranks of the greatest—of proving a major theorem or solving a difficult problem that had baffled the best minds for decades.

Some of my contemporaries succeeded. In 1963, only a few years ahead of me, the young American mathematician Paul Cohen solved Cantor's Continuum Problem, a puzzle that had resisted all attempts at resolution for more than 60 years. But as is true for the vast majority of mathematicians, eventually I had to settle for far less.

During the course of my career, like most of the world's 25,000 professional mathematicians listed in the *International Directory of Mathematicians*, I solved a number of minor problems and proved several respectable but largely unremarkable theorems. I taught at various universities, in Scotland, Norway, Germany, Canada, and the United States (where I moved permanently in 1987), and I wrote a number of textbooks for mathematicians and students. Again, these are all fairly typical career moves for many academic mathematicians, though perhaps I moved around more than many and ended up writing more books than most.

But along the way, almost by accident, I discovered another talent, perhaps my true calling: an ability to explain often obscure, advanced mathematical ideas to a general audience. I found that, through my words, I could make mathematics come alive for others not versed in the subject.

An unplanned sequence of events resulted in my discovering this ability and thereby embarking on a second career path as a public expositor of mathematics. In the early 1980s, having returned to the UK after four years in Norway and Germany, I grew increasingly frustrated by the fact that magazines and newspapers often carried articles on science—biology, physics, chemistry, and so on—but hardly ever on mathematics. On the few occasions when they did cover mathematics, they did so badly, often getting the main idea day, and he replied, "Oh, that's what we put you down as on our scheduling board." And so the NPR Math Guy was born.

Each new step brought me further pleasure, as more and more people came up to me after a talk, or wrote or emailed me after reading an article I had written or hearing me on the radio. They would tell me they found my words inspiring, challenging, thought-provoking, or enjoyable. Parents, teachers, stay-at-home moms, business people, and retired people would thank me for awakening in them an interest and a new appreciation of a subject they had long ago abandoned for being either dull and boring or beyond their understanding. I came to realize that I was touching people's lives, opening their eyes to the marvelous world of mathematics.

None of this was planned. I had become a "mathematics expositor" by accident. Only after I realized I had been born with a talent that others appreciated—and that by all accounts is fairly rare—did I begin to work on developing and improving my "gift."

In taking mathematical ideas developed by others and explaining them in a way that the layperson can understand, I was following in the footsteps of others who had also made efforts to organize and communicate mathematical ideas to people outside the discipline. Among that very tiny subgroup of mathematics communicators, the two who I regarded as the greatest and most influential mathematical expositors of all time are Euclid and Leonardo Fibonacci. Each wrote a mammoth book that influenced the way mathematics developed, and with it society as a whole.[2]

Euclid's classic work *Elements* presented ancient Greek geometry and number theory in such a well-organized and understandable

[2] A close third, by my reckoning, would be Abū ʿAbdallāh Muḥammad ibn Mūsā al-Khwārizmī, whose ninth-century Arabic books on Hindu-Arabic arithmetic and on algebra were also written for a wide audience, though I did not know much about him back then. But because of the cultural stagnation that overcame the Arabic-speaking world in medieval times, and that continues to this day, it was left to Leonardo to make that body of knowledge available to the world.

way that even today some instructors use it as a textbook. It is not known if any of the results or proofs Euclid describes in the book are his, although it is reasonable to assume that some are, maybe even many. What makes *Elements* such a great and hugely influential work, however, is the way Euclid organized and presented the material. He did such a good job of it that his text has formed the basis of school geometry teaching ever since. Present-day high school geometry texts still follow *Elements* fairly closely, and translations of the original remain in print.

Because geometry was an obligatory part of the school mathematics curriculum until a few years ago, most people have been exposed to Euclid's teaching during their childhood, and many recognize his name and that of his great book. In contrast, Leonardo of Pisa and his book *Liber abbaci* are much less well known. Yet their impact on present-day life is far greater. *Liber abbaci* was the first comprehensive book on modern practical arithmetic in the Western world. While few of us ever *use* geometry, people all over the world make daily use of the methods of arithmetic that Leonardo described in *Liber abbaci*.

In contrast to the widespread availability of the original Euclid's *Elements*, the only version of Leonardo's *Liber abbaci* we can read today is a second edition he completed in 1228, not his original 1202 text. Moreover, there is just one translation from the original Latin, in English, published as recently as 2002.

For all its rarity, *Liber abbaci* is an impressive work. Although its great fame rests on its treatment of Hindu-Arabic arithmetic, it is a mathematically solid book that covers not just arithmetic, but the beginnings of algebra and some applied mathematics, all firmly based on the theoretical foundations of Euclid's mathematics.

I will describe my own reaction on first reading *Liber abbaci* in my fairly lengthy chapter 10 of this text, and, for readers who want to know more, I provide a summary of the entire contents of *Liber abbaci* in the appendix. For now, however, let me set the scene for the story I will tell by giving you the overall flavor of Leonardo's book.

Leonardo established a range of general methods for solving arithmetical problems (some using the geometric algebra of Book II of *Elements*), providing rigorous proofs to justify the methods, in the fashion of the ancient Greeks.

In particular, he explained—and provided justification for—some non-algebraic methods for solving problems that were well known in the medieval world, such as the checking procedure of "casting out nines," various "rules of proportion," and methods called "single false position" and "double false position," none of which are taught to today's calculator-carrying students. Indeed, these methods had fallen out of fashion by the time I learned arithmetic in the 1950s, a decade before the arrival of the digital desk calculator! (I did look up some of those methods when I was carrying out my Leonardo research, but I have already forgotten what they are.)

The real impact of the book came from its examples. Leonardo included a wealth of applications of mathematics to business and trade. These include conversions of money, weight, and content, methods of barter, business partnerships, and allocation of profit, alloying of money, investment of money, and simple and compound interest.

Presumably to add some variety and keep his readers' engagement, he also peppered his account with a number of highly artificial, cutely formulated, "fun" problems designed to illustrate various aspects of the mathematics he was describing.[3] For some of these "fun problems" he presented ingenious solutions that may have

[3] Including "fun" problems is a literary device that I, and all other mathematicians who write for a broad audience, use frequently.

been of his own devising. One of his fun problems would prove to be forever identified with the name Fibonacci.

Incidentally, the unusual spelling of abbaci, with two b's, seems to have been introduced by Leonardo, to distinguish it from the name for the various kinds of devices merchants used to perform their calculations. For what *Liber abbaci* described was how to compute without using such aids. (It is definitely not the "book of the abacus" in the modern interpretation of the word "abacus"—with one b.)

After completing the first edition of *Liber abbaci*, Leonardo wrote several other mathematics books, his writing making him something of a celebrity throughout Italy—on one occasion he was summonsed to an audience with the Emperor Frederick II. Yet very little was written about his life.

In 2001, I decided to embark on a quest to try to collect what little was known about him and bring his story to a wider audience. My motivation? I saw in Leonardo someone who, like me, devoted a lot of time and effort trying to make the mathematics of the day accessible to the world at large. (Known today as "mathematical outreach," very few mathematicians engage in that activity.) He was the giant whose footsteps I had been following.

I was not at all sure I could succeed. Over the years, I had built up a good reputation as an expositor of mathematics, but writing a book on Leonardo would be a new endeavor. I would have to become something of an archival scholar, trying to make sense of thirteenth-century Latin manuscripts. I was definitely stepping outside my comfort zone.

The dearth of hard information about Leonardo in the historical record meant that a traditional biography was impossible—which is probably why no medieval historian had written one. To tell my story, I would have to rely heavily on the *mathematical* thread that connects today's world to that of Leonardo—an approach unique to mathematics, made possible by the timeless nature of the discipline. Even so, it would be a stretch.

In the end, I got lucky. Very lucky. And not just once, but several times. Three of my lucky breaks—and they were big ones—occurred very early on in the project.

My first stroke of luck, the biggest of all, came my way just as I was embarking on my quest. In 2001, an Italian historian of medieval mathematics at the University of Siena, Professor Rafaella Franci, was commencing the first-ever study of a late thirteenth-century manuscript in the collection in an archival library in Florence. Franci's analysis eventually determined (and other scholars subsequently confirmed) that the manuscript provided the long sought-after "missing link" to prove that Leonardo, and in particular *Liber abbaci*, was a major trigger for the arithmetical and financial revolution that began in Tuscany not long after the book's appearance, and in due course spread throughout northern Europe—all of which more anon.

As a result of that good fortune, when my historical account *The Man of Numbers: Fibonacci's Arithmetic Revolution* was published in 2011, I was able to compensate for the unavoidable paucity of information about Leonardo's life with the first-ever account of Franci's discovery showing that my medieval role-model expositor had indeed played a pivotal role in creating the modern world.

My second stroke of luck was the publication, in 2002, of a complete English translation of *Liber abbaci*, the first and hitherto only translation of the classic work into a modern language. This meant I was spared the task of brushing up on my school Latin in order to make sense of Leonardo's writing, and could focus instead on the mathematical content. Franci told me of this soon-to-be-published book when I first met her earlier in 2002, and I immediately pre-ordered a copy on Amazon.

The men of learning in Bugia who showed Leonardo the invention were not its discoverers; they merely passed it on. The invention itself was much older, having its origins in India some time before 700 CE. Arab traders had transported it northward across land to the shores of the Mediterranean.

Leonardo took the page he had just written and placed it carefully on top of the pile. Now the book had a title. He read through that top page again:

> Here begins the Book of Calculation
> *Composed by Leonardo Pisano, Family Bonaci,*
> *In the Year 1202*

(Like all European scholars at the time, Leonardo wrote in Latin. The title of his book in Latin was *Liber Abbaci.*)

The invention Leonardo had written about was a remarkable new way of writing numbers and calculating with them. So remarkable and complete would be the transformation of human life brought about by this new system that generations that came long after him would eventually take it for granted, relegating numbers and arithmetic to a tedious but necessary routine chore that had to be learned by repetition (or, for later generations still, left to machines). It is the system for writing numbers and doing basic arithmetic that is taught in schools today the world over. By making that system available to western European businessmen, Leonardo's *Liber abbaci* would play a major role in creating the modern world.

Part of the reason for the huge influence of *Liber abbaci* surely was circumstantial, as were the events that led Leonardo to write it in the first place.

To be born in Pisa in the twelfth century was to be born at the hub of the Western world. And to grow up in a Pisan merchant family was to be a member of what was then the most important

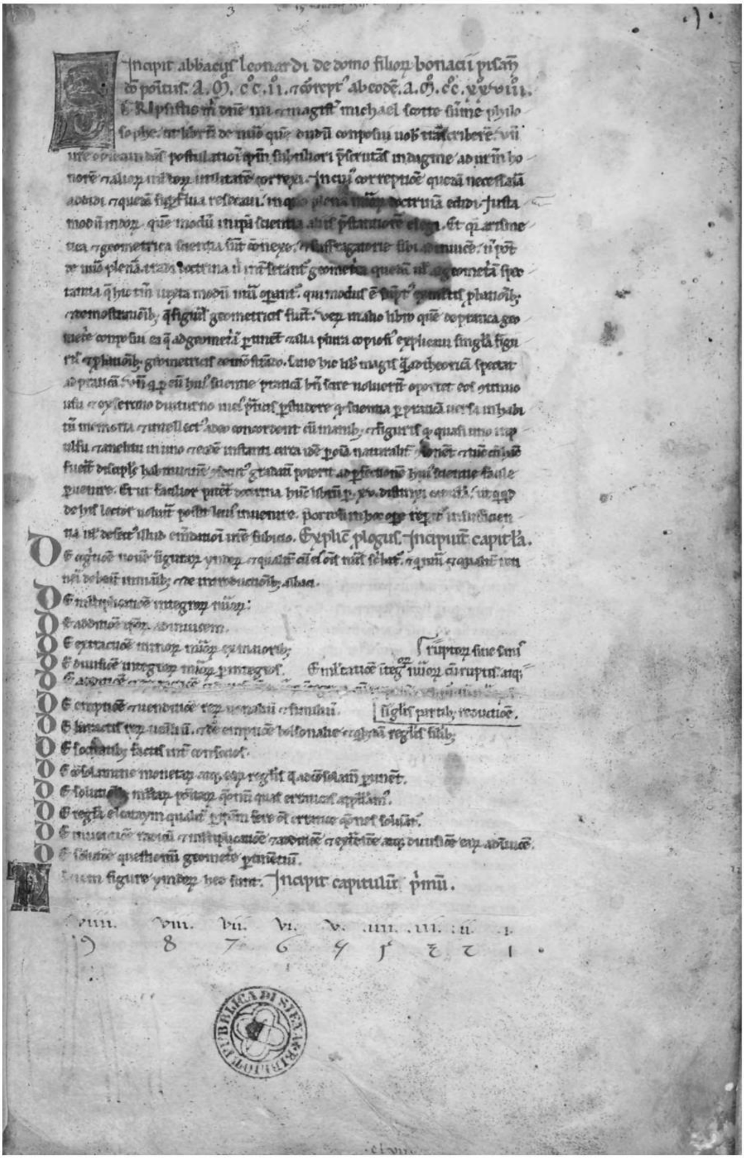

FIGURE 2. The first page of *Liber abbaci*, from the copy in the Siena Public Library.

sector of society. When Leonardo was born, Italy was a center of the vastly important and still rapidly growing international trade between the countries that fanned out from the Mediterranean Sea. Pisa, along with Italy's other maritime cities—Genoa to the north and Venice on the northeastern coast of Italy—dominated the trade, and their ships sailed constantly from one Mediterranean port to another. The merchants in those three cities were the key figures who were developing radical new ways of conducting international business, and thereby shaping the development of a new, more cosmopolitan world.

Yet time and place are only part of the picture. It usually takes true genius to see the greatness in the commonplace, and to recognize the enormous potential to change the world in what seems to most people to be a mundane or obscure idea. In the late sixteenth century, when Galileo Galilei gazed at the oil lamp hanging from the high ceiling in Pisa's Cathedral, swaying in the breeze, he made the key observation that led directly to the invention of the pendulum clock, the first accurate mechanical device to tell the time. In the seventeenth century, Newton saw an apple fall and realized there must be an invisible force—gravity—that not only governs all life on Earth but controls the motion of the entire universe. In the twentieth century, Albert Einstein wondered what it would be like to ride on the front of a beam of light, a seemingly childish question that led him to invent relativity—and with it nuclear power and the atomic bomb.

The pendulum, gravity, relativity—all of them single ideas, in retrospect simple ideas—that changed the world. So too with numbers. When the young Leonardo was living for several years in North Africa, he observed Arabic merchants using the system for writing and computing with numbers that they had acquired from the Hindus, and was convinced of the enormous potential of this invention, in particular for global commerce. As a result of that insight, he spent the next several years of his life writing an enormous instruction manual for using the new system.

Though Leonardo was born into a wealthy family with influential friends, and was well known for *Liber abbaci* and several other books when he died, there are almost no historical records relating to his life. While we know he was born sometime around 1170 CE, we do not know the exact year, and we are not completely sure where. Most likely it was Pisa, which is where he spent most of his childhood. According to the custom for naming at that time, in his later years, after he had become famous, he would have been known publicly as Leonardo Pisano. In 1838, the historian Guillaume Libri gave him the nickname "Fibonacci," a contraction of the Latin phrase *filius Bonacci* that Leonardo used to describe himself in the introduction to his book. (Although the literal meaning of *filius Bonacci* is "son of Bonacci," Bonacci was not the name of Leonardo's father, so the phrase should perhaps be translated as "of the family Bonacci.") The last contemporary reference to Leonardo was in 1240, in Pisa, but we have no idea how much longer he lived, or where or how he died.[2]

When I first became interested in Leonardo, I was surprised to find out that he had been largely forgotten just 200 years after he died. Yet, during his lifetime he had become famous throughout Italy and was honored by his native Pisa. But, as I learned from the medieval scholars I got to know, this was not an unusual occurrence, given the practice of the time. Other than the nobility, few people had anything recorded about them, even those who had achieved great things. The fame Leonardo enjoyed in his lifetime was because of his books, *Liber abbaci* among them. He was known as a brilliant mathematician and expositor of mathematics, and, later in life, a respected public servant. Clearly he was an accomplished

[2] See chapter 12.

man. But no one at that time had any idea that their compatriot had made a unique contribution that ultimately would change the world. That was for history to judge.

Far more surprising—and I realized this only much later, after I had learned a lot more about the man—was that history did not make that judgment about Leonardo's greatness until the 1960s, and was able to make it with complete confidence only at the start of the twenty-first century, one year after I embarked on my quest to tell his story.

So it was that, with no realization in the thirteenth century of what Leonardo's work would lead to, after one or two generations, virtually all mention of Leonardo of Pisa in the historical record dried up, and his name did not appear in any book on the history of science or mathematics for 400 years.

Then, in the late eighteenth century, an Italian mathematician, Pietro Cossali, came across a single reference to Leonardo in an early printed text by the famous Italian mathematician Luca Pacioli, *Summa de arithmetica geometria proportioni et proportionalità,* published in 1494.

Fortunately for history, Pacioli listed his sources, with Leonardo Pisano prominent among them, declaring in the book's introduction:

> And since we follow for the most part Leonardo Pisano, I intend to clarify now that any enunciation mentioned without the name of the author is to be attributed to Leonardo.

That reference led Cossali to wonder why Pacioli was famous while the man whose work he had apparently built on so extensively was unknown. When he followed up by doing some research into this mysterious "Leonardo Pisano," he came to understand what a major role the thirteenth-century Pisan played in the spread of modern arithmetic. Leonardo's 400 years of historical anonymity were over.

Particularly in the case of the more poorly written books, we don't know why they were written, or for whom. It is likely that, in many cases, they were written as personal study notebooks, not intended for use by anyone but the anonymous writer. After all, in an era when books had to be copied by hand, the most efficient (and arguably the only effective) way to learn practical arithmetic was to locate a local copy of a book on the topic, make your own copy (by hand, of course), and then work through it. That many of the abbacus books were created and used in just that way is clear from their appearance. The pages of carefully written main text are adorned with scribbled calculations, marginal notes, and diagrams, sometimes in a different colored ink.

This, incidentally, is why we can only guess at how many such manuscripts were written. The odds are clearly stacked against a personally written, teach-yourself-arithmetic notebook surviving 800 years. Yet 400 of them did just that. If, say, as many as one in ten were to survive, that would mean 4,000 must have been written. And one in ten strikes me as a very optimistic hypothesis for the survival over 800 years of a handwritten, personal notebook.

The disparity between the main text and the annotations found in many abbacus books, together with some significant copying errors, suggests that in many cases the manuscript's author had very little mathematical skill or understanding.

Others, though, are more scholarly, and may well have been written by teachers to use in classes on practical arithmetic. For, in addition to the sudden appearance of the abbacus books, the decades following the appearance of *Liber abbaci* also saw the growth, throughout Italy, of "abbacus schools" (*scuole d'abbaco* or *botteghe d'abbaco*), where young children, and perhaps adults, were taught practical arithmetic. Of course, along with the schools came perforce a whole new profession, that of "abbacus teachers" (*maestri d'abbaco*).

There was, then, a huge demand for books and courses in practical arithmetic, starting in the Pisa area in the decades following the appearance of *Liber abbaci*, and eventually spreading throughout Italy and beyond, into Mediterranean and Northern Europe.

Given the timing, and the geographic origin, of these new instructional manuscripts, it is hard to resist the conclusion that the appearance of *Liber abbaci* was the cause—the "gunshot that started the revolution."

In terms of the mathematics covered, the methods in the abbacus books could, for the most part, all be found in *Liber abbaci*, though some texts included a chapter on geometry, which Leonardo covered in a separate book, *De Practica Geometrie*. But, while *Liber abbaci* and *De Practica Geometrie* were lengthy, scholarly texts, written in Latin, the abbacus books were, as I noted earlier, far shorter, written in vernacular Italian, with much simpler examples than those Leonardo presented in his two masterpiece volumes.

You might think that some of the more mathematically able, early abbacus book authors took *Liber abbaci* (and perhaps *De Practica Geometrie*) as their starting point, and wrote their own simplified versions. But there is a problem with this conclusion—or rather, until recently there was.

For one thing, there is considerable overlap among all the abbacus books—in arithmetical content, book structure and organization, mathematical style, and example problems—showing that they were all produced by one author copying from another, with at most superficial changes. On the other hand, in terms of exposition, none of them had virtually anything in common with *Liber abbaci* (or *De Practica Geometrie*).

Yet they do all appear to stem from a common root. Here is how we know that. Because the abbacus authors used local currencies, with current value in their worked examples (and in some cases local weights and measures as well), present-day scholars have been able to trace the extent of the abbacus books, both geographically and over time. That enabled them to compare one abbacus text with another, making it possible to construct an ancestral tree of books, showing how the genre grew by way of localized and contemporized copying. When you trace that tree backwards in time, you cannot escape the conclusion that the entire genre began with a single "Abbacus Eve," the mother of all abbacus books.

But, since none of the abbacus books had anything in common with *Liber abbaci* (or *De Practica Geometrie*), that would be true of Abbacus Eve as well. So who wrote Eve? Whoever it was did so before 1290 or thereabouts, since that is the date ascribed to the first abbacus book that has been found, and the evident lack of mathematical ability of its author makes it clear that *it* cannot have been Eve. (This is the book Franci examined in Florence in 2001–2003.)

To write Eve, the author had to be capable not only of reading Leonardo's two massive, scholarly, Latin tomes *Liber abbaci* and *De Practica Geometrie*, but of understanding their contents sufficiently well to be able to produce a greatly simplified account of the same material. Moreover, since the ancestral tree leads back toward Pisa, the author most likely lived somewhere in the vicinity of that city.

Based on the historical record, there is only one possible candidate: Leonardo himself. Another mathematician as accomplished as Leonardo would surely have left his own collection of writings.

Moreover, we know for certain that Leonardo *did* write a much simpler, shorter book on arithmetic. On a number of occasions, he referred to having written a *liber minoris guise* ("book in a smaller manner") to *Liber abbaci*. One such reference is in *Liber abbaci* itself;[5] another is in another of his books, *Liber quadratorum*, and there is a third reference in still another book he wrote, *Flos*.

In addition, the author of a later abbacus book refers to Leonardo's *libro di minor guisa o Libro di merchanti* ("book in a smaller

5 Page 154 of the Boncompagni edition of *Liber abbaci*.

manner or book for merchants"). That phrase "book for merchants" is significant, since it suggests that the *Libro di minor guisa* would most likely have comprised material from the first ten chapters of *Liber abbaci* together with parts of *De Practica Geometrie*.

The evidence pointing to Leonardo as the author of Eve was substantial. The one remaining problem was that no copy of Leonardo's simpler book had ever been found. So, whereas scholars were confident that it was, indeed, Leonardo who initiated the arithmetical revolution (and thereby the commercial revolution that followed—see momentarily), they did not know with much certainty what it looked like. Or rather, that was the state of affairs before Franci made her study of a particular abbacus book in the Florence archive.

Because of the date the abbacus manuscript was written—sometime around 1290—coupled with the results of a number of additional literary forensic studies carried out by other scholars since Franci published her conclusions,[6] we can now say with historical certainty that the manuscript Franci studied is indeed an early copy of Leonardo's simpler book on practical arithmetic, perhaps copied directly from Leonardo's original. (It was very clearly a "slavish copy," since the author's errors—presumably copying errors—and annotations indicate that it was done by someone who was not able to do much more than make a direct copy.)

It is possible that Leonardo wrote his simpler text in Latin and someone else translated it into vernacular Italian, but my money is on Leonardo having written it in Italian. His purpose in writing it was, after all, to make the "new arithmetic" as widely accessible as possible.

One obvious question that arises as a result of the discovery of the abbacus books is, what created the demand for all of these books and courses on practical arithmetic? That question turns

6 I summarize the more significant studies in *The Man of Numbers* (2011).

out to be easy to answer. The key factor was the rapidly growing importance of arithmetic in the fast-changing world of the thirteenth century, particularly in the Italian regions of Tuscany, Lombardy, and to some extent Umbria.

The century after the appearance of *Liber abbaci* saw the birth of the modern world of global trade, commerce, and finance, with the introduction and development of, in particular, banking, insurance, and double-entry bookkeeping, together with the growth of ever larger trading conglomerates. All of those activities depend on the ability of an efficient way to perform arithmetical calculations that can be mastered by all.

Prior to the adoption of the Hindu-Arabic system of arithmetic, traders used one of two methods to carry out their calculations: finger arithmetic or a mechanical abacus.The former was a highly sophisticated system using all the fingers and thumbs of both hands—the general English term is "digit," of course, from the Latin *digitus* meaning finger or toe. (This is how finger arithmetic led to our present-day use of the word "digit" for the basic number symbols 0, 1, 2, through 9.)

Ancient finger arithmetic was capable of representing numbers up to 10,000. As you can imagine, it required considerable training to master the system and a lot of practice to become fluent. Medieval textbooks, including some surviving early copies of *Liber abbaci*, often had an initial page with carefully drawn diagrams showing the positions of the fingers to represent various numbers and perform different operations.

In the Muslim world and much of Europe, the abacus was a flat board with ruled lines on which small pebbles were placed and moved around. This form of early calculation gave rise to our present word "calculus" for a system of calculation procedures (and hence led to the word "calculate" itself), since the Latin word for a pebble is *calculus*. The more familiar form of abacus now found in toy stores, with beads on wires instead of pebbles on a rules