The Foundations of Mathematics and other Logical Essays

Frank Plumpton Ramsey

THE FOUNDATIONS OF MATHEMATICS

and other Logical Essays

by

FRANK PLUMPTON RAMSEY

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PREFACE

THE author of the papers collected in this volume seemed to me to combine very exceptional brilliance with very great soundness of judgment in philosophy. He was an extraordinarily clear thinker: no-one could avoid more easily than he the sort of confusions of thought to which even the best philosophers are liable, and he was capable of apprehending clearly, and observing consistently, the subtlest distinctions. He had, moreover, an exceptional power of drawing conclusions from a complicated set of facts: he could see what followed from them all taken together, or at least what might follow, in cases where others could draw no conclusions whatever. And, with all this, he produced the impression of also possessing the soundest common sense: his subtlety and ingenuity did not lead him, as it seems to have led some philosophers, to deny obvious facts. He had, moreover, so it seemed to me, an excellent sense of proportion: he could see which problems were the most fundamental, and it was these in which he was most interested and which he was most anxious to solve. For all these reasons, and perhaps for others as well, I almost always felt, with regard to any subject which we discussed, that he understood it much better than I did, and where (as was often the case) he failed to convince me, I generally thought the probability was that he was right and I wrong, and that my failure to agree with him was due to lack of mental power on my part.

Ramsey was not only exceptionally capable of thinking clearly himself; he also had a most uncommon power of explaining clearly to others what he thought and why he thought it. There are many good examples in this volume of his great capacity for lucid exposition. But sometimes I feel that he fails to explain things as clearly as he could have done, simply because he does not see that any explanation is needed: he does not realize that what to him seems perfectly clear and straightforward may to others, less gifted, offer many puzzles. I must confess that I personally often find a difficulty in understanding at all clearly what he means, in cases where he does not seem to have been aware than any difficulty whatever would be found. No doubt, in many of these cases, some readers will understand him without difficulty; but I suspect that many will be in my case. In the last two sections of the volume (the notes of 1928 and 1929), where he was writing chiefly for himself and not expanding and explaining as he would have done if writing for publication, the difficulty of following him with adequate comprehension is naturally specially great. But even where you cannot understand him completely you can often understand him enough to find him extraordinarily interesting; and I am convinced that it is well worth while to try to understand him. No doubt sometimes he may make mere mistakes; but in general I think he himself knew very well indeed what he was about, and, even if he was wrong, had very good reasons for the opinions at which he had arrived. It is a great misfortune that his early death prevented him from making these opinions, and the reasons for them, as clear as he, and perhaps he alone, could have made them.

G. E. MOORE.

December, 1930.

EDITOR'S INTRODUCTION

FRANK PLUMPTON RAMSEY was born on 22nd February, 1903, and died on 19th January, 1930. The son of the President of Magdalene, he spent nearly all his life in Cambridge, where he was successively Scholar of Trinity, Fellow of King's, and Lecturer in Mathematics in the University. His death at the height of his powers deprives Cambridge of one of its intellectual glories and contemporary philosophy of one of its profoundest thinkers.

Though mathematical teaching was Ramsey's profession, philosophy was his vocation. Reared on the logic of *Principia Mathematica*, he was early to see the importance of Dr. Wittgenstein's work (in the translation of which he assisted); and his own published papers were largely based on this. But the previously unprinted essays and notes collected in this volume show him moving towards a kind of pragmatism, and the general treatise on logic upon which at various times he had been engaged was to have treated truth and knowledge as purely natural phenomena to be explained psychologically without recourse to distinctively logical relations. Ramsey's philosophy, however, was always tentative and experimental—his calmness in infanticide frequently amazed his friends—and the papers in this volume are published as important in themselves and as likely to lead to work on similar lines and not as the exposition of a consistent and complete philosophical system.

The subtlety and fertility of Ramsey's philosophical work as shown here need no advertisement; but since his two papers on mathematical economics are not included, I have obtained Mr. J. M. Keynes' permission to quote from his notice in *The Economic Journal* of March, 1930 :—

" The death at the age of 26 of Frank Ramsey is a heavy loss—though his primary interests were in Philosophy and Mathematical Logic—to the pure theory of Economics. From a very early age, about 16 I think, his precocious mind was intensely interested in economic problems. Economists living in Cambridge have been accustomed from his undergraduate days to try their theories on the keen edge of his critical and logical faculties. If he had followed the easier path of mere inclination, I am not sure that he would not have exchanged the tormenting exercises of the foundations of thought and of psychology, where the mind tries to catch its own tail, for the delightful paths of our own most agreeable branch of the moral sciences, in which theory and fact, intuitive imagination and practical judgment, are blended in a manner comfortable to the human intellect.

" When he did descend from his accustomed stony heights, he still lived without effort in a rarer atmosphere than most economists care to breathe, and handled the technical apparatus of our science with the easy grace of one accustomed to something far more difficult. But he has left behind him in print (apart from his philosophical papers) only two witnesses to his powers—his papers published in *The Economic Journal* on 'A Contribution to the Theory of Taxation' in March, 1927, and on 'A Mathematical Theory of Saving' in December, 1928. The latter of these is, I think, one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject. The article is terribly difficult reading for an economist, but it is not difficult to appreciate how scientific and æsthetic qualities are combined in it together.

" The loss of Ramsey is, therefore, to his friends, for whom his personal qualities joined most harmoniously with his intellectual powers, one which it will take them long to forget. His bulky Johnsonian frame, his spontaneous gurgling laugh, the simplicity of his feelings and reactions, half-alarming sometimes and occasionally almost cruel in their directness and literalness, his honesty of mind and heart, his modesty, and the amazing, easy efficiency of the intellectual machine which ground away behind his wide temples and broad, smiling face, have been taken from us at the height of their excellence and before their harvest of work and life could be gathered in."

The essays collected in this volume range in date from 1923 to 1929, and present the development of Ramsey's thought from the age of 20 to his death. The papers on mathematical logic are placed first. I, on the Foundations of Mathematics, is an attempt to reconstruct the system of *Principia Mathematica* so that its blemishes may be avoided but its excellencies retained. By what he calls an " objective " theory of predicative functions, Ramsey shows how the notorious contradictions (I am lying, etc.) can be removed by the use of a Theory of Types which is simpler than that proposed by Mr. Bertrand Russell and which makes it unnecessary to assume an Axiom of Reducibility in order to save irrational numbers. Moreover a " complete extensionalizing " of mathematics solves the difficulties connected with identity and with the Multiplicative Axiom. Ramsey's paper is thus in the great tradition of Frege, Peano, Whitehead and Russell; and in a sense may be said to complete their work on the logical foundations of mathematics.

In II—a semi-popular paper read before the British Association in 1926— this " logical " treatment of mathematics is defended against the formalism of Hilbert and the intuitionism of Brouwer. The end of this paper shows that Ramsey was not completely satisfied with his theory, particularly with regard to the Axiom of Infinity ; and in 1929 he was converted to a finitist view which rejects the existence of any actual infinite aggregate and to which allusions are made in some of the later notes. Ramsey's profound disagreement with Hilbert's doctrine of mathematics as a game with meaningless marks did not prevent him from giving a good deal of attention to the formalists' chief problem—that of finding a general procedure for determining the consistency of a logical formula (the *Entscheidungsproblem*)—and III is his solution of the problem for a particularly interesting set of cases.

A relatively smaller proportion of Ramsey's purely philosophical work has been previously published. IV consists of an article denying that there is any ultimate distinction between particulars and universals, and VI—" Facts and Propositions "—is a logical analysis of belief. Ramsey's review of Wittgenstein's book is printed as an Appendix. This review was Ramsey's first important philosophical paper and it contains things of the greatest interest : but it was written before Ramsey had discussed the book with its author, and he admitted that on many points he had misunderstood it ; so the paper must be taken neither as exposition nor as criticism of the views of the *Tractatus Logico-Philosophicus* itself.

The previously unpublished papers that are printed here all deal with philosophical topics. VII is a long essay on Truth and Probability written at the end of 1926, a large portion of which was read to the Moral Science Club at Cambridge. It elaborates a thoroughly subjective theory of probability and a thoroughly pragmatic view of induction.

At one time Ramsey contemplated publishing this essay separately, and it is in a much more finished state than are the other unpublished papers. Its final chapter—on probability in science—was never written. I have supplemented the essay by some notes on relevant topics written in the spring of 1928 (VIII). The last section of the book (IX) is made up of a series of papers written in the summer of 1929. The first of these is a very serious attempt to provide a theory of theories and of their use in reasoning. There follow an exceedingly subtle theory of the nature of causal propositions, further remarks on probability and knowledge, and a note on the essence of philosophy. These essays, fragmentary and tentative though they be, seem to me to display Ramsey's mind in its highest power.

The short paper printed as Epilogue was read to a Cambridge discussion society in 1925 : Ramsey did not change the attitude towards life that he has so happily and characteristically expressed in it.

It is of interest to state which of Ramsey's important papers have been omitted from this volume. These are (1) the two articles on economics, (2) most of a symposium contribution which was mainly occupied in criticizing the previous symposiast (but I have reprinted the rest as V), (3) the notes for his annual course of lectures at Cambridge on the Foundations of Mathematics, (4) some notes on general propositions, causality, and knowledge written in the spring of 1928 and superseded by those on the same subjects of the summer of 1929 (allusion is made in these to his earlier causal theory), (5) fragments of 1929 on the occasion of his conversion to mathematical finitism, further attempts on the *Entscheidungsproblem* and fragments about theories, and (6) the draft of the first four chapters of a general treatise on logic. This work had occupied Ramsey intermittently during 1927 and 1928, but he was profoundly dissatisfied with it, and the preliminary matter which remains is quite unsuited for publication.

I am deeply grateful to Mrs. Lettice Ramsey for the privilege of editing this book as well as for assistance at every stage of its production. Mr. Alister Watson also has helped most ably in the proof-reading. Messrs. G. H. Hardy, J. M. Keynes, G. E. Moore, M. H. A. Newman, and L. Wittgenstein (with others of Ramsey's friends) have given me valued advice as to the selection of the papers, though I alone am responsible for the final choice. The authorities of the London Mathematical Society, the Mathematical Association, the Mind Association, and the Aristotelian Society have kindly given permission for the previously published papers to be reprinted. In these, and in the papers published for the first time, I have occasionally made slight verbal and symbolic alterations to assist the reader. But I have not attempted to modify to any extent the informality of many of the notes. Nor (beyond adding a note on the symbolism and a few references) have I attempted to alleviate the difficulties of the subject. This book is presented in the hope that it will stimulate others to think about the hardest things in the world with some of that singleness of mind which characterized Frank Ramsey.

R. B. BRAITHWAITE.

CAMBRIDGE.
June and December, 1930.

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[1] Only part of this symposium contribution is reprinted.

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NOTE ON SYMBOLISM

In some of these essays Ramsey uses the symbolism of A. N. Whitehead and Bertrand Russell's *Principia Mathematica*. Its most important features are :—

p, q, r used for *propositions*.

a, b, c used for *individuals*.

f, g, ϕ, χ, ψ used for *propositional functions*.

[These are sometimes written $\phi\hat{x}$, $\psi(\hat{x}, \hat{y}, \hat{z})$, etc., to show how many arguments they take.]

Then ϕa [sometimes written $\phi(a)$], $\psi(a, b, c)$, etc., are propositions.

x, y, z used for *variables* in expressions like

$(x) . \phi x$ meaning *For every* x, ϕx *is true*.

$(\exists x) . \phi x$ meaning *There is an* x *for which* ϕx *is true*.

Logical constants :—

$\sim$ meaning *not*.

$\vee$ meaning *or*.

. meaning *and*.

$\supset$ meaning *implies* [$\supset_x$ *implies for every* x].

$\equiv$ meaning *is equivalent to* [$\equiv_x$ *is equivalent to for every* x].

Other expressions sometimes used in this book :—

$\hat{x}(\phi x)$ meaning *the class of* ϕ's.

ϵ meaning *is a member of the class*.

$\subset$ meaning *is contained in* (relation between classes).

Nc meaning *the cardinal number of.*

$(\imath x)(\phi x)$ meaning *the one and only thing satisfying* ϕ.

$E ! (\imath x) (\phi x)$ meaning *One and only one thing satisfies* ϕ.

Points, colons, etc., . : :. are used for bracketing.

Ramsey also uses the following symbols not used by Whitehead and Russell :—

A stroke ‾ above the proposition or function to denote its contradictory $[\bar{p} = \sim p]$.

(a) meaning *the class whose only member is a.*.

Occasionally Ramsey uses ordinary mathematical notations [$m \equiv n \pmod{l}$ means *m and n when divided by l have the same remainder*], and in discussing probability J. M. Keynes' symbolism p/h meaning *the probability of proposition p given proposition h.*

R. B. B.

I

THE FOUNDATIONS OF MATHEMATICS (1925)

Preface

The object of this paper is to give a satisfactory account of the Foundations of Mathematics in accordance with the general method of Frege, Whitehead and Russell. Following these authorities, I hold that mathematics is part of logic, and so belong to what may be called the logical school as opposed to the formalist and intuitionist schools. I have therefore taken *Principia Mathematica* as a basis for discussion and amendment; and believe myself to have discovered how, by using the work of Mr Ludwig Wittgenstein, it can be rendered free from the serious objections which have caused its rejection by the majority of German authorities, who have deserted altogether its line of approach.

CONTENTS

I. Introduction

In this chapter we shall be concerned with the general nature of pure mathematics,[1] and how it is distinguished from

[1] In future by 'mathematics' will always be meant 'pure mathematics'.

other sciences. Here there are really two distinct categories of things of which an account must be given—the ideas or concepts of mathematics, and the propositions of mathematics. This distinction is neither artificial nor unnecessary, for the great majority of writers on the subject have concentrated their attention on the explanation of one or other of these categories, and erroneously supposed that a satisfactory explanation of the other would immediately follow.

Thus the formalist school, of whom the most eminent representative is now Hilbert, have concentrated on the propositions of mathematics, such as '$2 + 2 = 4$'. They have pronounced these to be meaningless formulae to be manipulated according to certain arbitrary rules, and they hold that mathematical knowledge consists in knowing what formulae can be derived from what others consistently with the rules. Such being the propositions of mathematics, their account of its concepts, for example the number 2, immediately follows. '2' is a meaningless mark occurring in these meaningless formulae. But, whatever may be thought of this as an account of mathematical propositions, it is obviously hopeless as a theory of mathematical concepts; for these occur not only in mathematical propositions, but also in those of everyday life. Thus '2' occurs not merely in '$2 + 2 = 4$', but also in 'It is 2 miles to the station', which is not a meaningless formula, but a significant proposition, in which '2' cannot conceivably be a meaningless mark. Nor can there be any doubt that '2' is used in the same sense in the two cases, for we can use '$2 + 2 = 4$' to infer from 'It is two miles to the station and two miles on to the Gogs' that 'It is four miles to the Gogs via the station', so that these ordinary meanings of two and four are clearly involved in '$2 + 2 = 4$'. So the hopelessly inadequate formalist theory is, to some extent, the result of considering only the propositions of mathematics and neglecting the analysis of its concepts, on which additional light can be thrown by their occurrence outside mathematics in the propositions of everyday life.

Apart from formalism, there are two main general attitudes to the foundation of mathematics: that of the intuitionists or finitists, like Brouwer and Weyl in his recent papers, and that of the logicians—Frege, Whitehead, and Russell. The theories of the intuitionists admittedly involve giving up many of the most fruitful methods of modern analysis, for no reason, as it seems to me, except that the methods fail to conform to their private prejudices. They do not, therefore, profess to give any foundation for mathematics as we know it, but only for a narrower body of truth which has not yet been clearly defined. There remain the logicians whose work culminated in *Principia Mathematica.* The theories there put forward are generally rejected for reasons of detail, especially the apparently insuperable difficulties connected with the Axiom of Reducibility. But these defects in detail seem to me to be results of an important defect in principle, first pointed out by Mr Wittgenstein.

The logical school has concentrated on the analysis of mathematical concepts, which it has shown to be definable in terms of a very small number of fundamental logical concepts; and, having given this account of the concepts of mathematics, they have immediately deduced an account of mathematical propositions—namely, that they were those true propositions in which only mathematical or logical concepts occurred. Thus Russell, in *The Principles of Mathematics,* defines pure mathematics as 'the class of all propositions of the form "p implies q" where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants'.[1] This reduction of mathematics to symbolic logic was rightly described by Mr Russell as one of the greatest discoveries of our

[1] Russell, *The Principles of Mathematics* (1903), p. 3

age[1]; but it was not the end of the matter, as he seemed to suppose, because he was still far from an adequate conception of the nature of symbolic logic, to which mathematics had been reduced. I am not referring to his naive theory that logical constants were names for real objects (which he has since abandoned), but to his belief that any proposition which could be stated by using logical terms[2] alone must be a proposition of logic or mathematics.[3] I think the question is made clearer by describing the class of propositions in question as the completely general propositions, emphasizing the fact that they are not about particular things or relations, but about some or all things and relations. It is really obvious that not all such propositions are propositions of mathematics or symbolic logic. Take for example 'Any two things differ in at least thirty ways'; this is a completely general proposition, it could be expressed as an implication involving only logical constants and variables, and it may well be true. But as a mathematical or logical truth no one could regard it; it is utterly different from such a proposition as 'Any two things together with any other two things make four things,' which is a logical and not merely an empirical truth. According to our philosophy we may differ in calling the one a contingent, the other a necessary proposition, or the one a genuine proposition, the other a mere tautology; but we must all agree that there is some essential difference between the two, and that a definition of mathematical propositions must include not merely their complete generality but some further property as well. This is pointed out, with a reference to Wittgenstein, in Russell's *Introduction to Mathematical Philosophy*[4]; but there is no trace of it in *Principia Mathematica*, nor does Mr Russell

[1] Loc. cit., p. 5.
[2] i.e. variables and logical constants.
[3] I neglect here, as elsewhere, the arbitrary and trivial proviso that the proposition must be of the form 'p implies q'.
[4] p. 205.

seem to have understood its tremendous importance, for example, in the consideration of primitive propositions. In the passage referred to in the *Introduction to Mathematical Philosophy*, Mr Russell distinguishes between propositions which can be enunciated in logical terms from those which logic can assert to be true, and gives as the additional characteristic of the latter that they are 'tautological' in a sense which he cannot define. It is obvious that a definition of this characteristic is essential for a clear foundation of our subject, since the idea to be defined is one of the essential sides of mathematical propositions—their content and their form. Their content must be completely generalized and their form tautological.

The formalists neglected the content altogether and made mathematics meaningless, the logicians neglected the form and made mathematics consist of any true generalizations; only by taking account of both sides and regarding it as composed of tautologous generalizations can we obtain an adequate theory.

We have now to explain a definition of tautology which has been given by Mr Wittgenstein in his *Tractatus Logico-Philosophicus* and forms one of the most important of his contributions to the subject. In doing this we cannot avoid some explanation of his theory of propositions in general.

We must begin with the notion of an *atomic proposition*[1]; this is one which could not be analysed in terms of other propositions and could consist of names alone without logical constants. For instance, by joining 'ϕ', the name of a quality, to 'a', the name of an individual, and writing 'ϕa', we have an atomic proposition asserting that the individual has the quality. Thus, if we neglect the fact that 'Socrates' and 'wise' are incomplete symbols and regard them as names,

[1] Wittgenstein calls these 'elementary propositions'; I have called them 'atomic' in order to follow Mr Russell in using 'elementary' with a different meaning.

' Socrates is wise ' is an atomic proposition ; but ' All men are wise ', ' Socrates is not wise ', are not atomic.

Suppose now we have, say, n atomic propositions p, q, r, ... With regard to their truth or falsity there are 2^n mutually exclusive ultimate possibilities, which we could arrange in a table like this (T signifies truth, and F falsity, and we have taken $n = 2$ for brevity).

p	q
T	T
F	T
T	F
F	F

These 2^n possibilities we will call the truth-possibilities of the n atomic propositions. We may wish to pick out any sub-set of them, and assert that it is a possibility out of this sub-set which is, in fact, realized—that is, to express our agreement with some of the possibilities and our disgreement with the remainder. We can do this by setting marks T and F against the possibilities with which we agree and disagree respectively. In this way we obtain a proposition.

Thus

is the proposition ' Not both p and q are true ', or 'p is incompatible with q ', for we have allowed all the possibilities except the first, which we have disallowed.

Similarly

is the proposition ' If p, then q '.

A proposition which expresses agreement and disagreement with the truth-possibilities of p, q, . . . (which need not be atomic) is called a truth-function of the arguments p, q, . . . Or, more accurately, P is said to be the same truth-function of p, q, . . . as R is of r, s, . . . if P expresses agreement with the truth-possibilities of p, q, . . . corresponding by the substitution of p for r, q for s, . . . to the truth-possibilities of r, s, . . . with which R expresses agreement. Thus 'p and q ' is the same truth-function of p, q as 'r and s ' is of r, s, in each case the only possibility allowed being that both the arguments are true. Mr Wittgenstein has perceived that, if we accept this account of truth-functions as expressing agreement and disagreement with truth-possiblities, there is no reason why the arguments to a truth-function should not be infinite in number.[1] As no previous writer has considered truth-functions as capable of more than a finite number of

[1] Thus the logical sum of a set of propositions is the proposition that one at least of the set is true, and it is immaterial whether the set is finite or infinite. On the other hand, an infinite algebraic sum is not really a sum at all, but a *limit*, and so cannot be treated as a sum except subject to certain restrictions.

arguments, this is a most important innovation. Of course if the arguments are infinite in number they cannot all be enumerated and written down separately; but there is no need for us to enumerate them if we can determine them in any other way, as we can by using propositional functions.

A propositional function is an expression of the form '$f\hat{x}$', which is such that it expresses a proposition when any symbol (of a certain appropriate logical type depending on f) is substituted for '$\hat{x}$'. Thus '$\hat{x}$ is a man' is a propositional function. We can use propositional functions to collect together the range of propositions which are all the values of the function for all possible values of x. Thus '$\hat{x}$ is a man' collects together all the propositions 'a is a man', 'b is a man', etc. Having now by means of a propositional function defined a set of propositions, we can, by using an appropriate notation, assert the logical sum or product of this set. Thus, by writing '$(x) . fx$' we assert the logical product of all propositions of the form 'fx'; by writing '$(\exists x). fx$' we assert their logical sum. Thus '(x) . x is a man' would mean 'Everything is a man'; '$(\exists x)$. x is a man', 'There is something which is a man'. In the first case we allow only the possibility that all the propositions of the form 'x is a man' are true; in the second we exclude only the possibility that all the propositions of the form 'x is a man' are false.

Thus general propositions containing 'all' and 'some' are found to be truth-functions, for which the arguments are not enumerated but given in another way. But we must guard here against a possible mistake. Take such a proposition as 'All men are mortal'; this is not as might at first sight be supposed the logical product of the propositions 'x is mortal' for such values of x as are men. Such an interpretation can easily be shown to be erroneous (see, for example, *Principia Matnematica*, I, 1st ed., p. 47, 2nd ed., p. 45). 'All men are mortal' must be interpreted as meaning '(x). if x is a man, x is mortal', i.e. it is the logical product of all the values of the function 'if x is a man, x is mortal'.

Mr Wittgenstein maintains that all propositions are, in the sense defined, truth-functions of elementary propositions. This is hard to prove, but is on its own merits extremely plausible; it says that, when we assert anything, we are saying that it is one out of a certain group of ultimate possibilities which is realized, not one out of the remaining possibilities. Also it applies to all the propositions which could be expressed in the symbolism of *Principia Mathematica*; since these are built up from atomic propositions by using firstly conjunctions like 'if', 'and', 'or', and secondly various kinds of generality (apparent variables). And both these methods of construction have been shown to create truth-functions.[1]

From this account we see when two propositional symbols are to be regarded as instances of the same proposition—namely, when they express agreement and disagreement with the same sets of truth-possibilities of atomic propositions. Thus in the symbolism of *Principia Mathematica*

$$`p \supset q : \sim p \,.\, \supset .\, q', \quad `q \vee : p \,.\, \sim p'$$

are both more complicated ways of writing 'q'.

Given any set if n atomic propositions as arguments, there are 2^n corresponding truth-possibilities, and therefore 2^{2^n} sub-classes of their truth-possibilities, and so 2^{2^n} truth-functions of n arguments, one expressing agreement with each sub-class and disagreement with the remainder. But among these 2^{2^n} there are two extreme cases of great importance: one in which we express agreement with all the truth-possibilities, the other in which we express agreement with none of them. A proposition of the first kind is called a *tautology*, of the second a *contradiction*. Tautologies and contradictions are

[1] The form 'A believes p' will perhaps be suggested as doubtful. This is clearly not a truth-function of 'p', but may nevertheless be one of other atomic propositions.

not real propositions, but degenerate cases. We may, perhaps, make this clear most easily by taking the simplest case, when there is only one argument.

The tautology is '

p	
T	T
F	T

', i.e. 'p or not-p'.

This really asserts nothing whatever; it leaves you no wiser than it found you. You know nothing about the weather, if you know that it is either raining or not raining.[1]

The contradiction is '

p	
T	F
F	F

',

i.e. 'p is neither true nor false'.

This is clearly self-contradictory and does not represent a possible state of affairs whose existence could be asserted.

Tautologies and contradictions can be of all degrees of complexity; to give other examples '$(x).\phi x : \supset : \phi a$' is a tautology, '$\sim.(\exists x).\phi x : \phi a$' a contradiction. Clearly by negating a contradiction we get a tautology, and by negating a tautology a contradiction. It is important to see that tautologies are not simply true propositions, though for many purposes they can be treated as true propositions. A genuine proposition asserts something about reality, and it is true if reality is as it is asserted to be. But a tautology is a symbol constructed so as to say nothing whatever about reality, but to express total ignorance by agreeing with every possibility.

[1] Wittgenstein, *Tractatus Logico-Philosophicus*, 4·461.

The assimilation of tautologies and contradictions with true and false propositions respectively results from the fact that tautologies and contradictions can be taken as arguments to truth-functions just like ordinary propositions, and for determining the truth or falsity of the truth-function, tautologies and contradictions among its arguments must be counted as true and false respectively. Thus, if '*t*' be a tautology, '*c*' a contradiction, '*t* and *p*', 'If *t*, then *p*', '*c* or *p*' are the same as '*p*', and '*t* or *p*', 'if *c*, then *p*' are tautologies.

We have here, thanks to Mr Wittgenstein, to whom the whole of this analysis is due, a clearly defined sense of tautology; but is this, it may be asked, the sense in which we found tautology to be an essential characteristic of the propositions of mathematics and symbolic logic? The question must be decided by comparison. Are the propositions of symbolic logic and mathematics tautologies in Mr Wittgenstein's sense?

Let us begin by considering not the propositions of mathematics but those of *Principia Mathematica*.[1] These are obtained by the process of deduction from certain primitive propositions, which fall into two groups—those expressed in symbols and those expressed in words. Those expressed in words are nearly all nonsense by the Theory of Types, and should be replaced by symbolic conventions. The real primitive propositions, those expressed in symbols, are, with one exception, tautologies in Wittgenstein's sense. So, as the process of deduction is such that from tautologies only tautologies follow, were it not for one blemish the whole structure would consist of tautologies. The blemish is of course the Axiom of Reducibility, which is, as will be shown below,[2] a genuine proposition, whose truth or falsity is a

[1] This distinction is made only because *Principia Mathematica* may be a wrong interpretation of mathematics; in the main I think it is a right one.
[2] See Chapter V.

matter of brute fact, not of logic. It is, therefore, not a tautology in any sense, and its introduction into mathematics is inexcusable. But suppose it could be dispensed with, and *Principia Mathematica* were modified accordingly, this would consist entirely of tautologies in Wittgenstein's sense. And therefore, if *Principia Mathematica* is on the right lines as a foundation and interpretation of mathematics, it is Wittgenstein's sense of tautology in which mathematics is tautologous.

But the adequacy of *Principia Mathematica* is a matter of detail; and, since we have seen it contains a very serious flaw, we can no longer be sure that mathematics is the kind of thing Whitehead and Russell suppose it to be, or therefore that it consists of tautologies in Wittgenstein's sense. One thing is, however, clear: that mathematics does not consist of genuine propositions or assertions of fact which could be based on inductive evidence, as it was proposed to base the Axiom of Reducibility, but is in some sense necessary or tautologous. In actual life, as Wittgenstein says, " it is never a mathematical proposition which we need, but we use mathematical propositions *only* in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics ".[1] Thus we use '$2 \times 2 = 4$' to infer from 'I have two pennies in each of my two pockets' to 'I have four pennies altogether in my pockets'. '$2 \times 2 = 4$' is not itself a genuine proposition in favour of which inductive evidence can be required, but a tautology which can be seen to be tautologous by anyone who can fully grasp its meaning. When we proceed further in mathematics the propositions become so complicated that we cannot see immediately that they are tautologous, and have to assure ourselves of this by deducing them from more obvious tautologies. The primitive propositions on which we fall back in the end must be such that no evidence could be required

[1] Wittgenstein, op. cit., 6·211.

for them, since they are patent tautologies like 'If p, then p'. But the tautologies of which mathematics consist may perhaps turn out not to be of Wittgenstein's kind, but of some other. Their essential use is to facilitate logical inference; this is achieved in the most obvious way by constructing tautologies in Wittgenstein's sense, for if 'If p, then q' is a tautology, we can logically infer 'q' from 'p', and, conversely, if 'q' follows logically from 'p', 'If p, then q' is a tautology.[1] But it is possible that there are other kinds of formulae which could be used to facilitate inference; for instance, what we may call identities such as '$a = b$', signifying that 'a', 'b' may be substituted for one another in any proposition without altering it. I do not mean without altering its truth or falsity, but without altering what proposition it is. '$2 + 2 = 4$' might well be an identity in this sense, since 'I have 2 + 2 hats' and 'I have 4 hats' are the same proposition, as they agree and disagree with the same sets of ultimate truth-possibilities.

Our next problem is to decide whether mathematics consists of tautologies (in the precise sense defined by Wittgenstein, to which we shall in future confine the word 'tautology') or of formulae of some other sort. It is fairly clear that geometry, in which we regard such terms as 'point', 'line' as meaning any things satisfying certain axioms, so that the only constant terms are truth-functions like 'or', 'some', consists of tautologies. And the same would be true of analysis if we regarded numbers as any things satisfying Peano's axioms. Such a view would however be certainly inadequate, because since the numbers from 100 on satisfy Peano's axioms, it would give us no means of distinguishing 'This equation has three roots' from 'This equation has a hundred and three roots'. So numbers must be defined not as variables but as

[1] This may perhaps be made clearer by remarking that if 'q' follows logically from 'p', '$p . \sim q$' must be self-contradictory, therefore '$\sim (p . \sim q)$' tautologous or '$p \supset q$' tautologous.

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constants, and the nature of the propositions of analysis becomes doubtful.

I believe that they are tautologies, but the proof of this depends on giving a detailed analysis of them, and the disproof of any other theory would depend on finding an insuperable difficulty in the details of its construction. In this chapter I propose to discuss the question in a general way, which must inevitably be rather vague and unsatisfactory. I shall first try to explain the great difficulties which a theory of mathematics as tautologies must overcome, and then I shall try to explain why the alternative sort of theory suggested by these difficulties seem hopelessly impracticable. Then in the following chapters I shall return to the theory that mathematics consists of tautologies, discuss and partially reject the method for overcoming the difficulties given in *Principia Mathematica*, and construct an alternative and, to my mind, satisfactory solution.

Our first business is, then, the difficulties of the tautology theory. They spring from a fundamental characteristic of modern analysis which we have now to emphasize. This characteristic may be called *extensionality*, and the difficulties may be explained as those which confront us if we try to reduce a calculus of extensions to a calculus of truth-functions. Here, of course, we are using ' extension ' in its logical sense, in which the extension of a predicate is a class, that of a relation a class of ordered couples ; so that in calling mathematics extensional we mean that it deals not with predicates but with classes, not with relations in the ordinary sense but with possible correlations, or " relations in extension " as Mr Russell calls them. Let us take as examples of this point three fundamental mathematical concepts—the idea of a real number, the idea of a function (of a real variable), and the idea of similarity of classes (in Cantor's sense).

Real numbers are defined as segments of rationals ; any segment of rationals is a real number, and there are $2^{\aleph_0}$ of them. It is not necessary that the segment should be defined by any property or predicate of its members in any ordinary sense of predicate. A real number is therefore an extension, and it may even be an extension with no corresponding intension. In the same way a function of a real variable is a relation in extension, which need not be given by any real relation or formula.

The point is perhaps most striking in Cantor's definition of similarity. Two classes are said to be *similar* (*i.e.* have the same cardinal number) when there is a one-one relation whose domain is the one class and converse domain the other. Here it is essential that the one-one relation need only be a relation in extension ; it is obvious that two classes could be similar, i.e. capable of being correlated, without there being any relation actually correlating them.

There is a verbal point which requires mention here ; I do not use the word ' class ' to imply a principle of classification, as the word naturally suggests, but by a ' class ' I mean any set of things of the same logical type. Such a set, it seems to me, may or may not be definable either by enumeration or as the extension of a predicate. If it is not so definable we cannot mention it by itself, but only deal with it by implication in propositions about all classes or some classes. The same is true of relations in extension, by which I do not merely mean the extensions of actual relations, but any set of ordered couples. That this is the notion occurring in mathematics seems to me absolutely clear from the last of the above examples, Cantor's definition of similarity, where obviously there is no need for the one-one relation in extension to be either finite or the extension of an actual relation.

Mathematics is therefore essentially extensional, and may be called a calculus of extensions, since its propositions assert relations between extensions. This, as we have said, is hard to reduce to a calculus of truth-functions, to which it must be reduced if mathematics is to consist of tautologies; for tautologies are truth-functions of a certain special sort, namely those agreeing with all the truth-possibilities of their arguments. We can perhaps most easily explain the difficulty by an example.

Let us take an extensional assertion of the simplest possible sort: the assertion that one class includes another. So long as the classes are defined as the classes of things having certain predicates ϕ and ψ, there is no difficulty. That the class of ψ's includes the class of ϕ's means simply that everything which is a ϕ is a ψ, which, as we have seen above is a truth-function. But we have seen that mathematics has (at least apparently) to deal also with classes which are not given by defining predicates. (Such classes occur not merely when mentioned separately, but also in any statement about 'all classes', 'all real numbers'.) Let us take two such classes as simple as possible—the class (a, b, c) and the class (a, b). Then that the class (a, b, c) includes the class (a, b) is, in a broad sense, tautological and apart from its triviality would be a mathematical proposition; but it does not seem to be a tautology in Wittgenstein's sense, that is a certain sort of truth-function of elementary propositions. The obvious way of trying to make it a truth-function is to introduce identity and write '(a, b) is contained in (a, b, c)' as '$(x) :. x = a . \vee . x = b : \supset : x = a . \vee . x = b . \vee . x = c$'. This certainly looks like a tautological truth-function, whose ultimate arguments are values of '$x = a$', '$x = b$', '$x = c$', that is propositions like '$a = a$', '$b = a$', '$d = a$'. But these are not real propositions at all; in '$a = b$' either 'a', 'b' are names of the same thing, in which case the proposition says nothing, or of different things, in which case it is absurd. In neither case is it the assertion of a fact; it only appears to be a real assertion by confusion with the case when 'a' or 'b' is not a name but a description.[1] When 'a', 'b' are both names, the only significance which can be placed on '$a = b$' is that it indicates that we use 'a', 'b' as names of the same thing or, more generally, as equivalent symbols.

The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I should prefer to substitute 'identities'. That is, formulae of the form '$a = b$' where 'a', 'b' are equivalent symbols. There is a certain plausibility in such an account of, for instance, '$2 + 2 = 4$.' Since 'I have $2 + 2$ hats', 'I have 4 hats' are the same proposition,[2] '$2 + 2$' and '4' are equivalent symbols. As it stands this is obviously a ridiculously narrow view of mathematics, and confines it to simple arithmetic; but it is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties. It would be out of place here to give a detailed survey of this blind alley, but I shall try to indicate in a general way the obstructions which block its end.

First of all we have to consider of what kind mathematical propositions will on such a theory be. We suppose the most primitive type to be the identity '$a = b$', which only becomes a real proposition if it is taken to be about not the things meant by 'a', 'b', but these symbols themselves; mathematics then consists of propositions built up out of identities by a process analogous to that by which ordinary propositions are constructed out of atomic ones; that is to say, mathematical propositions are (on this theory), in some sense, truth-functions of identities. Perhaps this is an overstatement,

[1] For a fuller discussion of identity see the next chapter

[2] In the sense explained above. They clearly are not the same sentence, but they are the same truth-function of atomic propositions and so assert the same fact.

and the theory might not assert all mathematical propositions to be of this form; but it is clearly one of the important forms that would be supposed to occur. Thus

$$‘ x^2 - 3x + 2 = 0 : \supset_x . x = 2 . \vee . x = 1 ’$$

would be said to be of this form, and would correspond to a verbal proposition which was a truth-function of the verbal propositions corresponding to the arguments ‘ $x = 2$ ’, etc. Thus the above proposition would amount to ‘ If “ $x^2 - 3x + 2$ ” means 0, “ x ” means 2 or 1 ’. Mathematics would then be, in part at least, the activity of constructing formulae which corresponded in this way to verbal propositions. Such a theory would be difficult and perhaps impossible to develop in detail, but there are, I think, other and simpler reasons for dismissing it. These arise as soon as we cease to treat mathematics as an isolated structure, and consider the mathematical elements in non-mathematical propositions. For simplicity let us confine ourselves to cardinal numbers, and suppose ourselves to know the analysis of the proposition that the class of ϕ's is n in number [$\hat{x}(\phi x) \epsilon n$]. Here ϕ may be any ordinary predicate defining a class, e.g. the class of ϕ's may be the class of Englishmen. Now take such a proposition as ‘ The square of the number of ϕ's is greater by two than the cube of the number of ψ's ’. This proposition we cannot, I think, help analysing in this sort of way :

$$(\exists m, n) . \hat{x}\,(\phi x)\,\epsilon m \,.\, \hat{x}\,(\psi x)\,\epsilon n \,.\, m^2 = n^3 + 2.$$

It is an empirical not a mathematical proposition, and is about the ϕ's and ψ's, not about symbols; yet there occurs in it the mathematical pseudo-proposition $m^2 = n^3 + 2$, of which, according to the theory under discussion, we can only make sense by taking it to be about symbols, thereby making the whole proposition to be partly about symbols. Moreover, being an empirical proposition, it is a truth-function of elementary propositions expressing agreement with those possibilities which give numbers of ϕ's and ψ's satisfying $m^2 = n^3 + 2$. Thus '$m^2 = n^3 + 2$' is not, as it seems to be, one of the truth-arguments in the proposition above. but rather part of the truth-function like '$\sim$' or '$\vee$' or '$\exists, m, n$,' which determine which truth-function of elementary propositions it is that we are asserting. Such a use of $m^2 = n^3 + 2$ the identity theory of mathematics is quite inadequate to explain.

On the other hand, the tautology theory would do everything which is required; according to it $m^2 = n^3 + 2$ would be a tautology for the values of m and n which satisfy it, and a contradiction for all others. So

$$\hat{x}\,(\phi x)\ \epsilon m\ .\ \hat{x}\,(\psi x)\ \epsilon n\ .\ m^2 = n^3 + 2$$

would for the first set of values of m, n be equivalent to

$$\hat{x}\,(\phi x)\ \epsilon m\ .\ \hat{x}\,(\psi x)\ \epsilon n$$

simply, '$m^2 = n^3 + 2$' being tautologous, and therefore superfluous; and for all other values it would be self-contradictory. So that

$$\text{'}(\exists m, n) : \hat{x}\,(\phi x)\ \epsilon m . \hat{x}\,(\psi x)\ \epsilon n . m^2 = n^3 + 2\text{'}$$

would be the logical sum of the propositions

$$\text{'}\hat{x}\,(\phi x)\ \epsilon m . \hat{x}\,(\psi x)\ \epsilon n\text{'}$$

for all m, n satisfying $m^2 = n^3 + 2$, and of contradictions for all other m, n; and is therefore the proposition we require, since in a logical sum the contradictions are superfluous. So this difficulty, which seems fatal to the identity theory, is escaped altogether by the tautology theory, which we are therefore encouraged to pursue and see if we cannot find a way of overcoming the difficulties which we found would confront us in attempting to reduce an extensional calculus to