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**General Systems
Theory:
Mathematical
Foundations**

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General Systems Theory: Mathematical Foundations

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PREFACE

This book reports on the development of a mathematical theory of general systems, initiated ten years ago. The theory is based on a broad and ambitious program aimed at formalizing all major systems concepts and the development of an axiomatic and general theory of systems. The present book provides the foundations for, and the initial steps toward, the fulfillment of that program. The interest in the present volume is strictly in the mathematical aspects of the theory. Applications and philosophical implications will be considered elsewhere.

The basic characteristics and the role of the proposed general systems theory are discussed in some detail in the first chapter. However, the unifying power of the proposed foundations ought to be specifically singled out; within the same framework, using essentially the same mathematical structure for the specification of a system, such diverse topics are considered and associated results proven as: the existence and minimal axioms for state-space construction; necessary and sufficient conditions for controllability of multivalued systems; minimal realization from input-output data; necessary and sufficient conditions for Lyapunov stability of dynamical systems; Goedel consistency and completeness theorem; feedback decoupling of multivariable systems; Krohn-Rhodes decomposition theorem; classification of systems using category theory.

A system can be described either as a transformation of inputs (stimuli) into outputs (responses)—the so-called input-output approach (also referred to as the causal or terminal systems approach), or in reference to the fulfillment of a purpose or the pursuit of a goal—the so-called goal-seeking or decision-making approach. In this book we deal only with the input-output approach. Originally, we intended to include a general mathematical theory of goal-seeking, but too many other tasks and duties have prevented us

from carrying out that intention. In fairness to our research already completed, we ought to point out that the theory of multilevel systems which has been reported elsewhere† although aimed in a different direction, does contain the elements of a general theory of complex goal-seeking systems. For the sake of completeness we have given the basic definition of a goal-seeking system and of an open system (another topic of major concern) in Appendix II.

We have discussed the material over the years with many colleagues and students. In particular the advice and help of Donald Macko and Seiji Yoshii were most constructive. The manuscript would have remained a scribble of notes on a pile of paper if it were not for tireless and almost nondenumerable series of drafts retyped by Mrs. Mary Lou Cantini.

† M. D. Mesarovic, D. Macko, and Y. Takahara, "Theory of Hierarchical Multilevel Systems." Academic Press, New York, 1970.

**GENERAL SYSTEMS THEORY:
MATHEMATICAL FOUNDATIONS**

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Chapter I

INTRODUCTION

1. GENERAL SYSTEMS THEORY: WHAT IS IT AND WHAT IS IT FOR?

Systems theory is a scientific discipline concerned with the explanations of various phenomena, regardless of their specific nature, in terms of the formal relationships between the factors involved and the ways they are transformed under different conditions; the observations are explained in terms of the relationships between the components, i.e., in reference to the organization and functioning rather than with an explicit reference to the nature of the mechanisms involved (e.g., physical, biological, social, or even purely conceptual). The subject of study in systems theory is not a “physical object,” a chemical or social phenomenon, for example, but a “system”: *a formal relationship between observed features or attributes*. For conceptual reasons, the language used in describing the behavior of systems is that of information processing and goal seeking (decision making, control).

General systems theory deals with the most fundamental concepts and aspects of systems. Many theories dealing with more specific types of systems (e.g., dynamical systems, automata, control systems, game-theoretic systems, among many others) have been under development for quite some time. General systems theory is concerned with the basic issues common to all of these specialized treatments. Also, for truly complex phenomena, such as those found predominantly in the social and biological sciences, the specialized descriptions used in classical theories (which are based on special mathematical structures such as differential or difference equations, numerical or abstract algebras, etc.) do not adequately and properly represent the actual

events. Either because of this inadequate match between the events and types of descriptions available or because of the pure lack of knowledge, for many truly complex problems one can give only the most general statements, which are qualitative and too often even only verbal. General systems theory is aimed at providing a description and explanation for such complex phenomena.

Our contention is that one and the same theory can serve both of these purposes. Furthermore, in order to do that, it ought to be *simple, elegant, general, and precise* (unambiguous). This is why the approach we have taken is both mathematical and perfectly general. At the risk of oversimplification, the principal characteristics of the approach whose foundation is presented in this book are:

(i) It is a *mathematical* theory of general systems; basic concepts are introduced axiomatically, and the system's properties and behavior are investigated in a precise manner.

(ii) It is concerned with the goal seeking (decision making, control) and similar representations of systems, as much as with the input-output or "transformational" (causal) representations. For example, the study of hierarchical, multilevel, decision-making systems was a major concern from the very beginning.

(iii) The mathematical structures required to formalize the basic concepts are introduced in such a way that precision is obtained without losing any generality. It is important to realize that nothing is gained by avoiding the use of a precise language, i.e., mathematics, in making statements about a system of concern. We take exception, therefore, to considering general systems theory as a *scientific philosophy*, but rather, consider it as a *scientific enterprise*, without denying, however, the impact of such a scientific development on philosophy in general and epistemology in particular. Furthermore, once a commitment to the mathematical method is made, logical inferences can be drawn about the system's behavior. Actually, the investigation of the logical consequences of systems having given properties should be of central concern for any general systems theory which cannot be limited solely to a descriptive classification of systems.

The decision-making or goal-seeking view of a system's behavior is of paramount importance. General systems theory is not a generalized circuit theory—a position we believe has introduced much confusion and has contributed to the rejection of systems theory and the systems approach in fields where goal-seeking behavior is central, such as psychology, biology, etc. Actually, the theory presented in this book can just as well be termed *general cybernetics*, i.e., a general theory of governing and governed systems. The

term “general systems theory” was adopted at the initiation of the theory as reflecting a broader concern. However, in retrospect, it appears that the choice might not have been the happiest one, since that term has already been used in a different context.

The application of the mathematical theory of general systems can play a major role in the following important problem areas.

(a) Study of Systems with Uncertainties

On too many occasions, there is not enough information about a given system and its operation to enable a detailed mathematical modeling (even if the knowledge about the basic cause–effect relationships exists). A general systems model can be developed for such a situation, thus providing a solid mathematical basis for further study or a more detailed analysis. In this way, general systems theory, as conceived in this book, significantly extends the domain of application of mathematical methods to include the most diverse fields and problem areas not previously amenable to mathematical modeling.

(b) Study of Large-Scale and Complex Systems

Complexity in the description of a system with a large number of variables might be due to the way in which the variables and the relationships between them are described, or the number of details taken into account, even though they are not necessarily germane to the main purpose of study. In such a case, by developing a model which is less structured and which concentrates only on the key factors, i.e., a general systems model in the set-theoretic or algebraic framework, one can make the analysis more efficient, or even make it possible at all. In short, one uses a mathematically more abstract, less structured description of a large-scale and complex system. Many structural problems, such as decomposition, coordination, etc., can be considered on such a level. Furthermore, even some more traditional problems such as Lyapunov stability can be analyzed algebraically using more abstract descriptions.

The distinction between classical methods of approximation and the abstraction approach should be noted. In the former, one uses the same mathematical structure, and simplification is achieved by omitting some parts of the model that are considered less important; e.g., a fifth-order differential equation is replaced with a second-order equation by considering only the two “dominant” state variables of the system. In the latter approach, however, one uses a different mathematical structure which is more abstract, but which still considers the system as a whole, although from a less detailed viewpoint. The simplification is not achieved by the omission of variables but by the suppression of some of the details considered unessential.

(c) Structural Considerations in Model Building

In both the analysis and synthesis of systems of various kinds, structural considerations are of utmost importance. Actually, the most crucial step in the model-building process is the selection of a structure for the model of a system under consideration. It is a rather poor strategy to start investigations with a detailed mathematical model before major hypotheses are tested and a better understanding of the system is developed. Especially when the system consists of a family of interrelated subsystems, it is more efficient first to delineate the subsystems and to identify the major interfaces before proceeding with a more detailed modeling of the mechanisms of how the various subsystems function. Traditionally, engineers have used block diagrams to reveal the overall composition of a system and to facilitate subsequent structural and analytical considerations. The principal attractiveness of block diagrams is their simplicity, while their major drawback is a lack of precision. General systems theory models eliminate this drawback by introducing the precision of mathematics, while preserving the advantage, i.e., the simplicity, of block diagrams. The role of general systems theory in systems analysis can be represented by the diagram in Fig. 1.1. General

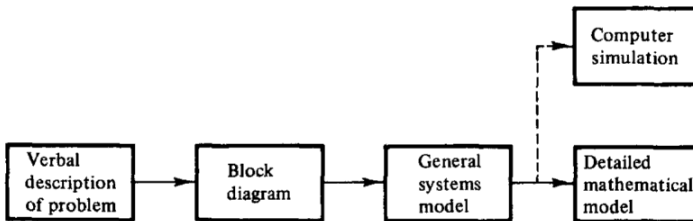


FIG. 1.1

systems models fall between block-diagram representation and a detailed mathematical (or computer) model. For complex systems, in particular, a general systems model might very well represent a necessary step, since the gulf between a block diagram and a detailed model can be too great. The fact that certain general systems techniques and results have become available to treat the systems problem on a general level makes it possible to add this step in practice.

(d) Precise Definition of Concepts and Interdisciplinary Communication

General systems theory provides a language for interdisciplinary communication, since it is sufficiently general to avoid introduction of constraints of its own, yet, due to its precision, it removes misunderstandings which can be quite misleading. (For example, the different notions of adaptation used in the fields of psychology, biology, engineering, etc., can first be formalized in

general systems theory terms and then compared.) It is often stated that systems theory has to reflect the “invariant” structural aspects of different real-life systems, i.e., those that remain invariant for similar phenomena from different fields (disciplines). This similarity can be truly established only if the relevant concepts are defined with sufficient care and precision. Otherwise, the danger of confusion is too great. It is quite appropriate, therefore, to consider the mathematical theory of general systems as providing a framework for the formalization of any systems concept. In this sense, general systems theory is quite basic for the application of the “systems approach” and systems theory in almost any situation. The important point to note when using general systems theory for concept definition is that when a concept has been introduced in a precise manner, what is crucial is not whether the definition is “correct” in any given interpretation, but rather, whether the concept is defined with sufficient precision so that it can be clearly and unambiguously understood and as such can be further examined and used in other disciplines. It is in this capacity that the general systems theory offers a language for interdisciplinary communication. Such an application of general systems theory might seem trivial from the purely mathematical standpoint, but is not so from the viewpoint of managing a large team effort in which specialists from different disciplines are working together on a complex problem, as is often found in the fields of environmental, urban, regional, and other large-scale studies.

(e) Unification and Foundation for More Specialized Branches of Systems Theory

Questions regarding basic systems problems which transcend many specialized branches of systems theory (e.g., the question of state-space representation) can be properly and successfully considered on the general systems level. This will be demonstrated many times in this book. The problems of foundations are of interest for extending and making proper use of systems theory in practice, for pedagogical reasons, and for providing a coherent framework to organize the facts and findings in the broad areas of systems research.

2. FORMALIZATION APPROACH FOR THE DEVELOPMENT OF THE MATHEMATICAL THEORY OF GENERAL SYSTEMS

The approach that we have used to develop the general systems theory reported in this book is the following:†

† A comparison with some other possible approaches is given in Appendix II.

(i) The basic systems concepts are introduced via *formalization*. By this we mean that starting from a verbal description of an intuitive notion, a precise mathematical definition for the concept is given using minimal mathematical structure, i.e., as few axioms as the correct interpretation would allow.

(ii) Starting from basic concepts introduced via formalization, the mathematical theory of general systems is further developed by adding more mathematical structure as needed for the investigation of various systems properties. Such a procedure allows us to establish how fundamental some particular systems properties really are and also what is the minimal set of assumptions needed in order that a given property or relationship holds.

The starting point for the entire development is the concept of a system defined on the set-theoretic level. Quite simply and most naturally for that level, a system is defined as a relation in the set-theoretic sense, i.e., it is assumed that a family of sets is given,

$$\bar{V} = \{V_i : i \in I\}$$

where I is the index set, and a system, defined on \bar{V} , is a proper subset of $\times \bar{V}$,

$$S \subset \times \{V_i : i \in I\}$$

The components of S , V_i , $i \in I$, are termed the systems objects. We shall primarily be concerned with a system consisting of two objects, the input object X and the output object Y :

$$S \subset X \times Y \tag{1.1}$$

Starting a mathematical theory of general systems on the set-theoretic level is fully consistent with the stated objective of starting with the least structured and most widely applicable concepts and then proceeding with the development of a mathematical theory in an axiomatic manner.

To understand better some of the reasons for adopting the concept of a system as a set-theoretic relation, the following remarks are pertinent.

A system is defined in terms of observed features or, more precisely, in terms of the relationship between those features rather than what they actually are (physical, biological, social, or other phenomena). This is in accord with the nature of the systems field and its concern with the organization and interrelationships of components into an (overall) system rather than with the specific mechanisms within a given phenomenological framework.

The notion of a system as given in (1.1) is perfectly general. On the one hand, if a system is described by more specific mathematical constructs, e.g., a set of

equations, it is obvious that these constructs define or specify a relation as given in (1.1). Different systems, of course, have different methods of specification, but they all are but relations as given in (1.1). On the other hand, in the case of the most incomplete information when the system can be described only in terms of a set of verbal statements, they still, by their linguistic function as statements, define a relation as in (1.1). Indeed, every statement contains two basic linguistic categories : nouns and functors—nouns denoting objects, functors denoting the relationship between them. For any proper set of verbal statements there exists a (mathematical) relation which represents the formal relationship between the objects denoted by nouns (technically referred to as a model for these statements). The adjective “proper” refers here, of course, to the conditions for the axioms of a set theory. In short, then, a system is always a relation, as given in (1.1), and various types of systems are more precisely defined by the appropriate methods, linguistic, mathematical, computer programs, etc.

A system is defined as a set (of a particular kind, i.e., a relation). It stands for the collection of all appearances of the object of study rather than for the object of study itself. This is necessitated by the use of mathematics as the language for the theory in which a “mechanism” (a function or a relation) is defined as a set, i.e., as a collection of all proper combinations of components. Such a characterization of a system ought not to create any difficulty since the set relation, with additional specifications, contains all the information about the actual “mechanism” we can legitimately use in the development of a formal theory.

The specification of a given system is often given in terms of some equations defined on appropriate variables. To every variable there corresponds a systems object which represents the range of the respective variable. Stating that a system is defined by a set of equations on a set of variables, one essentially states that the system is a relation on the respective systems objects specified by the variables (each one with a corresponding object as a range) such that for any combination of elements from the objects, i.e., the values for the variables, the given set of equations is satisfied.

To develop any kind of theory starting from (1.1), it is necessary to introduce more structure into the system as a relation. This can be done in two ways :

- (i) by introducing the additional structure into the elements of the system objects, i.e., to consider an element $v_i \in V_i$ as a set itself with additional appropriate structure ;
- (ii) by introducing the structure in the object sets, $V_i, i \in I$, themselves.

The first approach leads to the (abstract) time system concept, the second to the concept of an algebraic system.

(a) Time Systems

This approach will be introduced precisely in Chapter II and will be used extensively throughout the book ; therefore, a brief sketch is sufficient here.

If the elements of an object are functions, e.g., $v: T_v \rightarrow A_v$, the object is referred to as a family object or a function-generated object. Of particular interest is the case when both the domain and codomain of all the functions in the given object V are the same, i.e., any $v \in V$ is a function on T into A , $v: T \rightarrow A$. T represents the *index set* for V , while A is referred to as the *alphabet* for V . Notice that A can be of arbitrary cardinality. If the index set is linearly ordered, it is called a *time set*. This term was selected because such an index set captures the minimal property necessary for the concept of time, particularly as it relates to the time evolution and dynamic behavior of systems.

A function defined on a time set is called an (*abstract*) *time function*. An object whose elements are time functions is referred to as a *time object*. A system defined on time objects represents a *time system*.

Of particular interest are time systems whose input and output objects are both defined on the same sets $X \subseteq A^T$ and $Y \subseteq B^T$. The system is then

$$S \subseteq A^T \times B^T$$

(b) Algebraic Systems

An alternative way to introduce mathematical structure in a system's object V necessary for constructive specification is to define one or more operations in V so that V becomes an algebra. In the simplest case, a binary operation is given, $R: V \times V \rightarrow V$, and it is assumed that there exists a subset W of V , often of finite cardinality, such that any element in V can be obtained by the application of R on the elements of W or previously generated elements. The set W is referred to as the set of generators, or also as an *alphabet* and its elements as the *symbols*; the elements of the object V are referred to as *words*. If R is concatenation, the words are simply sequences of elements from the alphabet W .

A distinction should be noticed between the alphabet for a time object and for an algebraic object. For objects with finite alphabets, these are usually the same sets, i.e., the object, whose elements are sequences from the given set, can be viewed either as a set of time functions (on different time intervals, though) or as a set generated by an algebraic operation from the same set of symbols. When the alphabet is infinite, complications arise, and the set of generators and the codomain of the time functions are different sets, generally even of different cardinality.

In a more general situation, an algebraic object is generated by a family of operations. Namely, given a set of elements, termed primitive elements, W , and a set of operations $\bar{R} = \{R_1, \dots, R_n\}$, the object V contains the primitive elements themselves, $W \subset V$, and any element that can be generated by a repeated application of the operations from \bar{R} .

We shall use primarily the time systems approach in this book because it allows a more appealing intuitive interpretation in particular for the phenomena of time evolution and state transition. Actually, it can be shown that the two approaches are, by and large, equivalent. It should be emphasized, however, that we shall be using the algebraic structure both within a general system, $S \subset X \times Y$, and a general time system, $S \subset A^T \times B^T$, although not necessarily for the problems related to time evolution.

It is interesting to note that the two methods mentioned above correspond to the two basic ways to define a set constructively: by (transfinite) induction on an ordered set and by algebraic induction. The implication and meaning of this interesting fact will not be pursued here any further.

Chapter II

BASIC CONCEPTS

In this chapter we shall introduce some basic systems notions on the set-theoretic level and establish some relationships between them. First, we shall define a general system as a relation on abstract sets and then define the general time and dynamical systems as general systems defined on the sets of abstract time functions.

In order to enable more specific definition of various types of systems, certain kinds of so-called auxiliary functions are introduced. They are abstract counterparts of relationships, often given in the form of a set of equations, in terms of which a system is defined. Auxiliary functions enable also a more detailed analysis of systems, in particular their evolution in time.

In order to define various auxiliary functions, new auxiliary objects, termed state objects, had to be introduced; the elements of such an object are termed states. The primary functions of the state, as introduced in this chapter, are:

(i) to enable a system or its restrictions, which are both in general relations, to be represented as functions;

(ii) to enable the determination of a future output solely on the basis of a given future input and the present state completely disregarding the past (the state at any given time embodies the entire past history of the system);

(iii) to relate the states at different times so that one can determine whether the state of a system has changed over time and in what way. This third requirement leads to the concept of a state space. A general dynamical system is defined in such a state space.

Some basic conditions are given for the existence of various types of auxiliary functions in general and in reference to such system properties as input completeness and linearity. A classification of systems in reference to various kinds of time invariance of certain auxiliary functions is given. Finally, some questions of time causality are considered. Two notions are introduced in this respect:

(i) A system is termed nonanticipatory if there exists a family of state objects so that the future values of any output are determined solely by the state at a previous time and the input in this time period.

(ii) A system is termed past-determined if, after a certain initial period of time, the values of any output are determined solely by the past input–output pair. Conditions are then given for time systems to be nonanticipatory or past-determined.

1. SET-THEORETIC CONCEPT OF A GENERAL SYSTEM

(a) General System, Global States, and Global-Response Function

Starting point for the entire development is provided by the following definitions.

Definition 1.1. A (general) system is a relation on nonempty (abstract) sets

$$S \subset \times \{V_i : i \in I\} \quad (2.1)$$

where \times denotes Cartesian product and I is the index set. A component set V_i is referred to as a system object. When I is finite, (2.1) is written in the form

$$S \subset V_1 \times \cdots \times V_n \quad (2.2)$$

Definition 1.2. Let $I_x \subset I$ and $I_y \subset I$ be a partition of I , i.e., $I_x \cap I_y = \phi$, $I_x \cup I_y = I$. The set $X = \times \{V_i : i \in I_x\}$ is termed the input object, while $Y = \times \{V_i : i \in I_y\}$ is termed the output object. The system S is then

$$S \subset X \times Y \quad (2.3)$$

and will be referred to as an input–output system.

The form (2.3) rather than (2.2) will be used throughout this book.

Definition 1.3. If S is a function

$$S : X \rightarrow Y \quad (2.4)$$

it is referred to as a function-type (or functional) system.

Notice that the same symbol S is used both in (2.2) and (2.3) although strictly speaking the elements of the relation in (2.2) are n -tuples while those in the relation (2.3) are pairs. This convention is adopted for the sake of simplicity of notation. Which of the forms for S is used will be clear from the context in which it is used. Analogous comment applies to the use of the same symbol S in (2.3) and (2.4).

For notational convenience, we shall adopt the following conventions: The brackets in the domain of any function, e.g., $F:(A) \rightarrow B$, will indicate that the function F is only partial, i.e., it is not defined for every element in the domain A . The domain of F will be denoted by $\mathcal{D}(F) \subset A$, and the range by $\mathcal{R}(F) \subset B$. Similarly, the domain and the range of $S \subset X \times Y$ will be denoted, respectively, by

$$\mathcal{D}(S) = \{x : (\exists y)((x, y) \in S)\} \quad \text{and} \quad \mathcal{R}(S) = \{y : (\exists x)((x, y) \in S)\}$$

For the sake of notational simplicity, $\mathcal{D}(S) = X$ is always assumed unless stated otherwise.

Definition 1.4. Given a general system S , let C be an arbitrary set and R a function, $R:(C \times X) \rightarrow Y$, such that

$$(x, y) \in S \leftrightarrow (\exists c)[R(c, x) = y]$$

C is then a global state object or set, its elements being global states, while R is a global (systems)-response function (for S).

Theorem 1.1. Every system has a global-response function which is not partial, i.e.,

$$R:C \times X \rightarrow Y$$

PROOF. Let $F = Y^X = \{f:f:X \rightarrow Y\}$. Let $G = \{f_c:c \in C\} \subseteq F$ such that $f_c \in G \leftrightarrow f_c \subseteq S$, where C is an index set of G . Let $R:C \times X \rightarrow Y$ be such that $R(c, x) = f_c(x)$. Then we claim that $S = \{(x, y):(\exists c)(y = R(c, x))\}$. Let $S' = \{(x, y):(\exists c)(y = R(c, x))\}$. Let $(x, y) \in S'$ be arbitrary. Then $y = R(c, x) = f_c(x)$ for some $c \in C$. Hence, $(x, y) \in S$ because $f_c \subseteq S$. Therefore, $S' \subseteq S$. Conversely, let $(x, y) \in S$ be arbitrary. Since $\mathcal{D}(S) = X \ni x$, S is nonempty. Let $f_c \in G$. Let $\hat{f} = (f_c \setminus \{(x, f_c(x))\}) \cup \{(x, y)\}$. Then $\hat{f} \in F$ and $\hat{f} \subseteq S$. Hence, $\hat{f} = f_{c'}$ for some $c' \in C$. Consequently, $y = f_{c'}(x)$ or $(x, y) \in S'$ and hence $S \subseteq S'$. Therefore, $S = S'$. Q.E.D.

In the preceding theorem, no additional requirements are imposed either on C or R . However, if R is required to have a certain property, the global response function, although it might still exist, cannot be defined on the entire $C \times X$, i.e., R remains a partial function. Such is the case, e.g., when R

is required to be causal. Since the case when R is not a partial function is of special importance, we shall adopt the following convention:

R will be referred to as the global-response function only if it is not a partial function. Otherwise, it will be referred to as a partial global-response function.

(b) Abstract Linear System

Although many systems concepts can be defined solely by using the notion of a general system, the development of meaningful mathematical results is possible often only if additional structure is introduced. In order to avoid proliferation of definitions, we shall, as a rule, introduce specific concepts on the same level of abstraction on which mathematical results of interest can be developed; e.g., the concept of a dynamical system will be introduced only in the context of time systems. The concept of linearity, however, can be introduced usefully on the general systems level. We shall first introduce the notion of linearity, which is used as standard in this book.

Definition 1.5. Let \mathcal{A} be a field, X and Y be linear algebras over \mathcal{A} and let S be a relation, $S \subset X \times Y$, S is nonempty, and

- (i) $s \in S$ & $s' \in S \rightarrow s + s' \in S$
- (ii) $s \in S$ & $\alpha \in \mathcal{A} \rightarrow \alpha s \in S$

where $+$ is the additive operation in $X \times Y$ and $\alpha \in \mathcal{A} \uparrow S$ is then an (abstract) complete linear system.

In various applications, one encounters linear systems that are not complete, e.g., a system described by a set of linear differential equations whose set of initial conditions is not a linear space. For the sake of simplicity, we shall consider in this book primarily the complete systems, and, therefore, *every linear system will be assumed to be complete unless explicitly stated otherwise*. This is hardly a loss of generality since any incomplete linear system can be made complete by a perfectly straightforward and natural completeness procedure.

The following theorem is fundamental for linear systems theory.

Theorem 1.2. Let X and Y be linear algebras over the same field \mathcal{A} . $S \subset X \times Y$ is then a linear system if and only if there exists a global response function $R: C \times X \rightarrow Y$ such that:

- (i) C is a linear algebra over \mathcal{A} ;
- (ii) there exists a pair of linear mappings

$$R_1: C \rightarrow Y \quad \text{and} \quad R_2: X \rightarrow Y$$

† The operation $+$ and the scalar multiplication on $X \times Y$ is defined by: $(x, y) + (\hat{x}, \hat{y}) = (x + \hat{x}, y + \hat{y})$ and $\alpha(x, y) = (\alpha x, \alpha y)$ where $(x, y), (\hat{x}, \hat{y}) \in X \times Y$ and $\alpha \in \mathcal{A}$.

such that for all $(c, x) \in C \times X$

$$R(c, x) = R_1(c) + R_2(x)$$

PROOF. The *if* part is clear. Let us prove the *only if* part. First, we shall show that there exists a linear mapping $R_2: X \rightarrow Y$ such that $\{(x, R_2(x)): x \in X\} \subset S$. Let X_s be a subspace of X and $L_s: X_s \rightarrow Y$ a linear mapping such that $\{(x, L_s(x)): x \in X_s\} \subset S$. Such X_s and L_s always exist. Indeed, let $(\hat{x}, \hat{y}) \in S \neq \phi$; then $X_s = \{\alpha\hat{x}: \alpha \in \mathcal{A}\}$ and $L_s: X_s \rightarrow Y$ such that $L_s(\alpha\hat{x}) = \alpha\hat{y}$ are the desired ones. If $X_s = X$, then L_s is the desired linear mapping. If $X_s \neq X$, then L_s can always be extended by Zorn's lemma so that $X_s = X$ is achieved. Let $\tilde{L} = \{L_p\}$ be the class of all linear mappings defined on the subspaces of X such that when the subspace X_p is the domain of L_p , $\{(x, L_p(x)): x \in X_p\} \subset S$ holds. Notice that \tilde{L} is not empty. Let \leq be an ordering on \tilde{L} defined by: When L' and L'' are in \tilde{L} , then $L' \leq L''$ iff $L' \subseteq L''$. Since a mapping is a relation between the domain and the codomain and since a relation is a set, the above definition is proper. Let $P \subset \tilde{L}$ be an arbitrary linearly ordered subset of \tilde{L} . Let $L_0 = \bigcup P$, where $\bigcup P$ is the union of elements of P . We shall show that L_0 is in \tilde{L} . Suppose (x, y) and (x, y') are elements of L_0 . Then, since $L_0 = \bigcup P$, there exist two mappings L and L' in P such that $(x, y) \in L$ and $(x, y') \in L'$. Since P is linearly ordered, e.g., $L \leq L'$, $(x, y) \in L'$ holds. Since L' is a mapping, $y = y'$; that is, L_0 is a mapping. Next, suppose (x', y') and (x'', y'') are in L_0 . Then the same argument implies that $(x', y') \in L''$ and $(x'', y'') \in L''$ for some $L'' \subset P$. Since L'' is linear, $(x' + x'', y' + y'') \in L'' \subset L_0$. Furthermore, if $(x', y') \in L_0$ and $\alpha \in \mathcal{A}$, then there exists L' in P such that $(x', y') \in L'$; that is, $(\alpha x', \alpha y') \in L' \subset L_0$ holds. Hence, L_0 is a linear mapping. Finally, if $(x', y') \in L_0$, then $(x', y') \in L'$ for some L' in P . Hence, $(x', y') \in S$, or $L_0 \subset S$. Therefore, $L_0 \in \tilde{L}$. Consequently, L_0 is an upper bound of P in \tilde{L} . We can, then, conclude by Zorn's lemma that there is a maximal element R_2 in \tilde{L} . We claim that $\mathcal{D}(R_2) = X$. If it is not so, $\mathcal{D}(R_2)$ is a proper subspace of X . Then there is an element \hat{x} in X such that \hat{x} is not an element of $\mathcal{D}(R_2)$. Then $X' = \{\alpha\hat{x} + x: \alpha \in \mathcal{A} \text{ \& } x \in \mathcal{D}(R_2)\}$ is a linear subspace which includes $\mathcal{D}(R_2)$ properly. Notice that every element x' of X' is expressed in the form $x' = \alpha\hat{x} + x$ uniquely. As a matter of fact, if $x' = \alpha\hat{x} + x = \beta\hat{x} + y$, then $(\alpha - \beta)\hat{x} = (y - x)$. If $\alpha \neq \beta$, then $\hat{x} = (\alpha - \beta)^{-1}(y - x) \in \mathcal{D}(R_2)$, which is a contradiction. By using this fact, we can define a linear mapping $L': X' \rightarrow Y$ such that $L'(\alpha\hat{x} + x) = \alpha\hat{y} + R_2(x)$, where $(\hat{x}, \hat{y}) \in S$ and $x \in \mathcal{D}(R_2)$. Then L' is linear and $\{(x', L'(x')): x' \in X'\} \subset S$, and R_2 is a proper subset of L' , which contradicts the maximality of R_2 . Hence, R_2 is the desired mapping. To complete the construction of R , let C be $C = \{(o, y): (o, y) \in S\}$. C is, apparently, a linear space over \mathcal{A} when the addition and the scalar multiplication are defined as: $(o, y) + (o, y') = (o, y + y')$ and $\alpha(o, y) = (o, \alpha y)$, where $\alpha \in \mathcal{A}$. Let $R_1: C \rightarrow Y$ such that $R_1((o, y)) = y$. Then R_1 is a linear mapping. Let

$R(c, x) = R_1(x) + R_2(x)$. We shall show that

$$S = \{(x, y) : (\exists c)(c \in C \ \& \ y = R(c, x))\} \equiv S'$$

Suppose $(x, y) \in S$. Then $(x, R_2(x)) \in S$. Since S is linear, $(x, y) - (x, R_2(x)) = (0, y - R_2(x)) \in S$. Hence, $(\exists c)(c \in C \ \& \ y = R_1(c) + R_2(x))$; that is, $S \subseteq S'$ holds. Conversely, suppose $(x, R_1(c) + R_2(x)) \in S'$. Since $(0, R_1(c)) \in S$ and $(x, R_2(x)) \in S$ and since S is linear,

$$(x, R_2(x)) + (0, R_1(c)) = (x, R_1(c) + R_2(x)) \in S$$

Hence, $S' \subseteq S$ holds.

Q.E.D.

The fundamental character of the preceding theorem is illustrated by the fact that every result on linear systems developed in this book is based on it. We can now introduce the following definition.

Definition 1.6. Let $S \subset X \times Y$ be a linear system and R a mapping $R : C \times X \rightarrow Y$. R is termed a linear global-response function if and only if

- (i) R is consistent with S , i.e.,

$$(x, y) \in S \leftrightarrow (\exists c)[y = R(c, x)]$$

- (ii) C is a linear algebra over the field of X and Y ;

(iii) there exist two linear mappings $R_1 : C \rightarrow Y$ and $R_2 : X \rightarrow Y$ such that for all $(c, x) \in C \times X$

$$R(c, x) = R_1(c) + R_2(x)$$

C is referred to as the *linear global state object*. The mapping $R_1 : C \rightarrow Y$ is termed the *global state response*, while $R_2 : X \rightarrow Y$ is the *global input response*.

Notice the distinction between the global-response function and the linear global-response function. The first concept requires only (i), while for the second, conditions (ii) and (iii) have to be satisfied. A linear system, therefore, can have a response function which is not linear.

From Theorem 1.2 we have immediately the following proposition.

Proposition 1.1. A system is linear if and only if it has a linear global-response function.

The concept of a linear system as given by Definition 1.5 uses more than a “minimal” mathematical structure. The most abstract notion of a linear system consistent with the formalization approach is actually given by the following definition.

Definition 1.7. Let X be an (abstract) algebra with a binary operation $\cdot : X \times X \rightarrow X$ and a family of endomorphisms $\bar{\alpha} = \{\alpha : X \rightarrow X\}$; similarly, let Y have a binary operation $*$: $Y \times Y \rightarrow Y$ and a family $\bar{\beta} = \{\beta : Y \rightarrow Y\}$. A function system $S : X \rightarrow Y$ is a general linear system if and only if there exists a one-to-one mapping $\psi : \bar{\alpha} \rightarrow \bar{\beta}$ such that:

- (i) $(\forall x, x')[S(x \cdot x') = S(x) * S(x')]$
- (ii) $(\forall x)(\forall \alpha)[S(\alpha(x)) = \psi(\alpha)(S(x))]$

There could be other concepts of a linear system with the structure between that in Definitions 1.5 and 1.7, e.g., by assuming that X and Y are modules rather than abstract linear spaces. We have not considered such “intermediate” concepts in this book because for some essential results the structure of an abstract linear space is needed. Actually, one might argue that the concept of linearity based on the module structure is not satisfactory because it is neither most abstract (such as Definition 1.7) nor sufficiently rich in structure to allow proofs of basic mathematical results such as Theorem 1.2.

2. GENERAL TIME AND DYNAMICAL SYSTEMS

(a) General Time System

In order to introduce the concept of a general time system, we have to formalize the notion of time. In accordance with the strategy pronounced in Chapter I, we have to define the notion of time by using minimal mathematical structure and such that it captures the most essential feature of an intuitive notion of time. This seems a very easy task, yet the decision at this junction is quite crucial. The selection of structure for such a basic concept as the time set has important consequences for the entire subsequent developments and the richness and elegance of the mathematical results. We shall use the following notion.

Definition 2.1. A time set (for a general time system) is a linearly ordered (abstract) set. The time set will be denoted by T and the ordering in T by \leq .

Apparently, the minimal property of a time set is considered to be that its elements follow each other in an orderly succession. This reflects our intended usage of the concept of time for the study of the evolution of systems. No restrictions regarding cardinality are imposed on the time set. However, the time set might have some additional structure, e.g., that of an Abelian group. We shall introduce such additional assumptions when needed.

For notational convenience, T will be assumed to have the minimal element o . In other words, we assume that there exists a superset \bar{T} with a linear ordering \leq and a fixed element denoted by o in \bar{T} such that T is defined by $T = \{t : t \geq o\}$.

We can introduce now the following definition.

Definition 2.2. Let A and B be arbitrary sets, T a time set, A^T and B^T the set of all maps on T into A and B , respectively, $X \subset A^T$ and $Y \subset B^T$. A general time system S on X and Y is a relation on X and Y , i.e., $S \subset X \times Y$. A and B are called alphabets of the input set X and output set Y , respectively. X and Y are also termed time objects, while their elements $x : T \rightarrow A$ and $y : T \rightarrow B$ are abstract time functions. The values of X and Y at t will be denoted by $x(t)$ and $y(t)$, respectively.

In order to study the dynamical behavior of a time system, we need to introduce the appropriate *time segments*. In this respect, we shall use the following notational convention.

For every $t, t' > t$,

$$\begin{aligned} T_t &= \{t' : t' \geq t\}, & T^t &= \{t' : t' < t\}, & T_{t'} &= \{t'' : t \leq t'' < t'\} \\ \bar{T}_{t'} &= T_{t'} \cup \{t'\}, & \bar{T}^t &= T^t \cup \{t\} \end{aligned}$$

Corresponding to various time segments, the restrictions of $x \in A^T$ will be defined as follows.

$$x_t = x|T_t, \quad x^t = x|T^t, \quad x_{t'} = x|T_{t'}, \quad \bar{x}_{t'} = x|\bar{T}_{t'}$$

$$\bar{x}^t = x|\bar{T}^t, \quad X_t = \{x_t : x_t = x|T_t \& x \in X\}$$

$$X^t = \{x^t : x^t = x|T^t \& x \in X\}, \quad X_{t'} = \{x_{t'} : x_{t'} = x|T_{t'} \& x \in X\}$$

$$X(t) = \{x(t) : x \in X\}$$

The following conventions will also be used:

$$x_{t'} = \phi, \quad X_{t'} = \{\phi\}$$

Based on the restriction operation, we shall introduce another operation called *concatenation*. Let $x \in A^T$ and $x^* \in A^T$. Then for any t we can define another element \hat{x} in A^T :

$$\hat{x}(\tau) = \begin{cases} x(\tau), & \text{if } \tau < t \\ x^*(\tau), & \text{if } \tau \geq t \end{cases}$$

\hat{x} is represented by $\hat{x} = x^t \cdot x_t^*$ and is called the concatenation of x^t and x_t^* .

Given a set $X \subset A^T$, the family of all restrictions of X as defined above will be denoted by \bar{X} , i.e.,

$$\bar{X} = \{\hat{x} : (\hat{x} = x \vee \hat{x} = x_t \vee \hat{x} = x^t \vee \hat{x} = x_{tt'} \vee \hat{x} = \bar{x}_{tt'} \vee \hat{x} = \bar{x}^t) \\ \& x \in X \& t, t' \in T \& t' \geq t\}$$

The restrictions in Y and the corresponding operations are defined in completely the same way as in X .

For the sake of technical convenience, we shall introduce also the following definition.

Definition 2.3. A time system $S \subset X \times Y$ is input complete if and only if

$$(\forall x)(\forall x^*)(\forall t)(x, x^* \in \mathcal{D}(S) \& t \in T \rightarrow x^t \cdot x_t^* \in \mathcal{D}(S))$$

and

$$(\forall t)(\{x(t) \mid x \in X\} = A)$$

In the succeeding discussions, every general time system is assumed input complete unless explicitly stated otherwise.

The restrictions of a time system S are defined in reference to the restrictions of inputs and outputs:

$$S_t = \{(x_t, y_t) : x_t = x \mid T_t \& y_t = y \mid T_t \& (x, y) \in S\} \\ S^t = \{(x^t, y^t) : x^t = x \mid T^t \& y^t = y \mid T^t \& (x, y) \in S\} \\ S_{tt'} = \{(x_{tt'}, y_{tt'}) : x_{tt'} = x \mid T_{tt'} \& y_{tt'} = y \mid T_{tt'} \& (x, y) \in S\} \\ \bar{S} = \{\hat{s} : \hat{s} = s \vee \hat{s} = s^t \vee \hat{s} = s_t \vee \hat{s} = s_{tt'}\}$$

We shall also use the following notational convention:

$$X_t = X \mid T_t, \quad X^t = X \mid T^t, \quad X_{tt'} = X \mid T_{tt'}$$

and completely analogous for Y , and for S , e.g.,

$$S_t = S \mid T_t, \quad S^t = S \mid T^t, \quad \text{and} \quad S_{tt'} = S \mid T_{tt'}$$

Definition 2.4. Let S be a time system $S \subset A^T \times B^T$. The initial state object for S and the initial systems-response function are the global state object and the global systems response of S , respectively. The initial systems response will be denoted by ρ_0 , i.e., $\rho_0 : C_0 \times X \rightarrow Y$ such that

$$(x, y) \in S \leftrightarrow (\exists c)[\rho_0(c, x) = y]$$

We can now introduce the following definition.

Definition 2.5. Let S be a time system and $t \in T$. The state object at t , denoted by C_t , is an initial state object for the restriction S_t ; i.e., it is an abstract set such that there exists a function $\rho_t: C_t \times X_t \rightarrow Y_t$ such that

$$(x_t, y_t) \in S_t \leftrightarrow (\exists c)[\rho_t(c, x_t) = y_t]$$

ρ_t is referred to as the (systems)-response function at t .

A family of all response functions for a given system, i.e.,

$$\bar{\rho} = \{\rho_t: C_t \times X_t \rightarrow Y_t \text{ \& } t \in T\}$$

is referred to as a *response family* for S , while $\bar{C} = \{C_t: t \in T\}$ is a family of state objects.

Definition 2.6. Let S be a time system $S \subset X \times Y$, and ρ_t an arbitrary function such that $\rho_t: C_t \times X_t \rightarrow Y_t$. ρ_t will be termed consistent with S if and only if it is a response function at t for S , i.e.,

$$(x_t, y_t) \in S_t \leftrightarrow (\exists c)[\rho_t(c, x_t) = y_t]$$

Let

$$S_t^\rho = \{(x_t, y_t): (\exists c)(y_t = \rho_t(c, x_t))\}$$

Then the consistency condition is expressed as

$$S_t^\rho = S_t$$

Let $\bar{\rho} = \{\rho_t: C_t \times X_t \rightarrow Y_t\}$ be a family of arbitrary functions. Then $\bar{\rho}$ is consistent with a time system S if and only if $\bar{\rho}$ is a (systems-)response family for S , i.e., for all $t \in T$

$$S_t^\rho = S_t = S_o^\rho | T_t$$

Regarding the existence of a response family, we have the following direct consequence of Theorem 1.1.

Proposition 2.1. Every time system has a response family.

A family of arbitrary maps, naturally, cannot be a response family of a time system as seen from the following theorem.

Theorem 2.1. Let $\bar{\rho} = \{\rho_t: C_t \times X_t \rightarrow Y_t \text{ \& } t \in T\}$ be a family of arbitrary maps. There exists a time system $S \subset X \times Y$ consistent with $\bar{\rho}$; that is, $\bar{\rho}$ is a response family of S if and only if for all $t \in T$ the following conditions hold:

$$(P1) \quad (\forall c_o)(\forall x^t)(\forall x_t)(\exists c_t)[\rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t]$$

$$(P2) \quad (\forall c_t)(\forall x_t)(\exists c_o)(\exists x^t)[\rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t]$$

PROOF. First, we shall prove the *if* part. We have to prove that $S_t^\rho = S_o^\rho | T_t$ is satisfied for every $t \in T$. Let $(x_t, y_t) \in S_t^\rho$ be arbitrary. Then $y_t = \rho_t(c_t, x_t)$ for some $c_t \in C_t$. Property (P2) implies that

$$y_t = \rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t$$

for some $(c_o, x^t) \in C_o \times X^t$. Hence,

$$(x_t, y_t) = (x^t \cdot x_t, \rho_o(c_o, x^t \cdot x_t)) | T_t$$

or $(x_t, y_t) \in S_o^\rho | T_t$. Therefore, we have $S_t^\rho \subseteq S_o^\rho | T_t$. Conversely, let $(x, y) \in S_o^\rho$ be arbitrary. Then

$$y = \rho_o(c_o, x) = \rho_o(c_o, x^t \cdot x_t)$$

for some c_o . Property (P1) implies, then, that

$$y | T_t = \rho_o(c_o, x^t \cdot x_t) | T_t = \rho_t(c_t, x_t)$$

for some $c_t \in C_t$, or $(x, y) | T_t \in S_t^\rho$. Hence, $S_o^\rho | T_t \subseteq S_t^\rho$. Combining the first result with the present one, we have $S_t^\rho = S_o^\rho | T_t$.

Next, consider the *only if* part. Let x and c_o be arbitrary. Then $(x, \rho_o(c_o, x)) \in S_o^\rho$. Since $S_o^\rho | T_t \subseteq S_t^\rho$, we have that

$$(x, \rho_o(c_o, x)) | T_t = (x_t, \rho_o(c_o, x^t \cdot x_t)) | T_t \in S_t^\rho$$

or

$$\rho_o(c_o, x^t \cdot x_t) | T_t = \rho_t(c_t, x_t)$$

for some c_t . Hence, we have

$$(\forall t)(\forall c_o)(\forall x^t)(\forall x_t)(\exists c_t)(\rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t)$$

Let c_t and x_t be arbitrary. Then $(x_t, \rho_t(c_t, x_t)) \in S_t^\rho$. Since $S_t^\rho \subseteq S_o^\rho | T_t$, we have

$$\rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t$$

for some c_o and x^t . Hence

$$(\forall t)(\forall c_t)(\forall x_t)(\exists c_o)(\exists x^t)(\rho_t(c_t, x_t) = \rho_o(c_o, x^t \cdot x_t) | T_t) \quad \text{Q.E.D.}$$

(b) General Dynamical System

The concept of a dynamical system is related to the way a system evolves in time. It is necessary, therefore, to establish a relationship between the values of the system objects at different times. For this purpose, the response function is not sufficient, and another family of functions has to be introduced.

Definition 2.7. A time system $S \subset X \times Y$ is a dynamical system (or has a dynamical system representation) if and only if there exist two families of mappings

$$\bar{\rho} = \{\rho_t : C_t \times X_t \rightarrow Y_t \ \& \ t \in T\}$$

and

$$\bar{\phi} = \{\phi_{tt'} : C_t \times X_{tt'} \rightarrow C_{t'} \ \& \ t, t' \in T \ \& \ t' > t\}$$

such that

- (i) $\bar{\rho}$ is a response family consistent with S ;
- (ii) the functions $\phi_{tt'}$ in the family $\bar{\phi}$ satisfy the following conditions
 - (α) $\rho_t(c_t, x_t) | T_{t'} = \rho_{t'}(\phi_{tt'}(c_t, x_{tt'}), x_{t'})$, where $x_t = x_{tt'} \cdot x_{t'}$.
 - (β) $\phi_{tt'}(c_t, x_{tt'}) = \phi_{t''t'}(\phi_{tt''}(c_t, x_{tt''}), x_{t''t'})$, where $x_{tt'} = x_{tt''} \cdot x_{t''t'}$.

$\phi_{tt'}$ is termed the state-transition function (on $T_{tt'}$), while $\bar{\phi}$ will be referred to as the state-transition family.

$\phi_{tt'}$ is defined for $t < t'$. However, the following convention :

$$(\gamma) \ \phi_{tt}(c_t, x_{tt}) = c_t, \quad \text{for every } t \in T$$

will be used.

Since a dynamical system is completely specified by the two families of mappings $\bar{\rho}$ and $\bar{\phi}$, the pair $(\bar{\rho}, \bar{\phi})$ itself will be referred to as a dynamical system representation or simply as a dynamical system. If a response family has a consistent state-transition family, it will be called a *dynamical systems-response family*. It will be shown that not every response family is a dynamical systems-response family.

Condition (α) represents the *consistency property of the state-transition family* (with the given response family), while (β) represents the *state-transition composition property* (also referred to as the semigroup property). Conditions (α) and (β) are rather strongly related. Actually, under fairly general conditions, property (β) is implied by (α) so that only the consistency of $\bar{\phi}$ with a response family $\bar{\rho}$ is required in order for $\bar{\phi}$ to be qualified as a state-transition family. To arrive at these conditions, we need the following definition.

Definition 2.8. Let $\bar{\rho}$ be a response family consistent with a time system S . $\bar{\rho}$ is a reduced response family if and only if for all $t \in T$

$$(\forall c_t)(\forall \hat{c}_t)[(\forall x_t)(\rho_t(c_t, x_t) = \rho_t(\hat{c}_t, x_t)) \rightarrow c_t = \hat{c}_t]$$

The reduction of $\bar{\rho}$, i.e., of associated state objects $\bar{C} = \{C_t : t \in T\}$, does not represent a significant restriction. It only requires that if the two states

at any time $t \in T$ lead to the identical future behavior of the system, they ought to be recognized as being the same.

We have now the following theorem.

Theorem 2.2. Let $\bar{\rho} = \{\rho_t: C_t \times X_t \rightarrow Y_t\}$ be a response family and $\bar{\phi} = \{\phi_{t'}: C_t \times X_{t'} \rightarrow C_{t'}\}$ a family of functions consistent with $\bar{\rho}$, i.e., satisfying condition (α) from Definition 2.7:

$$\rho_t(c_t, x_t) | T_{t'} = \rho_{t'}(\phi_{t'}(c_t, x_{t'}), x_{t'})$$

Then if $\bar{\rho}$ is reduced, $\bar{\phi}$ has the state-transition composition property, i.e., condition (β) from Definition 2.7 is satisfied.

PROOF. From the consistency of $\bar{\phi}$, i.e., condition (α), it follows, for $t \leq t' \leq t''$,

$$\rho_t(c_t, x_t) | T_{t''} = \rho_{t''}(\phi_{t''}(c_t, x_{t''}), x_t)$$

$$\rho_t(c_t, x_t) | T_{t'} = \rho_{t'}(\phi_{t'}(c_t, x_{t'}), x_t)$$

$$\rho_{t'}(\phi_{t'}(c_t, x_{t'}), x_{t'}) | T_{t''} = \rho_{t''}(\phi_{t''}(\phi_{t'}(c_t, x_{t'}), x_{t''}), x_{t'})$$

Since

$$\rho_{t'}(\phi_{t'}(c_t, x_{t'}), x_{t'}) | T_{t''} = (\rho_t(c_t, x_t) | T_{t'}) | T_{t''} = \rho_t(c_t, x_t) | T_{t''}$$

we have

$$\rho_{t''}(\phi_{t''}(c_t, x_{t''}), x_{t''}) = \rho_{t''}(\phi_{t''}(\phi_{t'}(c_t, x_{t'}), x_{t''}), x_{t''})$$

for every $x_{t''} \in X_{t''}$. Since $\{\rho_t\}$ is reduced, we have

$$\phi_{t''}(c_t, x_{t''}) = \phi_{t''}(\phi_{t'}(c_t, x_{t'}), x_{t''}) \quad \text{Q.E.D.}$$

It should be pointed out that in the definition of a time system, both input and output objects are defined on the same time set. This is obviously not the most general case (e.g., an output can be defined as a point rather than as a function). We have selected this approach because it provides a convenient framework for the results of traditional interest in systems theory (e.g., the realization theory presented in Chapter III). The properties and behavior of systems that have input and output objects defined on different time sets can be derived from the more complete case we are considering in this book.

(c) General Dynamical Systems in State Space

The concept of a state object as introduced so far has a major deficiency because there is no explicit requirement that the states at any two different times are related; i.e., it is possible, in general, that for any $t \neq t'$, $C_t \cap C_{t'} = \phi$.

To use the potential of the state concept fully, the states at different times ought to be represented as related in an appropriate manner. It should be possible, e.g., to recognize when the system has returned into the “same” state it was before, or has remained in the same state, i.e., did not change at all. In short, the equivalence between states at different times ought to be recognized. What is needed then is a set C such that $C_t = C$ for every $t \in T$. Such a set would represent a state space for the system. At any time the state of the system is then an element of the state space, and dynamics (i.e., change in time) of the system for any given input can be represented as mapping of the state space into itself.

This consideration leads to the following definition.

Definition 2.9. Let S be a time system $S \subset X \times Y$ and C an arbitrary set. C is a state space for S if and only if there exist two families of functions $\bar{\rho} = \{\rho_t: C \times X_t \rightarrow Y_t\}$ and $\bar{\phi} = \{\phi_{t''}: C \times X_{t''} \rightarrow C\}$ such that

- (i) for all $t \in T$, $S_t \subset S_t^\rho$ and $S_0^\rho = \{(x, y) : (\exists c)(y = \rho_0(c, x))\} = S$
- (ii) for all $t, t', t'' \in T$
 - (α) $\rho_t(c, x_t) \mid T_t = \rho_t(\phi_{t''}(c, x_{t''}), x_{t'})$
 - (β) $\phi_{t''}(c, x_{t''}) = \phi_{t'''}(\phi_{t'''}(c, x_{t''''}), x_{t'''})$
 - (γ) $\phi_{t''}(c, x_{t''}) = c$

where $x_t = x_{t'} \cdot x_{t'}$ and $x_{t''} = x_{t'''} \cdot x_{t''''} \cdot x_{t''}$. S is then a dynamical system in the state space C .

Notice that, in general, S_t is a proper subset of S_t^ρ . This is so because $\bar{\rho}$ is defined on the entire state space C , while the system might not accept all states at any particular time; i.e., the set of possible states might be restricted at a specified instant of time. This leads to the following definition.

Definition 2.10. A dynamic system in a state space C is full if and only if it has $S_t = S_t^\rho$ for all $t \in T$.

Many full dynamical systems, e.g., those that are also linear and time invariant, will be considered in this book. However, in general, a system need not be full; e.g., even a finite automaton is, in general, not a full dynamical system. A simple example of such a case is given by a Mealy type automaton specified by:

Input alphabet: $\{1\}$, Output alphabet: $\{1, 2\}$, State set: $\{1, 2\}$

The state transition and the output function of the system are given by the state-transition diagram in Fig. 2.1.

(ii) Output Function

Time evolution of a dynamical system is customarily described in terms of the state transition, and it is of interest to relate the changes in states to the changes in outputs ; specifically, the state at any time $t \in T$ ought to be related with the value of the output at that time. This leads to the following definition.

Definition 3.2. Let S be a time system with the response family $\bar{\rho}$ and λ_t a relation

$$\lambda_t \subset C_t \times X(t) \times Y(t)$$

such that

$$(c_t, x(t), y(t)) \in \lambda_t \leftrightarrow (\exists x_t)(\exists y_t)[y_t = \rho_t(c_t, x_t) \ \& \ x(t) = x_t \ \& \ y(t) = y_t(t)]$$

When λ_t is a function such that

$$\lambda_t : C_t \times X(t) \rightarrow Y(t)$$

it is termed an output function at t , while $\bar{\lambda} = \{\lambda_t : t \in T\}$ is an output-function family (Fig. 3.2).

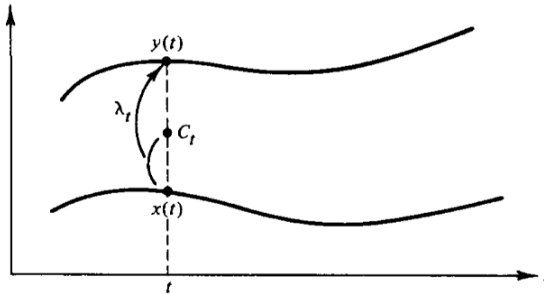


FIG. 3.2

Apparently, λ_t is a well-defined relation and exists for any general time system. The conditions for the existence of an output function will be presented in the next section.

(iii) State-Generating Function

For a dynamical system, the state at any time t is determined by the initial state c_0 and the initial input restriction x^i . However, for certain classes of systems, there exists a time $\hat{t} \in T$ such that the state at any subsequent time is determined solely by the past input and output restrictions ; i.e., no reference to the state is needed. This leads to the following definition.

Definition 3.3. Let $\bar{\rho}$ be a response family for a time system S and η^t a relation

$$\eta^t \subset X^t \times Y^t \times C_t$$

such that

$$(x^t, y^t, c_t) \in \eta^t \leftrightarrow (\forall x_t)(\forall y_t)[(x^t \cdot x_t, y^t \cdot y_t) \in S \rightarrow y_t = \rho_t(c_t, x_t)]$$

When η^t is a function such that

$$\eta^t : X^t \times Y^t \rightarrow C_t$$

it is termed a state-generating function at t , while $\bar{\eta} = \{\eta^t : X^t \times Y^t \rightarrow C_t, \& t \in T\}$ is termed a state-generating family (Fig. 3.3).

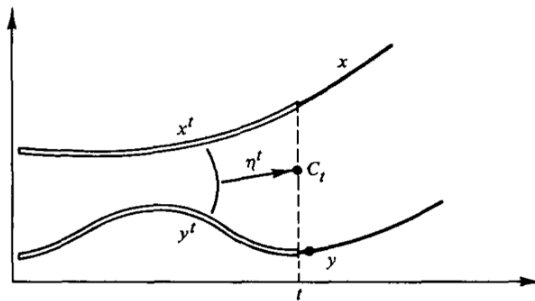


FIG. 3.3

Again, although η^t is always defined, the existence of a state-generating family requires certain conditions which, in this case, are of a more special kind.

(b) Some Classification of Time Systems

The auxiliary functions for any $t \in T$ are, in general, different. However, when some of them are the same for all $t \in T$ or are obtained from the same function by appropriate restrictions, various forms of time invariance can be introduced.

(i) Static and Memoryless Systems

The first type of time invariance refers to the relationship between the system objects at any given time and is intimately related with the response function.

Definition 3.4. A system S is static if and only if there exists an initial response function $\rho_o : C_o \times X \rightarrow Y$ for S such that for all $t \in T$

$$(\forall c_o)(\forall x)(\forall \hat{x})[x(t) = \hat{x}(t) \rightarrow \rho_o(c_o, x)(t) = \rho_o(c_o, \hat{x})(t)]$$

In other words, the system is static if and only if for any $t \in T$ there exists a map $K_t: C_o \times X(t) \rightarrow Y(t)$ such that

$$(x, y) \in S \leftrightarrow (\exists c_o \in C_o)(\forall t)(y(t) = K_t(c_o, x(t)))$$

Any time system that is not static is termed a dynamic system.

Intuitively, a system is static if the value of its output at any time t depends solely on the current value of the input and the state from which the evolution has initially started; i.e., if $x(t)$ becomes constant over a period of type, $y(t)$ becomes constant too. On the other hand, the output of a dynamic system depends not only on the current value of the input but also on the past "history" of that input as well. Notice that, in general, reference to the initial state had to be made (Fig. 3.4).

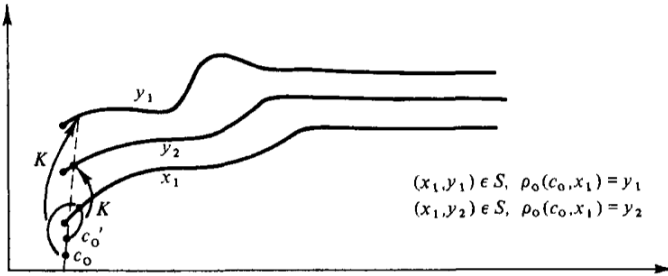


FIG. 3.4

It should be noticed that a distinction is made between a dynamic and dynamical system. For the former, it is sufficient that the system is not static, while the latter requires that a state-transition family is defined. This, perhaps, is not the most fortunate choice of terminology; however, it has been selected because it corresponds to the common usage in the already established specialized theories.

A related notion to a static system is the following definition.

Definition 3.5. A time system S is memoryless if and only if it is a static system such that

$$(\forall x)(\forall \hat{x})(\forall c_o)(\forall \hat{c}_o)[x(t) = \hat{x}(t) \rightarrow \rho_o(c_o, x)(t) = \rho_o(\hat{c}_o, \hat{x})(t)]$$

or, in terms of the mappings $K_t: C_o \times X(t) \rightarrow Y(t)$,

$$(\forall c_o)(\forall \hat{c}_o)(\forall x)(\forall \hat{x})[x(t) = \hat{x}(t) \rightarrow K_t(c_o, x(t)) = K_t(\hat{c}_o, \hat{x}(t))]$$

i.e., there exists a mapping $K_t^*: X(t) \rightarrow Y(t)$ such that $K_t^*(x(t)) = K_t(c_0, x(t))$ (Fig. 3.5).

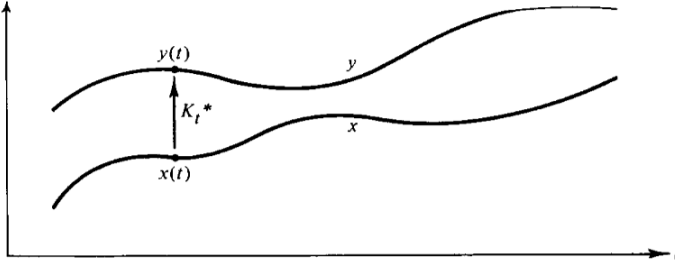


FIG. 3.5

Apparently, a memoryless system is completely characterized by the map $K_t^*: A \rightarrow B$. A system that does not satisfy Definition 3.5 is termed a *system with memory*.

(ii) Time-Invariant Dynamical Systems

The second kind of time invariancy refers to how system responses at two different times compare. To introduce appropriate concepts, we shall assume for this kind of time invariancy that the time set T is a right segment of a linearly ordered Abelian group \bar{T} whose group operation (addition) will be denoted by $+$. More precisely, $T = \{t : t \geq o\}$, where o is the identity element of \bar{T} and the addition is related with the linear ordering as

$$t \leq t' \leftrightarrow t' - t \geq o$$

The time set T defined above will be referred to as the *time set for stationary systems*.

For each $t \in \bar{T}$, let $F^t : \bar{X} \rightarrow \bar{X}$ denote an operator such that

$$(\forall t')[F^t(x)(t') = x(t' - t)]$$

Notice that F^t is defined for $t < o$ as well as $t \geq o$ and that whether or not F^t is meaningful depends on its argument. In general, $F^t(x_{t',t''}) \in X_{(t'+t)(t''+t)}$ holds whenever $F^t(x_{t',t''})$ is defined.

F^t is termed the *shift operator*; it simply shifts a given time function for the time interval indicated by the superscript, leaving it otherwise completely unchanged (Fig. 3.6). We shall use the same symbol F^t for the shift operator in \bar{Y} and define F^t also in \bar{S} , $F^t : \bar{S} \rightarrow \bar{S}$, such that

$$F^t(x, y) = (F^t(x), F^t(y))$$

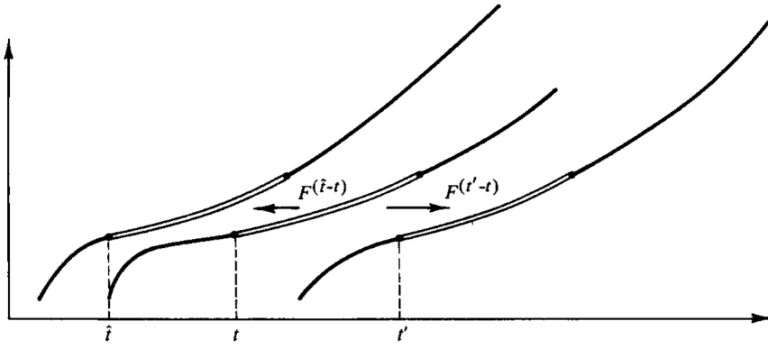


FIG. 3.6

We can introduce now the following definition.

Definition 3.6. A time system defined on the time set for stationary systems is fully stationary if and only if (Fig. 3.7)

$$(\forall t)[t \in T \rightarrow F^t(S) = S_t]$$

and stationary if and only if

$$(\forall t)(\forall t' \geq t)(t, t' \in T \rightarrow S_{t'} \subset F^{t'-t}(S_t))$$

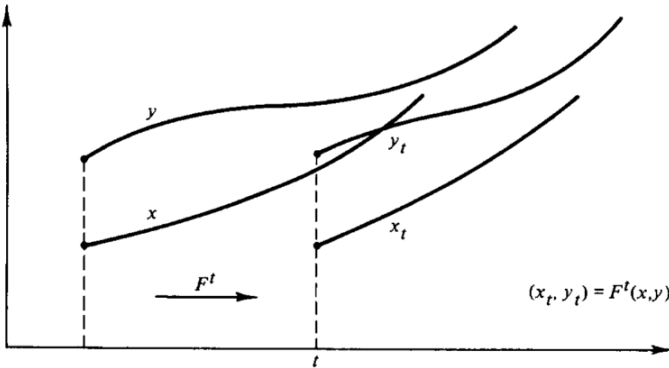


FIG. 3.7

Apparently, if a system is fully stationary, $F^{-t}(S_{t'} | T_t) = F^{-t'}(S_{t'})$ for any $t \geq t' \in T$; i.e., starting from any given time, its future evolution is the same except for the shift for the appropriate time interval.

When some given input and output objects X and Y satisfy the condition

$$(\forall t)(X_t = F^t(X)) \quad \text{and} \quad (\forall t)(Y_t = F^t(Y))$$

they will be referred to as the objects for a stationary system.

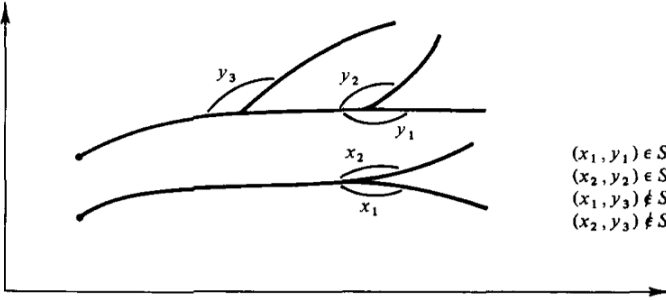


FIG. 4.1

Definition 4.2. An initial systems-response function $\rho_o: C_o \times X \rightarrow Y$ is strongly nonanticipatory if and only if

$$(\forall t)(\forall c_o)(\forall x)(\forall \hat{x})[x | T^t = \hat{x} | T^t \rightarrow \rho_o(c_o, x) | \bar{T}^t = \rho_o(c_o, \hat{x}) | \bar{T}^t]$$

Notice that Definitions 4.1 and 4.2 refer to time systems rather than to dynamical systems.

The difference between the nonanticipatory and strongly nonanticipatory response functions is that in the latter the present value of the output, $y(t)$, does not depend upon the present value of the input, $x(t)$, since the restriction in the antecedent in Definition 4.2 is on T^t while in Definition 4.1 is on $\bar{T}^t = T^t \cup \{t\}$; however, in both cases the output is restricted to \bar{T}^t .

It should also be pointed out that ρ_o is defined as a full function. This is an important restriction in the case of nonanticipation because it prevents some systems of having a nonanticipatory systems-response function. We shall introduce therefore the following definition.

Definition 4.3. Let $R \subset C_o \times X$ and $\rho_o: (R) \rightarrow Y$. ρ_o is an incomplete non-anticipatory initial systems response of S if and only if

- (i) ρ_o is consistent with S , i.e.,

$$(x, y) \in S \leftrightarrow (\exists c_o)[\rho_o(c_o, x) = y \ \& \ (c_o, x) \in R]$$
- (ii) $(\forall t)(\forall c_o)(\forall x)(\forall \hat{x})[(c_o, x) \in R \ \& \ (c_o, \hat{x}) \in R \ \& \ x | \bar{T}^t = \hat{x} | \bar{T}^t \rightarrow \rho_o(c_o, x) | \bar{T}^t = \rho_o(c_o, \hat{x}) | \bar{T}^t]$

Definition 4.4. We shall say that the system S is (strongly) nonanticipatory if and only if it has a complete (strongly) nonanticipatory initial-response function.

(ii) **Past-Determinacy**

Definition 4.5. A time system $S \subset A^T \times B^T$ is past-determined from \hat{t} if and only if there exists $\hat{t} \in T$ such that (see Fig. 4.2)

- (i) $(\forall(x, y) \in S)(\forall(x', y') \in S)(\forall t \geq \hat{t})((x^t, y^t) = (x'^t, y'^t) \ \& \ x_{it} = x'_{it}) \rightarrow \bar{y}_{it} = \bar{y}'_{it}$
- (ii) $(\forall(x^i, y^i))(\forall x_i)(\exists y_i)((x^i, y^i) \in S^i \rightarrow (x^i \cdot x_i, y^i \cdot y_i) \in S)$

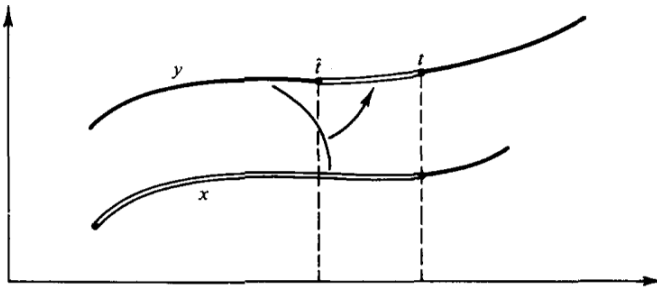


FIG. 4.2

Past-determinacy means that there exists $\hat{t} \in T$ such that for any $t \geq \hat{t}$, the future evolution of the system is determined solely by the past observations, and there is no need to refer to an auxiliary set as, e.g., the initial state object.

Condition (ii), which will be referred to as the *completeness property*, is introduced as a mathematical convenience.

(b) **Existence of Causal-Response Family**

Theorem 4.1. Every time system has an incomplete nonanticipatory initial systems response.

PROOF. Let $\equiv \subseteq S \times S$ be a relation such that $(x, y) \equiv (x', y')$ if and only if $y = y'$. Then, apparently, \equiv is an equivalence relation. Let $S/\equiv = \{[s]\} \equiv C_o$, where $[s] = \{s^* \mid s^* \equiv s \ \& \ s^* \in S\}$.[†] Let $\rho_o: C_o \times X \rightarrow Y$ such that

$$\rho_o([s], x) = \begin{cases} y & \text{if } (x, y) \in [s] \\ \text{undefined} & \text{otherwise} \end{cases}$$

[†] In this book the following convention will be used for a quotient set. Let E be an equivalence relation on a set X . Then the quotient set X/E will be represented by $X/E = \{[x]\}$, where $[x]$ is the equivalence class of x , i.e.,

$$[x] = \{x^* : (x, x^*) \in E \ \& \ x^* \in X\}$$

and $[\]$ will be considered as the natural mapping, i.e., $[\]: X \rightarrow X/E$.

Notice that ρ_o is properly defined, because if $(x, y) \in [s]$ and $(x, y') \in [s]$, then $y = y'$. In general, ρ_o is a partial function. First, we shall show that

$$S = \{(x, y) : (\exists c)(c \in C_o \text{ \& } y = \rho_o(c, x))\} \equiv S'$$

If $(x, y) \in S$, then $\rho_o([s], x) = y$ by definition. Hence, $(x, y) \in S'$, or $S \subseteq S'$. Conversely, if $(x, y) \in S'$, then $y = \rho_o([s], x)$ for some $[s] \in C_o$. Then $(x, y) \in [s]$ follows from the definition of ρ_o . Hence, $S' \subseteq S$. Furthermore, if the values of $\rho_o([s], x)$ and $\rho_o([s], x')$ are defined, then $\rho_o([s], x) = \rho_o([s], x')$ for any x and x' . Hence, condition (ii) in Definition 4.3 is trivially satisfied. Q.E.D.

Theorem 4.1 cannot be extended for the full initial systems-response function, i.e., when ρ_o is a full function. There are time systems that do not have a (complete) nonanticipatory initial systems response as defined in Definition 4.1; in other words, requirement for the initial response to be a full function prevents a causal representation of the systems in the sense of nonanticipation. Such systems can either be considered to be essentially noncausal or it can be assumed that only an incomplete description of the system is available and that noncausality is due to having only partial information. This can be best shown by an example as given in Fig. 4.3.

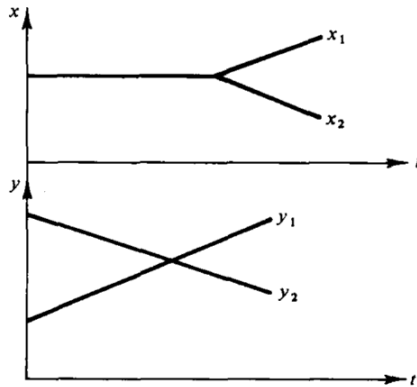


FIG. 4.3

Consider a system S which has only two elements, $S = \{(x_1, y_1), (x_2, y_2)\}$, which are as shown in Fig. 4.3. Since the initial segments of both x_1 and x_2 are the same, while those of y_1 and y_2 are different, the initial state object ought to have at least two elements if we want to have a nonanticipatory initial-response function. Let $C_o = \{c, c'\}$ and $\rho_o(c, x_1) = y_1$, $\rho_o(c', x_2) = y_2$. If ρ_o is a full function, (c, x_2) is also in the domain of ρ_o . Therefore, either $\rho_o(c, x_2) = y_1$ or $\rho_o(c, x_2) = y_2$. But $\rho_o(c, x_2) = y_1$ implies $(x_2, y_1) \in S$, i.e., ρ_o is not consistent with S , while $\rho_o(c, x_2) = y_2$ violates the nonanticipation

condition in Definition 4.1 since the initial segments of x_1 and x_2 are the same, while y_1 and y_2 are not. The system S does not have, therefore, a complete nonanticipatory response function.

Definition 4.6. A response family $\bar{\rho} = \{\rho_t : t \in T\}$ is called nonanticipatory if and only if every ρ_t is an initial nonanticipatory response function of S_t .

Theorem 4.2. A time system has a nonanticipatory response family if and only if it has an initial nonanticipatory response function.

PROOF. The *only if* part is obvious. Let us consider the *if* part. Let $\rho_0 : C_0 \times X \rightarrow Y$ be an initial nonanticipatory systems response. Let $C_t = C_0 \times X^t$ and $\rho_t : C_t \times X_t \rightarrow Y_t$ such that if $c_t = (c_0, \hat{x}^t)$, then

$$\rho_t(c_t, x_t) = \rho_0(c_0, \hat{x}^t \cdot x_t) | T_t, \quad \text{for } t \in T$$

Then the consistency conditions in Theorem 2.1 are trivially satisfied, i.e., $\bar{\rho} = \{\rho_t : t \in T\}$ is a response family. Furthermore, suppose $x_t | \bar{T}_{t'} = x_{t'} | \bar{T}_{t'}$ for $t' \geq t$. Then if $c_t = (c_0, \hat{x}^t)$,

$$\rho_t(c_t, x_t) | \bar{T}_{t'} = \rho_0(c_0, \hat{x}^t \cdot x_t) | \bar{T}_{t'}$$

and

$$\rho_t(c_t, x_t') | \bar{T}_{t'} = \rho_0(c_0, \hat{x}^t \cdot x_t') | \bar{T}_{t'}$$

Since ρ_0 is nonanticipatory and $\hat{x}^t \cdot x_t | \bar{T}_{t'} = \hat{x}^t \cdot x_t' | \bar{T}_{t'}$, we have

$$\begin{aligned} \rho_t(c_t, x_t) | \bar{T}_{t'} &= (\rho_0(c_0, \hat{x}^t \cdot x_t) | \bar{T}_{t'}) | \bar{T}_{t'} \\ &= (\rho_0(c_0, \hat{x}^t \cdot x_t') | \bar{T}_{t'}) | \bar{T}_{t'} \\ &= \rho_0(c_0, \hat{x}^t \cdot x_t') | \bar{T}_{t'} \\ &= \rho_t(c_t, x_t') | \bar{T}_{t'} \end{aligned}$$

Hence, ρ_t is nonanticipatory.

Q.E.D.

(c) Causality and Output Functions

Conditions for the existence of an output function and an output-generating function can be given directly in terms of the nonanticipation of systems response.

Dependence of the output function upon nonanticipation is clearly indicated by the following proposition.

Proposition 4.1. A time system has an output-function family $\bar{\lambda} = \{\lambda_t : C_t \times X(t) \rightarrow Y(t)\}$ if the system is nonanticipatory.

PROOF. Since the system is nonanticipatory, there exists, by Theorem 4.2, a nonanticipatory response family $\bar{\rho} = \{\rho_t : t \in T\}$. Let λ_t be defined for the nonanticipatory response family. Suppose $(c_t, x(t), y(t)) \in \lambda_t$ and $(c_t, x'(t), y'(t)) \in \lambda_t$ where $x(t) = x'(t)$. Since

$$x_t | \bar{T}_t = x(t) = x'(t) = x'_t | \bar{T}_t$$

and since ρ_t is nonanticipatory, we have that

$$\rho_t(c_t, x_t) | \bar{T}_t = \rho_t(c_t, x'_t) | \bar{T}_t$$

or $y(t) = y'(t)$. Hence λ_t is a mapping such that $\lambda_t : C_t \times X(t) \rightarrow Y(t)$. Q.E.D.

The concept of an output function illustrates one of the important roles of the concept of state: If a state is given and the system is nonanticipatory, all information about the past of the system, necessary to specify the present value of output, is contained in the state itself.

Dependence of the output-generating function upon the nonanticipation of the system is quite similar to the dependence of the output function and is given by the following proposition.

Proposition 4.2. A time system has an output-generating family $\bar{\mu} = \{\mu_{t'} : C_t \times X_{t'} \rightarrow Y_t(t')\}$ if the system is nonanticipatory.

PROOF. The proof is similar to that of Proposition 4.1. Q.E.D.

From Proposition 4.1 it is obvious that the output of a nonanticipatory time system can be determined solely by the present state and the present value of the input. For some systems, however, the present output depends solely upon the present state and does not depend upon the present value of the input. For the analysis of these systems, a somewhat stronger notion, namely, that of strong nonanticipation, is needed.

Proposition 4.3. A time system has an output function such that for all $t \in T$

$$(\forall x(t))(\forall \hat{x}(t))[c_t = \hat{c}_t \rightarrow \lambda_t(c_t, x(t)) = \lambda_t(\hat{c}_t, \hat{x}(t))]$$

if it is a strongly nonanticipatory system.

PROOF. Since the system is strongly nonanticipatory, there exists a strongly nonanticipatory initial system response ρ_0 . Then the same procedure as used in Proposition 4.1 proves the statement. Q.E.D.

When the system is strongly nonanticipatory, for every $t \in T$ there apparently exists a map

$$K_t : C_t \rightarrow B$$

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