AMS / MAA | DOLCIANI MATHEMATICAL EXPOSITIONS VOL 7

HOWARD EVES

Great Moments in Mathematics After 1650

MAA PRESS

An Imprint of the AMERICAN MATHEMATICAL SOCIETY

Library of Congress Catalog Card Number 81-86186

Hardcover ISBN 978-0-88385-307-8

Paperback ISBN 978-0-88385-311-5

eISBN 978-1-61444-215-8

Printed in the United States of America

Current printing (last digit):
10 9 8 7 6

CONTENTS

ORDER WITHIN DISORDER

Among the bones of the intricate skeletal structure of the foot is one lying in the heel just above the talus bone and known as the *astragalus.* In man, and in animals with a developed foot, the astragalus is quite irregular, but in the hoofed animals, like sheep, goats, and all kinds of deer, the astragalus has a rough symmetry, being squarish in cross section with two rounded ends, one slightly convex and the other slightly concave. These bones are solid and essentially marrowless, hard and durable, somewhat cubical with edges measuring an inch or less, and, with handling, capable of taking on a high polish.

It is not uncommon for archaeologists excavating at prehistoric sites to find sizable collections of astragalus bones of small hoofed animals, and sometimes collections of small stones of various colors. It seems reasonable to conjecture that these bones and pebbles may have been used by prehistoric man as tally-stones or counters, and as toys for himself and his children. While such a use for astragali in prehistoric times is only conjectural, there is no doubt that among the ancient Babylonians and Egyptians, and the Greeks and Romans of the pre-Christian era, one of the uses of astragali was as children's toys. We are informed that schoolboys played with them everywhere, sometimes by balancing four of the astragali on the knuckles of a hand, tossing them by a flip of the hand into the air, and then endeavoring to recapture them as they fell. Also, from Greek vase-painting, the astragali were sometimes tossed into a ring drawn on the ground, much as children of today play with marbles. Whether man adopted the toys of his children or the children copied the man is impossible to say, but by the First Dynasty in Egypt (about 3500 B.C.), astragali were used in a variety of games, in some of which "men" were moved about on a board according to the fall of a tossed astragalus bone. There is an Egyptian tomb-painting showing a nobleman in after-life with a playing board set out before him and an astragalus delicately balanced on his finger tip prior to being tossed. Children of today in France and Italy still play games with astragalus bones, and metallic versions of the bones can be purchased in village shops.

This is not the place to enter into the shadowy history of game-playing, nor into the cloudy origin of *gaming*. Did gaming develop from game-playing, or did it arise from wagering and the drawing of lots, or from religious divination and the consultation of oracles? In any case, by approximately 1200 B.C. the cubical marked die had evolved as a more suitable randomizing agent in games than the astragalus. This idealization occurred simultaneously in different parts of the world, and it is quite likely that the first primitive dice were made by rubbing flat the two opposite rounded faces of astragalus bones. The faces of a die were variously marked by drilling into them numbers of small shallow depressions with some sort of a circular engraving tool.

It was natural that gaming, as a game using only dice with no accompanying playing board and pieces, should arise and that players should concern themselves with the chances, or probabilities, of obtaining sums with the throw of two or more dice. Thus, although Greek philosophers of antiquity discussed necessity and contingency at some length, it is perhaps correct to say that the beginnings of a study of probability arose in attempts to evaluate the chances in certain gambling games, particularly the game of dice.

It is hard for historians of the calculus of probability to account for the extremely tardy conceptual growth of the subject. Of course, a realization of the equally likely possibilities in dice-throwing would of necessity be delayed until "honest" dice were made. So long as astragalus bones, or simple handy pieces of wood, ivory, or stone, smoothed off and appropriately marked, were used for either play or divination, the regularity of fall of the different faces would be quite obscured. Also, long series of trials are needed to calculate empirical probabilities, and there would have been few persons capable of keeping a tally of throws and of making the required enumerations. There seemed little alternative to the feeling that the fall of dice or astragali was completely controlled by the whimsies of the gods.

We know that a passion for gaming possessed the Roman Emperors and the surrounding leisured rich. It is said, for example, that Claudius (10 B.C.–A.D. 54) was greatly devoted to dicing and had even published a book, which unfortunately has not survived, entitled *How to Win at Dice.* But a real start in the calculation of random events did not take place until the Renaissance, when the ability to write and calculate with numbers had become widespread and simple algebra had developed.

It seems proper to say that there was no truly mathematical treatment of probability until the latter part of the fifteenth century and the early part of the sixteenth century, when some of the Italian mathematicians attempted to evaluate the chances in certain gambling games, like that of dice. Girolamo Cardano (1501–1576), as was noted in LECTURE 16, wrote a brief gambler's manual in which some of the simpler aspects of mathematical probability are involved. But it is generally agreed that the one problem to which can be credited the origin of the science of probability is the so-called *problem of the points.* This problem requires the determination of the division of the stakes of an interrupted game of chance between two supposedly equally skilled players, knowing the scores of the players at the time of interruption and the number of points needed to win the game. Fra Luca Pacioli (1445–1509), in his popular *Sūma** of 1494, was one of the first writers to introduce the problem of the points into a work on mathematics. The problem was subsequently discussed by Cardano and Tartaglia (ca. 1499–1557). All these men arrived at incorrect answers. A real advance was not made until the problem was proposed, in 1654, to Blaise Pascal, by the Chevalier de Méré, an able and experienced gambler whose theoretical reasoning on the problem did not agree with his observations. Pascal became interested in the problem and communicated it to Pierre de Fermat. There ensued a remarkable correspondence between these two French mathematicians,† in which the problem was correctly but differently solved by each man. It was in this correspondence of 1654 that Pascal and Fermat jointly laid the foundations of the theory of mathematical probability—a GREAT MOMENT IN MATHEMATICS had arrived.

*More completely, *Summa de arithmetica, geometria, proportioni e proportionalità.*

†This correspondence appears in D. E. Smith, *A Source Book in Mathematics.*

Blaise Pascal was born in 1623 in the French province of Auvergne and very early showed exceptional ability in mathematics. When only 12 he discovered, entirely on his own, many of the theorems of elementary plane geometry. At 14 he took part in the informal weekly sessions of a group of mathematicians from which the French Academy eventually arose in 1666. At 16 he discovered, among other things, his singularly rich "mystic hexagram" theorem* of projective geometry. A few years later he invented and constructed the first adding machine and began to apply his unusual talents to physics and mechanics. In 1648 he wrote a comprehensive, but now lost, treatise on projective geometry.

This astonishing and precocious activity came to a sudden halt in 1650, when, suffering from fragile health, Pascal decided to abandon his work in mathematics and science and to devote himself to religious meditation. Three years later, however, he transitorily returned to mathematics, at which time he wrote his *Traité du triangle arithmétique*, which, as we shall shortly see, played an important part in the matter that concerns us in the present lecture. He conducted a number of experiments on fluid pressure, which led to the invention of the hydraulic press, and, in 1654, carried on the historic correspondence with Fermat that laid the foundations of the mathematical theory of probability.

Then, late in 1654, Pascal received what he felt to be a strong intimation that his renewed activities in mathematics and science were displeasing to God. The divine hint occurred when his runaway horses dashed to their deaths over the high parapet of the bridge at Neuilly, and his own life was miraculously preserved only by the last minute breaking of the traces. Morally fortified by a reference to the accident written on a small piece of parchment henceforth carried next to his heart, he dutifully returned to his religious contemplations.

It was only once again, in 1658, that Pascal reverted to mathematics. While suffering from excruciating toothache, some geometrical ideas occurred to him, and his teeth forthwith ceased to ache. Interpreting this as a sign of divine will, he assiduously applied himself for eight days expanding his ideas, producing a fairly complete account of the geometry of the cycloid curve.

*The three points of intersection of the three pairs of opposite sides of any hexagon inscribed in any conic are collinear.

Pascal's famous *Provincial Letters* and his *Pensées,* both dealing with religious matters and read today as models of early French literature, were composed toward the close of his brief life. He died in Paris, after a lingering and complicated illness, in 1662 at the pathetically young age of 39.

Pascal has been called the greatest "might-have-been" in the history of mathematics. Possessing such remarkable talents and such keen geometrical intuition, he should have produced a great deal more. Unfortunately, much of his life was spent suffering the racking physical pains of acute neuralgia and the distressing mental torments of religious neuroticism.

In contrast to the short, disturbed, tortured, and only spasmodically productive life of Blaise Pascal, Pierre de Fermat's life was moderately long, peaceful, enjoyable, and almost continuously productive. Fermat was born at Beaumont de Lomagne, near Toulouse, in 1601(?), as the son of a well-to-do leather merchant. He received his early education at home, as did Pascal.

In 1631, Fermat was installed at Toulouse as commissioner of requests, and in 1648 was promoted to the post of King's councilor to the local parliament at Toulouse. In this latter capacity he spent the rest of his life discharging his duties with modesty and punctiliousness. While thus serving as a humble and retiring lawyer, he devoted the bulk of his leisure time to the study and creation of mathematics. Although he published very little during his lifetime, he was in scientific correspondence with many of the leading mathematicians of his day, and in this way considerably influenced his contemporaries.

Fermat enriched so many branches of mathematics with so many important contributions that he has been called the greatest French mathematician of the seventeenth century. We have seen in our preceding lecture that he was an independent inventor of analytic geometry; in the present lecture we shall see how he helped lay the foundations of the mathematical theory of probability, and in our next lecture we shall see that he contributed noteworthily to the early development of the calculus. But of all his varied contributions to mathematics, by far the most outstanding is his founding of the modern theory of numbers, a field in which he possessed extraordinary intuition and an awesomely impressive ability, putting him among the top number theorists of all time.

Fermat died in Castres (or perhaps Toulouse), quite suddenly, in

1665. His tombstone, originally in the Church of the Augustines in Toulouse, was later moved to the local museum.

Let us now turn to the problem of the points, the solutions of which by Pascal and Fermat, in their correspondence of 1654, commenced a sound mathematical study of probability. An illustrative case discussed by the two French mathematicians was that in which one seeks the division of the stakes in a game of chance between two equally skilled players A and B where player A needs 2 more points to win and player B needs 3 more points to win. We first consider Fermat's solution to the problem, since it is the simpler and more direct of the two; and then Pascal's solution, which perhaps is more refined and more capable of generalization.

Inasmuch as it is clear, in the illustrative example, that four more trials will decide the game, Fermat let a indicate a trial where A wins and b a trial where B wins, and considered the 16 possible permutations of the two letters a and b taken 4 at a time:

aaaa	*aaab*	*abba*	*bbab*
baaa	*bbaa*	*abab*	*babb*
abaa	*baba*	*aabb*	*abbb*
aaba	*baab*	*bbba*	*bbbb*

The cases where a appears 2 or more times are favorable to A; there are 11 of them. The cases where b appears 3 or more times are favorable to B; there are 5 of them. Therefore the stakes should be divided in the ratio 11:5. For the general case, where A needs m points to win and B needs n, one writes down the 2^{m+n-1} possible permutations of the two letters a and b taken $m + n - 1$ at a time. One then finds the number α of cases where a appears m or more times and the number β of cases where b appears n or more times. The stakes are then to be divided in the ratio $\alpha : \beta$.

Pascal solved the problem of the points by utilizing his "arithmetic triangle," an array of numbers discussed by him in his *Traité du triangle arithmétique*, which, though not published until 1665, was written in 1653. He constructed his "arithmetic triangle" as indicated in Figure 1. Any element (in the second or a following row) is obtained as the sum of all those elements of the preceding row lying just above or to the left of the desired element. Thus, in the fourth row,

$$35 = 15 + 10 + 6 + 3 + 1.$$

The triangle, which may be of any order, is obtained by drawing a diagonal as shown in the figure. The student of college algebra will recognize that the numbers along such a diagonal are the successive coefficients in a binomial expansion. For example, the numbers along the fifth diagonal, namely 1, 4, 6, 4, 1, are the successive coefficients in the expansion of $(a + b)^4$. The finding of binomial coefficients was one of the uses to which Pascal put his triangle. He also used it for finding the number of combinations of n things taken r at a time, which he correctly stated to be

$$C(n, r) = n!/r!(n - r)!,$$

where $n!$ is our present-day notation* for the product

$$n(n - 1)(n - 2) \cdots (3)(2)(1).$$

1	1	1	1	1	1	...
1	2	3	4	5	6	...
1	3	6	10	15	21	...
1	4	10	20	35	56	...
1	5	15	35	70	126	...
1	6	21	56	126	252	...
.	.	.	.	.	.	...

FIG. 1

*The symbol $n!$, called *factorial n*, was introduced in 1808 by Christian Kramp (1760–1826) of Strasbourg, who chose this symbol so as to circumvent printing difficulties incurred by a previously used symbol. For convenience one defines $0! = 1$.

One can easily show that the elements along the fifth diagonal are, respectively,

$$C(4, 4) = 1, \qquad C(4, 3) = 4, \qquad C(4, 2) = 6,$$
$$C(4, 1) = 4, \qquad C(4, 0) = 1.$$

Since $C(4, 4)$ is the number of ways to obtain 4 a's, $C(4, 3)$ is the number of ways to obtain 3 a's, etc., it follows that the solution of the illustrative problem of the points is given by

$$[C(4, 4) + C(4, 3) + C(4, 2)]:[C(4, 1) + C(4, 0)]$$
$$= (1 + 4 + 6):(4 + 1) = 11:5.$$

In the general case, where A needs m points to win and B needs n, one chooses the $(m + n)$th diagonal of Pascal's arithmetic array. One then finds the sum α of the first n elements of this diagonal and the sum β of the last m elements. The stakes are to be divided in the ratio $\alpha : \beta$.

There are many relations involving the numbers of the arithmetic triangle, several of which were developed by Pascal. Pascal was not the originator of the arithmetic triangle, for such an array had been anticipated several centuries earlier by Chinese and Persian writers and had been considered by a number of Pascal's European predecessors. It is because of Pascal's development of many of the triangle's properties and because of the applications which he made of these properties that the array has become known as *Pascal's triangle.* In Pascal's treatise on the triangle appears one of the earliest acceptable statements of the method of mathematical induction.

Pascal and Fermat, in their historic correspondence of 1654, reflected upon other problems related to the problem of the points, such as the division of stakes where there are more than two players, or where there are two unevenly skilled players. With this work by Pascal and Fermat, marking a GREAT MOMENT IN MATHEMATICS, the mathematical theory of probability was well launched. In 1657, the great Dutch genius Christiaan Huygens (1629–1695) wrote the first formal treatise on probability, basing his work on the Pascal-Fermat correspondence. This was the best account of the subject until the posthumous appearance, in 1713, of the *Ars conjectandi* of Jacob Bernoulli (1654–1705), which contained a reprint of the earlier treatise by Huygens. After these pioneering efforts, we find the subject carried forward by such men as Abraham De Moivre (1667–1754), Daniel Bernoulli (1700–1782), Leonhard Euler (1707–1783), Joseph Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), and a host of other contributors.

It is fascinating, and at the same time somewhat astonishing, to contemplate that mathematicians have been able to develop a science, namely the mathematical theory of probability, that establishes rational laws that can be applied to situations of pure chance. This science is far from being impractical, as is attested by experiments performed in great laboratories, by the existence of highly respected insurance companies, and by the logistics of big businesses and of war. About this science of probability, the eminent French mathematician Pierre-Simon Laplace remarked that, though it started with the consideration of certain lowly games of chance, it rose to become one of the most important areas of human knowledge. The great British mathematical-physicist, James Clerk Maxwell (1831–1879), claimed that it is the "mathematics for practical men." And the English logician and economist William Stanley Jevons (1835–1882) said it is "the very guide of life and hardly can we take a step or make a decision without correctly or incorrectly making an estimate of probability."

Exercises

21.1. Find the division of the stakes in a game of chance between two equally skilled players A and B where

(a) A needs 1 more point to win and B needs 4 more points to win, using Fermat's enumeration method.

(b) A needs 3 more points to win and B needs 4 more points to win, using Pascal's triangle method.

21.2. Show that

(a) $C(n, r) = n!/r!(n - r)!$

(b) $C(n, n - r) = C(n, r)$

(c) $C(n, r) = C(n - 1, r) + C(n - 1, r - 1)$.

21.3. (a) Show that the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$ is $C(n, r)$.

(b) Show that $C(n, 0) + C(n, 1) + C(n, 2) + \cdots + C(n,n) = 2^n$.

21.4. Establish the following relations, all of which were developed by Pascal, involving the numbers of the arithmetic triangle.

(a) Any element (not in the first row or the first column) of the arithmetic triangle is equal to the sum of the element just above it and the element just to the left of it.

(b) Any given element of the arithmetic triangle, decreased by 1, is equal to the sum of all the elements above the row and to the left of the column containing the given element.

(c) The mth element in the nth row is $C(m + n - 2, n - 1)$.

(d) The element in the mth row and nth column is equal to the element in the nth row and mth column.

(e) The sum of the elements along any diagonal is twice the sum of the elements along the preceding diagonal.

(f) The sum of the elements along the nth diagonal is 2^{n-1}.

(g) Show that $C(n, r)$ appears at the intersection of the $(n + 1)$st diagonal and the $(r + 1)$st column of the arithmetic triangle.

Further Reading

Bell, E. T., *Men of Mathematics*. New York: Simon and Schuster, 1937.

David, F. N., *Games, Gods and Gambling*. New York: Hafner, 1962.

Smith, D. E., *A Source Book in Mathematics*. New York: Dover, 1958.

Todhunter, I., *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace*. New York: Chelsea, 1949.

LECTURE **22**

MOVING PICTURES VERSUS STILL PICTURES

The prime stimulus to the invention of new mathematical procedures is the presence of problems whose solutions have evaded known methods of mathematical attack. Indeed, the continual appearance of unsolved problems constitutes the life blood that maintains the health and growth of mathematics. In our previous lecture we saw an example of this—it was an elusive problem, the so-called *problem of the points,* that led to the creation of the field of mathematical probability.

In earlier lectures we have seen that the problem of finding certain areas, volumes, and arc lengths gave rise to summation processes that led to the creation of the integral calculus. In the present lecture we shall see that the problem of drawing tangents to curves and the problem of finding maximum and minimum values of functions led to the creation of the differential calculus. Each of these creations certainly constitutes a GREAT MOMENT IN MATHEMATICS.

It is interesting that, whereas the origins of the integral calculus go back to classical Greek times, it is not until the seventeenth century that we find significant contributions to the differential calculus. Not that there was no prior attempt at drawing tangents to curves and no prior employment of maximum and minimum considerations. For example, the Greeks of antiquity were able to draw tangents to circles and to the conic sections. Apollonius, in his *Conic Sections,* treated normals to a conic as the maximum and minimum line segments drawn from a point to the curve, and other maximum and minimum considerations can be found in the works of the ancient Greeks. Again, many centuries later, something of a more general approach to drawing tangents to curves was given by Gilles Persone de Roberval (1602–1675). He endeavored to consider a curve as generated by a point whose motion is compounded from two known motions. Then the resultant of the velocity vectors of the two known motions gives the tangent line to the curve. For example, in the case of a parabola, we may consider the two motions as away from the focus and away from the directrix. Since the distances of the moving point from the focus and the directrix are always equal to each other, the velocity vectors of the two motions must also be of equal magnitude. It follows that the tangent at a point of the parabola bisects the angle between the focal radius to the point and the perpendicular through the point to the directrix (see Figure 2). This idea of tangents was also held by Evangelista Torricelli (1608–1647), and an argument of priority of invention ensued between Roberval and Torricelli. Attractive as the method is, however, it seems quite limited in application.

Another method of constructing tangents to certain curves was given by René Descartes in the second part of his *La géométrie* of 1637. Though he applied his method to a number of different

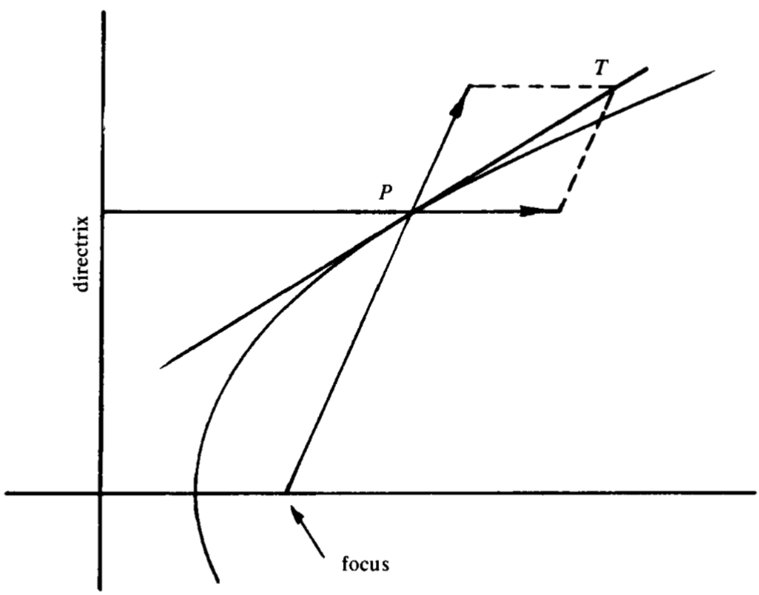

FIG. 2

curves, including one of the quartic ovals named after him,* the method is restricted to algebraic curves and, even at that, too often leads to forbidding algebra.

None of the methods above has general application, nor does any of them contain the procedure of differentiation. The first really marked anticipation of differentiation stems from ideas set forth by Fermat in 1629, though not much publicized until some eight or nine years later. Kepler had observed that the increment of a function becomes vanishingly small in the neighborhood of an ordinary maximum or minimum value. Fermat translated this fact into a process for determining such a maximum or minimum. In brief his method is this. If $f(x)$ has an ordinary maximum or minimum at x, and if e is very small, then the value of $f(x + e)$ is almost equal to that of $f(x)$. Therefore, we tentatively set $f(x + e) = f(x)$ and then make the equality correct by letting e assume the value zero. The roots of the resulting equation then give those values of x for which $f(x)$ is a maximum or a minimum.

Let us illustrate and clarify this procedure by considering Fermat's first example—to divide a quantity into two parts such that their product is a maximum. Fermat used Viète's notation, where constants are designated by upper-case consonants and variables by upper-case vowels. Employing this notation, let B be the given quantity and denote the desired parts by A and $B - A$. Forming the product

$$(A + E)[B - (A + E)]$$

and equating it to $A(B - A)$ we have

$$A(B - A) = (A + E)(B - A - E)$$

or

$$BE - 2AE - E^2 = 0.$$

After dividing by E, one obtains

$$B - 2A - E = 0.$$

*A *Cartesian oval* is the locus of a point whose distances, r_1 and r_2, from two fixed points satisfy the relation $r_1 + mr_2 = a$, where m and a are constants. The central conics will be recognized as special cases.

Now setting $E = 0$ we obtain $2A = B$, and thus find that the required division demands that the two parts each be half of B.

Although the logic of Fermat's exposition leaves much to be desired, it is seen that his method is equivalent to setting

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=0,$$

that is, to setting the derivative of $f(x)$ equal to zero. This is the customary method for finding ordinary maxima and minima of a function $f(x)$, and is sometimes referred to in our elementary textbooks as *Fermat's method*. Fermat, however, did not realize that the vanishing of the derivative of $f(x)$ is only a necessary, but not a sufficient, condition for an ordinary maximum or minimum. Also, Fermat's method does not distinguish between a maximum and a minimum value.

Fermat also devised a general procedure for finding the tangent at a point of a curve whose Cartesian equation is given. His idea is to find the *subtangent* for the point, that is, the segment on the x-axis between the foot of the ordinate drawn to the point of contact and the intersection of the tangent line with the x-axis. The method employs the idea of a tangent as the limiting position of a secant when two of the points of intersection of the secant with the curve tend to coincide. Using modern notation the method is as follows. Let the equation of the curve (see Figure 3) be $f(x,y) = 0$, and let us seek the subtangent t of the curve for the point (x, y) of the curve. By similar triangles we easily find the coordinates of a near point on the tangent to be

$$\left[x+e,\ y\left(1+\frac{e}{t}\right)\right].$$

This point is tentatively treated as if it were also on the curve, giving us

$$f\left[x+e,\ y\left(1+\frac{e}{t}\right)\right]=0.$$

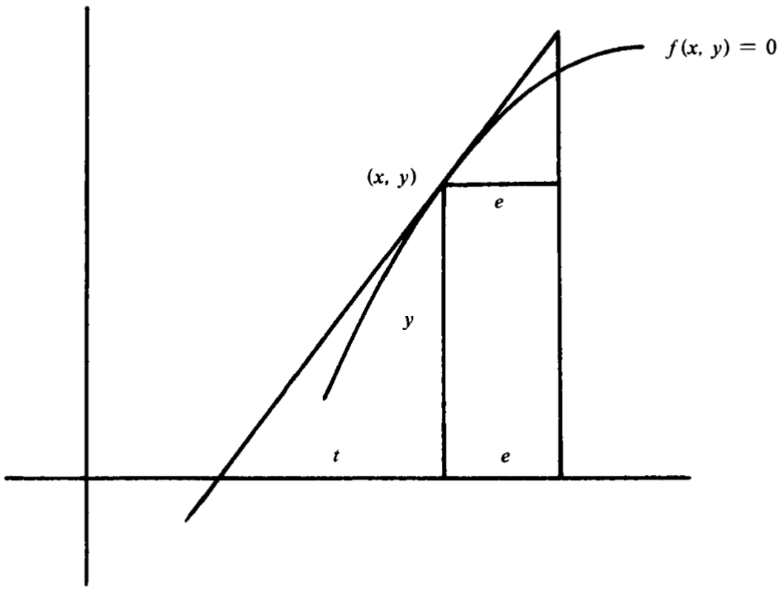

FIG. 3

The equality is then made correct by letting e assume the value zero. We then solve the resulting equation for the subtangent t in terms of the coordinates x and y of the point of contact. This, of course, is equivalent to setting

$$t = -y \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}},$$

a general formula that appeared later in 1652, naturally without the modern notation, in the work of René François Walter de Sluze (1622–1685), a canon in the Church who wrote numerous tracts in mathematics. Fermat, using his method, found tangents to the ellipse, cycloid, cissoid, conchoid, quadratrix, and folium of Descartes. Let us illustrate the method by finding the subtangent at a general point on the folium of Descartes:

$$x^3 + y^3 = nxy.$$

Here we have

$$(x + e)^3 + y^3 \left(1 + \frac{e}{t}\right)^3 - ny(x + e)\left(1 + \frac{e}{t}\right) = 0,$$

or

$$e\left(3x^2 + \frac{3y^3}{t} - \frac{nxy}{t} - ny\right) + e^2\left(3x + \frac{3y^3}{t^2} - \frac{ny}{t}\right) + e^3\left(1 + \frac{y^3}{t^3}\right) = 0.$$

Now, dividing by e and then setting $e = 0$, we find

$$t = -\frac{3y^3 - nxy}{3x^2 - ny}.$$

Another man who played a part in anticipating differentiation was Isaac Barrow. Barrow was born in London in 1630 and completed his education at Cambridge University. He was a man of high academic caliber, achieving recognition in mathematics, physics, astronomy, and theology. He was also renowned as one of the best Greek scholars of his day. He was the first to occupy the Lucasian chair at Cambridge, from which he magnanimously resigned in 1669 in favor of his great pupil, Isaac Newton, whose remarkable abilities he was one of the first to recognize and acknowledge. He died in Cambridge in 1677.

Barrow's most important mathematical work is his *Lectiones opticae et geometricae,* which appeared in the year he resigned his chair at Cambridge. The preface to the treatise admits indebtedness to Newton for some of the material of the book, probably the parts dealing with optics. It is in this book that we find a very near approach to the modern process of differentiation, utilizing essentially the so-called *differential triangle* which we find in our present-day calculus textbooks. Let it be required to find the tangent at a point P on the given curve represented in Figure 4. Let Q be a neighboring point on the curve. Then triangles PTM and PQR are very nearly similar to one another, and, Barrow argued, as the little triangle becomes indefinitely small, we have

$$RP/QR = MP/TM.$$

Let us set $QR = e$ and $RP = a$.* Then if the coordinates of P are x and y, those of Q are $x - e$ and $y - a$. Substituting these values in

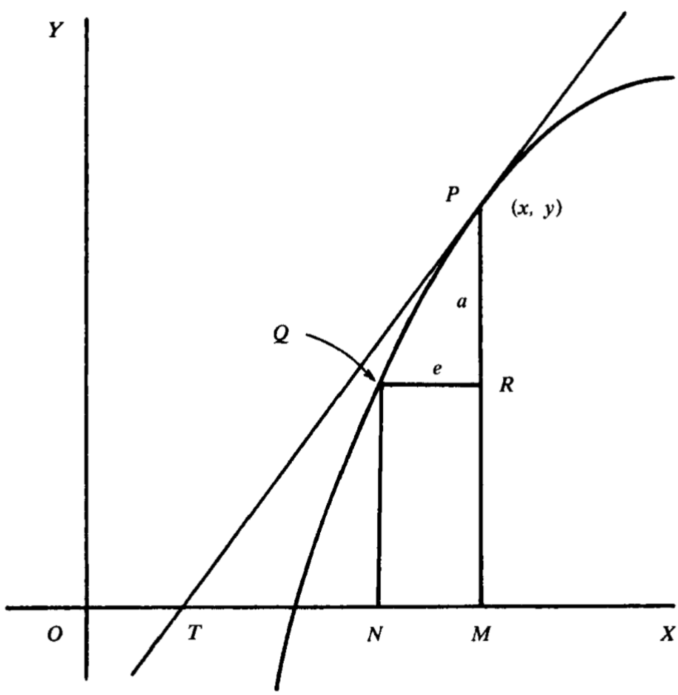

FIG. 4

*It is to be noted that a and e are the Δy and Δx of present-day treatments, whence the ratio a/e becomes dy/dx when $e \to 0$.

the equation of the curve and neglecting squares and higher powers of both e and a, we find the ratio a/e. We then have

$$OT = OM - TM = OM - MP(QR/RP) = x - y(e/a),$$

and the tangent line is determined. Barrow applied this method of constructing tangents to the curves:

(a) $x^2(x^2 + y^2) = r^2y^2$ (the *kappa curve*),
(b) $x^3 + y^3 = r^3$ (a special *Lamé curve*),
(c) $x^3 + y^3 = rxy$ (the *folium of Descartes*, but called *la galande* by Barrow),
(d) $y = (r - x)\tan(\pi x/2r)$ (the *quadratrix*),
(e) $y = r \tan(\pi x/2r)$ (a *tangent curve*).

As an illustration, let us apply the method to the curve (b). Here we have

$$(x - e)^3 + (y - a)^3 = r^3,$$

or

$$x^3 - 3x^2e + 3xe^2 - e^3 + y^3 - 3y^2a + 3ya^2 - a^3 = r^3.$$

Neglecting the square and higher powers of e and a, and using the fact that $x^3 + y^3 = r^3$, this reduces to

$$3x^2e + 3y^2a = 0,$$

from which we obtain

$$a/e = -x^2/y^2.$$

This ratio a/e is, of course, our modern dy/dx, and Barrow's dubious procedure can easily be made rigorous by use of the theory of limits.

With the work of Fermat, Barrow, and some of their contemporaries, a process of differentiation had been evolved and applied to the resolution of a number of maxima and minima problems and to the construction of tangents to many curves. What more in the development of the differential calculus remained to be done? There still remained the creation of a general symbolism with a systematic set of formal analytical rules for the calculation of derivatives, and also a consistent and rigorous redevelopment of the foundations of the subject. It is precisely the first of these, the creation of a suitable and workable *calculus*, that was furnished by Newton and Leibniz, working independently of one another. The redevelopment of the fundamental concepts on an acceptably rigorous basis had to outwait the period of energetic application of the subject, and was the work of the great French analyst Augustin-Louis Cauchy (1789–1857) and his nineteenth-century successors. This story, which is another GREAT MOMENT IN MATHEMATICS, will be told in a later lecture.

The first to publish a general and workable differential calculus was the great German mathematician and philosopher Gottfried Wilhelm Leibniz (1646–1716). In the journal *Acta eruditorum** of 1684, in an article entitled "A new method for maxima and minima as well as tangents, which is not restricted by fractional or irrational quantities, and a remarkable type of calculus for this," Leibniz published a concise exposition of his differential calculus, the formulation of which he says dated from 1676. In spite of several obscure points and some careless errors, the paper proved to be a landmark in the further advancement of mathematics. The notation of the differential calculus and many of the general rules for calculating derivatives that are in use today were given by Leibniz in this paper. Leibniz wrote as follows:

> Let an axis AX [see Figure 5, which is Leibniz's figure simplified and slightly augmented] and several curves VV, WW, YY, ZZ be given, of which the ordinates VX, WX, YX, ZX, perpendicular to the axis, are called v, w, y, z respectively. The segment AX cut off from the axis is called x. Let the tangents be VB, WC, YD, ZE, respectively intersecting the axis at B, C, D, E. Now some arbitrarily selected segment is called dx, and the line segment which is to dx as v (or w, or y, or z) is to XB (or XC, or XD, or XE) is called dv (or dw, or dy, or dz), for the difference of these v (or w, or y, or z).

Leibniz then goes on to derive a number of familiar differentiation rules, such as:

(1) If a is a constant, then $da = 0$.

---

*This journal was founded in 1682 by Leibniz and Otto Mencke, with Leibniz serving as editor-in-chief.

(2) $d(ax) = a\,dx$.

(3) $d(w - y + z) = dw - dy + dz$.

(4) $d(x^n) = nx^{n-1}dx$ (n a natural number).

(5) $d(1/x^n) = -\dfrac{n\,dx}{x^{n+1}}$.

(6) $d(\sqrt[b]{x^a}) = \dfrac{a}{b}\sqrt[b]{x^{a-b}}\,dx$.

(7) $d(vy) = v\,dy + y\,dv$.

(8) $d(v/y) = \dfrac{y\,dv - v\,dy}{y^2}$.

This is Leibniz's differential calculus, which makes differentiation an almost mechanical operation, whereas previously one had to go through the limiting procedure in each individual case. Further-

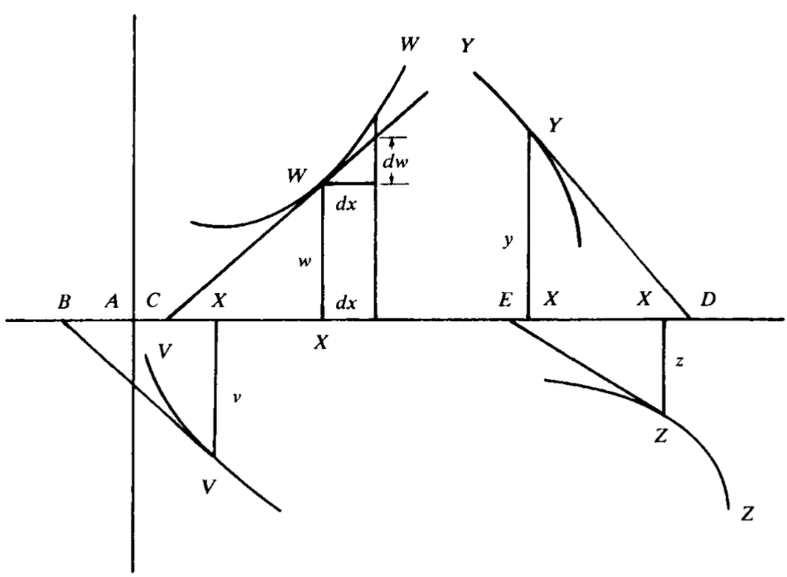

FIG. 5

more, Leibniz had introduced a fortunate, well-devised, and highly satisfying symbolism. The *differential calculus* had been invented.*

Leibniz had a happy knack for choosing convenient symbolism. Not only did he give us our present-day supple notation of the differential calculus, but in 1675 he introduced the modern integral sign, as a long letter *S* derived from the first letter of the Latin word *summa* (sum), to indicate the sum of Cavalieri's indivisibles. Though the invention of our symbolism of the differential calculus is to be credited solely to Leibniz, the invention of the calculus itself must be shared with the preëminent British mathematician and physicist Isaac Newton (1643–1727). As a matter of fact, Newton devised his *fluxional calculus,* as he called it, earlier than Leibniz had devised his *differential calculus,* but he did not publish his work until 1687. This delay of publication by Newton led to the greatest quarrel on priority of discovery in the history of mathematics.

The facts of the case are these. Newton developed his fluxional calculus as early as 1665, with the initial intention that it be applied to problems in physics, and only a few close colleagues knew of his creation. Many years later, in a letter to Leibniz sent via Henry Oldenburg, the secretary of the British Royal Society, Newton briefly and somewhat obscurely described his method, whereupon Leibniz, who by that time had already developed his own method, in a reply described his calculus to Newton. There the interchange of correspondence ceased. In the following years, Leibniz's differential calculus spread, by word of mouth, among the leading mathematicians of continental Europe, who applied it to many different problems with outstanding success. But not until 1684 did Leibniz actually put his invention into print, in the paper cited above and with no mention of Newton's corresponding achievement. Consequently Newton mentioned, in a scholium in his great scientific treatise *Philosophiae naturalis principia mathematica* of 1687, the exchange of letters that had earlier taken place. Thereupon contemporaries and successors of Newton and Leibniz started a priority quarrel, with accusations and counteraccusations of plagiarism, that often became highly undignified and degenerated into a political dispute

*We are not here concerned with any epistemological distinction between *discovery* and *invention.*

between England and Germany. So great was the English national pride that, a century or so after the quarrel, the English mathematicians steadfastly stuck to Newtonian terminology and symbolism, much to the detriment of British mathematics. Historical research has concluded that Newton and Leibniz, traveling different routes, each on his own arrived at essentially the same goal, and therefore the two men are to be regarded as independent inventors of the differential calculus.

Newton's approach to the calculus was a physical one. He considered a curve as generated by the continuous motion of a point. It follows that the abscissa and the ordinate of the generating point are, in general, changing quantities. He called a changing quantity a *fluent* (a flowing quantity), and its rate of change he called the *fluxion* of the fluent. If a fluent, such as the ordinate of the point generating a curve, is represented by y, then the fluxion of the fluent is represented by $\dot{y}$. In modern notation we see that this is equivalent to dy/dt, where t represents time. The fluxion of $\dot{y}$ is denoted by $\ddot{y}$, and so on, for higher-ordered fluxions. On the other hand, the fluent of y is denoted by the symbol y with a small square drawn about it, or sometimes by y. Newton also introduced another concept, which he called the *moment* of the fluent; it is the infinitely small amount by which a fluent increases in an infinitely small time interval o. Thus the moment of the fluent x is given by the product $\dot{x}o$. Newton remarked that we may, in any problem, neglect all terms that are multiplied by the second or higher power of o, and thus obtain an equation between the coordinates x and y of the generating point of a curve and their fluxions $\dot{x}$ and $\dot{y}$.

As an illustration of Newton's method, let us consider an example given by Newton in his work *Method of Fluxions and Infinite Series,* written in 1671 but not published until 1736, nine years after he had died. Here Newton considers the cubic curve

$$x^3 - ax^2 + axy - y^3 = 0.$$

Replacing x by $x + \dot{x}o$ and y by $y + \dot{y}o$, we get

$$\begin{aligned} &x^3 + 3x^2(\dot{x}o) + 3x(\dot{x}o)^2 + (\dot{x}o)^3 \\ &\quad - ax^2 - 2ax(\dot{x}o) - a(\dot{x}o)^2 \\ &\quad + axy + ay(\dot{x}o) + a(\dot{x}o)(\dot{y}o) + ax(\dot{y}o) \\ &\quad - y^3 - 3y^2(\dot{y}o) - 3y(\dot{y}o)^2 - (\dot{y}o)^3 = 0. \end{aligned}$$

Now, using the fact that $x^3 - ax^2 + axy - y^3 = 0$, dividing the remaining terms by o, and then rejecting all terms still containing o as a factor, we find

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.$$

If, in the last equation, we should divide by $\dot{x}$ and then solve for $\dot{y}/\dot{x}$, we would find

$$\dot{y}/\dot{x} = (3x^2 - 2ax + ay)/(3y^2 - ax).$$

Of course, in our modern notation,

$$\dot{y}/\dot{x} = (dy/dt)/(dx/dt) = dy/dx.$$

The rejection of terms containing the second and higher powers of o was much criticized by some of Newton's contemporaries. Newton later justified the process by introducing the notion of *ultimate ratios,* which is a primitive conception of the limit idea.

Newton considered two types of problems. In the first type we are given a relation connecting some fluents, and we are asked to find a relation connecting these fluents and their fluxions. This is what we did above, and is, of course, equivalent to differentiation. In the second type we are given a relation connecting some fluents and their fluxions, and we are asked to find a relation connecting the fluents alone. This is the inverse problem and is equivalent to solving a differential equation. Newton made numerous and remarkable applications of his method of fluxions. He determined maxima and minima, tangents to curves, curvature of curves, points of inflection, and convexity and concavity of curves, and he applied his theory to numerous quadratures and to rectification of curves. In the integration of some differential equations he showed extraordinary ability.

The creation of the differential calculus marks a watershed, or turning point, in the history of mathematics. The new mathematics inspired by this great invention differs markedly from the old mathematics that had been largely inherited from the ancient Greeks. The older mathematics appears static while the newer appears dynamic, so that the older mathematics compares to the still-picture stage of photography while the newer mathematics compares to the moving-picture stage. Again, the older mathematics is to the newer as anatomy is to physiology, wherein the former studies the dead body and the latter studies the living body. Once more, the older mathematics concerned itself with the fixed and the finite while the newer mathematics embraces the changing and the infinite.

Needless to say, the creation of the differential calculus was a truly GREAT MOMENT IN MATHEMATICS, and, to be fair to all involved, we perhaps should assign it to the period running from the initiatory efforts of Fermat in 1629 through the epoch-making work of Newton and Leibniz consummated over fifty years later. We shall have more to say about the calculus in our next lecture.

Exercises

22.1. Apply Roberval's method to the drawing of tangents to (a) an ellipse, (b) a hyperbola.

22.2. Following is Descartes' method of drawing tangents (see Figure 6). Let the equation of the given curve be $f(x, y) = 0$ and let (x_1, y_1) be the coordinates of the point P of the curve at which we wish to construct a tangent. Let Q, having coordinates $(x_2, 0)$, be a point on the x-axis. Then the equation of the circle with center Q and radius QP is

$$(x - x_2)^2 + y^2 = (x_1 - x_2)^2 + y_1^2.$$

Eliminating y between this equation and the equation $f(x, y) = 0$ yields an equation in x leading to the abscissas of the points where the circle cuts the given curve. Now determine x_2 so that this equation in x will have a *pair* of roots equal to x_1. This condition fixes Q as the intersection of the x-axis and the normal to the curve at P, since the circle is now tangent to the given curve at P. The required tangent is the perpendicular through P to PQ.

Construct, by Descartes' method, the tangent to the parabola $y^2 = 4x$ at the point (1,2).

22.3. Show that the slope of the tangent to the curve $y = f(x)$ at the point having abscissa x_1 is given by $f'(x_1)$, where $f'(x)$ denotes the derivative of $f(x)$.

22.4. Find the slope of the tangent at the point (3, 4) on the circle $x^2 + y^2 = 25$ by:

(a) Fermat's method,

(b) Barrow's method,
(c) Newton's method of fluxions,
(d) the method taught in calculus classes today.

22.5. The following procedure is known as the *four-step rule,* or *ab initio process,* for finding the derivative of a given function $y = f(x)$.

I. In $y = f(x)$, replace x by $x + \Delta x$, letting y become $y + \Delta y$.
II. Subtract the original relation to obtain

$$\Delta y = f(x + \Delta x) - f(x).$$

III. Divide both sides by Δx, to obtain

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

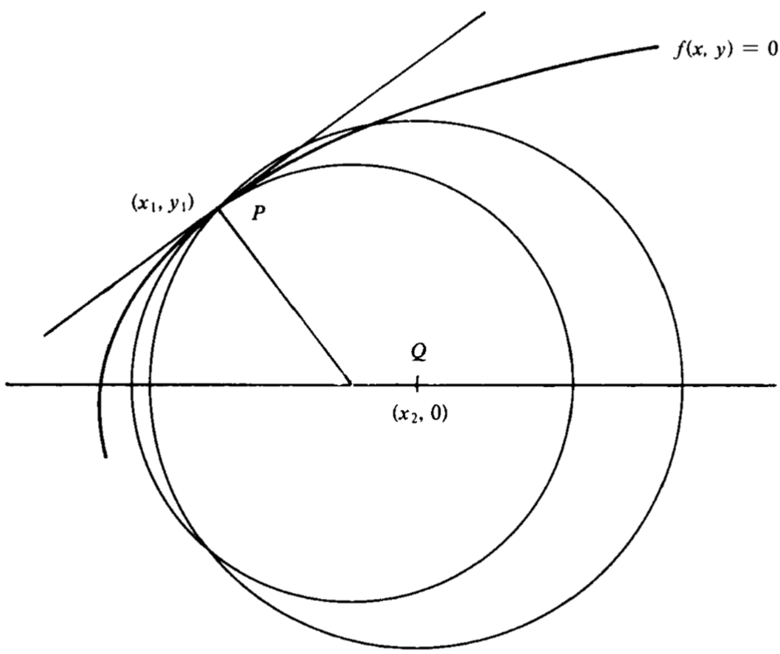

Fig. 6

IV. Take the limit of both sides as $\Delta x \to 0$ to obtain

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Using the four-step rule, obtain the following differentiation rules:

(a) If $y = a$, where a is a constant, then $dy/dx = 0$.
(b) If $y = ax$, then $dy/dx = a$.
(c) If $y = v - w + z$, where v, w, z are functions of x, then

$$dy/dx = dv/dx - dw/dx + dz/dx.$$

(d) If $y = x^n$, where n is a natural number, then $dy/dx = nx^{n-1}$.
(e) If $y = uv$, then $dy/dx = u(dv/dx) + v(du/dx)$.
(f) If $y = u/v$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

(g) If $y = 1/x^n$, where n is a natural number, then $dy/dx = -n/x^{n+1}$.

22.6. If $y = uv$, where u and v are functions of x, show that the nth derivative, $y^{(n)}$, of y with respect to x is given by

$$y^{(n)} = uv^{(n)} + nu'v^{(n-1)} + \frac{n(n-1)}{2!}u''v^{(n-2)} + \frac{n(n-1)(n-2)}{3!}u'''v^{(n-3)} + \cdots + u^{(n)}v.$$

This is known as *Leibniz's rule.*

22.7. If $s = f(t)$, where s represents the distance a body has traveled along a straight-line path in time t, show that ds/dt measures the velocity and d^2y/dt^2 the acceleration of the body at time t.

22.8. Show that at an ordinary (turning-point) maximum or minimum of a differentiable function $y = f(x)$ we must have $dy/dx = 0$. Show that, though this is a necessary condition, it is not a sufficient condition.

Further Reading

BOYER, C. B., *The History of the Calculus and Its Conceptual Development.* New York: Dover, 1959.

CHILD, J. M., *The Geometrical Lectures of Isaac Barrow.* Chicago: Open Court, 1916.

SMITH, D. E., *A Source Book in Mathematics.* New York: Dover, 1958.

STRUIK, D. J., *A Source Book in Mathematics, 1200–1800.* Cambridge, Mass.: Harvard University Press, 1969.