HOWARD EVES

Great Moments in Mathematics

Before 1650

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GREAT MOMENTS IN MATHEMATICS (BEFORE 1650)

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LECTURE 1

SCRATCHES AND GRUNTS

In the Homeric legends it is narrated that when Ulysses left the land of the Cyclops, after blinding the one-eyed giant Polyphemus, that unfortunate old giant would sit each morning by the entrance to his cave and from a heap of pebbles would pick up one pebble for each ewe that he let pass out of the cave. Then, in the evening, when the ewes returned, he would drop one pebble for each ewe that he admitted to the cave. In this way, by exhausting the supply of pebbles he had picked up in the morning, he was assured that all his flock had returned in the evening.

The story of Polyphemus is one of the earliest literary references to the notion of a one-to-one correspondence as the basis of counting. Many illustrations of the principle involved can be given. Thus, on a somewhat gruesome note, certain American Indians kept count of the number of enemies slain by collecting the scalp of each vanquished foe, and certain primitive African hunters, in proving their manhood, still keep count of the number of wild boars killed by collecting the tusks of each animal. The young unmarried girls of the tribe of Masai herdsman who live on the slopes of Mt. Kilimanjaro used to wear a number of brass rings about their necks equal to their ages. The English idiom "to chalk one up" arose from the custom of early bartenders keeping count of a customer's drinks by making chalk marks on a slate, and the Spanish idiom "echai chinas" ("to toss a pebble") arose from the similar custom of early Spanish bartenders keeping count by tossing pebbles in the customer's hood. It is also apparent in the body counting of some primitive peoples, wherein certain parts of the body are used to indicate various numbers. It is again apparent in the once widespread use of tally sticks, in which accounts were recorded by appropriate notches cut in pieces of wood; the tally sticks employed by the British Exchequer remained legal registers until as late as 1826. The ancient Peruvians maintained population and other counts on a *quipu*—a device consisting of a cord with attached knotted strings of various colors. And, of course, children today keep count of the days till Christmas or a vacation from school by checking the days off on a calendar. Almost anyone will, at one time or another, keep a small tally by ticking off on his fingers.

The oldest extant artifact of mathematical significance is a bone tool-handle, bearing notches arranged in definite numerical patterns, with a piece of quartz fitted into a narrow cavity at the head of the handle. Known as the *Ishango bone*, it was found in 1962 by Jean de Heinzelin at the fishing site of Ishango, on the shore of Lake Edward in the Democratic Republic of the Congo, and dates back to the period between 9000 and 6500 B.C. The meaning of the tally notches can only be conjectured, and there is a difference of opinion among the examining experts.

Quite likely the earliest GREAT MOMENT IN MATHEMATICS occurred when, many thousands of years ago, primitive man began to keep count of certain collections by making scratches in the dirt or on a stone. Society had evolved to the point where simple counting became imperative. A tribe, a clan, or a family had to apportion food among its members, or had to keep track of the size of a flock or herd. The process was a simple tally method employing the principle of one-to-one correspondence and was probably the beginning of the science of writing.

It seems fair to surmise that in keeping a count of a small collection, one finger was either raised or turned down per member of the collection. Tally counts for larger collections could, as indicated by the examples above, be made by assembling pebbles or sticks, by making scratches in the dirt or on a stone, by cutting notches on a bone or in a piece of wood, or by tying knots in a string. Perhaps later an assortment of grunts was developed as a vocal tally against the number of objects in small collections. Still later, an assortment of written symbols (*numerals*) was evolved to represent these numbers.

Although this development of early counting is largely conjectural, it is supported by reports of anthropologists in their studies of present-day primitive peoples and by certain artifacts unearthed in various parts of the world. It is the way small children of today begin to keep count.

In the earlier stages of the vocal period of counting, different grunts (words) were used, for example, for *two* sheep and *two* men. One merely has to recall that in English we still use *team* of horses, *span* of mules, *yoke* of oxen, *brace* of partridge, *pair* of shoes. The ultimate abstraction of the common property of *two*, represented by some sound considered independently of any concrete association, probably was a long time in arriving. Our present number words in all likelihood originally referred to sets of certain concrete objects, but these connections, except for that perhaps relating *five* and *hand*, are now lost to us.

The relation of certain number words to a concrete tally association still lingers in some primitive societies of today. Thus, because of a peculiar system of counting among a Papuan tribe in southeast New Guinea, it was found necessary to translate the Bible passage (John 5:5): "And a certain man was there, which had an infirmity thirty and eight years" into "A man lay ill one man (20), both hands (10), 5 and 3 years." Again, since primitive peoples count on their fingers, sometimes the names of the fingers are actually used by the people as number words. Thus the South American Kamayura tribe use the word "peak-finger" (middle finger) as their word for "three," and "three days" comes out as "peak-finger days." Again, the Dene-Dinje Indians of South America, who count by successively folding down the fingers of their hands, count by the following literal equivalents:

"one"—"the end is bent" (the little finger is folded)
"two"—"it is bent once more" (the ring finger is also folded)
"three"—"the middle is bent" (the middle finger is also folded)
"four"—"only one remains" (only the thumb is still extended)
"five"—"my hand is ended" or "my hand is dead" (all fingers and thumb are folded)
"ten"—"my hands are dead"
"four days"—"only-one-remains days"

Interesting is the word *kononto*, for "nine," of the Mandingo tribe of West Africa; the word literally means "to the one in the belly"—

a reference to the nine months of pregnancy. The concrete stage in counting is also evident in the Malay and Aztec tongues, where the numbers "one," "two," "three" are, literally, "one stone," "two stones," "three stones." Similarly, among the Niuès of the Southern Pacific, the first three number words are, literally, "one fruit," "two fruits," "three fruits," and among the Javanese they are, literally, "one grain," "two grains," "three grains."

There are instances where a silent language, in the form of appropriate gestures, may be employed in the one-to-one correspondence used for counting. Thus there is a Papuan body counting wherein to indicate small numbers one touches the appropriate part of the body according to the following scheme:

1	right little finger	12	nose
2	right ring finger	13	mouth
3	right middle finger	14	left ear
4	right index finger	15	left shoulder
5	right thumb	16	left elbow
6	right wrist	17	left wrist
7	right elbow	18	left thumb
8	right shoulder	19	left index finger
9	right ear	20	left middle finger
10	right eye	21	left ring finger
11	left eye	22	left little finger

One notes the mirrorlike repetition in reverse, interrupted by "nose" and "mouth" for 12 and 13.

It is common among primitive people, and even among sophisticated people, to accompany verbal counting with gestures. For example, in some tribes the word "ten" is frequently accompanied by clapping one hand against the palm of the other, and the word "six" is sometimes accompanied by passing one hand rapidly over the other. Karl Menninger says that certain African tribes can be identified and ethnically classified by observing whether they begin to count on the left hand or the right hand, whether they unfold the fingers or bend them in, or whether they turn the palm toward the body or away from the body.

The Englishman R. Mason has related a charming anecdote about World War II. A Japanese girl was in India, which at the time was at war with Japan. To avoid a possibly embarrassing situation, her friend introduced her as Chinese to an English resident of India. The Englishman was skeptical and asked the girl to count to five on her fingers, which, after some hesitation, she did. Then:

> Mr. Headley burst out delightedly: "There you are! Did you see that? Did you see how she did it? Began with her hand open and bent her fingers in one by one. Did you ever see a Chinese do such a thing? Never! The Chinese count like the English. Begin with the fist closed. She's Japanese!" he cried triumphantly.

The notion of one-to-one correspondence has long been realized as the basis for counting finite collections. In an extraordinary series of articles, beginning in 1874 and published for the most part in the mathematics journals *Mathematische Annalen* and *Journal für Mathematik*, the German mathematician Georg Cantor applied the same basic notion to the counting of infinite collections, and thereby created the remarkable theory of *transfinite numbers*. But this is another, and of course much more recent, GREAT MOMENT IN MATHEMATICS; it will be properly considered in its own place in a later lecture.

Exercises

1.1. Explain the Papuan translation of the Bible passage John 5:5 cited in the lecture text.

1.2. Explain how "peak-finger" became the word for "three" among the Kamayura tribe of South America.

1.3. The Zulus of South Africa use the following equivalents:

"six"—"taking the thumb"
"seven"—"he pointed"

Can you furnish an explanation for this?

1.4. The Malinké of West Sudan use the word *dibi* for "forty." The word literally means "a mattress." Can you give an explanation for this?

1.5. In British New Guinea, the number "ninety-nine" comes out as "four men die, two hands come to an end, one foot ends, and four." Explain this.

1.6. Two sets are said to be *equivalent* if and only if they can be placed in one-to-one correspondence. Show that

(a) the set of all letters of the alphabet is equivalent to the set of the first 26 positive integers;

(b) the set of all positive integers is equivalent to the set of all even positive integers;

(c) equivalence of sets is reflexive, symmetric, and transitive.

1.7. Two sets that are equivalent are said to have the *same cardinal number*. Let A be a set of cardinal number α and B a set of cardinal number β, where A and B have no element in common. Then, by $\alpha + \beta$, called the *sum* of α and β, we mean the cardinal number of the set $A \cup B$. This binary operation on cardinal numbers is called *addition*. Prove that addition of cardinal numbers is commutative and associative.

1.8. The set C whose elements are all ordered pairs (a, b), where a is an element of set A and b is an element of set B, is called the *Cartesian product* of A and B, and is denoted by $A \times B$. If A has cardinal number α and B has cardinal number β, then, by $\alpha\beta$, called the product of α and β, we mean the cardinal number of the set $C = A \times B$. This binary operation on cardinal numbers is called *multiplication*. Prove that multiplication of cardinal numbers is commutative, associative, and distributive over addition of cardinal numbers.

1.9. Show that a set A consisting of five elements contains 2^5 subsets (including itself and the null set). Generalize to the case of any finite set A.

1.10. Let A be a set with seven elements and B a set with five elements. What can be said about the number of elements in the sets $A \cap B$ and $A \cup B$? Generalize to the case of any two finite sets A and B.

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THE GREATEST EGYPTIAN PYRAMID

The first geometrical considerations of man must be very ancient and must have subconsciously originated in simple observations stemming from human ability to recognize physical form and to compare shapes and sizes. Certainly one of the earliest geometrical notions to thus impinge itself on even the least reflective mind would be that of distance, in particular the concept that the straight line is the shortest path connecting two points; for most animals seem instinctively to realize this. Another early notion that would gradually emerge from the subconscious to the conscious mind would be that of simple rectilinear forms, such as the triangle and the quadrilateral. Indeed, it seems almost instinctive in laying out boundaries first to locate the corners and then to connect the successive corners by straight-line walls or fences. In building walls the notions of vertical, parallel, and perpendicular would gradually emerge. Many special curves, standing out among the generally haphazard shapes of nature, would impress themselves on man's subconscious mind. Thus the discs of the sun and full moon are circular, as is an arc of a rainbow and the cross-section of a log. The parabolic trajectory of a hurled stone, the catenary curve of a hanging vine, the spiral curve of a coiled rope, and the helical curve of certain tendrils would similarly be noticed by even the least observant mind. Certain spiders spin webs that closely approximate regular polygons. The swelling set of concentric circles caused by a stone cast into a pond and the attractive flutings on many shells suggest families of associated curves. Many fruits and pits are spherical; tree trunks are circular cylinders; conical shapes appear here and there in nature. Surfaces and solids of revolution, observed in nature, as among melons, or from work on a potter's wheel, would subconsciously strike an inquisitive mind. Man, animals, and many leaves possess a bilateral symmetry. The notion of volume would be encountered every time a container was filled at the spring or river bank. The conception of space and of points in space is involved whenever one looks at the stars in the sky at nighttime. The list is easily extended.

This first nebulous acquaintance with many geometrical concepts may be called *subconscious geometry.* It was employed by early peoples, as it is by children today, in their primitive art work.

The second stage in geometry arose when human intelligence was able to extract from a set of concrete geometrical relationships a general abstract relationship containing the former as particular cases. One thus arrives at a geometrical law or rule. For example, in measuring the areas of various rectangles drawn upon quadrille-ruled paper by counting the number of little squares of the paper found inside the rectangles, a young grade-school pupil would soon induce that the area of any rectangle is probably given by the product of its two dimensions. Again, in measuring, by a tape measure, the circumferences of a number of wooden circular discs, the young pupil would induce that the circumference of any circle is somewhat more than three times the diameter of the circle.

As a more sophisticated example, consider a horizontal wooden circular disc with an upright nail driven part way into its center, and a wooden hemisphere of the same radius as the disc with a nail driven part way into its pole. Now coil a thick cord on the disc, in spiral fashion from the nail, until the disc is covered, noting the length of cord required to do this. Next coil the same kind of cord in spiral fashion about the nail in the hemisphere until the hemisphere is covered, again noting the length of cord required. In comparing the lengths of the cords used for the disc and for the hemisphere, it will be found that the latter is always (very closely) twice the former. From this one could induce that the area of the hemisphere is twice that of the disc, or that the area of a sphere is equal to four times the area of one of its great circles—a fact that was first rigorously established by Archimedes in the third century B.C. With such experiments, geometry became a laboratory study.

The laboratory stage in geometry is known as *scientific* (or *experimental,* or *empirical,* or *inductive*) *geometry.* As far back as history allows us to grope into the past, we find already present a sizable body of scientific geometry. This type of geometry seems to have arisen in certain advanced pockets of the ancient Orient (the world east of Greece) in the fifth to the third millennium B.C., to assist in engineering, agricultural, and business pursuits, and in religious ritual.

It is interesting that all recorded geometry prior to 600 B.C. is essentially scientific geometry. Geometry developed into a large bundle of rules of thumb, some correct and some only approximately correct. In a course in the history of mathematics, considerable attention is given to examining the laboratory nature of the geometry of the ancient Babylonians, Egyptians, Hindus, and Chinese. To illustrate, consider an early Chinese formula for the area of a segment of a circle. The formula is found in the *Arithmetic in Nine Sections,* dating from the second century B.C. but, because of the burning of the books in 213 B.C., believed to be a restoration of a much earlier work. In Figure 1, let c represent the chord and s the sagitta* of the circular segment. If from the midpoint of the arc of the segment one draws secants cutting the extensions of c so that the extended parts are each equal to half of s, our eyes tell us that the circular segment is approximately equal in area to the isosceles triangle formed by the line of c and the two secant lines. Assuming the areas are actually equal, we find the old Chinese formula $A = s(c + s)/2$ for the area of the circular segment. Applying this to a semicircular segment, it is easily shown that, in this case, the formula is equivalent to taking $\pi = 3$, an approximation of π frequently found in ancient mathematics.

In the Rhind papyrus, an Egyptian work on mathematics dating back at least to 1650 B.C., we find the area of a circle taken as equal to that of a square having eight-ninths of the circle's diameter as a side. It can be shown that this empirical formula is equivalent to taking $\pi = (4/3)^4 = 3.1604 \ldots$.

Although most of the baked clay mathematical tablets lifted in Mesopotamia show that the ancient Babylonians took $\pi = 3$, a

*The distance from the midpoint of the chord of the segment to the midpoint of the arc of the segment.

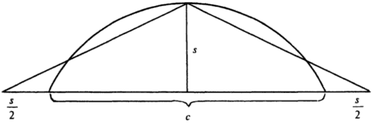

FIG. 1

recently discovered tablet dating from 1900 to 1600 B.C. and unearthed in 1936 at Susa, about 200 miles from Babylon, gives the better estimate of $3\frac{1}{8} = 3.125$.

Many other examples of the scientific nature of very early geometry are known. One is impressed by the amount of geometry that can be discovered by purely laboratory methods.

If, from the accumulation of examples of scientific geometry that have come down to us from antiquity, one were to pick an outstanding instance that might serve as a GREAT MOMENT IN MATHEMATICS, one could scarcely do better than to settle on Problem 14 of the Moscow papyrus. The Moscow papyrus, dating back to approximately 1850 B.C., is a mathematical text containing 25 problems which were already old when the manuscript was compiled. The papyrus was purchased in Egypt in 1893 and now resides in a museum in Moscow. In Problem 14 of the papyrus we find the following numerical example:

"You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right."

Now what are we to make of this? First of all we are to realize that, following the custom in illustrative problems in antiquity, a general procedure is being described and the specific numbers employed are only incidental. Since all the extant Egyptian pyramids of ancient times are regular square pyramids, we assume that in the problem we are given a frustum of a pyramid (a pyramid with its top cut off by a plane parallel to the base) whose lower base is a square of side $a = 4$, whose upper base is a square of side $b = 2$, and whose altitude is $h = 6$. We are told, in turn, to find $a^2 = (4)(4) = 16$, $ab = (4)(2) = 8$, $b^2 = (2)(2) = 4$. We are then told to obtain the sum $a^2 + ab + b^2 = 16 + 8 + 4 = 28$. Next we are told to find $\frac{1}{3}h = \frac{1}{3}(6) = 2$. Finally, we are told to compute the product $\frac{1}{3}h(a^2 + ab + b^2) = (2)(28) = 56$. This product, however, is the *volume* of the given truncated pyramid found by the correct formula

$$V = \frac{1}{3}h(B_1 + \sqrt{B_1B_2} + B_2)$$

for the volume of any frustum of a pyramid with lower base of area B_1, upper base of area B_2, and altitude h.

Let us pause a moment to consider, assuming our interpretation of Problem 14 is correct, the remarkableness of the above. The ancient Babylonians knew that the area of a trapezoid (which can be regarded as a truncated triangle) is given by the product of its altitude and half the sum of its two bases. Analagous to this, the ancient Babylonians took the volume of a frustum of a pyramid as the product of the altitude of the frustum and half the sum of the areas of its two bases, or, in the notation introduced above,

$$V = \frac{1}{2}h(B_1 + B_2).$$

Now, though it is natural to conjecture that this formula yields the volume of the frustum, the formula is incorrect. To find the volume of the frustum we expect, of course, to multiply the altitude h by some sort of mean or average of the areas B_1 and B_2. But the *arithmetic* mean of B_1 and B_2, namely, $\frac{1}{2}(B_1 + B_2)$, is not correct. What one needs here (and which is not at all obvious) is the *heronian* mean of B_1 and B_2, namely,

$$\frac{1}{3}(B_1 + \sqrt{B_1B_2} + B_2).$$

The ancient Egyptian author of Problem 14 of the Moscow papyrus, somehow or other and unlike the ancient Babylonians, made the correct conjecture. Surely this induction is a truly remarkable piece of empirical work in geometry. So remarkable did it seem to Eric Temple Bell that he named Problem 14 of the Moscow papyrus "the greatest Egyptian pyramid"; to Bell, the induction involved in the problem is far more remarkable than the actual physical construction of any of the massive stone pyramids of Egyptian antiquity still standing today. It was a GREAT MOMENT IN MATHEMATICS.

Exercises

2.1. (a) Follow through the empirical procedure, described in the lecture text, leading to the old Chinese formula for the area of a segment of a circle.

(b) Show that applying the formula to a semicircular segment is equivalent to taking $\pi = 3$.

(c) Derive a correct formula for the area of a circular segment in terms of the chord c and the sagitta s of the segment.

2.2. (a) Show that the ancient Egyptian method of finding the area of a circle is equivalent to taking $\pi = (4/3)^4 = 3.1604 \ldots$.

(b) Form an octagon from a square of side 9 units by trisecting the sides of the square and then cutting off the four triangular corners. The area of the octagon looks, by eye, to differ very little from the area of the circle inscribed in the square. Show that the area of the octagon is 63 square units, whence the area of the circle cannot be far from that of a square of 8 units on a side. There is evidence, in the form of a crudely drawn figure accompanying Problem 48 of the Rhind papyrus, that the Egyptian formula for the area of a circle may have been arrived at in this way.

2.3. On an old Babylonian baked clay tablet lifted at Susa in 1936, the ratio of the perimeter of a regular hexagon to the circumference of the circumscribed circle is given as $57/60 + 36/3600$. Show that this leads to $3\frac{1}{8}$ as an approximation of π.

2.4. The idea of averaging is common in empirical work. Thus we find, in the Rhind papyrus, the area of a quadrilateral having successive sides a, b, c, d given by

$$K = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right).$$

(a) Show that, actually, the formula above gives too large a result for all nonrectangular quadrilaterals.

(b) If the Egyptian formula above is assumed correct, show that the area of a triangle would be given by half the sum of two sides multiplied by half the third side. We find this incorrect formula for the area of a triangle in an extant deed from Edfu dating some 1500 years after the Rhind papyrus.

2.5. Interpret the following, found on a Babylonian tablet believed to date from about 2600 B.C.:

"60 is the circumference, 2 is the sagitta, find the chord.

"Thou, double 2 and get 4, dost thou not see? Take 4 from 20, thou gettest 16. Square 20, thou gettest 400. Square 16, thou gettest 256. Take 256 from 400, thou gettest 144. Find the square root of 144. 12, the square root, is the chord. Such is the procedure."

2.6. The *Śulvasūtras*, ancient Hindu religious writings dating from about 500 B.C., are of interest in the history of mathematics because they embody certain geometrical rules for the construction of altars and show an acquaintance with the Pythagorean theorem. Among the rules furnished there appear empirical solutions to the circle-squaring problem which are equivalent to taking $d = (2 + \sqrt{2})s/3$ and $s = 13d/15$, where d is the diameter of the circle and s is the side of the equivalent square. These formulas are equivalent to taking what values for π?

2.7. If m and n are two positive numbers, we define the *arithmetic mean*, the *heronian mean*, and the *geometric mean* of m and n to be $A = (m + n)/2$, $H = (m + \sqrt{mn} + n)/3$, $G = \sqrt{mn}$. Show that $A \geq H \geq G$, the equality signs holding if and only if $m = n$.

2.8. Assuming the familiar formula for the volume of any pyramid (volume equals one-third the product of base and altitude), show that the volume of any frustum T of a pyramid is given by the product of the height of T and the heronian mean of the two bases of T.

2.9. Let a, b, and h denote the lengths of an edge of the lower base, an edge of the upper base, and the altitude of a frustum T of a regular square pyramid. Dissect T into: (1) a rectangular parallelepiped P of upper base b^2 and altitude h, (2) four right triangular prisms A, B, C, and D each of volume $b(a - b)h/4$, (3) four square pyramids E, F, G, and H each of volume $(a - b)^2h/12$. Now obtain the formula

$$V = h(a^2 + ab + b^2)/3$$

for the volume of T.

2.10. Consider the dissected frustum T of Exercise 2.9. Horizontally slice P into three equal parts each of altitude $h/3$ and designate one of these slices by U. Combine A, B, C, D into a rectangular parallelepiped Q of base $b(a - b)$ and altitude h, and horizontally slice Q into three equal parts of altitude $h/3$. Replace E, F, G, H by a rectangular parallelepiped R of base $(a - b)^2$ and altitude $h/3$. Combine one slice of P with one slice of Q to form a rectangular parallelepiped V of base ab and altitude $h/3$. Combine one slice of P, two slices of Q, and R to form a rectangular parallelepiped W of base a^2 and altitude $h/3$. The volume of T is then equal to the sum of the volumes of the three rectangular parallelepipeds U, V, W. Using this fact find the formula of Exercise 2.9 for the volume of T. It has been suggested that the procedure in Problem 14 of the Moscow papyrus may have been obtained in this fashion.

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LECTURE 3

FROM THE LABORATORY INTO THE STUDY

It was about 600 B.C. that geometry entered a third stage of development. Historians of mathematics are unanimous in accrediting this further advancement to the Greeks of the period, and the earliest pioneering efforts to Thales of Miletus, one of the "seven wise men" of antiquity. Thales, it seems, spent the early part of his life as a merchant, becoming wealthy enough to devote much of his later life to study and some travel. He visited Egypt and brought back with him to Miletus knowledge of Egyptian accomplishments in geometry. His many-sided genius won him a reputation as a statesman, counselor, engineer, business man, philosopher, mathematician, and astronomer. He is the first individual known by name in the history of mathematics, and the first individual with whom deductive geometrical discoveries are associated. He is credited with the following elementary results:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. Vertical angles formed by two intersecting lines are equal.
4. Two triangles are congruent if they have two angles and one side in each respectively equal.
5. An angle inscribed in a semicircle is a right angle.

Now all five of the results above were undoubtedly known long before Thales' time, and all five are easily arrived at in a laboratory. So the value of these results is not to be measured by their content, but rather by the belief that Thales supported each of them by some logical reasoning instead of by intuition and experiment. Take, for example, the third result, which, in the laboratory, would easily be verified by cutting out a pair of vertical angles with scissors and applying one of the angles to the other. Thales, however, probably reasoned out the result much as we do today in a beginning geometry class. In Figure 2, we want to show that angle x = angle y. Now angle x is the supplement of angle z; also, angle y is the supplement of angle z. Therefore, since things equal to the same thing are equal to one another, it follows that angle x = angle y. The desired result has been obtained by a small chain of deductive reasoning, stemming from a more fundamental result. This type of geometry is known as *demonstrative* (or *deductive*, or *systematic*) *geometry*, and was considerably developed by the Greeks from 600 B.C. on. These early Greeks removed the establishment of geometrical, and similarly all mathematical, results from the laboratory into the study. This conscious and deliberate effort was certainly a GREAT MOMENT IN MATHEMATICS, and, if tradition is correct, Thales of Miletus was the original motivator.

Just why, of all the peoples of the time, the Greeks decided that geometrical facts must be assured by logical demonstration rather than by laboratory experimentation is sometimes referred to as the *Greek mystery*. Scholars have tried to furnish explanations of the Greek mystery, and though no one explanation by itself seems wholly satisfying, it may be that all of them together are acceptable. The most commonly given explanation finds the reason in the peculiar mental bias of the Greeks of classical times toward philosophical inquiries. In philosophy one is concerned with

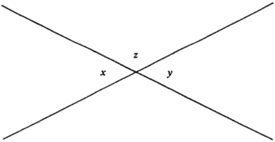

FIG. 2

inevitable conclusions that follow from assumed premises, and the empirical method affords only a measure of probability in favor of a given result. It is deductive reasoning that philosophers find to be their indispensable tool, and so the Greeks naturally gave preference to this method when they began to consider geometry.

Another explanation of the Greek mystery lies in the Hellenic love of beauty, as is manifest in their art, their writing, their sculpture, and their architecture. Now appreciation of beauty is an intellectual as well as an emotional experience, and from this point of view the orderliness, the consistency, the completeness, and the conviction found in deductive argument are very satisfying.

A still further explanation of the Greek mystery has been found in the slave-based nature of Greek society in classical times. The privileged class was supported by a large slave class that ran the businesses, managed the industries, took care of the households, and did both the technological and the unskilled work of the time. This slave basis naturally fostered a separation of theory from practice and led members of the privileged class to a preference for deduction and abstraction and a disdain for experimentation and practical application.

Finally, the explanation may lie essentially in the sweeping economic and political changes that occurred at the time. The Iron Age had been ushered in, the alphabet had been invented, coins were introduced, and geographic discoveries were made. The world was ready for a new type of civilization, and this new civilization made its appearance among the more forward-looking and imaginative people in the trading towns that sprang up along the coast of Asia Minor and later on the mainland of Greece, in Sicily, and on the Italian shore. These trading towns were largely Greek settlements. Under a developing atmosphere of rationalism, men began to ask *why* as well as *how*. Now empirical processes are quite adequate for the question how, but they do not suffice to answer inquiries of why, and attempts at demonstrative methods were bound to assert themselves, with the result that the deductive feature, which modern scholars regard as a fundamental characteristic of mathematics, came into prominence.

But whatever may be the true explanation of the Greek mystery, it must be conceded that the Greeks of classical times converted geometry into something vastly different from the collection of empirical rules handed down by their predecessors. Moreover, the fact that the first deductive thinking was done in the field of geometry instead of algebra inaugurated a tradition in mathematics that was maintained until quite recent times.

It must not be thought that the Greeks shunned all preliminary empirical and experimental methods in mathematics, for it is probably quite true that few, if any, significant mathematical facts have ever been found without some preliminary empirical work of one form or another. Before a mathematical statement can be proved or disproved by deduction, it must first be conjectured, and a conjecture is nothing but a guess made more or less plausible by intuition, observation, analogy, experimentation, or some other form of empirical procedure. Deduction is a convincing formal mode of exposition, but it is hardly a means of discovery. It is a set of complicated machinery that needs material to work upon, and the material is usually furnished by empirical considerations. Even the steps of a deductive proof or disproof are not dictated to us by the deductive apparatus itself but must be arrived at by trial and error, experience, and shrewd guessing. Indeed, skill in the art of good guessing is one of the prime ingredients in the makeup of a worthy mathematician. What is important here is that the Greeks insisted that a conjectured or laboratory-obtained mathematical statement must be followed up with a rigorous proof or disproof by deduction and that no amount of verification by experiment is sufficient to *establish* the statement.

To succeed in geometry, either as a creator or simply as a problem-solver, one must be willing to experiment, to draw and test innumerable figures, to try this and to try that. Galileo (1564–1642), in 1599, attempted to ascertain the area under one arch of the cycloid curve* by balancing a cycloidal template against circular templates of the size of the generating circle. Because of a slight flaw in his platform balance, he incorrectly concluded that the area under an arch is very nearly, but not exactly, three times the area of the circle. The first published mathematical demon-

*A *cycloid* is the curve traced by a fixed point on the circumference of a circle that rolls, without slipping, along a straight line.

stration that the area is exactly three times that of the generating circle was furnished, in 1644, by his pupil, Evangelista Torricelli (1608–1647), with the use of early integration methods.

Blaise Pascal (1623–1662), when a very young boy, "discovered" that the sum of the angles of a triangle is a straight angle by a simple experiment involving the folding of a paper triangle.

Archimedes (287?–212 B.C.), in his treatise *Method*, has described how he first came to realize, by mechanical considerations, that the volume of a sphere is given by $4\pi r^3/3$, where r is the radius of the sphere. But Archimedes' mathematical conscience would not permit him to accept his mechanical argument as a proof, and he accordingly supplied a rigorous demonstration.

By actually constructing a right circular cone, three times filling it with sand and then emptying the contents into a right circular cylinder of the same radius and height, one would conjecture that the volume of a right circular cone is one-third the product of its altitude and the area of its circular base.

Many first-rate conjectures concerning maxima and minima problems in the calculus of variations were first obtained by soap-film experiments.

One should not deprecate experiments and approaches of this kind, for there is no doubt that much geometry has been "discovered" by such means. Of course, once a geometrical conjecture has been formulated, one must, like Archimedes, establish or refute it by deductive reasoning, and thus completely settle the matter one way or the other. Many a geometrical conjecture has been discarded by the outcome of just one carefully drawn figure or by the examination of some extreme case.

A very fruitful way of making geometrical conjectures is by the employment of analogy, though it must be confessed that many conjectures so made are ultimately proved false. An astonishing amount of space geometry has been discovered via analogy from similar situations in the plane, and in the geometry of higher dimensional spaces analogy has played a very successful role.

There is a pedagogical principle based on the famous law pithily stated by biologists in the form: "Ontogeny recapitulates phylogeny," which simply means that, in general, "the individual repeats the development of the group." The pedagogical principle is that, at least in broad outline, a student should be taught a subject pretty much in the order in which the subject developed over the ages. Take geometry, for example. We have seen that historically geometry progressed through three stages—first subconscious geometry, then scientific geometry, and finally demonstrative geometry. The pedagogical principle claims, then, that geometry should first be presented to young children in its subconscious form, probably through simple art work and simple observations of nature. In this manner the young pupils will subconsciously become aware of a large number of geometrical concepts, such as distance, angle, triangle, quadrilateral, vertical, perpendicular, parallel, straight line, circle, spiral, sphere, cylinder, cone, and so on. Then, somewhat later, this subconscious basis should be evolved into scientific geometry, wherein the pupils induce a considerable array of geometrical facts through experimentation with compasses and straightedge, with ruler and protractor, with scissors and paste, with simple models, and so on. Still later, when the student has become sufficiently sophisticated, geometry can be presented in its demonstrative, or deductive, form, and the advantages and disadvantages of the earlier process can be pointed out.

The weakest part of this geometrical study program in our schools today seems to lie in the second, or scientific, stage of geometry. Not enough time is spent on this stage. There is much to be said for empirical, or experimental, geometry. The time spent here solidifies the students' grasp of many geometrical concepts. It shows them the importance and essential necessity of preliminary inductive procedures in mathematics, at the same time pointing out the shortcomings when the work is not followed up by rigorous demonstrations. What the schoolteachers need in order to make this phase of geometrical learning more extended and more valuable is a good collection of simple but significant geometrical experiments employing inexpensive and easily constructed models. The assembling of a booklet of such experiments is highly recommended to anyone interested in the venture.

Exercises

3.1. The Hindu mathematician Āryabhata the Elder wrote early in the sixth century. His work is a poem of 33 couplets called the *Ganita*. Following are translations of two of the couplets:

The area of a triangle is the product of the altitude and half the base; half of the product of this area and the height is the volume of the solid of six edges.

Half the circumference multiplied by half the diameter gives the area of the circle; this area multiplied by its own square root gives the volume of the sphere.

Show that in each of these couplets Āryabhata is correct in two dimensions but wrong in three. We note that Hindu mathematics remained empirical long after the Greeks had introduced the deductive feature.

3.2. There are two versions of how Thales, when in Egypt, evoked admiration by calculating the height of a pyramid by shadows. The earlier account, given by Hieronymus, a pupil of Aristotle, says that Thales determined the height of the pyramid by measuring the shadow it cast at the moment a man's shadow was equal to his height. The later version, given by Plutarch, says that he set up a stick and then made use of similar triangles. Both versions fail to mention the very real difficulty, in either case, of obtaining the length of the shadow of the pyramid—that is, the distance from the shadow of the apex of the pyramid to the center of the base of the pyramid.

The unaccounted-for difficulty above has given rise to what has become known as the *Thales puzzle*: Devise a method, based on shadow observations and similar triangles and independent of latitude and specific time of day or year, for determining the height of a pyramid. (There is a neat solution employing *two* shadow observations spaced a few hours apart.)

3.3. Assuming the equality of alternate interior angles formed by a transversal cutting a pair of parallel lines, prove the following:

(a) The sum of the angles of a triangle is equal to a straight angle.

(b) The sum of the interior angles of a convex polygon of n sides is equal to $n - 2$ straight angles.

3.4. Assuming the area of a rectangle is given by the product of its two dimensions, establish the following chain of theorems:

(a) The area of a parallelogram is equal to the product of its base and altitude.

(b) The area of a triangle is equal to half the product of any side and the altitude on that side.

(c) The area of a right triangle is equal to half the product of its two legs.

(d) The area of a triangle is equal to half the product of its perimeter and the radius of its inscribed circle.

(e) The area of a trapezoid is equal to the product of its altitude and half the sum of its bases.

(f) The area of a regular polygon is equal to half the product of its perimeter and its apothem.*

(g) The area of a circle is equal to half the product of its circumference and its radius.

3.5. Assuming (1) a central angle of a circle is measured by its intercepted arc, (2) the sum of the angles of a triangle is equal to a straight angle, (3) the base angles of an isosceles triangle are equal, (4) a tangent to a circle is perpendicular to the radius drawn to the point of contact, establish the following chain of theorems:

(a) An exterior angle of a triangle is equal to the sum of the two remote interior angles.

(b) An inscribed angle in a circle is measured by one-half its intercepted arc.

(c) An angle inscribed in a semicircle is a right angle.

(d) An angle formed by two intersecting chords in a circle is measured by one-half the sum of the two intercepted arcs.

(e) An angle formed by two intersecting secants of a circle is measured by one-half the difference of the two intercepted arcs.

(f) An angle formed by a tangent to a circle and a chord through the point of contact is measured by one-half the intercepted arc.

(g) An angle formed by a tangent and an intersecting secant of a circle is measured by one-half the difference of the two intercepted arcs.

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*The *apothem* of a regular polygon is the perpendicular from the center to any one of its sides.

(h) An angle formed by two intersecting tangents to a circle is measured by one-half the difference of the two intercepted arcs.

3.6. Show empirically, by a simple experiment involving the folding of a paper triangle, that the sum of the angles of a triangle is a straight angle.

3.7. To trisect a central angle AOB of a circle, someone suggests that we trisect the chord AB and then join these points of trisection with O. While this construction may look somewhat reasonable for small angles, show, by taking an angle almost equal to $180°$, that the construction is patently false.

3.8. Two ladders, 60 feet long and 40 feet long, lean from opposite sides across an alley lying between two buildings, the feet of the ladders resting against the bases of the buildings. If the ladders cross each other at a distance of 10 feet above the alley, how wide is the alley?

Find an approximate solution from drawings. An algebraic treatment of this problem requires the solution of a quartic equation. If a and b represent the lengths of the ladders, c the height at which they cross, and x the width of the alley, one can show that

$$(a^2 - x^2)^{-1/2} + (b^2 - x^2)^{-1/2} = c^{-1}.$$

3.9. Let F, V, E denote the number of faces, vertices, and edges of a polyhedron. For the tetrahedron, cube, triangular prism, pentagonal prism, square pyramid, pentagonal pyramid, cube with one corner cut off, cube with a square pyramid erected on one face, we find $V - E + F = 2$. Do you feel that this formula holds for *all* polyhedra?

3.10. There are convex polyhedra all faces of which are triangles (for instance, a tetrahedron), all faces of which are quadrilaterals (for instance, a cube), all faces are pentagons (for instance, a regular dodecahedron). Do you think the list can be continued?

3.11. (a) Consider a convex polyhedron P and let C be any point in its interior. We can imagine a suitable heterogeneous distribution of mass within P such that the center of gravity of P will coincide with C. If this weighted polyhedron should be thrown upon a horizontal floor, it will come to rest on one of its faces (since otherwise we would have perpetual motion). Show that these considerations yield a mechanical argument for the geometrical proposition: "Given a convex polyhedron P and a point C in its interior, then there exists a face F of P such that the foot of the perpendicular from C to the plane of F lies in the interior of F."

(b) Give a geometrical proof of the proposition of part (a).

3.12. Consider an ellipse with semiaxes a and b. If $a = b$ the ellipse becomes a circle and the two expressions

$$P = \pi(a + b) \quad \text{and} \quad P' = 2\pi(ab)^{1/2}$$

each becomes $2\pi a$, which gives the perimeter of the circle. This suggests that P or P' may give the perimeter E of any ellipse. Discuss.

3.13. If the inside of a race track is a noncircular ellipse, and the track is of constant width, is the outside of the track also an ellipse?

3.14. The three altitudes of a triangle are concurrent. Are the four altitudes of a tetrahedron concurrent?

3.15. Formulate theorems in three-space that are analogs of the following theorems in the plane.

(a) The bisectors of the angles of a triangle are concurrent at the center of the inscribed circle of the triangle.

(b) The area of a circle is equal to the area of a triangle the base of which has the same length as the circumference of the circle and the altitude of which is equal to the radius of the circle.

(c) The foot of the altitude of an isosceles triangle is the midpoint of the base of the triangle.

Further Reading

VAN DER WAERDEN, B. L., *Science Awakening*, tr. by Arnold Dresden. New York: Oxford University Press, 1961; New York: John Wiley, 1963 (paperback ed.).

THE FIRST GREAT THEOREM

One of the most attractive, and certainly one of the most famous and most useful, theorems of elementary geometry is the so-called *Pythagorean theorem*, which asserts that "in any right triangle the square on the hypotenuse is equal to the sum of the squares on the two legs." If there is a theorem whose birth merits inclusion as a GREAT MOMENT IN MATHEMATICS, the Pythagorean theorem is probably the prime candidate, for it is perhaps the first truly great theorem in mathematics. But when we come to consider the origin of the theorem, we find ourselves treading on anything but solid ground. Although legend has ascribed the famous theorem to Pythagoras, twentieth-century examination of cuneiform baked clay tablets excavated in Mesopotamia has revealed that the ancient Babylonians of over a thousand years prior to Pythagoras' time were aware of the theorem. Knowledge of the theorem also appears in some ancient Hindu and Chinese works that may go back to the time of Pythagoras, if not earlier. These non-Hellenic and possibly pre-Hellenic references to the theorem, however, contain no proofs of the relationship, and it may well be that Pythagoras, or some member of his renowned fraternity, was the first to furnish a logical demonstration of the theorem.

Let us pause for a moment to say something about Pythagoras and his semimystical brotherhood. Pythagoras is the second person to be mentioned by name in the history of mathematics. Peering through the mythical haze of the past, we gather that Pythagoras was born about 572 B.C. on the Aegean island of Samos, not far from Miletus, the home of the illustrious Thales. Being about fifty years younger than Thales and living so near to him, it may be that Pythagoras studied under the older man. At any rate, he appears, like Thales, to have sojourned at one time in Egypt, and then to have indulged in more extensive travel, probably going as far as India. Returning home after two years of wandering, he found Samos under the tyranny of Polycrates and much of Ionia under Persian dominion, and accordingly he migrated to the Greek seaport of Crotona, located in the boot of southern Italy. There he founded the famous Pythagorean school, which, in addition to being an academy for the study of philosophy, mathematics, and natural science, developed into a closely knit brotherhood with secret rites and observances. In time the political power and aristocratic tendencies of the brotherhood became so great that the democratic forces of southern Italy destroyed the buildings of the school and caused the society to disperse. According to report, Pythagoras fled to Metapontum, where he died, maybe through murder by his pursuers, at the advanced age of 75 or 80. The brotherhood, although scattered, continued to exist for at least two more centuries.

The Pythagorean philosophy, smacking of Hindu origin, rested on the assumption that the whole numbers are the cause of the various qualities of man and matter; in short, the whole numbers rule the universe qualitywise as well as quantitywise. This concept and exaltation of the whole numbers led to their deep study; for, who knows, maybe by unveiling the intricate properties of the whole numbers man might be able, to some degree, to guide or ameliorate his own destiny. Accordingly numbers, and, because of their intimate connection with geometry, geometry too, were assiduously studied. Because the teaching of Pythagoras was entirely oral, and because of the custom of the brotherhood to refer all discoveries back to the revered founder, it is now difficult to know just what mathematical findings should be credited to Pythagoras himself, and which to other members of the fraternity.

Returning to the GREAT MOMENT IN MATHEMATICS under consideration, it is natural to wonder as to the nature of the proof Pythagoras might have given of the great theorem named after him. There has been much conjecture on this, and it is generally felt that the proof was probably a dissection type of proof like the following. Let a, b, c denote the legs and the hypotenuse of the given right triangle, and consider the two squares of Figure 3, each

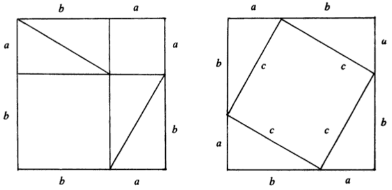

FIG. 3

having $a + b$ as its side. The first square is dissected into six pieces, namely, the two squares on the legs and four right triangles congruent to the given triangle. The second square is dissected into five pieces, namely, the square on the hypotenuse and again four right triangles congruent to the given triangle. By subtracting equals from equals, it now follows that the square on the hypotenuse is equal to the sum of the squares on the legs.

To prove that the central piece of the second dissection is actually a square of side c, we need to employ the fact that the sum of the angles of a right triangle is equal to two right angles. But this fact for the general triangle has been attributed to the Pythagoreans. Since a proof of this general fact requires, in turn, a knowledge of some properties of parallels, the early Pythagoreans are also credited with the development of that theory.

Perhaps no theorem in all of mathematics has received more diverse proofs than has the Pythagorean theorem. In the second edition of his book, *The Pythagorean Proposition*,* E. S. Loomis has collected and classified 370 demonstrations of this famous theorem.

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*Ann Arbor, Mich.: private printing, Edward Brothers, 1940. Reprint available from The National Council of Teachers of Mathematics, Washington, D.C.

Two areas, or two volumes, P and Q, are said to be *congruent by addition* if they can be dissected into corresponding pairs of congruent pieces. They are said to be *congruent by subtraction* if corresponding pairs of congruent pieces can be adjoined to P and Q to give two new figures that are congruent by addition. There are many proofs of the Pythagorean theorem which achieve their end by showing that the square on the hypotenuse of the right triangle is congruent either by addition or subtraction to the combined squares on the legs of the right triangle. The proof sketched above, and thought perhaps due to Pythagoras, is a congruency-by-subtraction proof.

Figures 4 and 5 suggest two congruency-by-addition proofs of the Pythagorean theorem, the first given by H. Perigal in 1873* and the second by H. E. Dudeney in 1917. Figure 6 suggests a congruency-by-subtraction proof said to have been devised by Leonardo da Vinci (1452–1519).

It is interesting that any two equal polygonal areas are congruent by addition, and the dissection can always be effected with straightedge and compasses. On the other hand, in 1901, Max Dehn showed that two equal polyhedral volumes are not necessarily congruent by either addition or subtraction. In particular, it is impossible to dissect a regular tetrahedron into polyhedral pieces that can be reassembled to form a cube. Euclid, in his *Elements* (ca. 300 B.C.), occasionally employs dissection methods to establish equivalence of areas.

The elegant proof of the Pythagorean theorem given by Euclid in Proposition 47 of Book I of his *Elements* depends upon the diagram of Figure 7, sometimes referred to as the Franciscan's cowl, or as the bride's chair. A précis of the proof runs as follows: $(AC)^2 = 2\ \triangle\ JAB = 2\ \triangle\ CAD = ADKL$. Similarly, $(BC)^2 = BEKL$. Therefore $(AC)^2 + (BC)^2 = ADKL + BEKL = (AB)^2$.

High school teachers sometimes show their students the curious proof of the Pythagorean theorem given by the Hindu mathematician and astronomer Bhāskara, who flourished around 1150. It is a dissection proof in which the square on the hypotenuse is cut up, as indicated in Figure 8, into four triangles, each congruent to

*This was a rediscovery, for the dissection was known to Ṯâbit ibn Qorra (826–901).

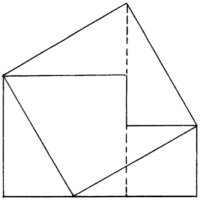

FIG. 4

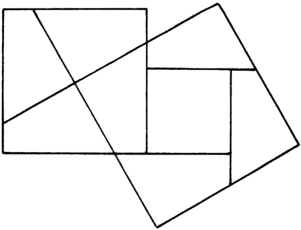

FIG. 5

the given right triangle, plus a square with side equal to the difference of the legs of the given right triangle. The pieces are easily rearranged to give the sum of the squares on the two legs. Bhāskara drew the figure and offered no further explanation than the word

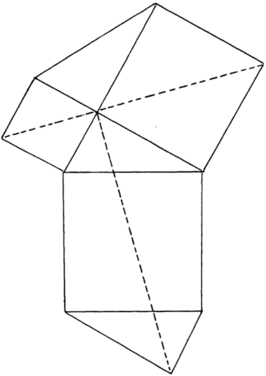

Fig. 6

"Behold!" Of course, a little algebra supplies a proof. For, if c is the hypotenuse and a and b are the legs of the given right triangle,

$$c^2 = 4(ab/2) + (b - a)^2 = a^2 + b^2.$$

Perhaps a better "behold" proof of the Pythagorean theorem would be a dynamical one on movie film wherein the square on the hypotenuse is continuously transformed into the sum of the squares on the legs by passing through the stages indicated in Figure 9.

Bhāskara also gave a second demonstration of the Pythagorean

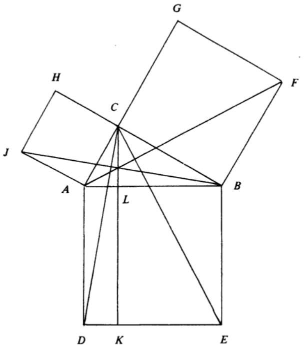

FIG. 7

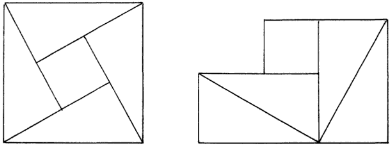

FIG. 8

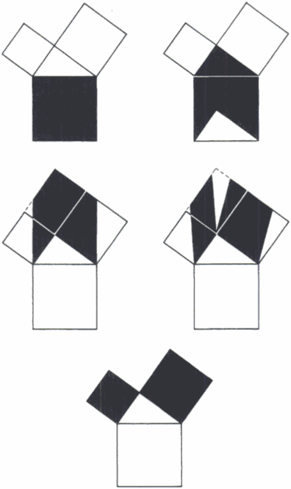

FIG. 9

theorem by drawing the altitude on the hypotenuse. From similar right triangles in Figure 10 we have

$$c/b = b/m \text{ and } c/a = a/n,$$

or

$$cm = b^2 \text{ and } cn = a^2.$$

Adding, we get

$$a^2 + b^2 = c(m + n) = c^2.$$

This proof was rediscovered in the seventeenth century by the English mathematician John Wallis (1616–1703).

A few of our country's presidents have been tenuously connected with mathematics. George Washington was a noted surveyor, Thomas Jefferson did much to encourage the teaching of higher mathematics in the United States, and Abraham Lincoln is said to have learned logic by studying Euclid's ***Elements.*** More creative was James Abram Garfield (1831-1881), the country's twentieth president, who in his student days developed a keen interest and fair ability in elementary mathematics. It was in 1876, while he was a member of the House of Representatives, and five years before he became President of the United States, that he independently discovered a very pretty proof of the Pythagorean theorem. He hit upon the proof in a mathematics discussion with some other members of Congress, and the proof was subsequently printed up in the *New England Journal of Education.* Students of high school

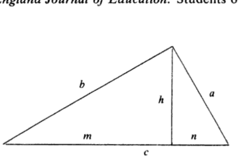

FIG. 10

geometry are always interested to see the proof, which can be presented immediately after the formula for the area of a trapezoid has been covered. The proof depends upon calculating the area of the trapezoid of Figure 11 in two different ways—first by the formula for the area of a trapezoid (as the product of half the sum of the parallel sides and the perpendicular distance between these sides) and then as the sum of three right triangles into which the trapezoid can be dissected. Equating the two expressions so found for the area of the trapezoid, we find (see Figure 11)

$$(a + b)(a + b)/2 = 2[(ab)/2] + c^2/2$$

or

$$a^2 + 2ab + b^2 = 2ab + c^2,$$

whence

$$a^2 + b^2 = c^2.$$

Since a trapezoid, as pictured, exists for any right triangle of legs a and b and hypotenuse c, the Pythagorean theorem has been established.

Like many other great theorems, the Pythagorean theorem has received numerous extensions. Even in Euclid's time certain generalizations of the theorem were known. For example, Proposition 31 of Book VI of the *Elements* states: *In a right triangle the area of a figure described on the hypotenuse is equal to the sum of the areas of similar figures similarly described on the two legs.* This generalization merely replaces the three squares on the three sides of the right triangle by any three similar and similarly described figures. A more worthy generalization stems from Propositions 12 and 13 of Book II. A combined and somewhat modernized statement of these two propositions is: *In a triangle, the square of the side opposite an obtuse (acute) angle is equal to the sum of the squares on the other two sides increased (decreased) by twice the product of one of these sides and the projection of the other side on it.* That is, in the notation of Figure 12.

$$(AB)^2 = (BC)^2 + (CA)^2 \pm 2(BC)(DC),$$

the plus or minus sign being taken according as angle C of triangle ABC is obtuse or acute. If we employ directed line segments we may combine Propositions 12 and 13 of Book II and Proposition 47 of Book I (the Pythagorean theorem) into the single statement: *If in triangle ABC, D is the foot of the altitude on side BC, then*

$$(AB)^2 = (BC)^2 + (CA)^2 - 2(BC)(DC).$$

Since $DC = CA \cos BCA$, we recognize this last statement as essentially the so-called *law of cosines*, and the law of cosines is indeed a fine generalization of the Pythagorean theorem.

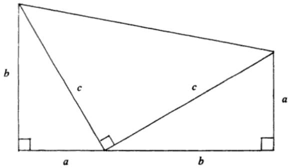

FIG. 11

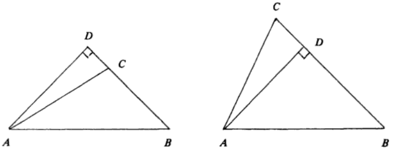

FIG. 12

But perhaps the most remarkable extension of the Pythagorean theorem that dates back to the days of Greek antiquity is that given by Pappus of Alexandria (ca. A.D. 300) at the start of Book IV of his *Mathematical Collection.* The Pappus extension of the Pythagorean theorem is as follows (see Figure 13): *Let ABC be any triangle and CADE, CBFG any parallelograms described externally on sides CA and CB. Let DE and FG meet in H and draw AL and BM equal and parallel to HC. Then the area of parallelogram ABML is equal to the sum of the areas of parallelograms CADE and CBFG.* The proof is easy, for we have *CADE* = *CAUH* = *SLAR* and *CBFG* = *CBVH* = *SMBR*. Hence *CADE* + *CBFG* = *SLAR* + *SMBR* = *ABML*. It is to be noted that the Pythagorean theorem has been generalized in two directions, for the right triangle in the Pythagorean theorem has been replaced by *any* triangle, and the squares on the legs of the right triangle have been replaced by *any* parallelograms.

The student of high school geometry can hardly fail to be interested in the Pappus extension of the Pythagorean theorem, and the proof of the extension can serve as a nice exercise for the student. Perhaps the more gifted student of geometry might like to try his hand at establishing the further extension (to three-space) of the Pappus extension: *Let ABCD* (see Figure 14) *be any tetrahedron and let ABD-EFG, BCD-HIJ, CAD-KLM be any three triangular prisms described externally on the faces ABD, BCD, CAD of ABCD. Let Q be the point of intersection of the planes EFG, HIJ, KLM, and let ABC-NOP be the triangular prism whose edges AN, BO, CP are translates of the vector QD. Then the volume of ABC-NOP is equal to the sum of the volumes of ABD-EFG, BCD-HIJ, CAD-KLM.* A proof analogous to the one given above for the Pappus extension can be supplied.

We give finally, without proof, a three-space analogue of the Pythagorean theorem that is often referred to as *de Gua's theorem.* *

*Named after J. P. de Gua de Malves (1712–1785), who presented the proposition to the Paris Academy of Sciences in 1783. The theorem, however, had been known to Descartes (1596–1650) and his contemporary J. Faulhaber (1580–1635). It is a special case of a more general theorem that Tinseau had presented to the Paris Academy of Sciences in 1774.

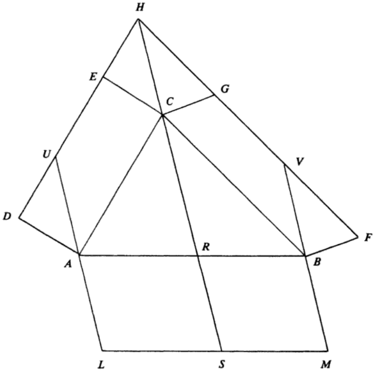

FIG. 13

We first formulate some definitions. A tetrahedron having a trihedral angle all face angles of which are right angles is called a *trirectangular tetrahedron*, and the trihedral angle is called the *right angle* of the tetrahedron. The face opposite the right angle is called the *base* of the tetrahedron. De Gua's theorem may now be stated as follows: ***The square of the area of the base of a trirectangular tetrahedron is equal to the sum of the squares of the areas of its other three faces.*** We leave the matter of proof to any enterprising reader.

With the mounting interest in space exploration and the possibility of life in other parts of the universe, suggestions have appeared from time to time concerning the construction on the earth

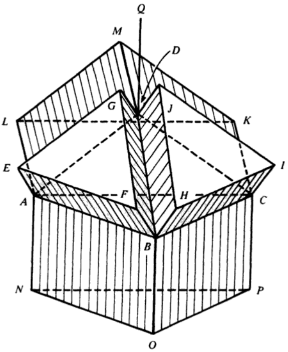

FIG. 14

of some enormous device that would indicate to possible outside observers that there is intelligence on our planet. The most favored device seems to be a mammoth configuration illustrating the Pythagorean theorem, built on the Sahara Desert, the Steppes of Russia, or some other vast area. All intelligent beings must be acquainted with this remarkable and certainly nontrivial theorem of Euclidean geometry, and it does seem difficult to think of a better visual device for the purpose under consideration.

In 1971 Nicaragua issued a series of stamps paying homage to the world's "ten most important mathematical formulas." Each stamp features a particular formula accompanied by an appropriate illustration and carries on its reverse side a brief statement in Spanish concerning the importance of the formula. One of the stamps in the series honors the Pythagorean relation "$a^2 + b^2 = c^2$." It must be pleasing to scientists and mathematicians to see these formulas so honored, for these formulas have certainly contributed far more to human development than did many of the kings and generals so often featured on stamps.

Exercises

4.1. Prove that two parallelograms having a common base and equal altitudes have equal areas by showing them to be either congruent by addition or congruent by subtraction. (This is the method employed by Euclid in Proposition 35 of Book I of his *Elements.*)

4.2. Show that any triangle is congruent by addition to the equivalent rectangle having for length a longest side of the triangle.

4.3. Fill in the details of the dissection proof of the Pythagorean theorem thought perhaps to have been given by Pythagoras.

4.4. Complete the details in the dissection proof of the Pythagorean theorem credited to: (a) H. Perigal, (b) H. E. Dudeney, (c) Leonardo da Vinci.

4.5. (a) There are reports that ancient Egyptian surveyors laid out right angles by constructing 3-4-5 triangles with a rope divided into 12 equal parts by 11 knots. Show how this can be done.

(b) Since there is no documentary evidence to the effect that the Egyptians were aware of even a particular case of the Pythagorean theorem, the following purely academic problem arises: Show, without using the Pythagorean theorem, its converse, or any of its consequences, that the 3-4-5 triangle is a right triangle. Solve this problem by means of Figure 15, which appears in the *Chóu-peï*, the oldest known Chinese mathematical work, which may date back to the second millennium B.C.

4.6. Supply a proof of the three-space analogue of the Pappus extension of the Pythagorean theorem.

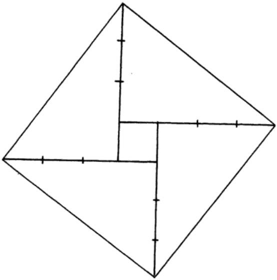

FIG. 15

4.7. The edges issuing from the right angle of a trirectangular tetrahedron are called the *legs* of the tetrahedron, and the perpendicular from the vertex of the right angle to the base is called the *altitude* of the tetrahedron.

(a) Prove that the sum of the squares of the reciprocals of the legs of a trirectangular tetrahedron is equal to the square of the reciprocal of the altitude of the tetrahedron.

(b) Prove de Gua's theorem.

4.8. Establish the following generalization of the Pythagorean theorem given by Ṯâbit ibn Qorra: If triangle ABC is any triangle, and if B' and C' are points on BC such that $\measuredangle AB'B = \measuredangle AC'C = \measuredangle A$, then $(AB)^2 + (AC)^2 = BC(BB' + CC')$.

Show that when $\measuredangle A$ is a right angle this theorem becomes the Pythagorean theorem.

4.9. What is the Pythagorean relation for a right spherical triangle of legs a and b and hypotenuse c, where a, b, and c are angular measurements?

4.10. State and prove the converse of the Pythagorean theorem. (This is Proposition 48, the final proposition, of Book I of Euclid's *Elements.*)

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LECTURE 5

PRECIPITATION OF THE FIRST CRISIS

The first numbers we encounter as we grow up from early childhood are the so-called *natural numbers*, or *positive integers*: 1, 2, 3, ... These numbers are abstractions that arise from the process of counting finite collections of objects. Somewhat later we realize that the needs of daily life require us, in addition to counting individual objects, to measure various quantities, such as length, weight, and time. To satisfy these simple measuring needs, *fractions* are required, for seldom will a length, to take an example, appear to contain an exact integral number of a prechosen linear unit. For some measurements, such as recording very low temperatures, the *zero* and *negative integers* and the *negative fractions* are found convenient. Our number system has been widened. But, if we define a *rational number* as the quotient of two integers, p/q, $q \neq 0$, then this system of rational numbers, since it contains all the integers and all the fractions, is quite sufficient for all our practical measuring purposes.

Now the rational numbers have a simple geometrical representation. Mark two distinct points O and I (see Figure 16) on a horizontal straight line, I to the right of O, and choose the segment OI as a unit of length. If we let O and I represent the numbers 0 and 1, respectively, then the positive and negative integers can be represented by a set of points on the line spaced at unit intervals apart, the positive integers being represented to the right of O and the negative integers to the left of O. The fractions with denominator q may then be represented by the points that divide each of the unit intervals into q equal parts. For each rational number, then, there is a unique point on the line. To the early mathematicians it seemed evident, as indeed it seems to anyone today who

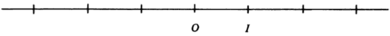

FIG. 16

has not yet been initiated into the deeper mysteries of the number line, that all the points on the line are in this way used up; ordinary common sense seems to indicate this to us.

It must have been a genuine mental shock for man to learn that there are points on the number line not corresponding to any rational number. This discovery was certainly one of the greatest achievements of the early Greeks, and it seems to have occurred some time in the fifth or sixth century B.C. among the ranks of the Pythagorean brotherhood. A truly GREAT MOMENT IN MATHEMATICS had arisen.

In particular, the Pythagoreans found that there is no rational number corresponding to the point P on the number line (see Figure 17) where the distance OP is equal to the diagonal of a square having a unit side. Later, many other points on the number line were found not corresponding to any rational number. New numbers had to be invented to correspond to such points, and since these numbers cannot be rational numbers (that is, *ratio* numbers), they came to be called *irrational numbers*.

Since, by the Pythagorean theorem, the length of a diagonal of a square of unit side is $\sqrt{2}$, in order to prove that the point P above cannot be represented by a rational number, it suffices to show that $\sqrt{2}$ is irrational. To this end, we first observe that, for a positive integer s, s^2 is even if and only if s is even. Now suppose, for the purpose of argument, that $\sqrt{2}$ is rational, that is, that $\sqrt{2} = p/q$, where p and q are relatively prime integers.* Then

$$p = q\sqrt{2},$$

or

$$p^2 = 2q^2.$$

*Two integers are *relatively prime* if they have no common positive integral factor other than unity. Thus 5 and 18 are relatively prime, whereas 12 and 18 are not relatively prime.

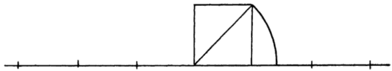

FIG. 17

Since p^2 is twice an integer, we see that p^2, and hence p, must be even. Put $p = 2r$. Then the last equation becomes

$$4r^2 = 2q^2,$$

or

$$2r^2 = q^2,$$

from which we conclude that q^2, and hence q, must be even. But this is impossible since p and q were assumed to be relatively prime. Thus the assumption that $\sqrt{2}$ is rational has led to an impossible situation, and the assumption must be abandoned.

This proof of the irrationality of $\sqrt{2}$ is essentially the traditional one reported by Aristotle (384–322 B.C.). According to Plato (427–347 B.C.), after $\sqrt{2}$ had been shown to be irrational, Theodorus of Cyrene (ca. 425 B.C.) showed that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, $\sqrt{17}$ are also irrational.

The discovery of the existence of irrational numbers upset another intuitive belief held by the early Greeks. Given any two line segments, common sense seemed to dictate that there must be some third line segment, perhaps very, very small, that can be marked off a whole number of times into each of the two given segments. Indeed, almost anyone today who has not yet learned otherwise intuitively feels the same way. But take as the two segments a side s and a diagonal d of a square. Now if there exists a third segment t that can be marked off a whole number of times into s and d we would have $s = qt$ and $d = pt$, where p and q are positive integers. But $d = s\sqrt{2}$, whence $pt = qt\sqrt{2}$. That is, $p = q\sqrt{2}$, or $\sqrt{2} = p/q$, a rational number. Contrary to intuition, then, there exist *incom-*

mensurable line segments, that is, line segments having no common unit of measure.

Let us sketch an alternative, geometrical, demonstration of the irrationality of $\sqrt{2}$ by showing that a side and diagonal of a square are incommensurable. Suppose the contrary. Then, according to this supposition, there exists a segment AP (see Figure 18) such that both the diagonal AC and the side AB of a square $ABCD$ are integral multiples of AP; that is, AC and AB are commensurable with respect to AP. On AC lay off $CB_1 = AB$ and draw B_1C_1 perpendicular to CA. One may easily prove that $C_1B = C_1B_1 = AB_1$. Then $AC_1 = AB - AB_1$ and AB_1 are commensurable with respect to AP. But AC_1 and AB_1 are a diagonal and a side of a square of dimensions less than half those of the original square. It follows that by repeating the process enough times we may finally obtain a square whose diagonal AC_n and side AB_n are commensurable with respect to AP, and $AC_n < AP$. This absurdity proves the theorem.

One notes that each of the above proofs of the irrationality of $\sqrt{2}$ employs the indirect, or *reductio ad absurdum*, method of proof. The eminent English mathematician G. H. Hardy (1877–1947) has made a delightful remark about this type of proof. In the game of chess a *gambit* is any maneuver in which a pawn or a piece is sacrificed in order to obtain an advantageous attack. Hardy pointed out that *reductio ad absurdum* "is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or a piece, but a mathematician offers *the game*."* *Reductio ad absurdum* emerges as the most stupendous gambit conceivable.

An interesting encounter with an irrational number arose in ancient times when Greek geometers tried to construct a regular polygon of five sides. They had easily succeeded in constructing regular polygons of three and four sides, namely, an equilateral triangle and a square, and, of course, the construction of a regular polygon of six sides presented no difficulty. But the construction of a regular polygon of five sides—that is, a regular pentagon—is quite another matter. Success would be assured if one can construct an angle of 36°, inasmuch as twice this angle, or 72°, is the

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*G. H. Hardy, *A Mathematician's Apology*. New York: Cambridge University Press, 1941, p. 34.

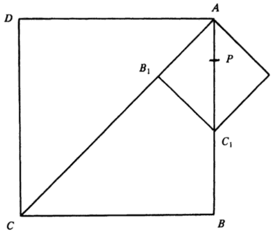

FIG. 18

central angle subtended by one side of a regular pentagon inscribed in a circle. Since, in an isosceles triangle each of whose base angles is twice the vertex angle of the triangle (see Figure 19), the base angles are 72° and the vertex angle is 36°, the problem is reduced to the construction of such an isosceles triangle. Let AC in Figure 19 bisect the base angle OAB. Then $OC = AC = AB$ and triangle BAC is similar to triangle AOB. Taking $OA = 1$ and setting $AB = x$, we then have, in turn,

$$AB/BC = OA/AB, \; x/(1-x) = 1/x, \; x^2 + x - 1 = 0.$$

It follows that $x = (\sqrt{5} - 1)/2$. The construction of this x is an easy matter, and is indicated in Figure 20, where $OA = 1$ and $MO = 1/2$, and, consequently, $AM = \sqrt{5}/2$ and

$$AB = AN = AM - MN = (\sqrt{5} - 1)/2 = x.$$

The construction of the inscribed regular pentagon now easily follows.

When a line segment OB (like OB in Figure 19) is divided by a point C such that the longer segment OC is a mean proportional

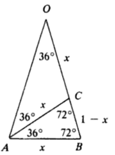

FIG. 19

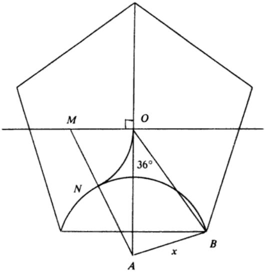

FIG. 20

between the shorter segment *CB* and the whole segment *OB*, that is, when

$$CB/OC = OC/OB,$$

the Greeks said that the line segment *OB* is divided into *golden section*. We found above that if x represents either of the ratios CB/OC or OC/OB, then $x = (\sqrt{5} - 1)/2$. This number, or sometimes its reciprocal

$$y = 1/x = (\sqrt{5} + 1)/2 \doteq 1.618,$$

is called the *golden ratio*, and this ratio seems to occur ubiquitously in nature and elsewhere.

We shall comment on the occurrences of the golden ratio in nature later on, in LECTURE 15. We here remark that psychological tests tend to show that to most people the rectangle that appears most pleasing to the eye is the one whose ratio of width to length is the golden ratio x. This rectangle, which is called the *golden rectangle*, is fundamental in an art technique known as "dynamic symmetry," which has been intensively studied by Jay Hambidge and others. The golden ratio and the golden rectangle have been observed in Greek architecture and Greek pottery, and have been applied to sculpture, painting, architectural design, furniture design, and type display. A number of artists, such as the well-known American painter George Bellows, have extensively used the principles of dynamic symmetry in their work.

A fundamental difference between rational and irrational numbers became manifest after the invention of decimal fractions. It is easily shown that any rational number possesses either a terminating or a repeating decimal expansion, and conversely, any terminating or repeating decimal expansion represents a rational number. For example: $7/4 = 1.75$, $47/22 = 2.1\overline{63}$, where the bar over the 63 means that the decimal segment 63 is endlessly repeated. It follows that the decimal expansion of an irrational number is nonterminating and nonrepeating, and conversely, any nonterminating and nonrepeating decimal expansion represents some irrational number.

The distinction between the decimal expansions of rational and irrational numbers is very useful in establishing certain properties of these numbers. Suppose, for example, we wish to show that there exists a rational number between any two distinct positive irrational numbers. Denote the two irrational numbers by a and b, $0 < a < b$, and let their decimal expansions be

$$a = a_0.a_1a_2 \ldots \quad \text{and} \quad b = b_0.b_1b_2 \ldots .$$

Let i be the first value of n for which $a_n \neq b_n$ $(n = 0, 1, 2, \ldots)$. Then

$$c = b_0.b_1b_2 \ldots b_i$$

is a rational number between a and b.

A real number is called *simply normal* if all ten digits occur with equal frequency in its decimal representation, and it is called *normal* if all blocks of digits of the same length occur with equal frequency. It is believed, but not known, that π, e, and $\sqrt{2}$, for example, are normal numbers. To obtain statistical evidence of the supposed normalcy of the above numbers, their decimal expansions have been carried out to great numbers of decimal places.

In 1967, British mathematicians, working with a computer, carried the decimal expansion of $\sqrt{2}$ to 100,000 places. In 1971, Jacques Dutka, of Columbia University, found $\sqrt{2}$ to over one million places—after 47.5 hours of computer time, the electronic machine ticked off the decimal expansion of $\sqrt{2}$ to at least 1,000,082 correct places, filling 200 pages of tightly spaced computer printout, each page containing 5000 digits. This is the longest approximation to an irrational number ever computed.

Exercises

5.1. (a) Fill in the details of the geometric proof of the irrationality of $\sqrt{2}$ sketched in the lecture text.

(b) Draw a 60°-30° right triangle; mark off the longer leg, from the 30° angle vertex, on the hypotenuse; draw a perpendicular to the hypotenuse from the dividing point. Using this figure, formulate a geometrical proof of the irrationality of $\sqrt{3}$.

5.2. (a) Prove that the straight line through the points (0, 0) and (1, $\sqrt{2}$) passes through no point, other than (0, 0), of the coordinate lattice.

(b) Show how the coordinate lattice may be used for finding rational approximations of $\sqrt{2}$.

5.3. If p is a prime number and n an integer greater than 1, show that $\sqrt[n]{p}$ is irrational.

5.4. (a) Show that $\log_{10}2$ is irrational.

(b) Generalize part (a) by showing that $\log_a b$ is irrational if a and b are positive integers, $a > 1$, and one of them contains a prime factor not contained in the other.

5.5. (a) Show that the sum of a rational and an irrational number is an irrational number.

(b) Show that the product of a rational and an irrational number is an irrational number.

5.6. (a) The symbol of the Pythagorean brotherhood was the *pentagram,* or five-pointed star formed by the five diagonals of a regular pentagon. Prove that each of the five sides of a pentagram divides into golden section the two sides of the pentagram that it intersects.

(b) Let point G divide line segment AB in golden section, where AG is the longer segment. On AB mark off $AH = GB$. Show that H divides AG in golden section.

(c) Show that if a square is cut off one end of a golden rectangle, the remaining piece is a golden rectangle.

(d) Show that 5/8 overestimates the golden ratio $x = (\sqrt{5} - 1)/2$, with an error which is less than 3 percent.

(e) If x is the golden ratio $(\sqrt{5} - 1)/2$, show that

$$x = \frac{1}{1+x} = \cfrac{1}{1+\cfrac{1}{1+x}} = \text{etc.}$$

5.7. (a) Construct, with straightedge and compasses, a regular pentagon given one side of the pentagon.

(b) Construct, with straightedge and compasses, a regular pentagon given one diagonal of the pentagon.

(c) Construct, with straightedge and compasses, a regular polygon of 15 sides.

(d) Suppose r and s are relatively prime positive integers and that a regular r-gon and a regular s-gon are constructible with straightedge and compasses. Show that a regular rs-gon is also so constructible.

(e) Establish Proposition XIII, 10, of Euclid's *Elements: A side of a regular pentagon, of a regular hexagon, and of a regular decagon inscribed in the same circle constitute the sides of a right triangle.*

5.8 (a) Prove that the decimal expansion of a rational number is either terminating or repeating.

(b) Prove that a terminating or repeating decimal expansion represents some rational number.

(c) Find the rational number having $3.2\overline{39}$ for its decimal expansion.

5.9. (a) Show that

$$0.101001000100001\ \ldots,$$

where the number of 0's between successive 1's increases each time by one, is an irrational number.

(b) Show that

$$0.12345678910111213\ \ldots,$$

in which the decimal expansion consists of the successive positive integers, is an irrational number.

5.10. (a) Prove that between any two distinct rational numbers there are infinitely many rational numbers.

(b) Prove that between any two distinct rational numbers there are infinitely many irrational numbers.

(c) Prove that between any two distinct irrational numbers there are infinitely many rational numbers.

(d) Prove that between any two distinct irrational numbers there are infinitely many irrational numbers.

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RESOLUTION OF THE FIRST CRISIS

The discovery of irrational numbers and of incommensurable magnitudes caused considerable consternation in the Pythagorean ranks. First of all, it seemed to deal a mortal blow to the Pythagorean philosophy that all depends upon the whole numbers—after all, how does an irrational number, like $\sqrt{2}$, depend on the whole numbers if it cannot be written as the ratio of two such numbers? Next, it seemed contrary to common sense, for it was strongly felt intuitively that any magnitude could be expressed by *some* rational number. The geometric counterpart was equally startling, for, again contrary to intuition, there exist line segments having no common unit of measure. But the whole Pythagorean theory of proportion and similar figures was built upon the seemingly obvious assumption that any two line segments are commensurable, that is, do have some common unit of measure. A large portion of geometry that the Pythagoreans had felt was established suddenly had to be scrapped as unsound because the proofs were invalid. A serious crisis in the foundations of mathematics was precipitated. So great was the "logical scandal" that, according to report, efforts were made for a while to keep the matter secret, and one legend has it that the Pythagorean Hippasus of Metapontum perished at sea for his impiety in disclosing the secret to outsiders, or (according to another version) was banished from the Pythagorean community and a tomb erected for him as though he were dead.

Let us see, by way of an example, how the early Pythagoreans believed they had established a basic proposition concerning areas of triangles.

THEOREM. *The areas of two triangles having the same altitude are to one another as their bases.*

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