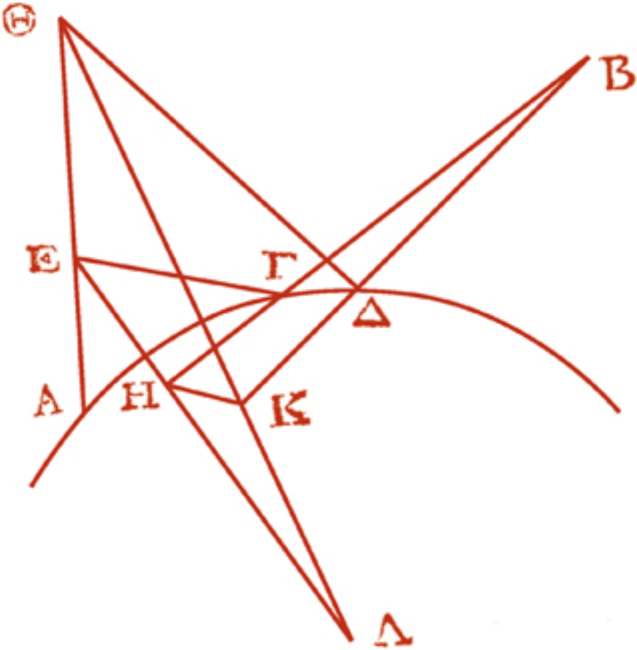
How To Prove It

A Structured Approach
Second Edition



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Preface to the Second Edition

I would like to thank all of those who have sent me comments about the first edition. Those comments have resulted in a number of small changes throughout the text. However, the biggest difference between the first edition and the second is the addition of more than 200 new exercises. There is also an appendix containing solutions to selected exercises. Exercises for which solutions are supplied are marked with an asterisk. In most cases, the solution supplied is a complete solution; in some cases, it is a sketch of a solution, or a hint.

Some exercises in Chapters 3 and 4 are also marked with the symbol PD. This indicates that these exercises can be solved using Proof Designer. Proof Designer is computer software that helps the user write outlines of proofs in elementary set theory, using the methods discussed in this book. Further information about Proof Designer can be found in an appendix, and at the Proof Designer Web site: `http://www.cs.amherst.edu/~djv/pd/pd.html`.

Preface

Students of mathematics and computer science often have trouble the first time they're asked to work seriously with mathematical proofs, because they don't know the "rules of the game." What is expected of you if you are asked to prove something? What distinguishes a correct proof from an incorrect one? This book is intended to help students learn the answers to these questions by spelling out the underlying principles involved in the construction of proofs.

Many students get their first exposure to mathematical proofs in a high school course on geometry. Unfortunately, students in high school geometry are usually taught to think of a proof as a numbered list of statements and reasons, a view of proofs that is too restrictive to be very useful. There is a parallel with computer science here that can be instructive. Early programming languages encouraged a similar restrictive view of computer programs as numbered lists of instructions. Now computer scientists have moved away from such languages and teach programming by using languages that encourage an approach called "structured programming." The discussion of proofs in this book is inspired by the belief that many of the considerations that have led computer scientists to embrace the structured approach to programming apply to proof-writing as well. You might say that this book teaches "structured proving."

In structured programming, a computer program is constructed, not by listing instructions one after another, but by combining certain basic structures such as the if-else construct and do-while loop of the Java programming language. These structures are combined, not only by listing them one after another, but also by *nesting* one within another. For example, a program constructed by nesting an if-else construct within a do-while loop would look like this:

```
do
   if [condition]
      [List of instructions goes here.]
   else
      [Alternate list of instructions goes here.]
while [condition]
```

The indenting in this program outline is not absolutely necessary, but it is a convenient method often used in computer science to display the underlying structure of a program.

Mathematical proofs are also constructed by combining certain basic proof structures. For example, a proof of a statement of the form "if P then Q" often uses what might be called the "suppose-until" structure: We *suppose* that P is true *until* we are able to reach the conclusion that Q is true, at which point we retract this supposition and conclude that the statement "if P then Q" is true. Another example is the "for arbitrary x prove" structure: To prove a statement of the form "for all x, $P(x)$," we *declare x to be an arbitrary object* and then *prove P(x)*. Once we reach the conclusion that $P(x)$ is true we retract the declaration of x as arbitrary and conclude that the statement "for all x, $P(x)$" is true. Furthermore, to prove more complex statements these structures are often combined, not only by listing one after another, but also by nesting one within another. For example, to prove a statement of the form "for all x, if $P(x)$ then $Q(x)$" we would probably nest a "suppose-until" structure within a "for arbitrary x prove" structure, getting a proof of this form:

Let x be arbitrary.
   Suppose $P(x)$ is true.
      [Proof of $Q(x)$ goes here.]
   Thus, if $P(x)$ then $Q(x)$.
Thus, for all x, if $P(x)$ then $Q(x)$.

As before, we have used indenting to make the underlying structure of the proof clear.

Of course, mathematicians don't ordinarily write their proofs in this indented form. Our aim in this book is to teach students to write proofs in ordinary English paragraphs, just as mathematicians do, and not in the indented form. Nevertheless, our approach is based on the belief that if students are to succeed at writing such proofs, they must understand the underlying structure that proofs have. They must learn, for example, that sentences like "Let x be arbitrary" and "Suppose P" are not isolated steps in proofs, but are used to introduce the "for arbitrary x prove" and "suppose-until" proof structures. It is not uncommon for beginning students to use these sentences inappropriately in other ways.

Introduction

What is mathematics? High school mathematics is concerned mostly with solving equations and computing answers to numerical questions. College mathematics deals with a wider variety of questions, involving not only numbers, but also sets, functions, and other mathematical objects. What ties them together is the use of *deductive reasoning* to find the answers to questions. When you solve an equation for x you are using the information given by the equation to *deduce* what the value of x must be. Similarly, when mathematicians solve other kinds of mathematical problems, they always justify their conclusions with deductive reasoning.

Deductive reasoning in mathematics is usually presented in the form of a *proof*. One of the main purposes of this book is to help you develop your mathematical reasoning ability in general, and in particular your ability to read and write proofs. In later chapters we'll study how proofs are constructed in detail, but first let's take a look at a few examples of proofs.

Don't worry if you have trouble understanding these proofs. They're just intended to give you a taste of what mathematical proofs are like. In some cases you may be able to follow many of the steps of the proof, but you may be puzzled about why the steps are combined in the way they are, or how anyone could have thought of the proof. If so, we ask you to be patient. Many of these questions will be answered later in this book, particularly in Chapter 3.

All of our examples of proofs in this introduction will involve prime numbers. Recall that an integer larger than 1 is said to be *prime* if it cannot be written as a product of two smaller positive integers. For example, 6 is not a prime number, since $6 = 2 \cdot 3$, but 7 is a prime number.

Before we can give an example of a proof involving prime numbers, we need to find something to prove – some fact about prime numbers whose correctness can be verified with a proof. Sometimes you can find interesting patterns in mathematics just by trying out a calculation on a few numbers. For example, consider the table in Figure 1. For each integer n from 2 to 10, the table shows whether or not both n and $2^n - 1$ are prime, and a surprising pattern emerges. It appears that $2^n - 1$ is prime in precisely those cases in which n is prime!

n	Is n prime?	$2^n - 1$	Is $2^n - 1$ prime?
2	yes	3	yes
3	yes	7	yes
4	no: $4 = 2 \cdot 2$	15	no: $15 = 3 \cdot 5$
5	yes	31	yes
6	no: $6 = 2 \cdot 3$	63	no: $63 = 7 \cdot 9$
7	yes	127	yes
8	no: $8 = 2 \cdot 4$	255	no: $255 = 15 \cdot 17$
9	no: $9 = 3 \cdot 3$	511	no: $511 = 7 \cdot 73$
10	no: $10 = 2 \cdot 5$	1023	no: $1023 = 31 \cdot 33$

Figure 1

Will this pattern continue? It is tempting to guess that it will, but this is only a guess. Mathematicians call such guesses *conjectures*. Thus, we have the following two conjectures:

Conjecture 1. *Suppose n is an integer larger than 1 and n is prime. Then $2^n - 1$ is prime.*

Conjecture 2. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

Unfortunately, if we continue the table in Figure 1, we immediately find that Conjecture 1 is incorrect. It is easy to check that 11 is prime, but $2^{11} - 1 = 2047 = 23 \cdot 89$, so $2^{11} - 1$ is not prime. Thus, 11 is a *counterexample* to Conjecture 1. The existence of even one counterexample establishes that the conjecture is incorrect, but it is interesting to note that in this case there are many counterexamples. If we continue checking numbers up to 30, we find two more counterexamples to Conjecture 1: Both 23 and 29 are prime, but $2^{23} - 1 = 8{,}388{,}607 = 47 \cdot 178{,}481$ and $2^{29} - 1 = 536{,}870{,}911 = 2{,}089 \cdot 256{,}999$. However, no number up to 30 is a counterexample to Conjecture 2.

Do you think that Conjecture 2 is correct? Having found counterexamples to Conjecture 1, we know that this conjecture is incorrect, but our failure to find a counterexample to Conjecture 2 does not show that it is correct. Perhaps there are counterexamples, but the smallest one is larger than 30. Continuing to check examples might uncover a counterexample, or, if it doesn't, it might increase our confidence in the conjecture. But we can never be sure that the conjecture is correct if we only check examples. No matter how many examples we check, there is always the possibility that the next one will be the first counterexample. The only way we can be sure that Conjecture 2 is correct is to *prove* it.

In fact, Conjecture 2 *is* correct. Here is a proof of the conjecture:

Proof of Conjecture 2. Since n is not prime, there are positive integers a and b such that $a < n$, $b < n$, and $n = ab$. Let $x = 2^b - 1$ and $y = 1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}$. Then

$$\begin{aligned} xy &= (2^b - 1) \cdot (1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}) \\ &= 2^b \cdot (1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}) - (1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}) \\ &= (2^b + 2^{2b} + 2^{3b} + \cdots + 2^{ab}) - (1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}) \\ &= 2^{ab} - 1 \\ &= 2^n - 1. \end{aligned}$$

Since $b < n$, we can conclude that $x = 2^b - 1 < 2^n - 1$. Also, since $ab = n > a$, it follows that $b > 1$. Therefore, $x = 2^b - 1 > 2^1 - 1 = 1$, so $y < xy = 2^n - 1$. Thus, we have shown that $2^n - 1$ can be written as the product of two positive integers x and y, both of which are smaller than $2^n - 1$, so $2^n - 1$ is not prime. □

Now that the conjecture has been proven, we can call it a *theorem*. Don't worry if you find the proof somewhat mysterious. We'll return to it again at the end of Chapter 3 to analyze how it was constructed. For the moment, the most important point to understand is that if n is any integer larger than 1 that can be written as a product of two smaller positive integers a and b, then the proof gives a method (admittedly, a somewhat mysterious one) of writing $2^n - 1$ as a product of two smaller positive integers x and y. Thus, if n is not prime, then $2^n - 1$ must also not be prime. For example, suppose $n = 12$, so $2^n - 1 = 4095$. Since $12 = 3 \cdot 4$, we could take $a = 3$ and $b = 4$ in the proof. Then according to the formulas for x and y given in the proof, we would have $x = 2^b - 1 = 2^4 - 1 = 15$, and $y = 1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b} = 1 + 2^4 + 2^8 = 273$. And, just as the formulas in the proof predict, we have $xy = 15 \cdot 273 = 4095 = 2^n - 1$. Of course, there are other ways of factoring 12 into a product of two smaller integers, and these might lead to other ways of factoring 4095. For example, since $12 = 2 \cdot 6$, we could use the values $a = 2$ and $b = 6$. Try computing the corresponding values of x and y and make sure their product is 4095.

Although we already know that Conjecture 1 is incorrect, there are still interesting questions we can ask about it. If we continue checking prime numbers n to see if $2^n - 1$ is prime, will we continue to find counterexamples to the conjecture – examples for which $2^n - 1$ is not prime? Will we continue to find examples for which $2^n - 1$ is prime? If there were only finitely many prime numbers, then we might be able to investigate these questions by simply checking $2^n - 1$ for every prime number n. But in fact there are infinitely many prime numbers. Euclid (circa 350 B.C.) gave a proof of this fact in Book IX of his *Elements*. His proof is one of the most famous in all of mathematics:

Theorem 3. *There are infinitely many prime numbers.*

Proof. Suppose there are only finitely many prime numbers. Let $p_1, p_2, \ldots, p_n$ be a list of all prime numbers. Let $m = p_1 p_2 \cdots p_n + 1$. Note that m is not divisible by p_1, since dividing m by p_1 gives a quotient of $p_2 p_3 \cdots p_n$ and a remainder of 1. Similarly, m is not divisible by any of $p_2, p_3, \ldots, p_n$.

We now use the fact that every integer larger than 1 is either prime or can be written as a product of primes. (We'll see a proof of this fact in Chapter 6.) Clearly m is larger than 1, so m is either prime or a product of primes. Suppose first that m is prime. Note that m is larger than all of the numbers in the list $p_1, p_2, \ldots, p_n$, so we've found a prime number not in this list. But this contradicts our assumption that this was a list of *all* prime numbers.

Now suppose m is a product of primes. Let q be one of the primes in this product. Then m is divisible by q. But we've already seen that m is not divisible by any of the numbers in the list $p_1, p_2, \ldots, p_n$, so once again we have a contradiction with the assumption that this list included all prime numbers.

Since the assumption that there are finitely many prime numbers has led to a contradiction, there must be infinitely many prime numbers. □

Once again, you should not be concerned if some aspects of this proof seem mysterious. After you've read Chapter 3 you'll be better prepared to understand the proof in detail. We'll return to this proof then and analyze its structure.

We have seen that if n is not prime then $2^n - 1$ cannot be prime, but if n is prime then $2^n - 1$ can be either prime or not prime. Because there are infinitely many prime numbers, there are infinitely many numbers of the form $2^n - 1$ that, based on what we know so far, *might* be prime. But how many of them *are* prime?

Prime numbers of the form $2^n - 1$ are called *Mersenne primes*, after Father Marin Mersenne (1588–1647), a French monk and scholar who studied these numbers. Although many Mersenne primes have been found, it is still not known if there are infinitely many of them. Many of the largest known prime numbers are Mersenne primes. As of this writing (April 2005), the largest known prime number is the Mersenne prime $2^{25,964,951} - 1$, a number with 7,816,230 digits.

Mersenne primes are related to perfect numbers, the subject of another famous unsolved problem of mathematics. A positive integer n is said to be *perfect* if n is equal to the sum of all positive integers smaller than n that divide n. (For any two integers m and n, we say that m *divides* n if n is divisible by m; in other words, if there is an integer q such that $n = qm$.) For example, the only positive integers smaller than 6 that divide 6 are 1, 2, and 3, and $1 + 2 + 3 = 6$. Thus, 6 is a perfect number. The next smallest perfect number is 28. (You should check for yourself that 28 is perfect by finding all the positive integers smaller than 28 that divide 28 and adding them up.)

Euclid proved that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. Thus, every Mersenne prime gives rise to a perfect number. Furthermore, about 2000 years after Euclid's proof, the Swiss mathematician Leonhard Euler (1707–1783), the most prolific mathematician in history, proved that every even perfect number arises in this way. (For example, note that $6 = 2^1(2^2 - 1)$ and $28 = 2^2(2^3 - 1)$.) Because it is not known if there are infinitely many Mersenne primes, it is also not known if there are infinitely many even perfect numbers. It is also not known if there are any odd perfect numbers.

Although there are infinitely many prime numbers, the primes thin out as we look at larger and larger numbers. For example, there are 25 primes between 1 and 100, 16 primes between 1000 and 1100, and only six primes between 1,000,000 and 1,000,100. As our last introductory example of a proof, we show that there are long stretches of consecutive positive integers containing no primes at all. In this proof, we'll use the following terminology: For any positive integer n, the product of all integers from 1 to n is called n *factorial* and is denoted $n!$. Thus, $n! = 1 \cdot 2 \cdot 3 \cdots n$. As with our previous two proofs, we'll return to this proof at the end of Chapter 3 to analyze its structure.

Theorem 4. *For every positive integer n, there is a sequence of n consecutive positive integers containing no primes.*

Proof. Suppose n is a positive integer. Let $x = (n + 1)! + 2$. We will show that none of the numbers $x, x + 1, x + 2, \ldots, x + (n - 1)$ is prime. Since this is a sequence of n consecutive positive integers, this will prove the theorem.

1

Sentential Logic

1.1. Deductive Reasoning and Logical Connectives

As we saw in the introduction, proofs play a central role in mathematics, and deductive reasoning is the foundation on which proofs are based. Therefore, we begin our study of mathematical reasoning and proofs by examining how deductive reasoning works.

Example 1.1.1. Here are three examples of deductive reasoning:

1. It will either rain or snow tomorrow.
   It's too warm for snow.
   Therefore, it will rain.
2. If today is Sunday, then I don't have to go to work today.
   Today is Sunday.
   Therefore, I don't have to go to work today.
3. I will go to work either tomorrow or today.
   I'm going to stay home today.
   Therefore, I will go to work tomorrow.

In each case, we have arrived at a *conclusion* from the assumption that some other statements, called *premises*, are true. For example, the premises in argument 3 are the statements "I will go to work either tomorrow or today" and "I'm going to stay home today." The conclusion is "I will go to work tomorrow," and it seems to be forced on us somehow by the premises.

But is this conclusion really correct? After all, isn't it possible that I'll stay home today, and then wake up sick tomorrow and end up staying home again? If that happened, the conclusion would turn out to be false. But notice that in that case the first premise, which said that I would go to work either tomorrow or today, would be false as well! Although we have no guarantee that the conclusion is true, it can only be false if at least one of the premises is also false. *If* both premises are true, we can be sure that the conclusion is also true. This is the sense in which the conclusion is forced on us by the premises, and this is the standard we will use to judge the correctness of deductive reasoning. We will say that an argument is *valid* if the premises cannot all be true without the conclusion being true as well. All three of the arguments in our example are valid arguments.

Here's an example of an invalid deductive argument:

Either the butler is guilty or the maid is guilty.
Either the maid is guilty or the cook is guilty.
Therefore, either the butler is guilty or the cook is guilty.

The argument is invalid because the conclusion could be false even if both premises are true. For example, if the maid were guilty, but the butler and the cook were both innocent, then both premises would be true and the conclusion would be false.

We can learn something about what makes an argument valid by comparing the three arguments in Example 1.1.1. On the surface it might seem that arguments 2 and 3 have the most in common, because they're both about the same subject: attendance at work. But in terms of the reasoning used, arguments 1 and 3 are the most similar. They both introduce two possibilities in the first premise, rule out the second one with the second premise, and then conclude that the first possibility must be the case. In other words, both arguments have the form:

P or Q.
not Q.
Therefore, P.

It is this form, and not the subject matter, that makes these arguments valid. You can see that argument 1 has this form by thinking of the letter P as standing for the statement "It will rain tomorrow," and Q as standing for "It will snow tomorrow." For argument 3, P would be "I will go to work tomorrow," and Q would be "I will go to work today."

Replacing certain statements in each argument with letters, as we have in stating the form of arguments 1 and 3, has two advantages. First, it keeps us from being distracted by aspects of the arguments that don't affect their validity. You don't need to know anything about weather forecasting or work habits to recognize that arguments 1 and 3 are valid. That's because both arguments have the form shown earlier, and you can tell that this argument form is valid without even knowing what P and Q stand for. If you don't believe this, consider the following argument:

> Either the framger widget is misfiring, or the wrompal mechanism is out of alignment.
> I've checked the alignment of the wrompal mechanism, and it's fine.
> Therefore, the framger widget is misfiring.

If a mechanic gave this explanation after examining your car, you might still be mystified about why the car won't start, but you'd have no trouble following his logic!

Perhaps more important, our analysis of the forms of arguments 1 and 3 makes clear what *is* important in determining their validity: the words *or* and *not*. In most deductive reasoning, and in particular in mathematical reasoning, the meanings of just a few words give us the key to understanding what makes a piece of reasoning valid or invalid. (Which are the important words in argument 2 in Example 1.1.1?) The first few chapters of this book are devoted to studying those words and how they are used in mathematical writing and reasoning.

In this chapter, we'll concentrate on words used to combine statements to form more complex statements. We'll continue to use letters to stand for statements, but only for unambiguous statements that are either true or false. Questions, exclamations, and vague statements will not be allowed. It will also be useful to use symbols, sometimes called *connective symbols*, to stand for some of the words used to combine statements. Here are our first three connective symbols and the words they stand for:

Symbol	Meaning
$\vee$	or
$\wedge$	and
$\neg$	not

Thus, if P and Q stand for two statements, then we'll write $P \vee Q$ to stand for the statement "P or Q," $P \wedge Q$ for "P and Q," and $\neg P$ for "not P" or "P is false." The statement $P \vee Q$ is sometimes called the *disjunction* of P and Q, $P \wedge Q$ is called the *conjunction* of P and Q, and $\neg P$ is called the *negation* of P.

Example 1.1.2. Analyze the logical forms of the following statements:

1. Either John went to the store, or we're out of eggs.
2. Joe is going to leave home and not come back.
3. Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't.

Solutions

1. If we let P stand for the statement "John went to the store" and Q stand for "We're out of eggs," then this statement could be represented symbolically as $P \lor Q$.
2. If we let P stand for the statement "Joe is going to leave home" and Q stand for "Joe is not going to come back," then we could represent this statement symbolically as $P \land Q$. But this analysis misses an important feature of the statement, because it doesn't indicate that Q is a negative statement. We could get a better analysis by letting R stand for the statement "Joe is going to come back" and then writing the statement Q as $\neg R$. Plugging this into our first analysis of the original statement, we get the improved analysis $P \land \neg R$.
3. Let B stand for the statement "Bill is at work" and J for the statement "Jane is at work." Then the first half of the statement, "Bill is at work and Jane isn't," can be represented as $B \land \neg J$. Similarly, the second half is $J \land \neg B$. To represent the entire statement, we must combine these two with *or*, forming their disjunction, so the solution is $(B \land \neg J) \lor (J \land \neg B)$.

Notice that in analyzing the third statement in the preceding example, we added parentheses when we formed the disjunction of $B \land \neg J$ and $J \land \neg B$ to indicate unambiguously which statements were being combined. This is like the use of parentheses in algebra, in which, for example, the product of $a + b$ and $a - b$ would be written $(a + b) \cdot (a - b)$, with the parentheses serving to indicate unambiguously which quantities are to be multiplied. As in algebra, it is convenient in logic to omit some parentheses to make our expressions shorter and easier to read. However, we must agree on some conventions about how to read such expressions so that they are still unambiguous. One convention is that the symbol $\neg$ always applies only to the statement that comes immediately after it. For example, $\neg P \land Q$ means $(\neg P) \land Q$ rather than $\neg(P \land Q)$. We'll see some other conventions about parentheses later.

Example 1.1.3. What English sentences are represented by the following expressions?

1. $(\neg S \land L) \lor S$, where S stands for "John is stupid" and L stands for "John is lazy."
2. $\neg S \land (L \lor S)$, where S and L have the same meanings as before.
3. $\neg(S \land L) \lor S$, with S and L still as before.

Solutions

1. Either John isn't stupid and he is lazy, or he's stupid.
2. John isn't stupid, and either he's lazy or he's stupid. Notice how the placement of the word *either* in English changes according to where the parentheses are.
3. Either John isn't both stupid and lazy, or John is stupid. The word *both* in English also helps distinguish the different possible positions of parentheses.

It is important to keep in mind that the symbols $\wedge$, $\vee$, and $\neg$ don't really correspond to all uses of the words *and*, *or*, and *not* in English. For example, the symbol $\wedge$ could not be used to represent the use of the word *and* in the sentence "John and Bill are friends," because in this sentence the word *and* is not being used to combine two statements. The symbols $\wedge$ and $\vee$ can only be used *between two statements*, to form their conjunction or disjunction, and the symbol $\neg$ can only be used *before a statement*, to negate it. This means that certain strings of letters and symbols are simply meaningless. For example, $P\neg \wedge Q$, $P \wedge/\vee Q$, and $P\neg Q$ are all "ungrammatical" expressions in the language of logic. "Grammatical" expressions, such as those in Examples 1.1.2 and 1.1.3, are sometimes called *well-formed formulas* or just *formulas*. Once again, it may be helpful to think of an analogy with algebra, in which the symbols $+$, $-$, $\cdot$, and $\div$ can be used *between two numbers*, as operators, and the symbol $-$ can also be used *before a number*, to negate it. These are the only ways that these symbols can be used in algebra, so expressions such as $x - \div y$ are meaningless.

Sometimes, words other than *and*, *or*, and *not* are used to express the meanings represented by $\wedge$, $\vee$, and $\neg$. For example, consider the first statement in Example 1.1.3. Although we gave the English translation "Either John isn't stupid and he is lazy, or he's stupid," an alternative way of conveying the same information would be to say "Either John isn't stupid *but* he is lazy, or he's stupid." Often, the word *but* is used in English to mean *and*, especially when there is some contrast or conflict between the statements being combined. For a more striking example, imagine a weather forecaster ending his forecast with the statement "Rain and snow are the only two possibilities for tomorrow's weather." This is just a roundabout way of saying that it will either rain or snow tomorrow. Thus, even though the forecaster has used the word *and*, the meaning expressed by his statement is a disjunction. The lesson of these examples is that to determine the logical form of a statement you must think about what the statement means, rather than just translating word by word into symbols.

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Figure 1

Q both stand for statements that are either true or false, we can summarize all the possibilities with the table shown in Figure 1. This is called a *truth table* for the formula $P \wedge Q$. Each row in the truth table represents one of the four possible combinations of truth values for the statements P and Q. Although these four possibilities can appear in the table in any order, it is best to list them systematically so we can be sure that no possibilities have been skipped. The truth table for $\neg P$ is also quite easy to construct because for $\neg P$ to be true, P must be false. The table is shown in Figure 2.

P	$\neg P$
F	T
T	F

Figure 2

The truth table for $P \vee Q$ is a little trickier. The first three lines should certainly be filled in as shown in Figure 3, but there may be some question about the last line. Should $P \vee Q$ be true or false in the case in which P and Q are both true? In other words, does $P \vee Q$ mean "P or Q, or both" or does it mean "P or Q but not both"? The first way of interpreting the word *or* is called the *inclusive or* (because it *includes* the possibility of both statements being true), and the second is called the *exclusive or*. In mathematics, *or* always means inclusive or, unless specified otherwise, so we will interpret $\vee$ as inclusive or. We therefore complete the truth table for $P \vee Q$ as shown in Figure 4. See exercise 3 for more about the exclusive or.

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	?

Figure 3

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Figure 4

Using the rules summarized in these truth tables, we can now work out truth tables for more complex formulas. All we have to do is work out the truth values of the component parts of a formula, starting with the individual letters and working up to more complex formulas a step at a time.

Example 1.2.1. Make a truth table for the formula $\neg(P \vee \neg Q)$.

Solution

P	Q	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$
F	F	T	T	F
F	T	F	F	T
T	F	T	T	F
T	T	F	T	F

The first two columns of this table list the four possible combinations of truth values of P and Q. The third column, listing truth values for the formula $\neg Q$, is found by simply negating the truth values for Q in the second column. The fourth column, for the formula $P \vee \neg Q$, is found by combining the truth values for P and $\neg Q$ listed in the first and third columns, according to the truth value rule for $\vee$ summarized in Figure 4. According to this rule, $P \vee \neg Q$ will be false only if both P and $\neg Q$ are false. Looking in the first and third columns, we see that this happens only in row two of the table, so the fourth column contains an F in the second row and T's in all other rows. Finally, the truth values for the formula $\neg(P \vee \neg Q)$ are listed in the fifth column, which is found by negating the truth values in the fourth column. (Note that these columns had to be worked out in order, because each was used in computing the next.)

Example 1.2.2. Make a truth table for the formula $\neg(P \wedge Q) \vee \neg R$.

Solution

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$\neg(P \wedge Q) \vee \neg R$
F	F	F	F	T	T	T
F	F	T	F	T	F	T
F	T	F	F	T	T	T
F	T	T	F	T	F	T
T	F	F	F	T	T	T
T	F	T	F	T	F	T
T	T	F	T	F	T	T
T	T	T	T	F	F	F

Note that because this formula contains three letters, it takes eight lines to list all possible combinations of truth values for these letters. (If a formula contains n different letters, how many lines will its truth table have?)

Here's a way of making truth tables more compactly. Instead of using separate columns to list the truth values for the component parts of a formula, just list those truth values below the corresponding connective symbol in the original formula. This is illustrated in Figure 5, for the formula from Example 1.2.1.

In the first step, we have listed the truth values for P and Q below these letters where they appear in the formula. In step two, the truth values for $\neg Q$ have been added under the $\neg$ symbol for $\neg Q$. In the third step, we have combined the truth values for P and $\neg Q$ to get the truth values for $P \vee \neg Q$, which are listed under the $\vee$ symbol. Finally, in the last step, these truth values are negated and listed under the initial $\neg$ symbol. The truth values added in the last step give the truth value for the entire formula, so we will call the symbol under which they are listed (the first $\neg$ symbol in this case) the *main connective* of the formula. Notice that the truth values listed under the main connective in this case agree with the values we found in Example 1.2.1.

Step 1

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		**F**			**F**
F	T		**F**			**T**
T	F		**T**			**F**
T	T		**T**			**T**

Step 2

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		F		**T**	F
F	T		F		**F**	T
T	F		T		**T**	F
T	T		T		**F**	T

Step 3

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		F	**T**	T	F
F	T		F	**F**	F	T
T	F		T	**T**	T	F
T	T		T	**T**	F	T

Step 4

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F	**F**	F	T	T	F
F	T	**T**	F	F	F	T
T	F	**F**	T	T	T	F
T	T	**F**	T	T	F	T

Figure 5

Now that we know how to make truth tables for complex formulas, we're ready to return to the analysis of the validity of arguments. Consider again our first example of a deductive argument:

It will either rain or snow tomorrow.
It's too warm for snow.
Therefore, it will rain.

As we have seen, if we let P stand for the statement "It will rain tomorrow" and Q for the statement "It will snow tomorrow," then we can represent the argument symbolically as follows:

$$P \vee Q$$
$$\underline{\neg Q}$$
$$\therefore P$$

(The symbol $\therefore$ means *therefore*.)

We can now see how truth tables can be used to verify the validity of this argument. Figure 6 shows a truth table for both premises and the conclusion of the argument. Recall that we decided to call an argument valid if the premises cannot all be true without the conclusion being true as well. Looking at Figure 6 we see that the only row of the table in which both premises come out true is row three, and in this row the conclusion is also true. Thus, the truth table confirms that if the premises are all true, the conclusion must also be true, so the argument is valid.

		Premises		Conclusion
P	Q	$P \vee Q$	$\neg Q$	P
F	F	F	T	F
F	T	T	F	F
T	F	T	T	T
T	T	T	F	T

Figure 6

Example 1.2.3. Determine whether the following arguments are valid.

1. Either John isn't stupid and he is lazy, or he's stupid.
   John is stupid.
   Therefore, John isn't lazy.
2. The butler and the cook are not both innocent.
   Either the butler is lying or the cook is innocent.
   Therefore, the butler is either lying or guilty.

Solutions

1. As in Example 1.1.3, we let S stand for the statement "John is stupid" and L stand for "John is lazy." Then the argument has the form:

$$\begin{array}{l} (\neg S \wedge L) \vee S \\ \underline{S \qquad\qquad} \\ \therefore \neg L \end{array}$$

Now we make a truth table for both premises and the conclusion. (You should work out the intermediate steps in deriving column three of this table to confirm that it is correct.)

		Premises		Conclusion
S	L	$(\neg S \wedge L) \vee S$	S	$\neg L$
F	F	F	F	T
F	T	T	F	F
T	F	T	T	T
T	T	T	T	F

Both premises are true in lines three and four of this table. The conclusion is also true in line three, but it is false in line four. Thus, it is possible for both premises to be true and the conclusion false, so the argument is invalid. In fact, the table shows us exactly why the argument is invalid. The problem occurs in the fourth line of the table, in which S and L are both true – in other words, John is both stupid and lazy. Thus, if John is both stupid and lazy, then both premises will be true but the conclusion will be false, so it would be a mistake to infer that the conclusion must be true from the assumption that the premises are true.

2. Let B stand for the statement "The butler is innocent," C for the statement "The cook is innocent," and L for the statement "The butler is lying." Then the argument has the form:

$$\begin{array}{l} \neg(B \wedge C) \\ \underline{L \vee C} \\ \therefore L \vee \neg B \end{array}$$

Here is the truth table for the premises and conclusion:

			Premises		Conclusion
B	C	L	$\neg(B \wedge C)$	$L \vee C$	$L \vee \neg B$
F	F	F	T	F	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	F	T	F
T	T	T	F	T	T

The premises are both true only in lines two, three, four, and six, and in each of these cases the conclusion is true as well. Therefore, the argument is valid.

If you expected the first argument in Example 1.2.3 to turn out to be valid, it's probably because the first premise confused you. It's a rather complicated statement, which we represented symbolically with the formula $(\neg S \wedge L) \vee S$. According to our truth table, this formula is false if S and L are both false, and true otherwise. But notice that this is exactly the same as the truth table for the simpler formula $L \vee S$! Because of this, we say that the formulas $(\neg S \wedge L) \vee S$ and $L \vee S$ are *equivalent*. Equivalent formulas always have the same truth value no matter what statements the letters in them stand for and no matter what the truth values of those statements are. The equivalence of the premise $(\neg S \wedge L) \vee S$ and the simpler formula $L \vee S$ may help you understand why

P, if $\neg\neg P$ occurs in any formula, you can always replace it with P and the resulting formula will be equivalent to the original.

Example 1.2.5. Find simpler formulas equivalent to these formulas:

1. $\neg(P \vee \neg Q)$.
2. $\neg(Q \wedge \neg P) \vee P$.

Solutions

1. $\neg(P \vee \neg Q)$

is equivalent to	$\neg P \wedge \neg\neg Q$	(DeMorgan's law),
which is equivalent to	$\neg P \wedge Q$	(double negation law).

You can check that this equivalence is right by making a truth table for $\neg P \wedge Q$ and seeing that it is the same as the truth table for $\neg(P \vee \neg Q)$ found in Example 1.2.1.

2. $\neg(Q \wedge \neg P) \vee P$

is equivalent to	$(\neg Q \vee \neg\neg P) \vee P$	(DeMorgan's law),
which is equivalent to	$(\neg Q \vee P) \vee P$	(double negation law),
which is equivalent to	$\neg Q \vee (P \vee P)$	(associative law),
which is equivalent to	$\neg Q \vee P$	(idempotent law).

Some equivalences are based on the fact that certain formulas are either always true or always false. For example, you can verify by making a truth table that the formula $Q \wedge (P \vee \neg P)$ is equivalent to just Q. But even before you make the truth table, you can probably see why they are equivalent. In every line of the truth table, $P \vee \neg P$ will come out true, and therefore $Q \wedge (P \vee \neg P)$ will come out true when Q is also true, and false when Q is false. Formulas that are always true, such as $P \vee \neg P$, are called *tautologies*. Similarly, formulas that are always false are called *contradictions*. For example, $P \wedge \neg P$ is a contradiction.

Example 1.2.6. Are these statements tautologies, contradictions, or neither?

$$P \vee (Q \vee \neg P), \quad P \wedge \neg(Q \vee \neg Q), \quad P \vee \neg(Q \vee \neg Q).$$

Solution

First we make a truth table for all three statements.

P	Q	$P \vee (Q \vee \neg P)$	$P \wedge \neg(Q \vee \neg Q)$	$P \vee \neg(Q \vee \neg Q)$
F	F	T	F	F
F	T	T	F	F
T	F	T	F	T
T	T	T	F	T

From the truth table it is clear that the first formula is a tautology, the second a contradiction, and the third neither. In fact, since the last column is identical to the first, the third formula is equivalent to P.

We can now state a few more useful laws involving tautologies and contradictions. You should be able to convince yourself that all of these laws are correct by thinking about what the truth tables for the statements involved would look like.

Tautology laws

$$P \wedge \text{(a tautology) is equivalent to } P.$$
$$P \vee \text{(a tautology) is a tautology.}$$
$$\neg\text{(a tautology) is a contradiction.}$$

Contradiction laws

$$P \wedge \text{(a contradiction) is a contradiction.}$$
$$P \vee \text{(a contradiction) is equivalent to } P.$$
$$\neg\text{(a contradiction) is a tautology.}$$

Example 1.2.7. Find simpler formulas equivalent to these formulas:

1. $P \vee (Q \wedge \neg P)$.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$.

Solutions

1. $P \vee (Q \wedge \neg P)$
   is equivalent to $(P \vee Q) \wedge (P \vee \neg P)$ (distributive law),
   which is equivalent to $P \vee Q$ (tautology law).
   The last step uses the fact that $P \vee \neg P$ is a tautology.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$
   is equivalent to $(\neg P \wedge \neg(Q \wedge \neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee \neg\neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee R)) \wedge Q$ (double negation law),
   which is equivalent to $\neg P \wedge ((\neg Q \vee R) \wedge Q)$ (associative law),
   which is equivalent to $\neg P \wedge (Q \wedge (\neg Q \vee R))$ (commutative law),
   which is equivalent to $\neg P \wedge ((Q \wedge \neg Q) \vee (Q \wedge R))$
   (distributive law),
   which is equivalent to $\neg P \wedge (Q \wedge R)$ (contradiction law).

   The last step uses the fact that $Q \wedge \neg Q$ is a contradiction. Finally, by the associative law for $\wedge$ we can remove the parentheses without making the formula ambiguous, so the original formula is equivalent to the formula $\neg P \wedge Q \wedge R$.

Exercises

*1. Make truth tables for the following formulas:
   (a) $\neg P \vee Q$.
   (b) $(S \vee G) \wedge (\neg S \vee \neg G)$.
2. Make truth tables for the following formulas:
   (a) $\neg[P \wedge (Q \vee \neg P)]$.
   (b) $(P \vee Q) \wedge (\neg P \vee R)$.
3. In this exercise we will use the symbol $+$ to mean *exclusive or*. In other words, $P + Q$ means "P or Q, but not both."
   (a) Make a truth table for $P + Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P + Q$. Justify your answer with a truth table.
4. Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.
*5. Some mathematicians use the symbol $\downarrow$ to mean *nor*. In other words, $P \downarrow Q$ means "neither P nor Q."
   (a) Make a truth table for $P \downarrow Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \downarrow Q$.
   (c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
6. Some mathematicians write $P \mid Q$ to mean "P and Q are not both true." (This connective is called *nand*, and is used in the study of circuits in computer science.)
   (a) Make a truth table for $P \mid Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \mid Q$.
   (c) Find formulas using only the connective $\mid$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
*7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.
8. Use truth tables to determine which of the following formulas are equivalent to each other:
   (a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
   (b) $\neg P \vee Q$.
   (c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$.
   (d) $\neg(P \vee Q)$.
   (e) $(Q \wedge P) \vee \neg P$.
*9. Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither:

(a) $(P \lor Q) \land (\neg P \lor \neg Q)$.
(b) $(P \lor Q) \land (\neg P \land \neg Q)$.
(c) $(P \lor Q) \lor (\neg P \lor \neg Q)$.
(d) $[P \land (Q \lor \neg R)] \lor (\neg P \lor R)$.

10. Use truth tables to check these laws:
    (a) The second DeMorgan's law. (The first was checked in the text.)
    (b) The distributive laws.

*11. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
    (a) $\neg(\neg P \land \neg Q)$.
    (b) $(P \land Q) \lor (P \land \neg Q)$.
    (c) $\neg(P \land \neg Q) \lor (\neg P \land Q)$.

12. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
    (a) $\neg(\neg P \lor Q) \lor (P \land \neg R)$.
    (b) $\neg(\neg P \land Q) \lor (P \land \neg R)$.
    (c) $(P \land R) \lor [\neg R \land (P \lor Q)]$.

13. Use the first DeMorgan's law and the double negation law to derive the second DeMorgan's law.

*14. Note that the associative laws say only that parentheses are unnecessary when combining *three* statements with $\land$ or $\lor$. In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that $[P \land (Q \land R)] \land S$ is equivalent to $(P \land Q) \land (R \land S)$.

15. How many lines will there be in the truth table for a statement containing n letters?

*16. Find a formula involving the connectives $\land$, $\lor$, and $\neg$ that has the following truth table:

P	Q	???
F	F	T
F	T	F
T	F	T
T	T	T

17. Find a formula involving the connectives $\land$, $\lor$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	T
T	F	T
T	T	F

18. Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

1.3. Variables and Sets

In mathematical reasoning it is often necessary to make statements about objects that are represented by letters called *variables*. For example, if the variable x is used to stand for a number in some problem, we might be interested in the statement "x is a prime number." Although we may sometimes use a single letter, say P, to stand for this statement, at other times we will revise this notation slightly and write $P(x)$, to stress that this is a statement *about* x. The latter notation makes it easy to talk about substituting some number for x in the statement. For example, $P(7)$ would represent the statement "7 is a prime number," and $P(a+b)$ would mean "$a+b$ is a prime number." If a statement contains more than one variable, our abbreviation for the statement will include a list of all the variables involved. For example, we might represent the statement "p is divisible by q" by $D(p,q)$. In this case, $D(12,4)$ would mean "12 is divisible by 4."

Although you have probably seen variables used most often to stand for numbers, they can stand for anything at all. For example, we could let $M(x)$ stand for the statement "x is a man," and $W(x)$ for "x is a woman." In this case, we are using the variable x to stand for a person. A statement might even contain several variables that stand for different kinds of objects. For example, in the statement "x has y children," the variable x stands for a person, and y stands for a number.

Statements involving variables can be combined using connectives, just like statements without variables.

Example 1.3.1. Analyze the logical forms of the following statements:

1. x is a prime number, and either y or z is divisible by x.
2. x is a man and y is a woman and x likes y, but y doesn't like x.

Solutions

1. We could let P stand for the statement "x is a prime number," D for "y is divisible by x," and E for "z is divisible by x." The entire statement would then be represented by the formula $P \wedge (D \vee E)$. But this analysis, though not incorrect, fails to capture the relationship between the statements

whereas x is a *bound* variable (or a *dummy* variable). The free variables in a statement stand for objects that the statement says something about. Plugging in different values for a free variable affects the meaning of a statement and may change its truth value. The fact that you can plug in different values for a free variable means that it is free to stand for anything. Bound variables, on the other hand, are simply letters that are used as a convenience to help express an idea and should not be thought of as standing for any particular object. A bound variable can always be replaced by a new variable without changing the meaning of the statement, and often the statement can be rephrased so that the bound variables are eliminated altogether. For example, the statements $y \in \{x \mid x^2 < 9\}$ and $y \in \{w \mid w^2 < 9\}$ mean the same thing, because they both mean "y is an element of the set of all numbers whose squares are less than 9." In this last statement, all bound variables have been eliminated, and the only variable mentioned is the free variable y.

Note that x is a bound variable in the statement $y \in \{x \mid x^2 < 9\}$ even though it is a free variable in the statement $x^2 < 9$. This last statement is a statement about x that would be true for some values of x and false for others. It is only when this statement is used inside the elementhood test notation that x becomes a bound variable. We could say that the notation $\{x \mid \ldots\}$ *binds* the variable x.

Everything we have said about the set $\{x \mid x^2 < 9\}$ would apply to any set defined by an elementhood test. In general, the statement $y \in \{x \mid P(x)\}$ means the same thing as $P(y)$, which is a statement about y but not x. Similarly, $y \notin \{x \mid P(x)\}$ means the same thing as $\neg P(y)$. Of course, the expression $\{x \mid P(x)\}$ is not a statement at all; it is a name for a set. As you learn more mathematical notation, it will become increasingly important to make sure you are careful to distinguish between expressions that are mathematical statements and expressions that are names for mathematical objects.

Example 1.3.3. What do these statements mean? What are the free variables in each statement?

1. $a + b \notin \{x \mid x \text{ is an even number}\}$.
2. $y \in \{x \mid x \text{ is divisible by } w\}$.
3. $2 \in \{w \mid 6 \notin \{x \mid x \text{ is divisible by } w\}\}$.

Solutions

1. This statement says that $a + b$ is not an element of the set of all even numbers, or in other words, $a + b$ is not an even number. Both a and b are free variables, but x is a bound variable. The statement will be true for some values of a and b and false for others.

2. This statement says that y is divisible by w. Both y and w are free variables, but x is a bound variable. The statement is true for some values of y and w and false for others.
3. This looks quite complicated, but if we go a step at a time, we can decipher it. First, note that the statement $6 \notin \{x \mid x \text{ is divisible by } w\}$, which appears inside the given statement, means the same thing as "6 is not divisible by w." Substituting this into the given statement, we find that the original statement is equivalent to the simpler statement $2 \in \{w \mid 6 \text{ is not divisible by } w\}$. But this just means the same thing as "6 is not divisible by 2." Thus, the statement has no free variables, and both x and w are bound variables. Because there are no free variables, the truth value of the statement doesn't depend on the values of any variables. In fact, since 6 *is* divisible by 2, the statement is false.

Perhaps you have guessed by now how we can use set theory to help us understand truth values of statements containing free variables. As we have seen, a statement, say $P(x)$, containing a free variable x, may be true for some values of x and false for others. To distinguish the values of x that make $P(x)$ true from those that make it false, we could form the set of values of x for which $P(x)$ is true. We will call this set the *truth set* of $P(x)$.

Definition 1.3.4. The *truth set* of a statement $P(x)$ is the set of all values of x that make the statement $P(x)$ true. In other words, it is the set defined by using the statement $P(x)$ as an elementhood test:

$$\text{Truth set of } P(x) = \{x \mid P(x)\}.$$

Note that we have defined truth sets only for statements containing *one* free variable. We will discuss truth sets for statements with more than one free variable in Chapter 4.

Example 1.3.5. What are the truth sets of the following statements?

1. Shakespeare wrote x.
2. n is an even prime number.

Solutions

1. $\{x \mid \text{Shakespeare wrote } x\} = \{\text{Hamlet, Macbeth, Twelfth Night}, \ldots\}$.
2. $\{n \mid n \text{ is an even prime number}\}$. Because the only even prime number is 2, this is the set $\{2\}$. Note that 2 and $\{2\}$ are not the same thing! The first is a number, and the second is a set whose only element is a number. Thus, $2 \in \{2\}$, but $2 \neq \{2\}$.

Suppose A is the truth set of a statement $P(x)$. According to the definition of truth set, this means that $A = \{x \mid P(x)\}$. We've already seen that for any object y, the statement $y \in \{x \mid P(x)\}$ means the same thing as $P(y)$. Substituting in A for $\{x \mid P(x)\}$, it follows that $y \in A$ means the same thing as $P(y)$. Thus, we see that in general, if A is the truth set of $P(x)$, then to say that $y \in A$ means the same thing as saying $P(y)$.

When a statement contains free variables, it is often clear from context that these variables stand for objects of a particular kind. The set of all objects of this kind – in other words, the set of all possible values for the variables – is called the *universe of discourse* for the statement, and we say that the variables *range over* this universe. For example, in most contexts the universe for the statement $x^2 < 9$ would be the set of all real numbers; the universe for the statement "x is a man" might be the set of all people.

Certain sets come up often in mathematics as universes of discourse, and it is convenient to have fixed names for them. Here are a few of the most important ones:

> $\mathbb{R} = \{x \mid x \text{ is a real number}\}$.
> $\mathbb{Q} = \{x \mid x \text{ is a rational number}\}$.
> (Recall that a *real* number is any number on the number line, and a *rational* number is a number that can be written as a fraction p/q, where p and q are integers.)
> $\mathbb{Z} = \{x \mid x \text{ is an integer}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
> $\mathbb{N} = \{x \mid x \text{ is a natural number}\} = \{0, 1, 2, 3, \ldots\}$.
> (Some books include 0 as a natural number and some don't. In this book, we consider 0 to be a natural number.)

The letters $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ can be followed by a superscript $+$ or $-$ to indicate that only positive or negative numbers are to be included in the set. For example, $\mathbb{R}^+ = \{x \mid x \text{ is a positive real number}\}$, and $\mathbb{Z}^- = \{x \mid x \text{ is a negative integer}\}$.

Although the universe of discourse can usually be determined from context, it is sometimes useful to identify it explicitly. Consider a statement $P(x)$ with a free variable x that ranges over a universe U. Although we have written the truth set of $P(x)$ as $\{x \mid P(x)\}$, if there were any possibility of confusion about what the universe was, we could specify it explicitly by writing $\{x \in U \mid P(x)\}$; this is read "the set of all x in U such that $P(x)$." This notation indicates that only elements of U are to be considered for elementhood in this truth set, and among elements of U, only those that pass the elementhood test $P(x)$ will actually be in the truth set. For example, consider again the statement $x^2 < 9$. If the universe of discourse for this statement were the set of all real numbers, then its truth set would be $\{x \in \mathbb{R} \mid x^2 < 9\}$, or in other words, the set of all real numbers between -3 and 3. But if the universe were the set of all integers, then the truth set would be $\{x \in \mathbb{Z} \mid x^2 < 9\} = \{-2, -1, 0, 1, 2\}$. Thus, for example, $1.58 \in \{x \in \mathbb{R} \mid x^2 < 9\}$ but $1.58 \notin \{x \in \mathbb{Z} \mid x^2 < 9\}$. Clearly, the choice of universe can sometimes make a difference!

Sometimes this explicit notation is used not to specify the universe of discourse but to restrict attention to just a part of the universe. For example, in the case of the statement $x^2 < 9$, we might want to consider the universe of discourse to be the set of all real numbers, but in the course of some reasoning involving this statement we might want to temporarily restrict our attention to only positive real numbers. We might then be interested in the set $\{x \in \mathbb{R}^+ \mid x^2 < 9\}$. As before, this notation indicates that only positive real numbers will be considered for elementhood in this set, and among positive real numbers, only those whose square is less than 9 will be in the set. Thus, for a number to be an element of this set, it must pass two tests: it must be a positive real number, and its square must be less than 9. In other words, the statement $y \in \{x \in \mathbb{R}^+ \mid x^2 < 9\}$ means the same thing as $y \in \mathbb{R}^+ \wedge y^2 < 9$. In general, $y \in \{x \in A \mid P(x)\}$ means the same thing as $y \in A \wedge P(y)$.

When a new mathematical concept has been defined, mathematicians are usually interested in studying any possible extremes of this concept. For example, when we discussed truth tables, the extremes we studied were statements whose truth tables contained only T's (tautologies) or only F's (contradictions). For the concept of the truth set of a statement containing a free variable, the corresponding extremes would be the truth sets of statements that are always true or always false. Suppose $P(x)$ is a statement containing a free variable x that ranges over a universe U. It should be clear that if $P(x)$ comes out true for every value of $x \in U$, then the truth set of $P(x)$ will be the whole universe U. For example, since the statement $x^2 \geq 0$ is true for every real number x, the truth set of this statement is $\{x \in \mathbb{R} \mid x^2 \geq 0\} = \mathbb{R}$. Of course, this is not unrelated to the concept of a tautology. For example, since $P \vee \neg P$ is a tautology, the statement $P(x) \vee \neg P(x)$ will be true for every $x \in U$, no matter what statement $P(x)$ stands for or what the universe U is, and therefore the truth set of the statement $P(x) \vee \neg P(x)$ will be U.

For a statement $P(x)$ that is false for every possible value of x, nothing in the universe can pass the elementhood test for the truth set of $P(x)$, and so this truth set must have no elements. The idea of a set with no elements may sound strange, but it arises naturally when we consider truth sets for statements that are always false. Because a set is completely determined once its elements have been specified, there is only one set that has no elements. It is called the *empty set*, or the *null set*, and is often denoted $\varnothing$. For example, $\{x \in \mathbb{Z} \mid x \neq x\} = \varnothing$.

Since the empty set has no elements, the statement $x \in \varnothing$ is an example of a statement that is always false, no matter what x is.

Another common notation for the empty set is based on the fact that any set can be named by listing its elements between braces. Since the empty set has no elements, we write nothing between the braces, like this: $\varnothing = \{\ \}$. Note that $\{\varnothing\}$ is not correct notation for the empty set. Just as we saw earlier that 2 and $\{2\}$ are not the same thing, $\varnothing$ is not the same as $\{\varnothing\}$. The first is a set with no elements, whereas the second is a set with one element, that one element being $\varnothing$, the empty set.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) 3 is a common divisor of 6, 9, and 15. (Note: You did this in exercise 2 of Section 1.1, but you should be able to give a better answer now.)
   (b) x is divisible by both 2 and 3 but not 4.
   (c) x and y are natural numbers, and exactly one of them is prime.

2. Analyze the logical forms of the following statements:
   (a) x and y are men, and either x is taller than y or y is taller than x.
   (b) Either x or y has brown eyes, and either x or y has red hair.
   (c) Either x or y has both brown eyes and red hair.

*3. Write definitions using elementhood tests for the following sets:
   (a) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}.
   (b) {Brown, Columbia, Cornell, Dartmouth, Harvard, Princeton, University of Pennsylvania, Yale}.
   (c) {Alabama, Alaska, Arizona, ..., Wisconsin, Wyoming}.
   (d) {Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland and Labrador, Northwest Territories, Nova Scotia, Nunavut, Ontario, Prince Edward Island, Quebec, Saskatchewan, Yukon}.

4. Write definitions using elementhood tests for the following sets:
   (a) $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$.
   (b) $\{1, 2, 4, 8, 16, 32, 64, \ldots\}$.
   (c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.

*5. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.
   (a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\}$.
   (b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\}$.
   (c) $5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\}$.

Sometimes it is helpful when working with operations on sets to draw pictures of the results of these operations. One way to do this is with diagrams like that in Figure 1. This is called a *Venn diagram*. The interior of the rectangle enclosing the diagram represents the universe of discourse U, and the interiors of the two circles represent the two sets A and B. Other sets formed by combining these sets would be represented by different regions in the diagram. For example, the shaded region in Figure 2 is the region common to the circles representing A and B, and so it represents the set $A \cap B$. Figures 3 and 4 show the regions representing $A \cup B$ and $A \setminus B$, respectively.

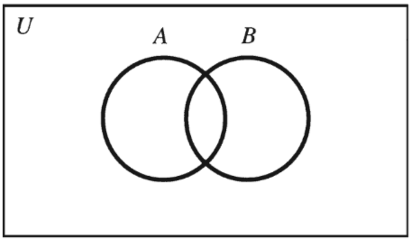
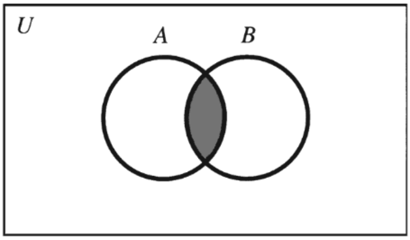
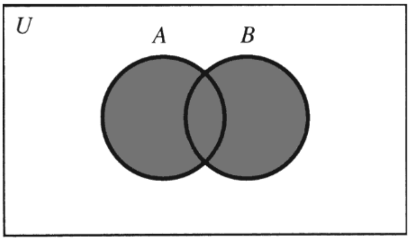


Figure 1



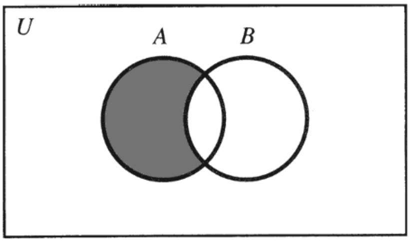
$A \cap B$

Figure 2



$A \cup B$

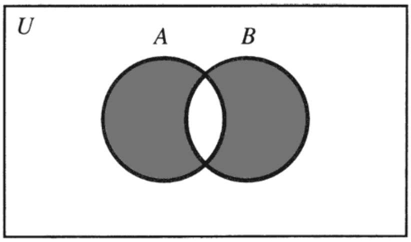
Figure 3



$A \setminus B$

Figure 4

Here's an example of how Venn diagrams can help us understand operations on sets. In Example 1.4.2 the sets $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ turned out to be equal, for a particular choice of A and B. You can see by making Venn diagrams for both sets that this was not a coincidence. You'll find that both Venn diagrams look like Figure 5. Thus, these sets will always be equal, no matter what the sets A and B are, because both sets will always be the set of objects that are elements of either A or B but not both. This set is called the *symmetric difference* of A and B and is written $A \triangle B$. In other words, $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. Later in this section we'll see another explanation of why these sets are always equal.



$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Figure 5

Let's return now to the question with which we began this section. If A is the truth set of a statement $P(x)$ and B is the truth set of $Q(x)$, then, as we saw in the last section, $x \in A$ means the same thing as $P(x)$ and $x \in B$ means the same thing as $Q(x)$. Thus, the truth set of $P(x) \wedge Q(x)$ is $\{x \mid P(x) \wedge Q(x)\} = \{x \mid x \in A \wedge x \in B\} = A \cap B$. This should make sense. It just says that the truth set of $P(x) \wedge Q(x)$ consists of those elements that the truth sets of $P(x)$ and $Q(x)$ have in common – in other words, the values of x that make both $P(x)$ and $Q(x)$ come out true. We have already seen an example of this. In Example 1.4.3 the sets A and B were the truth sets of the statements "x is a man" and "x has brown hair," and $A \cap B$ turned out to be the truth set of "x is a man and x has brown hair."

Similar reasoning shows that the truth set of $P(x) \vee Q(x)$ is $A \cup B$. To find the truth set of $\neg P(x)$, we need to talk about the universe of discourse U. The truth set of $\neg P(x)$ will consist of those elements of the universe for which $P(x)$ is false, and we can find this set by starting with U and removing from it those elements for which $P(x)$ is true. Thus, the truth set of $\neg P(x)$ is $U \setminus A$.

These observations about truth sets illustrate the fact that the set theory operations $\cap$, $\cup$, and $\setminus$ are related to the logical connectives $\wedge$, $\vee$, and $\neg$. This shouldn't be surprising, since after all the words *and*, *or*, and *not* appear in their definitions. (The word *not* doesn't appear explicitly, but it's there, hidden in the mathematical symbol $\notin$ in the definition of the difference of two sets.) It is important to remember, though, that although the set theory operations and logical connectives are related, they are not interchangeable. The logical connectives can only be used to combine *statements*, whereas the set theory operations must be used to combine *sets*. For example, if A is the truth set of $P(x)$ and B is the truth set of $Q(x)$, then we can say that $A \cap B$ is the truth set of $P(x) \wedge Q(x)$, but expressions such as $A \wedge B$ or $P(x) \cap Q(x)$ are completely meaningless and should never be used.

The relationship between set theory operations and logical connectives also becomes apparent when we analyze the logical forms of statements about intersections, unions, and differences of sets. For example, according to the definition of intersection, to say that $x \in A \cap B$ means that $x \in A \wedge x \in B$. Similarly, to say that $x \in A \cup B$ means that $x \in A \vee x \in B$, and $x \in A \setminus B$ means $x \in A \wedge x \notin B$, or in other words $x \in A \wedge \neg(x \in B)$. We can combine these rules when analyzing statements about more complex sets.

Example 1.4.4. Analyze the logical forms of the following statements:

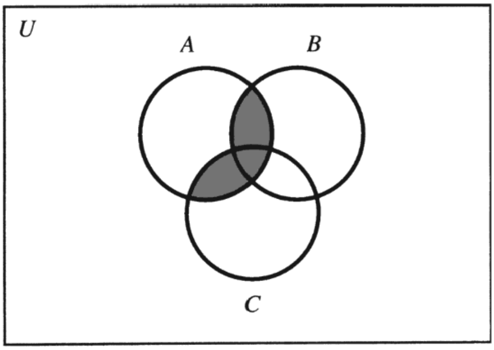
1. $x \in A \cap (B \cup C)$.
2. $x \in A \setminus (B \cap C)$.
3. $x \in (A \cap B) \cup (A \cap C)$.

Solutions

1. $x \in A \cap (B \cup C)$
   is equivalent to $x \in A \wedge x \in (B \cup C)$ (definition of $\cap$),
   which is equivalent to $x \in A \wedge (x \in B \vee x \in C)$ (definition of $\cup$).
2. $x \in A \setminus (B \cap C)$
   is equivalent to $x \in A \wedge \neg(x \in B \cap C)$ (definition of $\setminus$),
   which is equivalent to $x \in A \wedge \neg(x \in B \wedge x \in C)$ (definition of $\cap$).
3. $x \in (A \cap B) \cup (A \cap C)$
   is equivalent to $x \in (A \cap B) \vee x \in (A \cap C)$ (definition of $\cup$),
   which is equivalent to $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$
   (definition of $\cap$).

Look again at the solutions to parts 1 and 3 of Example 1.4.4. You should recognize that the statements we ended up with in these two parts are equivalent. (If you don't, look back at the distributive laws in Section 1.2.) This equivalence means that the statements $x \in A \cap (B \cup C)$ and $x \in (A \cap B) \cup (A \cap C)$ are equivalent. In other words, the objects that are elements of the set $A \cap (B \cup C)$ will be precisely the same as the objects that are elements of $(A \cap B) \cup (A \cap C)$, no matter what the sets A, B, and C are. But recall that sets with the same elements are equal, so it follows that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Another way to see this is with the Venn diagram in Figure 6. Our earlier Venn diagrams had two circles, because in previous examples only two sets were being combined. This Venn diagram has three circles, which represent the three sets A, B, and C that are being combined in this case. Although it is possible to create Venn diagrams for more than three sets, it is rarely done, because it cannot be done with overlapping circles. For more on Venn diagrams for more than three sets, see exercise 10.

Thus, we see that a distributive law for logical connectives has led to a distributive law for set theory operations. You might guess that because there



$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Figure 6

were *two* distributive laws for the logical connectives, with $\wedge$ and $\vee$ playing opposite roles in the two laws, there might be two distributive laws for set theory operations too. The second distributive law for sets should say that for any sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. You can verify this for yourself by writing out the statements $x \in A \cup (B \cap C)$ and $x \in (A \cup B) \cap (A \cup C)$ using logical connectives and verifying that they are equivalent, using the second distributive law for the logical connectives $\wedge$ and $\vee$. Another way to see it is to make a Venn diagram.

We can derive another set theory identity by finding a statement equivalent to the statement we ended up with in part 2 of Example 1.4.4:

$x \in A \setminus (B \cap C)$
is equivalent to $x \in A \wedge \neg(x \in B \wedge x \in C)$ (Example 1.4.4),
which is equivalent to $x \in A \wedge (x \notin B \vee x \notin C)$ (DeMorgan's law),
which is equivalent to $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$
(distributive law),
which is equivalent to $(x \in A \setminus B) \vee (x \in A \setminus C)$ (definition of $\setminus$),
which is equivalent to $x \in (A \setminus B) \cup (A \setminus C)$ (definition of $\cup$).

Thus, we have shown that for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. Once again, you can verify this with a Venn diagram as well.

Earlier we promised an alternative way to check the identity $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. You should see now how this can be done. First, we write out the logical forms of the statements $x \in (A \cup B) \setminus (A \cap B)$ and $x \in (A \setminus B) \cup (B \setminus A)$:

$$x \in (A \cup B) \setminus (A \cap B) \text{ means } (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B);$$
$$x \in (A \setminus B) \cup (B \setminus A) \text{ means } (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A).$$

You can now check, using equivalences from Section 1.2, that these statements are equivalent. An alternative way to check the equivalence is with a truth table. To simplify the truth table, let's use P and Q as abbreviations for the statements $x \in A$ and $x \in B$. Then we must check that the formulas $(P \vee Q) \wedge \neg(P \wedge Q)$ and $(P \wedge \neg Q) \vee (Q \wedge \neg P)$ are equivalent. The truth table in Figure 7 shows this.

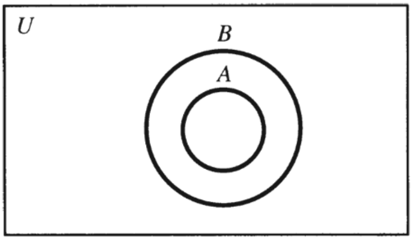
P	Q	$(P \vee Q) \wedge \neg(P \wedge Q)$	$(P \wedge \neg Q) \vee (Q \wedge \neg P)$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	F

Figure 7

Definition 1.4.5. Suppose A and B are sets. We will say that A is a *subset* of B if every element of A is also an element of B. We write $A \subseteq B$ to mean that A is a subset of B. A and B are said to be *disjoint* if they have no elements in common. Note that this is the same as saying that the set of elements they have in common is the empty set, or in other words $A \cap B = \varnothing$.

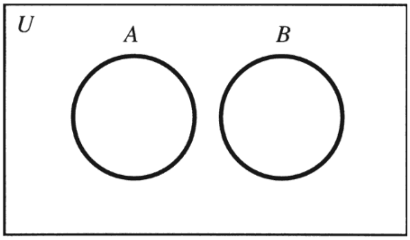
Example 1.4.6. Suppose $A = \{\text{red, green}\}$, $B = \{\text{red, yellow, green, purple}\}$, and $C = \{\text{blue, purple}\}$. Then the two elements of A, red and green, are both also in B, and therefore $A \subseteq B$. Also, $A \cap C = \varnothing$, so A and C are disjoint.

If we know that $A \subseteq B$, or that A and B are disjoint, then we might draw a Venn diagram for A and B differently to reflect this. Figures 8 and 9 illustrate this.



$A \subseteq B$

Figure 8

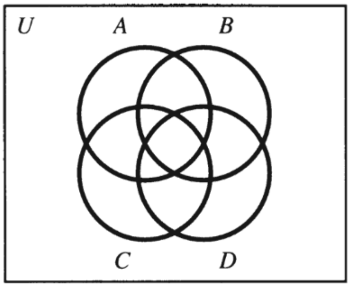


$A \cap B = \varnothing$

Figure 9

Just as we earlier derived identities showing that certain sets are always equal, it is also sometimes possible to show that certain sets are always disjoint, or that one set is always a subset of another. For example, you can see in a Venn

(a) What's wrong with the following diagram? (Hint: Where's the set $(A \cap D) \setminus (B \cup C)$?)



(b) Can you make a Venn diagram for four sets using shapes other than circles?

11. (a) Make Venn diagrams for the sets $(A \cup B) \setminus C$ and $A \cup (B \setminus C)$. What can you conclude about whether one of these sets is necessarily a subset of the other?
    (b) Give an example of sets A, B, and C for which $(A \cup B) \setminus C \neq A \cup (B \setminus C)$.

*12. Use Venn diagrams to show that the associative law holds for symmetric difference; that is, for any sets A, B, and C, $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

13. Use any method you wish to verify the following identities:
    (a) $(A \triangle B) \cup C = (A \cup C) \triangle (B \setminus C)$.
    (b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.
    (c) $(A \triangle B) \setminus C = (A \setminus C) \triangle (B \setminus C)$.

14. Use any method you wish to verify the following identities:
    (a) $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$.
    (b) $(A \cap B) \triangle C = (A \triangle C) \triangle (A \setminus B)$.
    (c) $(A \setminus B) \triangle C = (A \triangle C) \triangle (A \cap B)$.

15. Fill in the blanks to make true identities:
    (a) $(A \triangle B) \cap C = (C \setminus A) \triangle$ ________.
    (b) $C \setminus (A \triangle B) = (A \cap C) \triangle$ ________.
    (c) $(B \setminus A) \triangle C = (A \triangle C) \triangle$ ________.

1.5. The Conditional and Biconditional Connectives

It is time now to return to a question we left unanswered in Section 1.1. We have seen how the reasoning in the first and third arguments in Example 1.1.1 can be understood by analyzing the connectives $\vee$ and $\neg$. But what about the reasoning in the second argument? Recall that the argument went like this:

> If today is Sunday, then I don't have to go to work today.
> Today is Sunday.
> Therefore, I don't have to go to work today.

What makes this reasoning valid?

It appears that the crucial words here are *if* and *then*, which occur in the first premise. We therefore introduce a new logical connective, $\rightarrow$, and write $P \rightarrow Q$ to represent the statement "If P then Q." This statement is sometimes called a *conditional* statement, with P as its *antecedent* and Q as its *consequent*. If we let P stand for the statement "Today is Sunday" and Q for the statement "I don't have to go to work today," then the logical form of the argument would be

$$\begin{array}{l} P \rightarrow Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

Our analysis of the new connective $\rightarrow$ should lead to the conclusion that this argument is valid.

Example 1.5.1. Analyze the logical forms of the following statements:

1. If it's raining and I don't have my umbrella, then I'll get wet.
2. If Mary did her homework, then the teacher won't collect it, and if she didn't, then he'll ask her to do it on the board.

Solutions

1. Let R stand for the statement "It's raining," U for "I have my umbrella," and W for "I'll get wet." Then statement 1 would be represented by the formula $(R \wedge \neg U) \rightarrow W$.
2. Let H stand for "Mary did her homework," C for "The teacher will collect it," and B for "The teacher will ask Mary to do the homework on the board." Then the given statement means $(H \rightarrow \neg C) \wedge (\neg H \rightarrow B)$.

To analyze arguments containing the connective $\rightarrow$ we must work out the truth table for the formula $P \rightarrow Q$. Because $P \rightarrow Q$ is supposed to mean that if P is true then Q is also true, we certainly want to say that if P is true and Q is false then $P \rightarrow Q$ is false. If P is true and Q is also true, then it seems reasonable to say that $P \rightarrow Q$ is true. This gives us the last two lines of the truth table in Figure 1. The remaining two lines of the truth table are harder to fill in, although some people might say that if P and Q are both false then

$P \to Q$ should be considered true. Thus, we can sum up our conclusions so far with the table in Figure 1.

P	Q	$P \to Q$
F	F	T?
F	T	?
T	F	F
T	T	T

Figure 1

To help us fill in the undetermined lines in this truth table, let's look at an example. Consider the statement "If $x > 2$ then $x^2 > 4$," which we could represent with the formula $P(x) \to Q(x)$, where $P(x)$ stands for the statement $x > 2$ and $Q(x)$ stands for $x^2 > 4$. Of course, the statements $P(x)$ and $Q(x)$ contain x as a free variable, and each will be true for some values of x and false for others. But surely, no matter what the value of x is, we would say it is true that *if* $x > 2$ then $x^2 > 4$, so the conditional statement $P(x) \to Q(x)$ should be true. Thus, the truth table should be completed in such a way that no matter what value we plug in for x, this conditional statement comes out true.

For example, suppose $x = 3$. In this case $x > 2$ and $x^2 = 9 > 4$, so $P(x)$ and $Q(x)$ are both true. This corresponds to line four of the truth table in Figure 1, and we've already decided that the statement $P(x) \to Q(x)$ should come out true in this case. But now consider the case $x = 1$. Then $x < 2$ and $x^2 = 1 < 4$, so $P(x)$ and $Q(x)$ are both false, corresponding to line one in the truth table. We have tentatively placed a T in this line of the truth table, and now we see that this tentative choice must be right. If we put an F there, then the statement $P(x) \to Q(x)$ would come out false in the case $x = 1$, and we've already decided that it should be true for all values of x.

Finally, consider the case $x = -5$. Then $x < 2$, so $P(x)$ is false, but $x^2 = 25 > 4$, so $Q(x)$ is true. Thus, in this case we find ourselves in the second line of the truth table, and once again, if the conditional statement $P(x) \to Q(x)$ is to be true in this case, we must put a T in this line. So it appears that all the questionable lines in the truth table in Figure 1 must be filled in with T's, and the completed truth table for the connective $\to$ must be as shown in Figure 2.

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Figure 2

Of course, there are many other values of x that could be plugged into our statement "If $x > 2$ then $x^2 > 4$"; but if you try them, you'll find that they all lead to line one, two, or four of the truth table, as our examples $x = 1, -5$, and 3 did. No value of x will lead to line three, because you could never have $x > 2$ but $x^2 \leq 4$. After all, that's why we said that the statement "If $x > 2$ then $x^2 > 4$" was always true, no matter what x was! The point of saying that this conditional statement is always true is simply to say that you will never find a value of x such that $x > 2$ and $x^2 \leq 4$ – in other words, there is no value of x for which $P(x)$ is true but $Q(x)$ is false. Thus, it should make sense that in the truth table for $P \to Q$, the only line that is false is the line in which P is true and Q is false.

As the truth table in Figure 3 shows, the formula $\neg P \lor Q$ is also true in every case except when P is true and Q is false. Thus, if we accept the truth table in Figure 2 as the correct truth table for the formula $P \to Q$, then we will be forced to accept the conclusion that the formulas $P \to Q$ and $\neg P \lor Q$ are equivalent. Is this consistent with the way the words *if* and *then* are used in ordinary language? It may not seem to be at first, but, at least for some uses of the words *if* and *then*, it is.

P	Q	$\neg P \lor Q$
F	F	T
F	T	T
T	F	F
T	T	T

Figure 3

For example, imagine a teacher saying to a class, in a threatening tone of voice, "You won't neglect your homework, or you'll fail the course." Grammatically, this statement has the form $\neg P \lor Q$, where P is the statement "You will neglect your homework" and Q is "You'll fail the course." But what message is the teacher trying to convey with this statement? Clearly the intended message is "If you neglect your homework, then you'll fail the course," or in other words $P \to Q$. Thus, in this example, the statements $\neg P \lor Q$ and $P \to Q$ seem to mean the same thing.

There is a similar idea at work in the first statement from Example 1.1.2, "Either John went to the store, or we're out of eggs." In Section 1.1 we represented this statement by the formula $P \lor Q$, with P standing for "John went to the store" and Q for "We're out of eggs." But someone who made this statement would probably be trying to express the idea that if John didn't go to the store, then we're out of eggs, or in other words $\neg P \to Q$. Thus, this example suggests that $\neg P \to Q$ means the same thing as $P \lor Q$. In fact, we can derive this equivalence from the previous one by substituting $\neg P$ for P. Because $P \to Q$

is equivalent to $\neg P \lor Q$, it follows that $\neg P \to Q$ is equivalent to $\neg\neg P \lor Q$, which is equivalent to $P \lor Q$ by the double negation law.

We can derive another useful equivalence as follows:

$$\begin{aligned} \neg P \lor Q \quad & \text{is equivalent to } \neg P \lor \neg\neg Q \quad \text{(double negation law)}, \\ & \text{which is equivalent to } \neg(P \land \neg Q) \quad \text{(DeMorgan's law)}. \end{aligned}$$

Thus, $P \to Q$ is also equivalent to $\neg(P \land \neg Q)$. In fact, this is precisely the conclusion we reached earlier when discussing the statement "If $x > 2$ then $x^2 > 4$." We decided then that the reason this statement is true for every value of x is that there is no value of x for which $x > 2$ and $x^2 \leq 4$. In other words, the statement $P(x) \land \neg Q(x)$ is never true, where as before $P(x)$ stands for $x > 2$ and $Q(x)$ for $x^2 > 4$. But that's the same as saying that the statement $\neg(P(x) \land \neg Q(x))$ is always true. Thus, to say that $P(x) \to Q(x)$ is always true means the same thing as saying that $\neg(P(x) \land \neg Q(x))$ is always true.

For another example of this equivalence, consider the statement "If it's going to rain, then I'll take my umbrella." Of course, this statement has the form $P \to Q$, where P stands for the statement "It's going to rain" and Q stands for "I'll take my umbrella." But we could also think of this statement as a declaration that I won't be caught in the rain without my umbrella – in other words, $\neg(P \land \neg Q)$.

To summarize, so far we have discovered the following equivalences involving conditional statements:

Conditional laws

$$P \to Q \text{ is equivalent to } \neg P \lor Q.$$

$$P \to Q \text{ is equivalent to } \neg(P \land \neg Q).$$

In case you're still not convinced that the truth table in Figure 2 is right, we give one more reason. We know that, using this truth table, we can now analyze the validity of deductive arguments involving the words *if* and *then*. We'll find, when we analyze a few simple arguments, that the truth table in Figure 2 leads to reasonable conclusions about the validity of these arguments. But if we were to make any changes in the truth table, we would end up with conclusions that are clearly incorrect. For example, let's return to the argument form with which we started this section:

$$\begin{array}{l} P \to Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

We have already decided that this form of argument should be valid, and the truth table in Figure 4 confirms this. The premises are both true only in line four of the table, and in this line the conclusion is true as well.

3. If the game has been canceled, then it's either raining or snowing.
4. If it's raining then the game has been canceled, and if it's snowing then the game has been canceled.
5. If it's neither raining nor snowing, then the game hasn't been canceled.

Solution

We translate all of the statements into the notation of logic, using the following abbreviations: R stands for the statement "It's raining," S stands for "It's snowing," and C stands for "The game has been canceled."

1. $(R \lor S) \to C$.
2. $\neg C \to (\neg R \land \neg S)$. By one of DeMorgan's laws, this is equivalent to $\neg C \to \neg(R \lor S)$. This is the contrapositive of statement 1, so they are equivalent.
3. $C \to (R \lor S)$. This is the converse of statement 1, which is *not* equivalent to it. You can verify this with a truth table, or just think about what the statements mean. Statement 1 says that rain or snow would result in cancelation of the game. Statement 3 says that these are the *only* circumstances in which the game will be canceled.
4. $(R \to C) \land (S \to C)$. This is also equivalent to statement 1, as the following reasoning shows:

   $(R \to C) \land (S \to C)$
   is equivalent to $(\neg R \lor C) \land (\neg S \lor C)$ (conditional law),
   which is equivalent to $(\neg R \land \neg S) \lor C$ (distributive law),
   which is equivalent to $\neg(R \lor S) \lor C$ (DeMorgan's law),
   which is equivalent to $(R \lor S) \to C$ (conditional law).

   You should read statements 1 and 4 again and see if it makes sense to you that they're equivalent.
5. $\neg(R \lor S) \to \neg C$. This is the contrapositive of statement 3, so they are equivalent. It is not equivalent to statements 1, 2, and 4.

Statements that mean $P \to Q$ come up very often in mathematics, but sometimes they are not written in the form "If P then Q." Here are a few other ways of expressing the idea $P \to Q$ that are used often in mathematics:

P implies Q.
Q, if P.
P only if Q.
P is a sufficient condition for Q.
Q is a necessary condition for P.