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How to Solve It

a new aspect of mathematical method

With a new foreword by John Conway

G. POLYA

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† Contains only cross-references.

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HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First.

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

Foreword
by John H. Conway

How to Solve It is a wonderful book! This I realized when I first read right through it as a student many years ago, but it has taken me a long time to appreciate just *how* wonderful it is. Why is that? One part of the answer is that the book is unique. In all my years as a student and teacher, I have never seen another that lives up to George Polya's title by teaching you how to go about solving problems. A. H. Schoenfeld correctly described its importance in his 1987 article "Polya, Problem Solving, and Education" in *Mathematics Magazine*: "For mathematics education and the world of problem solving it marked a line of demarcation between two eras, problem solving before and after Polya."

It is one of the most successful mathematics books ever written, having sold over a million copies and been translated into seventeen languages since it first appeared in 1945. Polya later wrote two more books about the art of doing mathematics, *Mathematics and Plausible Reasoning* (1954) and *Mathematical Discovery* (two volumes, 1962 and 1965).

The book's title makes it seem that it is directed only toward students, but in fact it is addressed just as much to their teachers. Indeed, as Polya remarks in his introduction, the first part of the book takes the teacher's viewpoint more often than the student's.

Everybody gains that way. The student who reads the book on his own will find that overhearing Polya's comments to his non-existent teacher can bring that desirable person into being, as an imaginary but very helpful figure leaning over one's shoulder. This is what happened to me, and naturally I made heavy use of the remarks I'd found most important when I myself started teaching a few years later.

But it was some time before I read the book again, and when I did, I suddenly realized that it was even more valuable than I'd thought! Many of Polya's remarks that hadn't helped me as a student now made me a better teacher of those whose problems had differed from mine. Polya had met many more students than I had, and had obviously thought very hard about how to best help all of them learn mathematics. Perhaps his most important point is that learning must be active. As he said in a lecture on teaching, "Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean [to be] doing mathematics? In the first place, it means to be able to solve mathematical problems."

It is often said that to teach any subject well, one has to understand it "at least as well as one's students do." It is a paradoxical truth that to teach mathematics well, one must also know how to misunderstand it at least to the extent one's students do! If a teacher's statement can be parsed in two or more ways, it goes without saying that some students will understand it one way and others another, with results that can vary from the hilarious to the tragic. J. E. Littlewood gives two amusing examples of assumptions that can easily be made unconsciously and misleadingly. First, he remarks that the description of the coordinate axes ("Ox and Oy as in 2 dimensions, Oz vertical") in Lamb's book *Mechanics* is incorrect for him, since he always worked in an armchair with his feet up! Then, after asking how his reader would present the picture of a closed curve lying all on one side of its tangent, he states that there are four main schools (to left or right of vertical tangent, or above or below horizontal one) and that by lecturing without a figure, presuming that the curve was to the right of its vertical tangent, he had unwittingly made nonsense for the other three schools.

I know of no better remedy for such presumptions than Polya's counsel: before trying to solve a problem, the student should demonstrate his or her understanding of its statement, preferably to a real teacher, but in lieu of that, to an imagined one. Experienced mathematicians know that often the hardest part of researching a problem is understanding precisely what that problem says. They often follow Polya's wise advice: "If you can't solve a problem, then there is an easier problem you can't solve: find it."

Readers who learn from this book will also want to learn about its author's life.[1]

George Polya was born György Pólya (he dropped the accents sometime later) on December 13, 1887, in Budapest, Hungary, to Jakab Pólya and his wife, the former Anna Deutsch. He was baptized into the Roman Catholic faith, to which Jakab, Anna, and their three previous children, Jenő, Ilona, and Flóra, had converted from Judaism in the previous year. Their fifth child, László, was born four years later.

Jakab had changed his surname from Pollák to the more Hungarian-sounding Pólya five years before György was born, believing that this might help him obtain a university post, which he eventually did, but only shortly before his untimely death in 1897.

At the Dániel Berzsenyi Gymnasium, György studied Greek, Latin, and German, in addition to Hungarian. It is surprising to learn that there he was seemingly uninterested in mathematics, his work in geometry deemed merely "satisfactory" compared with his "outstanding" performance in literature, geography, and other subjects. His favorite subject, outside of literature, was biology.

He enrolled at the University of Budapest in 1905, initially studying law, which he soon dropped because he found it too boring. He then obtained the certification needed to teach Latin and Hungarian at a gymnasium, a

[1]The following biographical information is taken from that given by J. J. O'Connor and E. F. Robertson in the MacTutor History of Mathematics Archive (www-gap.dcs.st-and.ac.uk/~history/).

certification that he never used but of which he remained proud. Eventually his professor, Bernát Alexander, advised him that to help his studies in philosophy, he should take some mathematics and physics courses. This was how he came to mathematics. Later, he joked that he "wasn't good enough for physics, and was too good for philosophy—mathematics is in between."

In Budapest he was taught physics by Eötvös and mathematics by Fejér and was awarded a doctorate after spending the academic year 1910–11 in Vienna, where he took some courses by Wirtinger and Mertens. He spent much of the next two years in Göttingen, where he met many more mathematicians—Klein, Caratheodory, Hilbert, Runge, Landau, Weyl, Hecke, Courant, and Toeplitz—and in 1914 visited Paris, where he became acquainted with Picard and Hadamard and learned that Hurwitz had arranged an appointment for him in Zürich. He accepted this position, writing later: "I went to Zürich in order to be near Hurwitz, and we were in close touch for about six years, from my arrival in Zürich in 1914 to his passing [in 1919]. I was very much impressed by him and edited his works."

Of course, the First World War took place during this period. It initially had little effect on Polya, who had been declared unfit for service in the Hungarian army as the result of a soccer wound. But later when the army, more desperately needing recruits, demanded that he return to fight for his country, his strong pacifist views led him to refuse. As a consequence, he was unable to visit Hungary for many years, and in fact did not do so until 1967, fifty-four years after he left.

In the meantime, he had taken Swiss citizenship and married a Swiss girl, Stella Vera Weber, in 1918. Between 1918 and 1919, he published papers on a wide range of mathematical subjects, such as series, number theory, combinatorics, voting systems, astronomy, and probability. He was made an extraordinary professor at the Zürich ETH in 1920, and a few years later he and Gábor Szegő published their book *Aufgaben und Lehrsatze aus der Analysis* ("Problems and Theorems in Analysis"), described by G. L. Alexanderson and L. H. Lange in their obituary of Polya as "a mathematical masterpiece that assured their reputations."

That book appeared in 1925, after Polya had obtained a Rockefeller Fellowship to work in England, where he collaborated with Hardy and Littlewood on what later became their book *Inequalities* (Cambridge University Press, 1936). He used a second Rockefeller Fellowship to visit Princeton University in 1933, and while in the United States was invited by H. F. Blichfeldt to visit Stanford University, which he greatly enjoyed, and which ultimately became his home. Polya held a professorship at Stanford from 1943 until his retirement in 1953, and it was there, in 1978, that he taught his last course, in combinatorics; he died on September 7, 1985, at the age of ninety-seven.

Some readers will want to know about Polya's many contributions to mathematics. Most of them relate to analysis and are too technical to be understood by non-experts, but a few are worth mentioning.

In probability theory, Polya is responsible for the now-standard term "Central Limit Theorem" and for proving that the Fourier transform of a probability measure is a characteristic function and that a random walk on the integer lattice closes with probability 1 if and only if the dimension is at most 2.

In geometry, Polya independently re-enumerated the seventeen plane crystallographic groups (their first enumeration, by E. S. Fedorov, having been forgotten) and together with P. Niggli devised a notation for them.

In combinatorics, Polya's Enumeration Theorem is now a standard way of counting configurations according to their symmetry. It has been described by R. C. Read as "a

It contains sixty-seven articles arranged alphabetically. For example, the meaning of the term HEURISTIC (set in small capitals) is explained in an article with this title on page 112. When the title of such an article is referred to within the text it will be set in small capitals. Certain paragraphs of a few articles are more technical; they are enclosed in square brackets. Some articles are fairly closely connected with the first part to which they add further illustrations and more specific comments. Other articles go somewhat beyond the aim of the first part of which they explain the background. There is a key-article on MODERN HEURISTIC. It explains the connection of the main articles and the plan underlying the Dictionary; it contains also directions how to find information about particular items of the list. It must be emphasized that there is a common plan and a certain unity, because the articles of the Dictionary show the greatest outward variety. There are a few longer articles devoted to the systematic though condensed discussion of some general theme; others contain more specific comments, still others cross-references, or historical data, or quotations, or aphorisms, or even jokes.

The Dictionary should not be read too quickly; its text is often condensed, and now and then somewhat subtle. The reader may refer to the Dictionary for information about particular points. If these points come from his experience with his own problems or his own students, the reading has a much better chance to be profitable.

The title of the fourth part is "Problems, Hints, Solutions." It proposes a few problems to the more ambitious reader. Each problem is followed (in proper distance) by a "hint" that may reveal a way to the result which is explained in the "solution."

We have mentioned repeatedly the "student" and the "teacher" and we shall refer to them again and again. It may be good to observe that the "student" may be a high school student, or a college student, or anyone else who is studying mathematics. Also the "teacher" may be a high school teacher, or a college instructor, or anyone interested in the technique of teaching mathematics. The author looks at the situation sometimes from the point of view of the student and sometimes from that of the teacher (the latter case is preponderant in the first part). Yet most of the time (especially in the third part) the point of view is that of a person who is neither teacher nor student but anxious to solve the problem before him.

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How to Solve It

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not to "problems to prove." If we have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS TO PROVE.

4. Common sense. The questions and suggestions of our list are general, but, except for their generality, they are natural, simple, obvious, and proceed from plain common sense. Take the suggestion: *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.* This suggestion advises you to do what you would do anyhow, without any advice, if you were seriously concerned with your problem. Are you hungry? You wish to obtain food and you think of familiar ways of obtaining food. Have you a problem of geometric construction? You wish to construct a triangle and you think of familiar ways of constructing a triangle. Have you a problem of any kind? You wish to find a certain unknown, and you think of familiar ways of finding such an unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted from our list. And you are on the right track, too; the suggestion is a good one, it suggests to you a procedure which is very frequently successful.

All the questions and suggestions of our list are natural, simple, obvious, just plain common sense; but they state plain common sense in general terms. They suggest a certain conduct which comes naturally to any person who is seriously concerned with his problem and has some common sense. But the person who behaves the right way usually does not care to express his behavior in clear words and, possibly, he cannot express it so; our list tries to express it so.

5. Teacher and student. Imitation and practice. There are two aims which the teacher may have in view when addressing to his students a question or a suggestion of the list: First, to help the student to solve the problem at hand. Second, to develop the student's ability so that he may solve future problems by himself.

Experience shows that the questions and suggestions of our list, appropriately used, very frequently help the student. They have two common characteristics, common sense and generality. As they proceed from plain common sense they very often come naturally; they could have occurred to the student himself. As they are general, they help unobtrusively; they just indicate a general direction and leave plenty for the student to do.

But the two aims we mentioned before are closely connected; if the student succeeds in solving the problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that our questions are general, applicable in many cases. If the same question is repeatedly helpful, the student will scarcely fail to notice it and he will be induced to ask the question by himself in a similar situation. Asking the question repeatedly, he may succeed once in eliciting the right idea. By such a success, he discovers the right way of using the question, and then he has really assimilated it.

The student may absorb a few questions of our list so well that he is finally able to put to himself the right question in the right moment and to perform the corresponding mental operation naturally and vigorously. Such a student has certainly derived the greatest possible profit from our list. What can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill some interest for problems into their minds and give them plenty of opportunity for imitation and practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions and suggestions of our list, he puts these questions and suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact.

MAIN DIVISIONS, MAIN QUESTIONS

6. Four phases. Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again and again. Our conception of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions and suggestions of our list, we shall distinguish four phases of the work. First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solution, we review and discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea and jumping all preparations blurts out with the solution. Such lucky ideas, of course, are most desirable, but something very undesirable and unfortunate may result if the student leaves out any of the four phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan*. Many mistakes can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may be lost if the student fails to reexamine and to *reconsider* the completed solution.

7. Understanding the problem. It is foolish to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish and sad things often happen, in and out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation.

First of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement, and the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principal parts of the problem attentively, repeatedly, and from various sides. If there is a figure connected with the problem he should *draw a figure* and point out on it the unknown and the data. If it is necessary to give names to these objects he should *introduce suitable notation;* devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory stage provided that we do not expect a definitive answer but just a provisional answer, a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] "Understanding the problem" is subdivided into two stages: "Getting acquainted" and "Working for better understanding.")

8. Example. Let us illustrate some of the points explained in the foregoing section. We take the following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, the width, and the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem of Pythagoras, and with some of its applications in plane geometry, but they may have very little systematic knowledge in solid geometry. The teacher may rely here upon the student's unsophisticated familiarity with spatial relations.

The teacher can make the problem interesting by making it concrete. The classroom is a rectangular parallelepiped whose dimensions could be measured, and can be estimated; the students have to find, to "measure indirectly," the diagonal of the classroom. The teacher points out the length, the width, and the height of the magic. If they do not work, we must look around for some other appropriate point of contact, and explore the various aspects of our problem; we have to vary, to transform, to modify the problem. *Could you restate the problem?* Some of the questions of our list hint specific means to vary the problem, as generalization, specialization, use of analogy, dropping a part of the condition, and so on; the details are important but we cannot go into them now. Variation of the problem may lead to some appropriate auxiliary problem: *If you cannot solve the proposed problem try to solve first some related problem.*

Trying to apply various known problems or theorems, considering various modifications, experimenting with various auxiliary problems, we may stray so far from our original problem that we are in danger of losing it altogether. Yet there is a good question that may bring us back to it: *Did you use all the data? Did you use the whole condition?*

10. Example. We return to the example considered in section 8. As we left it, the students just succeeded in understanding the problem and showed some mild interest in it. They could now have some ideas of their own, some initiative. If the teacher, having watched sharply, cannot detect any sign of such initiative he has to resume carefully his dialogue with the students. He must be prepared to repeat with some modification the questions which the students do not answer. He must be prepared to meet often with the disconcerting silence of the students (which will be indicated by dots . . . . .).

"*Do you know a related problem?*"

. . . . .

"*Look at the unknown! Do you know a problem having the same unknown?*"

. . . . .

"Well, *what is the unknown?*"

"The diagonal of a parallelepiped."

"Do you know any *problem with the same unknown?*"

"No. We have not had any problem yet about the diagonal of a parallelepiped."

"Do you know any *problem with a similar unknown?*"

. . . . .

"You see, the diagonal is a segment, the segment of a straight line. Did you never solve a problem whose unknown was the length of a line?"

"Of course, we have solved such problems. For instance, to find a side of a right triangle."

"Good! *Here is a problem related to yours and solved before. Could you use it?*"

. . . . .

"You were lucky enough to remember a problem which is related to your present one and which you solved

FIG. 1

before. Would you like to use it? *Could you introduce some auxiliary element in order to make its use possible?*"

. . . . .

"Look here, the problem you remembered is about a triangle. Have you any triangle in your figure?"

Let us hope that the last hint was explicit enough to provoke the idea of the solution which is to introduce a right triangle, (emphasized in Fig. 1) of which the required diagonal is the hypotenuse. Yet the teacher should be prepared for the case that even this fairly explicit hint is insufficient to shake the torpor of the students; and so he should be prepared to use a whole gamut of more and more explicit hints.

"Would you like to have a triangle in the figure?"

"What sort of triangle would you like to have in the figure?"

"You cannot find yet the diagonal; but you said that you could find the side of a triangle. Now, what will you do?"

"Could you find the diagonal, if it were a side of a triangle?"

When, eventually, with more or less help, the students succeed in introducing the decisive auxiliary element, the right triangle emphasized in Fig. 1, the teacher should convince himself that the students see sufficiently far ahead before encouraging them to go into actual calculations.

"I think that it was a good idea to draw that triangle. You have now a triangle; but have you the unknown?"

"The unknown is the hypotenuse of the triangle; we can calculate it by the theorem of Pythagoras."

"You can, if both legs are known; but are they?"

"One leg is given, it is c. And the other, I think, is not difficult to find. Yes, the other leg is the hypotenuse of another right triangle."

"Very good! Now I see that you have a plan."

11. Carrying out the plan. To devise a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is mainly patience.

The plan gives a general outline; we have to convince ourselves that the details fit into the outline, and so we have to examine the details one after the other, patiently, till everything is perfectly clear, and no obscure corner remains in which an error could be hidden.

If the student has really conceived a plan, the teacher has now a relatively peaceful time. The main danger is that the student forgets his plan. This may easily happen if the student received his plan from outside, and accepted it on the authority of the teacher; but if he worked for it himself, even with some help, and conceived the final idea with satisfaction, he will not lose this idea easily. Yet the teacher must insist that the student should *check each step.*

We may convince ourselves of the correctness of a step in our reasoning either "intuitively" or "formally." We may concentrate upon the point in question till we see it so clearly and distinctly that we have no doubt that the step is correct; or we may derive the point in question according to formal rules. (The difference between "insight" and "formal proof" is clear enough in many important cases; we may leave further discussion to philosophers.)

The main point is that the student should be honestly convinced of the correctness of each step. In certain cases, the teacher may emphasize the difference between "seeing" and "proving": *Can you see clearly that the step is correct?* But can you also *prove that the step is correct?*

12. Example. Let us resume our work at the point where we left it at the end of section **10.** The student, at last, has got the idea of the solution. He sees the right triangle of which the unknown x is the hypotenuse and the given height c is one of the legs; the other leg is the diagonal of a face. The student must, possibly, be urged to introduce suitable notation. He should choose y to denote that other leg, the diagonal of the face whose sides are a and b. Thus, he may see more clearly the idea of the solution which is to introduce an auxiliary problem whose unknown is y. Finally, working at one right triangle after the other, he may obtain (see Fig. 1)

$$x^2 = y^2 + c^2$$
$$y^2 = a^2 + b^2$$

and hence, eliminating the auxiliary unknown y,

$$x^2 = a^2 + b^2 + c^2$$
$$x = \sqrt{a^2 + b^2 + c^2} \,.$$

The teacher has no reason to interrupt the student if he carries out these details correctly except, possibly, to warn him that he should *check each step*. Thus, the teacher may ask:

"Can you *see clearly* that the triangle with sides x, y, c is a right triangle?"

To this question the student may answer honestly "Yes" but he could be much embarrassed if the teacher, not satisfied with the intuitive conviction of the student, should go on asking:

"But can you *prove* that this triangle is a right triangle?"

Thus, the teacher should rather suppress this question unless the class has had a good initiation in solid geometry. Even in the latter case, there is some danger that the answer to an incidental question may become the main difficulty for the majority of the students.

13. Looking back. Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consoliparallelepiped becomes a parallelogram. If you put $c = 0$ in your formula, do you obtain the correct formula for the diagonal of the rectangular parallelogram?"

"If the height c increases, the diagonal increases. Does your formula show this?"

"If all three measures a, b, c of the parallelepiped increase in the same proportion, the diagonal also increases in the same proportion. If, in your formula, you substitute $12a, 12b, 12c$ for a, b, c respectively, the expression of the diagonal, owing to this substitution, should also be multiplied by 12. Is that so?"

"If a, b, c are measured in feet, your formula gives the diagonal measured in feet too; but if you change all measures into inches, the formula should remain correct. Is that so?"

(The two last questions are essentially equivalent; see TEST BY DIMENSION.)

These questions have several good effects. First, an intelligent student cannot help being impressed by the fact that the formula passes so many tests. He was convinced before that the formula is correct because he derived it carefully. But now he is more convinced, and his gain in confidence comes from a different source; it is due to a sort of "experimental evidence." Then, thanks to the foregoing questions, the details of the formula acquire new significance, and are linked up with various facts. The formula has therefore a better chance of being remembered, the knowledge of the student is consolidated. Finally, these questions can be easily transferred to similar problems. After some experience with similar problems, an intelligent student may perceive the underlying general ideas: use of all relevant data, variation of the data, symmetry, analogy. If he gets into the habit of directing his attention to such points, his ability to solve problems may definitely profit.

Can you check the argument? To recheck the argument step by step may be necessary in difficult and important cases. Usually, it is enough to pick out "touchy" points for rechecking. In our case, it may be advisable to discuss retrospectively the question which was less advisable to discuss as the solution was not yet attained: Can you *prove* that the triangle with sides x, y, c is a right triangle? (See the end of section **12.**)

Can you use the result or the method for some other problem? With a little encouragement, and after one or two examples, the students easily find applications which consist essentially in giving some *concrete interpretation* to the abstract mathematical elements of the problem. The teacher himself used such a concrete interpretation as he took the room in which the discussion takes place for the parallelepiped of the problem. A dull student may propose, as application, to calculate the diagonal of the cafeteria instead of the diagonal of the classroom. If the students do not volunteer more imaginative remarks, the teacher himself may put a slightly different problem, for instance: "Being given the length, the width, and the height of a rectangular parallelepiped, find the distance of the center from one of the corners."

The students may use the *result* of the problem they just solved, observing that the distance required is one half of the diagonal they just calculated. Or they may use the *method,* introducing suitable right triangles (the latter alternative is less obvious and somewhat more clumsy in the present case).

After this application, the teacher may discuss the configuration of the four diagonals of the parallelepiped, and the six pyramids of which the six faces are the bases, the center the common vertex, and the semidiagonals the lateral edges. When the geometric imagination of the students is sufficiently enlivened, the teacher should come back to his question: *Can you use the result, or the method, for some other problem?* Now there is a better chance that the students may find some more interesting concrete interpretation, for instance, the following:

"In the center of the flat rectangular top of a building which is 21 yards long and 16 yards wide, a flagpole is to be erected, 8 yards high. To support the pole, we need four equal cables. The cables should start from the same point, 2 yards under the top of the pole, and end at the four corners of the top of the building. How long is each cable?"

The students may use the *method* of the problem they solved in detail introducing a right triangle in a vertical plane, and another one in a horizontal plane. Or they may use the *result*, imagining a rectangular parallelepiped of which the diagonal, x, is one of the four cables and the edges are

$$a = 10.5 \qquad b = 8 \qquad c = 6.$$

By straightforward application of the formula, $x = 14.5$.

For more examples, see CAN YOU USE THE RESULT?

15. Various approaches. Let us still retain, for a while, the problem we considered in the foregoing sections **8, 10, 12, 14.** The main work, the discovery of the plan, was described in section **10.** Let us observe that the teacher could have proceeded differently. Starting from the same point as in section **10,** he could have followed a somewhat different line, asking the following questions:

"Do you know any related problem?"

"Do you know an *analogous* problem?"

"You see, the proposed problem is a problem of solid geometry. Could you think of a simpler analogous problem of plane geometry?"

"You see, the proposed problem is about a figure in space, it is concerned with the diagonal of a rectangular parallelepiped. What might be an analogous problem about a figure in the plane? It should be concerned with —the diagonal—of—a rectangular—"

"Parallelogram."

The students, even if they are very slow and indifferent, and were not able to guess anything before, are obliged finally to contribute at least a minute part of the idea. Besides, if the students are so slow, the teacher should not take up the present problem about the parallelepiped without having discussed before, in order to prepare the students, the analogous problem about the parallelogram. Then, he can go on now as follows:

"Here is a problem related to yours and solved before. Can you use it?"

"Should you introduce some auxiliary element in order to make its use possible?"

Eventually, the teacher may succeed in suggesting to the students the desirable idea. It consists in conceiving the diagonal of the given parallelepiped as the diagonal of a suitable parallelogram which must be introduced into the figure (as intersection of the parallelepiped with a plane passing through two opposite edges). The idea is essentially the same as before (section **10**) but the approach is different. In section **10**, the contact with the available knowledge of the students was established through the unknown; a formerly solved problem was recollected because its unknown was the same as that of the proposed problem. In the present section analogy provides the contact with the idea of the solution.

16. The teacher's method of questioning shown in the foregoing sections **8, 10, 12, 14, 15** is essentially this: Begin with a general question or suggestion of our list, and, if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind. If you