How to Think Like a Mathematician

A Companion to Undergraduate Mathematics

KEVIN HOUSTON

CAMBRIDGE

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Preface

Question: How many months have 28 days?
Mathematician's answer: All of them.

The power of mathematics

Mathematics is the most powerful tool we have. It controls our world. We can use it to put men on the moon. We use it to calculate how much insulin a diabetic should take. It is hard to get right.

And yet. And yet . . . And yet people who use or like mathematics are considered geeks or nerds.[1] And yet mathematics is considered useless by most people – throughout history children at school have whined 'When am I ever going to use this?'

Why would anyone want to become a mathematician? As mentioned earlier mathematics is a very powerful tool. Jobs that use mathematics are often well-paid and people do tend to be impressed. There are a number of responses from non-mathematicians when meeting a mathematician, the most common being 'I hated maths at school. I wasn't any good at it', but another common response is 'You must be really clever.'

The concept

The aim of this book is to divulge the secrets of how a mathematician actually thinks. As I went through my mathematical career, there were many instances when I thought, 'I wish someone had told me that earlier.' This is a collection of such advice. Well, I hope it is more than such a collection. I wish to present an attitude – a way of thinking and doing mathematics that works – not just a collection of techniques (which I will present as well!)

If you are a beginner, then studying high-level mathematics probably involves using study skills new to you. I will not be discussing generic study skills necessary for success – time management, note taking, exam technique and so on; for this information you must look elsewhere.

I want you to be able to think like a mathematician and so my aim is to give you a book jam-packed with practical advice and helpful hints on how to acquire skills specific to

[1] Add your own favourite term of abuse for the intelligent but unstylish.

thinking like a mathematician. Some points are subtle, others appear obvious when you have been told them. For example, when trying to show that an equation holds you should take the most complicated side and reduce it until you get to the other side (page 143). Some advice involves high-level mathematical thinking and will be too sophisticated for a beginner – so don't worry if you don't understand it all immediately.

How to use this book

Each part has a different style as it deals with a different idea or set of ideas. The book contains a lot of information and, like most mathematics books, you can't read it like a novel in one sitting.

Some friendly advice

And now for some friendly advice that you have probably heard before – but is worth repeating.

- *It's up to you* – Your actions are likely to be the greatest determiner of the outcome of your studies. Consider the ancient proverb: The teacher can open the door, but you must enter by yourself.
- *Be active* – Read the book. Do the exercises set.
- *Think for yourself* – Always good advice.
- *Question everything* – Be sceptical of all results presented to you. Don't accept them until you are sure you believe them.
- *Observe* – The power of Sherlock Holmes came not from his deductions but his observations.
- *Prepare to be wrong* – You will often be told you are wrong when doing mathematics. Don't despair; mathematics is hard, but the rewards are great. Use it to spur yourself on.
- *Don't memorize – seek to understand* – It is easy to remember what you truly understand.
- *Develop your intuition* – But don't trust it completely.
- *Collaborate* – Work with others, if you can, to understand the mathematics. This isn't a competition. Don't merely copy from them though!
- *Reflect* – Look back and see what you have learned. Ask yourself how you could have done better.

To instructors and lecturers – a moment of your valuable time

One of my colleagues recently complained to me that when a student is given a statement of the form A implies B to prove their method of proof is generally wholly inadequate. He jokingly said, the student assumes A, works with that for a bit, uses the fact that B is true and so concludes that A is true. How can it be that so many students have such a hard time constructing logical arguments that form the backbone of proofs?

I wish I had an answer to this. This book is an attempt at an answer. It is not a theoretical manifesto. The ideas have been tried and tested from years of teaching to improve mathematical thinking in my students. I hope I have provided some good techniques to get them onto the path of understanding.

If you want to use this book, then I suggest you take your favourite bits or pick some techniques that you know your own students find hard, as even I think that students cannot swallow every piece of advice in this book in a single course. One aim in my own teaching is to be inspirational to students. Mathematics should be exciting. If the students feel this excitement, they are motivated to study and, as in the proverb quoted above, will enter by themselves. I aim to make them free to explore, give them the tools to climb the mountains, and give them their own compasses so they can explore other mathematical lands. Achieving this is hard, as you know, and it is often not lack of time, resources, help from the university or colleagues that is the problem. Often, through no fault of their own, it is the students themselves. Unfortunately, they are not taught to have a questioning nature, they are taught to have an answering nature. They expect us to ask questions and for them to give the answers because that is they way they have been educated. This book aims to give them the questions they need to ask so they don't need me anymore.

I'd just like to thank . . .

This book has had a rather lengthy genesis and so there are many people to thank for influencing me or my choice of contents. Some of the material appeared in a booklet of the same name, given to all first-year Mathematics students at the University of Leeds, and so many students and staff have given their opinions on it over the years. The booklet was available on the web, and people from around the world have sent unsolicited comments. My thanks go to Ahmed Ali, John Bibby, Garth Dales, Tobias Gläßer, Chris Robson, Sergey Klokov, Katy Mills, Mike Robinson and Rachael Smith, and to students at the University of Leeds and at the University of Warwick who were first subjected to my wild theories and experiments (and whose names I have forgotten). Many thanks to David Franco, Margit Messmer, Alan Slomson and Maria Veretennikova for reading a preliminary draft. Particular thanks to Margit and Alan with whom I have had many fruitful discussions. My thanks to an anonymous referee and all the people at the Cambridge University Press who were involved in publishing this book, in particular, Peter Thompson.

Lastly, I would like to thank my gorgeous wife Carol for putting up with me while I was writing this book and for putting the sunshine in my life.

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CHAPTER **1**

Sets and functions

Everything starts somewhere, although many physicists disagree.
Terry Pratchett, *Hogfather*, 1996

To think like a mathematician requires some mathematics to think about. I wish to keep the number of prerequisites for this book low so that any gaps in your knowledge are not a drag on understanding. Just so that we have some mathematics to play with, this chapter introduces sets and functions. These are very basic mathematical objects but have sufficient abstraction for our purposes.

A set is a collection of objects, and a function is an association of members of one set to members of another. Most high-level mathematics is about sets and functions between them. For example, calculus is the study of functions from the set of real numbers to the set of real numbers that have the property that we can differentiate them. In effect, we can view sets and functions as the mathematician's building blocks.

While you read and study this chapter, think about *how* you are studying. Do you read every word? Which exercises do you do? Do you, in fact, do the exercises? We shall discuss this further in the next chapter on reading mathematics.

Sets

The set is the fundamental object in mathematics. Mathematicians take a set and do wonderful things with it.

Definition 1.1

A ***set*** *is a well-defined collection of objects.*[1]

The objects in the set are called the ***elements*** *or* ***members*** *of the set.*

We usually define a particular set by making a list of its elements between brackets. (We don't care about the ordering of the list.)

[1] The proper mathematical definition of set is much more complicated; see almost any text book on set theory. This definition is intuitive and will not lead us into many problems. Of course, a pedant would ask what does well-defined mean?

If x is a member of the set X, then we write $x \in X$. We read this as 'x is an element (or member) of X' or 'x is in X'.[2] If x is not a member, then we write $x \notin X$.

Examples 1.2

(i) The set containing the numbers 1, 2, 3, 4 and 5 is written $\{1, 2, 3, 4, 5\}$. The number 3 is an element of the set, i.e. $3 \in \{1, 2, 3, 4, 5\}$, but $6 \notin \{1, 2, 3, 4, 5\}$. Note that we could have written the set as $\{3, 2, 5, 4, 1\}$ as the order of the elements is unimportant.

(ii) The set {dog, cat, mouse} is a set with three elements: dog, cat and mouse.

(iii) The set {1, 5, 12, {dog, cat}, {5, 72}} is the set containing the numbers 1, 5, 12 and the sets {dog, cat} and {5, 72}. Note that sets can contain sets as members. Realizing this now can avoid a lot of confusion later.

It is vitally important to note that {5} and 5 are not the same. That is, we must distinguish between being a set and being an element of a set. Confusion is possible since in the last example we have {5, 72}, which is a set in its own right but can also be thought of as an element of a set, i.e. {5, 72} ∈ {1, 5, 12, {dog, cat}, {5, 72}}.

Let's have another example of a set created using sets.

Example 1.3

The set $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$ has five elements. It has the the four elements, 1, 2, dog, mouse; and the other element is the set $\{3, 4\}$. We can write $1 \in X$, and $\{3, 4\} \in X$. It is vitally important to note that $3 \notin X$ and $4 \notin X$, i.e. the numbers 3 and 4 are not members of X, the set $\{3, 4\}$ is.

Some interesting sets of numbers

Let's look at different types of numbers that we can have in our sets.

Natural numbers

The set of **natural numbers** is $\{1, 2, 3, 4, \ldots\}$ and is denoted by $\mathbb{N}$. The dots mean that we go on forever and can be read as 'and so on'.

Some mathematicians, particularly logicians, like to include 0 as a natural number. Others say that the natural numbers are the counting numbers and you don't start counting with zero (unless you are a computer programmer). Furthermore, how natural is a number that was not invented until recently?

On the other hand, some theorems have a better statement if we take $0 \in \mathbb{N}$. One can get round the argument by specifying that we are dealing with non-negative integers or positive integers, which we now define.

[2] Of course, to distinguish the x and X we read it out loud as 'little x is an element of capital X.'

Integers

The set of **integers** is $\{\dots, -4, -3, -2, 0, 1, 2, 3, 4, \dots\}$ and is denoted by $\mathbb{Z}$. The $\mathbb{Z}$ symbol comes from the German word Zahlen, which means numbers. From this set it is easy to define the **non-negative integers**, $\{0, 1, 2, 3, 4, \dots\}$, often denoted $\mathbb{Z}^+$. Note that all natural numbers are integers.

Rational numbers

The set of **rational numbers** is denoted by $\mathbb{Q}$ and consists of all fractional numbers, i.e. $x \in \mathbb{Q}$ if x can be written in the form p/q where p and q are integers with $q \neq 0$. For example, $1/2$, $6/1$ and $80/5$. Note that the representation is not unique since, for example, $80/5 = 16/1$. Note also that all integers are rational numbers since we can write $x \in \mathbb{Z}$ as $x/1$.

Real numbers

The **real numbers**, denoted $\mathbb{R}$, are hard to define rigorously. For the moment let us take them to be any number that can be given a decimal representation (including infinitely long representations) or as being represented as a point on an infinitely long number line.

The real numbers include all rational numbers (hence integers, hence natural numbers). Also real are π and e, neither of which is a rational number.[3] The number $\sqrt{2}$ is not rational as we shall see in Chapter 23.

The set of real numbers that are not rational are called **irrational numbers**.

Complex numbers

We can go further and introduce **complex numbers**, denoted $\mathbb{C}$, by pretending that the square root of -1 exists. This is one of the most powerful additions to the mathematician's toolbox as complex numbers can be used in pure and applied mathematics. However, we shall not use them in this book.

More on sets

The empty set

The most fundamental set in mathematics is perhaps the oddest – it is the set with no elements!

[3] The proof of these assertions are beyond the scope of this book. For π see Ian Stewart, *Galois Theory*, 2nd edition, Chapman and Hall 1989, p. 62 and for e see Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill 1976, p. 65.

Definition 1.4

The set with no elements is called the ***empty set*** *and is denoted* $\emptyset$*.*

It may appear to be a strange object to define. The set has no elements so what use can it be? Rather surprisingly this set allows us to build up ideas about counting. We don't have time to explain fully here but this set is vital for the foundations of mathematics. If you are interested, see a high level book on set theory or logic.

Example 1.5

The set $\{\emptyset\}$ is the set that contains the empty set. This set has one element. Note that we can then write $\emptyset \in \{\emptyset\}$, but we *cannot* write $\emptyset \in \emptyset$ as the empty set has, by definition, no elements.

Definition 1.6

Two sets are ***equal*** *if they have the same elements. If set* X *equals set* Y*, then we write* $X = Y$*. If not we write* $X \neq Y$*.*

Examples 1.7

(i) The sets $\{5, 7, 15\}$ and $\{7, 15, 5\}$ are equal, i.e. $\{5, 7, 15\} = \{7, 15, 5\}$.
(ii) The sets $\{1, 2, 3\}$ and $\{2, 3\}$ are not equal, i.e. $\{1, 2, 3\} \neq \{2, 3\}$.
(iii) The sets $\{2, 3\}$ and $\{\{2\}, 3\}$ are not equal.
(iv) The sets $\mathbb{R}$ and $\mathbb{N}$ are not equal.

Note that, as used in the above, if we have a symbol such as $=$ or $\in$, then we can take the opposite by drawing a line through it, such as $\neq$ and $\notin$.

Definition 1.8

If the set X *has a finite number of elements, then we say that* X *is a* ***finite set****. If* X *is finite, then the number of elements is called the* ***cardinality*** *of* X *and is denoted* $|X|$*.*

If X has an infinite number of elements, then it becomes difficult to define the cardinality of X. We shall see why in Chapter 30. Essentially it is because there are different sizes of infinity! For the moment we shall just say that the cardinality is undefined for infinite sets.

Examples 1.9

(i) The set $\{\emptyset, 3, 4, \text{cat}\}$ has cardinality 4.
(ii) The set $\{\emptyset, 3, \{4, \text{cat}\}\}$ has cardinality 3.

Exercises 1.10

What is the cardinality of the following sets?

(i) $\{1, 2, 5, 4, 6\}$
(ii) $\{\pi, 6, \{\pi, 5, 8, 10\}\}$
(iii) $\{\pi, 6, \{\pi, 5, 8, 10\}, \{\text{dog}, \text{cat}, \{5\}\}\}$
(iv) $\emptyset$
(v) $\mathbb{N}$
(vi) $\{\text{dog}, \emptyset\}$
(vii) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$
(viii) $\{\emptyset, \{20, \pi, \{\emptyset\}\}, 14\}$

Now we come to another crucial definition, that of being a subset.

Definition 1.11

*Suppose X is a set. A set Y is a **subset** of X if every element of Y is an element of X. We write $Y \subseteq X$.*

This is the same as saying that, if $x \in Y$, then $x \in X$.

Examples 1.12

(i) The set $Y = \{1, \{3, 4\}, \text{mouse}\}$ is a subset of $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$.
(ii) The set of even numbers is a subset of $\mathbb{N}$.
(iii) The set $\{1, 2, 3\}$ is not a subset of $\{2, 3, 4\}$ or $\{2, 3\}$.
(iv) For any set X, we have $X \subseteq X$.
(v) For any set X, we have $\emptyset \subseteq X$.

Remark 1.13

It is vitally important that you distinguish between being an *element* of a set and being a *subset* of a set. These are often confused by students. If $x \in X$, then $\{x\} \subseteq X$. Note the brackets. Usually, and I stress usually, if $x \in X$, then $\{x\} \notin X$, but sometimes $\{x\} \in X$, as the following special example shows.

Example 1.14

Consider the set $X = \{x, \{x\}\}$. Then $x \in X$ and $\{x\} \subseteq X$ (the latter since $x \in X$) but we also have $\{x\} \in X$.

Therefore we *cannot* state any simple rule such as 'if $a \in A$, then it would be wrong to write $a \subseteq A$', and vice versa.

If you felt a bit confused by that last example, then go back and think about it some more, until you really understand it. This type of precision and the nasty examples that go against intuition, and prevent us from using simple rules, are an important aspect of high-level mathematics.

Definition 1.15

*A subset Y of X is called a **proper subset** of X if Y is not equal to X. We denote this by $Y \subset X$. Some people use $Y \subsetneq X$ for this.*

Examples 1.16

(i) $\{1, 2, 5\}$ is a proper subset of $\{-6, 0, 1, 2, 3, 5\}$.
(ii) For any set X, the subset X is not a proper subset of X.
(iii) For any set $X \neq \emptyset$, the empty set $\emptyset$ is a proper subset of X. Note that, if $X = \emptyset$, then the empty set $\emptyset$ is *not* a proper subset of X.
(iv) For numbers, we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Note that we can use the symbols $\nsubseteq$ to denote 'not a subset of' and $\not\subset$ to denote 'not a proper subset of'.

Now let's consider where the notation came from. It is is obvious that for a finite set the two statements

$$\text{If } X \subseteq Y, \text{ then } |X| \leq |Y|,$$

Products of sets

Here's another example of mathematicians creating new objects from old ones.

Definition 1.27

*Let X and Y be two sets. The **product** of X and Y, denoted $X \times Y$ is the set of all possible pairs (x, y) where $x \in X$ and $y \in Y$, i.e.*

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Note that here (x, y) denotes a pair and has nothing to do with Example 1.17(iv).

Examples 1.28

(i) Let $X = \{0, 1\}$ and $Y = \{1, 2, 3\}$. Then $X \times Y$ has six elements:

$$X \times Y = \{(0, 1),\ (0, 2),\ (0, 3),\ (1, 1),\ (1, 2),\ (1, 3)\}.$$

(ii) The set $\mathbb{R} \times \mathbb{R}$ is denoted $\mathbb{R}^2$. The set $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ is denoted $\mathbb{R}^3$. This is because its elements can be given by triples of real numbers, i.e. its elements are of the form (x, y, z) where x, y and z are real numbers.

Note that $X \times Y$ is not a subset of either X or Y.

Maps and functions

We have defined sets. Now we make a definition for relating elements of sets to elements of other sets.

Definition 1.29

*Suppose that X and Y are sets. A **function** or **map** from X to Y is an association between the members of the sets. More precisely, for every element of X there is a unique element of Y.*

*If f is a function from X to Y, then we write $f : X \to Y$, and the unique element in Y associated to x is denoted $f(x)$. This element is called the **value of** x **under** f or called a **value** of f. The set X is called the **source** (or **domain**) of f and Y is called the **target** (or **codomain**) of f.*

To describe a function f we usually use a formula to define $f(x)$ for every x and talk about applying f to elements of a set, or to a set.

A schematic picture is shown in Figure 1.1. Note that every element of X has to be associated to one in Y but not vice versa and that two distinct elements of X may map to the same one in Y.

Examples 1.30

(i) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{Z}$. Then the value of x under f is the square of x. Note that there are elements in the target which are not values of f. For example -1 is not a value since there is no integer x such that $x^2 = -1$.

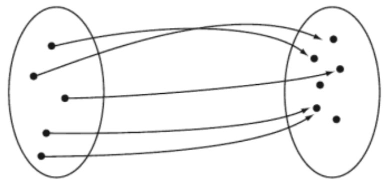


Figure 1.1 A function from X to Y

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 0$. Then the only value of f is 0.
(iii) The cardinality of a set is a function on the set of finite sets. That is $|\,|$: Finite Sets $\to \{0\} \cup \mathbb{N}$. Note that we need 0 in the codomain as the set could be the empty set.
(iv) The **identity map** on X is the map id : $X \to X$ given by $\mathrm{id}(x) = x$ for all $x \in X$.

Having a formula does not necessarily define a function, as the next example shows.

Example 1.31

The formula $f(x) = 1/(x-1)$ does not define a function from $\mathbb{R}$ to $\mathbb{R}$ as it is not defined for $x = 1$.

We can rescue this example by restricting the source to $\mathbb{R}$ without the element 1. That is, define $X = \{x \in \mathbb{R} \mid x \neq 1\}$, then $f : X \to \mathbb{R}$ defined by $f(x) = 1/(x-1)$ is a function.

Polynomials provide a good source of examples of functions.

Examples 1.32

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2x + 3$. Notice again that, although the target is all of $\mathbb{R}$, not every element of the target is a value of f. For example there is no x such that $f(x) = -2$. This is something you can check by attempting to solve $x^2 + 2x + 3 = -2$.
(ii) More generally, from a polynomial we can define a function $f : \mathbb{R} \to \mathbb{R}$ by defining

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some real numbers $a_0, \ldots, a_n$ and a real variable x.
(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ can be differentiated, for example a polynomial. Then the derivative, denoted f', is a function.

Exercises 1.33

(i) Find the largest domain that makes $f(x) = x/(x^2 - 5x + 3)$ a function.
(ii) Find the largest domain that makes $f(x) = (x^3 + 2)/(x^2 + x + 2)$ a function.
(iii) Construct an example of a polynomial so that its graph goes through the points $(-1, 5)$ and $(3, -2)$.

Exercises

Exercises 1.34

(i) Let $X = \{x \in \mathbb{Z} \mid 0 \leq x \leq 10\}$ and A and B be subsets such that $A = \{0, 2, 4, 6, 8, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$, $A \cup B$, $A \backslash B$, $B \backslash A$, $A \times B$, $X \times A$, A^{c}, and B^{c}.

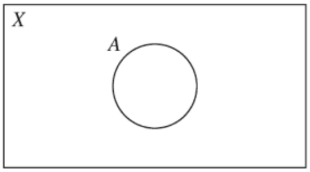
(ii) Find the union and intersection of $\{x \in \mathbb{R} \mid x^2 - 9x + 14 = 0\}$ and $\{y \in \mathbb{Z} \mid 3 \leq y < 10\}$.

(iii) Suppose that A, B and C are subsets of X. Use examples of these sets to investigate the following:

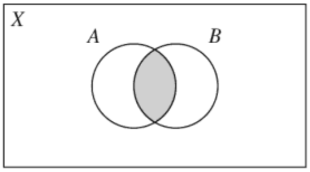
(a) $(A \cap B) \cup (A \cap C)$ and $A \cap (B \cup C)$,
(b) $(A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C)$,
(c) $(A \cup B)^{\mathrm{c}}$ and $A^{\mathrm{c}} \cap B^{\mathrm{c}}$,
(d) $(A \cup B)^{\mathrm{c}}$ and $A^{\mathrm{c}} \cup B^{\mathrm{c}}$,
(e) $(A \cap B)^{\mathrm{c}}$ and $A^{\mathrm{c}} \cup B^{\mathrm{c}}$,
(f) $(A \cap B)^{\mathrm{c}}$ and $A^{\mathrm{c}} \cap B^{\mathrm{c}}$.

Do you notice anything?

(iv) A **Venn diagram** is useful way of representing sets. If A is a subset of X, then we can draw the following in the plane:



In fact, the precise shape of A is unimportant but we often use a circle. If B is another subset, then we can draw B in the diagram as well. In the following we have shaded the intersection $A \cap B$.



(a) Draw a Venn diagram for the case that A and B have no intersection.
(b) Draw Venn diagrams and shade the sets $A \cup B$, A^{c}, and $(A \cap B)^{\mathrm{c}}$.
(c) Draw three (intersecting) circles to represent the sets A, B and C. Shade in the intersection $A \cap B \cap C$.
(d) Using exercise (iii) construct Venn diagrams and shade in the relevant sets.

(v) Analyse how you approached the reading of this chapter.

(a) If you had not met the material in this chapter before, then did you attempt to understand everything?
(b) If you had met the material before, did you check to see that I had not made any mistakes?

Summary

- A set is a well-defined collection of objects.
- The empty set has no elements.
- The cardinality of a finite set is the number of elements in the set.
- The set Y is a subset of X if every element of Y is in X.
- A subset Y of X is a proper subset if it is not equal to X.
- The union of X and Y is the collection of elements that are in X or in Y.
- The intersection of X and Y is the collection of elements that are in X and in Y.
- The product of X and Y is the set of all pairs (x, y) where $x \in X$ and $y \in Y$.
- A function assigns elements of one set to another.

CHAPTER **2**

Reading mathematics

Don't believe everything you read.
Anon

Obviously you can read and probably you have been taught reading skills for academic purposes as part of a study skills course. Unfortunately, mathematics has some special subtleties which often get missed in classes or books on how to study. For example, speed reading is recommended as a valuable tool for learning in many subjects. In mathematics, however, this is not a good method. Mathematics is rarely overwritten; there are few superfluous adjectives, every word and symbol is important and their omission would render the material incomprehensible or incorrect.

The hints and tips here, which include a systematic method for breaking down reading into digestible pieces, are practical suggestions, not a rigid list of instructions. The main points are the following:

- You should be flexible in your reading habits – read many different treatments of a subject.
- Reading should be a dynamic process – you should be an active, not passive, reader, working with a pen and paper at hand, checking the text and verifying what the author asserts is true.

The last point is where thinking mathematically diverges from thinking in many other subjects, such as history and sociology. You really do need to be following the details as you go along – check them. In history (assuming you don't have a time machine) you can't check that Caesar invaded Britain in 55 BC, you can only check what other people have claimed he did. In mathematics you really can, and should, verify the truth.

The following applies to reading lecture notes and web pages, not just to books, but to make a simpler exposition I shall refer only to books. Tips on specific situations, such as reading a definition, theorem[1] or proof are given in later chapters.

[1] A theorem is a mathematical statement that is true. Theorems will be discussed in greater detail in Part III.

Look for theorems or formulas that allow you to calculate because calculation is an effective way to get into a subject. Stop and reread that last sentence – I think it's one of the most useful pieces of advice given to me. Often when I am stuck trying to understand some theory attempting to calculate makes it clear. Noticing what allows you to calculate is thus very important.

In Chapter 1 the most obvious notion involving calculation was the cardinality of a set. However, there were no theorems involving it. Nonetheless, you should mark it as something that will be of use later because it involves the possibility of calculation. And in fact we look at calculation of cardinality in Chapter 5.

A more general example is the product rule and chain rule, etc. in calculus. These allow us to calculate the derivative of a function without using the definition of derivative (which is hard to work with).

Ask questions

At this stage it is helpful to pose some questions about the text, such as, Why does the theory hinge on this particular definition or theorem? What is the important result that the text is leading up to and how does it get us there? From your questions you can make a detailed list of what you want from the text.

In the last chapter the main point of the text was to lay the groundwork for material we will use later as examples.

Careful reading

It is now that the careful reading is undertaken. This should be systematic and combined with thinking, doing exercises and solving problems.

Reading is more than just reading the words, you must think about what they mean. In particular, ensure that you know the meaning of every word and symbol; if you don't know or have forgotten, then look back and find out.

For example, one needs to read carefully to ensure that the difference between being a set and being an element of a set is truly grasped.

Stop periodically to review

Do not try to read too much in one go. Stop periodically to review and think about the text. Keep thinking about the big picture, where are we going and how is a particular result getting us there?

Read statements first – proofs later

Many mathematical texts are written so that proofs can be ignored on an initial reading. This is not to say that proofs are unimportant; they are at the heart of mathematics, but usually – not always – can be read later. You *must* tackle the proof at some point.

There were no proofs in the previous chapter. Don't worry, we will produce many proofs later in the book.

Check the text

The necessity to check the text is why you need pen and paper at hand. There are two reasons. First, to fill in the gaps left by the writer. Often we meet phrases like 'By a straightforward calculation' or 'Details are left to the reader'. In that case, do that calculation or produce those details. This really allows you to get inside the theory.

For example, on page 7 in Chapter 1, I stated 'It is obvious that ... if $X \subseteq Y$, then $|X| \leq |Y|$.' Did you check to see that it really was 'obvious'? Did you try some examples? Similarly did you focus on the non-intuitive facts such as the fact that it is possible to have $\{x\} \in X$ and $\{x\} \subseteq X$ at the same time?

The second reason is to see how theorems, formulas, etc. apply. If the text says use Theorem 3.5 or equation Y, then check that Theorem 3.5 can be applied or check what happens to equation Y in this situation. Verify the formulas and so on. Be a sceptic – don't just take the author's word for it.

Do the exercises and problems

Most modern mathematics books have exercises and problems. It is hard to overplay the importance of doing these. Mathematics is an activity. Think of yourself as not studying mathematics, but *doing* mathematics.

Imagine yourself as having a mathematics muscle. It needs exercise to become developed. Passive reading is like watching someone else training with weights; it won't build your muscles – *you* have to do the exercises.

Furthermore, just because you have read something it does not mean you truly understand it. Answering the exercises and problems identifies your misconceptions and misunderstandings. Regularly I hear from students that they can understand a topic; it's just that they can't do the exercises, or can't apply the material. Basically, my rule is: if I can't do the exercises, then I don't understand the topic.

Reflect

In order to understand something fully we need to relate it to what we already know. Is it analogous to something else? For example, note how the $\subseteq$ notation made sense when it was compared with $\leq$ via cardinality. Can you think what intersection and union might be analogous to?

Another question to ask is 'What does this tell us or allow us to do that other work does not?' For example, the empty set allows us to count (something that was not explained but was alluded to in Chapter 1). Functions allow us to connect sets to sets. Cardinality allows us to talk about the relative sizes of sets. So when you meet a topic ask 'What does it allow me to do?'

What to do afterwards

Don't reread and reread – move on

It is unlikely that understanding will come from excessive rereading of a difficult passage. If you are rereading, then it is probably a sign that you are not active – so do some exercises, ask some questions and so on.

If that fails, it is time to look for an alternative approach, such as consulting another book. Ultimately, it is acceptable to give up and move on to the next part; you can always come back.

By moving on, you may encounter difficulty in understanding the subsequent material, but it might clarify the difficult part by revealing something important.

Also mathematics is a subject that requires time to be absorbed by the brain; ideas need to percolate and have time to grow and develop.

Reread

The assertion to reread may seem strange as the previous piece of advice was not to reread. The difference here is that one should come back much later and reread, for example, when you feel that you have learned the material. This often reveals many subtle points missed or gives a clearer overview of the subject.

Write a summary

The material may appear obvious once you have finished reading, but will that be true at a later date? It is a good time to make a summary – written in your own words.

Exercises

Exercises 2.1

(i) Look back at Chapter 1 and analyse how you attempted to read and understand it.

(ii) Find a journal or a science magazine that includes some mathematics in articles, for example *Scientific American*, *Nature*, or *New Scientist*.

Read an article. What is the aim of the article and who is the audience? How is the maths used? In one sentence what is the aim? Give three main points.

(iii) Find three textbooks of a similar level and within your mathematical ability. Briefly look through the books and decide which is most friendly and explain your reasons why.

(iv) Find three books tackling the same subject. Find a mathematical object in all three book, say, a set. Are the definitions different in the different books? Which is the best definition? Or rather, which is your favourite definition?

Are any diagrams used to illustrate the concept? What understanding do the diagrams give? How are the diagrams misleading?

Summary

- Read with a purpose.
- Read actively. Have pen and paper with you.
- You do not have to read in sequence but read systematically.
- Ask questions.
- Read the definitions, theorems and examples first. The proofs can come later.
- Check the text by applying formulas etc.
- Do exercises and problems.
- Move on if you are stuck.
- Write a summary.
- Reflect – What have you learned?

CHAPTER **3**

Writing mathematics I

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on.

Richard Feynman, Nobel Lecture, 1966

As a lecturer my toughest initial task in turning enthusiastic students into able mathematicians is to force them (yes, force them) to write mathematics correctly. Their first submitted assessments tend to be incomprehensible collections of symbols, with no sentences or punctuation. 'What's the point of writing sentences?', they ask, 'I've got the correct answer. There it is – see, underlined – at the bottom of the page.' I can sympathize but in mathematics we have to get to the right answer in a rigorous way and we have to be able to show to others that our method is rigorous.

A common response when I indicate a nonsensical statement in a student's work is 'But you are a lecturer, you know what I meant.' I have sympathy with this view too, but there are two problems with it.

(i) If the reader has to use their intelligence to work out what was intended, then the student is getting marks because of the reader's intelligence, not their own intelligence.[1]

(ii) This second point is perhaps more important for students. Sorting through a jumble of symbols and half-baked poorly expressed ideas is likely to frustrate and annoy any assessor – not a good recipe for obtaining good marks.

My students performed well at school and are frustrated at losing marks over what seems to them unimportant details. However, by the end of the year they generally accept that writing well has improved their performance. You have to trust me that this works! Besides, writing well in any subject is a useful skill to possess.

[1] To be honest, students don't mind this!

solutions exist – is certainly included. The student also showed that he knows that the empty set is denoted by $\emptyset$. However, the inclusion of this symbol is unnecessary; it serves no purpose.

But what he wrote is not a sentence – it is a string of symbols and conveys no meaning in itself.

The answer could be better expressed as

> 'Since the equation $0 = 1$ is present, the system of equations is inconsistent and so no solutions exist.'

We could add 'That is, the solution set is empty', but it is not necessary. Understanding is clearly shown in this answer, and so more marks will be forthcoming.

All the other usual rules of written English apply, for example the use of paragraphs and punctuation. The rules of grammar are just as important: every sentence should have a verb, subjects should agree with verbs, and so on.

Let us look at the example in Figure 3.1 of the proof of the cosine formula. Examine the first two lines below the student's diagram.

If I read from left to right in the standard fashion, I read

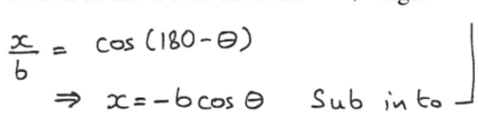
$$\Delta CBL\ \Delta CLA\ a^2 = (c + x)^2 + h^2\ b^2 = h^2 + x^2.$$

Now what does that mean? It is obvious what is intended. But why should we have to work out what was intended? It would be better to say what was meant from the start:

> In triangle ΔCBL we have $a^2 = (c+x)^2+h^2$ and in ΔCLA we have $b^2 = h^2+x^2$.

This is now a proper sentence. As an aside, notice how I explained my notation Δ by using the word 'triangle'.

Now look at the words after the $\Longrightarrow$ sign:



This is a perfect example of where we can understand what the student had intended but it is not well written. It is much clearer as

> … $x = -b\cos\theta$. Substituting this into …

Use punctuation

The purpose of punctuation is to make the sentence clear. Punctuation should be used in accordance with standard practice. In particular, all sentences begin with a capital letter and end with a full stop. The latter holds even if the sentence ends in a mathematical expression. For example,

> 'Let $x = y^4 + 2y^2$ Then x is positive.'

needs a full stop after the expression $y^4 + 2y^2$ as it is obvious that the second part is a new sentence – it begins with a capital letter. This is true for a list of equal expressions:

$$\begin{aligned} x &= y^2 + 2y \\ &= y(y+2) \end{aligned}$$

This should end with a full stop. Note that some authors do not adhere to this rule of punctuation. They are wrong.[2]

Mathematical expressions need to be punctuated. For example,

> 'Let $x = 4a + 3b$ where $a \in \mathbb{R}$ $b \in \mathbb{Z}$'

should have commas like so

> 'Let $x = 4a + 3b$, where $a \in \mathbb{R}$, $b \in \mathbb{Z}$.'

Notice the three commas and the final full stop in the following example.

$$\text{Let } f(x) = \begin{cases} x^2, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Look at the example of the proof of the cosine formula. As you can see there is is no punctuation! Presumably a sentence starts at 'In ΔCLA …' but it is not proceeded by a full stop so who knows?

Keep it simple

Mathematics is written in a very economical way. To achieve this, use short words and sentences. Short sentences are easy to read. To eliminate ambiguities avoid complicated sentences with lots of negations.

Consider the following hard-to-read example:

> 'The functions f and g are defined to be equal to the function defined on the set of non-positive integers given by x maps to its square and x maps to the negative of its square respectively.'

This would be better as:

> 'Let $\mathbb{Z}^{\leq 0} = \{\ldots, -5, -4, -3, -2, -1, 0\}$ be the set of non-positive integers. Let $f : \mathbb{Z}^{\leq 0} \to \mathbb{R}$ be given by $f(x) = x^2$ and $g : \mathbb{Z}^{\leq 0} \to \mathbb{R}$ be given by $g(x) = -x^2$.'

Note that we separated the definition of the domains of the maps into a separate sentence.

[2] A number of people think this is a controversial statement. 'What does it matter, as long as you are consistent?' Well, we could apply that argument to any sentence and we can get rid of all full stops! The majority opinion is that sentences end with a full stop – go with that.

Also we defined the set in words and clarified by writing it in a different way. The definitions of f and g are mixed together in the first sentence due to the use of 'respectively', while in the second sentence they are separated and defined using symbols. Sometimes using symbols is clearer, sometimes not; see page 28.

Expressing yourself clearly

The purpose of writing is communication – you are supposed to be transferring a thought to someone else (or yourself at a later date). Unfortunately – and I have lots of experience of this – it is easy to communicate an incorrect or unintended idea. The following advice is offered to prevent this from happening.

Explain what you are doing – keeping the reader informed

Readers are not psychic. It is crucial to explain what you are doing. To do this imagine that you are giving a running commentary. As stated earlier, it is not sufficient to produce a list of symbols, formulas, or unconnected statements. A good explanation will help gain marks as it demonstrates understanding.

You can introduce an argument by saying what you are about to do, e.g.

> 'We now show that X is a finite set',
> 'We shall prove that . . .'.

Similarly you can end by

> 'This concludes the proof that X is a finite set', or
> 'We have proved . . .'.

Make clear, bold assertions. Avoid phrases like 'it should be possible'; either it is possible or it isn't, so claim 'it is possible'. Be positive.

Of course, avoid going to the extreme of explaining every last detail. A balance, which will come from practice and having your written work criticized, needs to be struck.

If we look at the end of the example in Figure 3.1, then we see the following.

$\Rightarrow x = -b\cos\theta$ Sub into ┘

$a^2 = b^2 + c^2 - 2cb\cos\theta$

This ending would be better as

> '. . . $x = -b\cos\theta$. Substituting this into the above we deduce that $a^2 = b^2 + c^2 - 2cb\cos\theta$.'

This is certainly much better as it implicitly makes the claim that what we had to prove has been proved. Otherwise it may look like we wrote the cosine formula at the end to fool the marker into thinking that the solution had been given. Also, using the word 'deduce' in the final sentence explains where the result came from.

Explain your assertions

Rather than merely make an assertion, say where it comes from. That is, use sentences containing

> 'as, because, since, due to, in view of, from, using, we have,' and so on.

For example,

> 'Using Theorem 4(i), we see that the solution set is non-empty'

is obviously preferable to

> 'The solution set is non-empty,'

and

> '$x^3 > 0$ because x is positive'

is better than the bare

> '$x^3 > 0$'

since, for a general $x \in \mathbb{R}$, we don't have $x^3 > 0$. The point is that the reader may be misled into thinking the statement is 'obviously false' if they had forgotten that x was positive. It doesn't hurt to include such helpful comments.

Another example is to say when a rule has been used:

> '$f'(x) = 2x\cos(x^2)$ by the Chain Rule.'

In this way, you demonstrate your understanding.

Returning to the first few lines of the student's proof of the cosine formula in Figure 3.1

$\triangle CBL$ $\quad$ $\triangle CLA$

$a^2 = (c+x)^2 + h^2$ $\quad$ $b^2 = h^2 + x^2$

we have already seen that it would be better to have said

> 'In triangle ΔCBL we have $a^2 = (c+x)^2+h^2$ and in ΔCLA we have $b^2 = h^2+x^2$.'

But what about the next line? It says simply

$$a^2 = c^2 + 2cx + h^2 + x^2$$

Is this a deduction from the diagram? Certainly the first two equalities were, i.e. $a^2 = (c+x)^2 + h^2$ and $b^2 = h^2 + x^2$. In this case the line is not deduced from the diagram but from the first equation by expanding the bracket. So we should say so.

> Expanding the brackets we get $a^2 = c^2 + 2cx + h^2 + x^2$.

We'll see that it is not necessary to phrase it this way when we look at the next line:

$$a^2 = b^2 + c^2 + 2cx$$

This comes from substituting the second equation, $b^2 = h^2+x^2$, into the expanded version of the first, $a^2 = c^2 + 2cx + h^2 + x^2$. Let's say so.

> 'In triangle ΔCBL we have $a^2 = (c + x)^2 + h^2$ and in ΔCLA we have $b^2 = h^2 + x^2$. Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$.'

Note that we have left out the expansion of the brackets. You can include it if you wish but the calculation is so trivial that it is not worth the ink. The reader can check it themselves if they don't believe us.

Say what you mean

In any writing, saying what you mean is important – and difficult. Precise use of grammar can help in this task.

The first rule is that the reader should not have to deduce what you mean from context; all the necessary information should be there. Nothing should be ambiguous.

The true mathematician is pedantic, and requires that mathematics is precise. Without precision mathematics is nothing. Without it we cannot build with one concept placed on top of another. If one of the ideas is vague or open to different interpretations by different parties, then errors can creep in and the endeavour is unsound. So, be precise!

As an example, use the quantifiers 'some' and 'all'. Rather than say

> '$f(x) = 5$',

which is ambiguous – the reader may ask 'Is it for one x? At least one x? All x?' – say

> '$f(x) = 5$ for some $x \in \mathbb{R}$', or '$f(x) = 5$ for all $x \in \mathbb{R}$',

depending on the situation.

More will be said in Chapter 10 on quantifiers to explain the importance of precision in this area.

Using symbols

We now come to tips concerning symbols. There is no escaping that mathematics is highly symbolic, but using lots of mathematical symbols does not make an argument a mathematical one.

Words or symbols?

Symbols are shorthand. For example, a famous theorem by Euler in the theory of complex numbers,

> $e^{2\pi\sqrt{-1}} = 1$,

Sometimes we need to indicate where a particular result came from. Avoid interrupting the flow of the argument like so:

$$\begin{aligned} &= x^2 + 5y \\ \text{by theorem 6} &= x^2 + 25 \ldots \end{aligned}$$

If the details of why a particular step is true need to be included, then do the following. For the sake of argument suppose that $y = 3$ by Theorem 4.6. Then we write

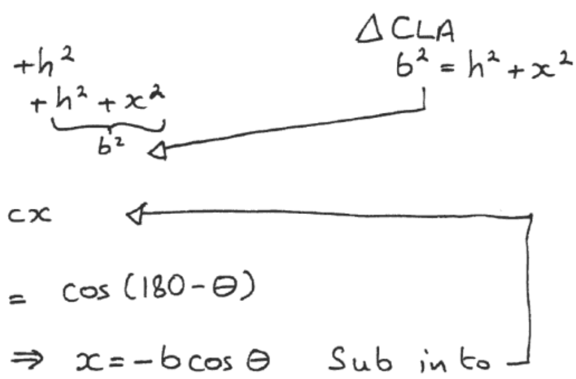
$$\begin{aligned} x^2 + 4x + y &= x^2 + 4x + 3, \text{ by Theorem 4.6,} \\ &= (x + 1)(x + 3) \ldots \end{aligned}$$

Note the punctuation after the symbolic expression on the first line and after the mention of the theorem. It doesn't read well, but is clear on the page.

Don't draw arrows everywhere

If a result requires an earlier one, it is tempting to draw a long arrow to point to it. Don't do this on aesthetic grounds. Instead, give the required result a name, number or symbol, so you can refer to it.

Our example in Figure 3.1 uses arrows.



We can change this to

> 'Expanding this first equation and substituting in b^2 from the second we get $a^2 = b^2 + c^2 + 2cx$. $\quad (*)$
>
> ⋮
>
> $\ldots \implies x = -b\cos\theta$. Substituting this into $(*)$ we deduce that $a^2 = b^2 + c^2 - 2cb\cos\theta$.'

Exercise 3.1

Rewrite the proof of the the Cosine Rule so that it follows the suggestions given.

Finishing off

Proofread

Always proofread your work. That is, read through it looking for errors. These could be typographical errors (also known as typos), where the wrong character is used, e.g. cay instead of cat, or spelling mistakes, e.g. parrallel instead of parallel, grammatical mistakes, e.g. 'A herd of cows are in the field', or even mathematical errors.

Read your work slowly. Reading aloud can help catch many errors as it stops you skimming. Get someone else to read your work as you will often read what you think is there, rather than what actually is there. If your checker misses mistakes, then you are not allowed to blame them. The final responsibility always rests with the writer!

A useful proofreading method is to concentrate on one aspect of proofreading at a time. That is, read through first for accuracy, i.e. is it true? Next, check for spelling, typos, are all the brackets closed?, etc. After that check that the order of the material is correct and that it flows as you read it.

Reflection

Reflection is an important part of the writing process. Put your work away for some time and come back to it with a fresh eye. Obviously, this is not possible for work with tight deadlines, but can be done with project work.

When reading through again, ask 'What can I take away?' (aim for economy of words) and 'What can I add?' (more examples might clarify). For the former remove unnecessary words and sentences. Also ask: 'Are all the symbols explained and are they necessary? Does it say what I mean and is it simple? Is it more than just a collection of symbols?' And of course, most importantly, 'Did I write in sentences?'

Exercises

Exercises 3.2

(i) **The Sine Rule:** Suppose that we have a triangle with sides of length a, b and c with the angles opposite these sides labelled α, β and γ respectively. Then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

In an exam a student answered the question 'State and prove the Sine Rule' with the following:

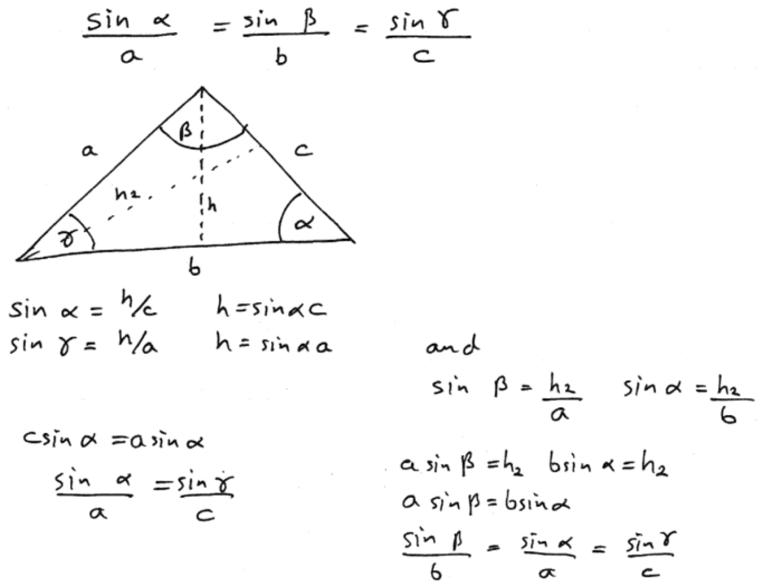


Figure 3.2 Student's proof of the Sine Rule

Rewrite this answer so that it is correctly written and easily comprehended.

(ii) If you know how to find maxima and minima as well as curve sketching you should rewrite the following answer to the exercise 'Find the maximum and minimum values of the function $f(x) = 2x^3 - 12x^2 + 18x$ and sketch its graph.'

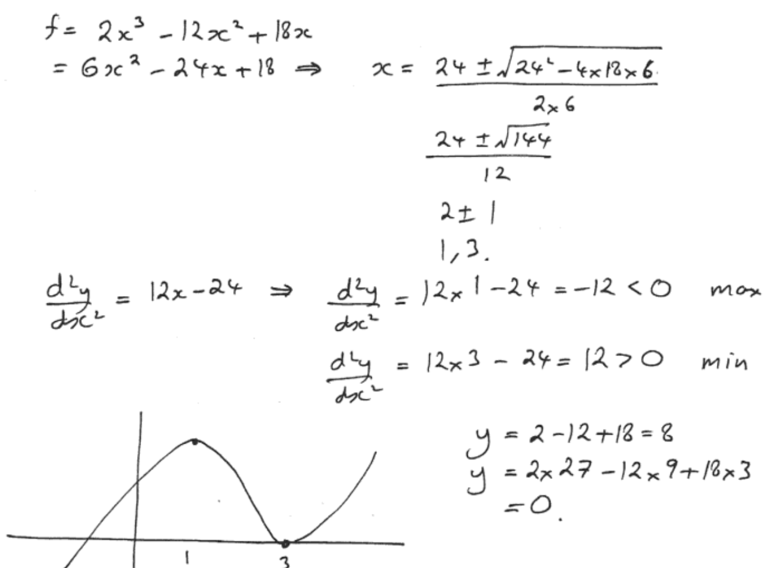


Figure 3.3 Student's answer to finding maximum and minimum values of $f(x) = 2x^3 - 12x^2 + 18x$