Anticipation Science 2

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Intangible Life

Functorial Connections in Relational Biology

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Prolegomenon
Category Theory for the Aspiring Relational Biologist

> We take the viewpoint that the study of natural systems is precisely the specification of the observables belonging to such a system, and a characterization of the manner in which they are linked. Indeed, for us *observables are the fundamental units of natural systems …*
>
> — Robert Rosen (1985)
> *Anticipatory Systems: Philosophical, Mathematical, and Methodological Foundations*
> 2.1 The Concept of a Natural System

Category

> Category theory asks of every type of Mathematical object: "What are the morphisms?"; it suggests that these morphisms should be described at the same time as the objects.
>
> — Saunders Mac Lane (1997)
> *Category Theory for the Working Mathematician*
> § I. Notes

Robert Rosen entered Nicolas Rashevsky's Committee on Mathematical Biology at the University of Chicago in the autumn of 1957. Engaged in his work on relational biology, Rosen quickly discovered the (M,R)-systems, and developed some of their extraordinary properties. A happy happenstance was when Rosen connected this relational theory of biological systems to the algebraic theory of categories (founded by Samuel Eilenberg and Saunders Mac Lane in 1945), thus equipping himself with a ready-made mathematical tool. Indeed, Rosen's first published scientific paper was on his (M,R)-systems [Rosen 1958a], and his

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second paper was on 'The representation of biological systems from the standpoint of the theory of categories' [Rosen 1958b].

The confluence of ideas, as can be seen from the above quotes of Rosen and Mac Lane, is that in describing systems, be it natural or formal, the material and efficient causes must be characterized together. The pairs of causes are variously manifested as

(a) objects and morphisms;
(b) states and observables;
(c) structure and function;
(d) material and functional entailments;
(e) sequential and hierarchical composites;
(f) metabolism and repair;
etc.

A category comprises of two collections: i. objects, and ii. morphisms. One may define a category in which the collection of morphisms is partitioned into hom-sets:

0.1 Definition A (*ML*: A.1; *RL*: 6.7) A *category* $\mathbf{C}$ consists of

i. A collection of *objects*.
ii. For each pair of $\mathbf{C}$-objects A, B, a set

$$\mathbf{C}(A,B), \tag{1}$$

the *hom-set* of *morphisms* from A to B. [If $f \in \mathbf{C}(A,B)$, one also writes $f : A \to B$ and $A \xrightarrow{f} B$. Often for simplicity, or when the category $\mathbf{C}$ need not be emphasized, the hom-set $\mathbf{C}(A,B)$ may be denoted by $H(A,B)$.]

iii. For any three objects A, B, C, a *mapping*

$$\circ : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \to \mathbf{C}(A,C) \tag{2}$$

taking $f : A \to B$ and $g : B \to C$ to its *composite* $g \circ f : A \to C$.

iv. For each object A, there exists a morphism

$$1_A \in \mathbf{C}(A,A), \tag{3}$$

called the *identity morphism* on A.

These entities satisfy the following three axioms:

($c1$) *Uniqueness*:

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$$\mathbf{C}(A,B) \cap \mathbf{C}(C,D) = \varnothing \tag{4}$$

unless $A = C$ and $B = D$. [Thus each morphism $f : A \to B$ uniquely determines its *domain* $A = \mathrm{dom}(f)$ and *codomain* $B = \mathrm{cod}(f)$: different hom-sets are mutually exclusive.]

($c2$) *Associativity*: If $f : A \to B$, $g : B \to C$, $h : C \to D$, so that both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined, then

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{5}$$

($c3$) *Identity*: For each object A, the identity morphism on A, $1_A : A \to A$, has the property that for any $f : A \to B$ and $g : C \to A$,

$$f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g \tag{6}$$

[which leads demonstrably to the uniqueness of 1_A in $\mathbf{C}(A,A)$].

Alternatively, one may define a category in terms of arrows, equipping the collection of morphisms with a pair of mappings that assign to each morphism a domain and a codomain:

0.2 Definition B (*RL*: 6.8) A *category* $\mathbf{C}$ consists of

i′. A set $\mathfrak{O}\mathbf{C}$ of *objects*.

ii′. A set $\mathfrak{A}\mathbf{C}$ of *arrows* (*morphisms*), equipped with two mappings dom and cod:

$$\begin{cases} \mathrm{dom}: & \mathfrak{A}\mathbf{C} \to \mathfrak{O}\mathbf{C} \\ \mathrm{cod}: & \mathfrak{A}\mathbf{C} \to \mathfrak{O}\mathbf{C} \end{cases}. \tag{7}$$

iii′. A (*sequential*) *composition* mapping

$$\circ : \mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{C} \tag{8}$$

(where the domain

$$\mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C} = \{(f,g) \in \mathfrak{A}\mathbf{C} \times \mathfrak{A}\mathbf{C} : \mathrm{dom}(g) = \mathrm{cod}(f)\} \tag{9}$$

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is a proper subset of $\mathfrak{A}\mathbf{C} \times \mathfrak{A}\mathbf{C}$, called the '*product over* $\mathfrak{O}\mathbf{C}$', and an ordered pair $(f, g) \in \mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C}$ is called a '*composable pair of morphisms*'), taking (f, g) to its *composite* $g \circ f$, such that

$$\text{dom}(g \circ f) = \text{dom}(f) \quad \text{and} \quad \text{cod}(g \circ f) = \text{cod}(g). \tag{10}$$

iv′. A mapping

$$\text{id} : \mathfrak{O}\mathbf{C} \to \mathfrak{A}\mathbf{C} \tag{11}$$

that sends a C-object A to the *identity morphism* $\text{id}(A) = 1_A$ on A, such that

$$\text{dom}(1_A) = \text{cod}(1_A) = A. \tag{12}$$

These entities satisfy the following two axioms:

($c2'$) *Associativity*: If $(f, g) \in \mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C}$ and $(g, h) \in \mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C}$, so that both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined, then

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{13}$$

($c3'$) *Identity*: For any $f : A \to B$, $g : C \to A$, one has

$$f \circ 1_A = f, \quad 1_A \circ g = g. \tag{14}$$

The hom-set $\mathbf{C}(A, B)$ is the inverse image of the pair of $\mathbf{C}$-objects A, B under the mapping $\text{dom} \times \text{cod} : \mathfrak{A}\mathbf{C} \to \mathfrak{O}\mathbf{C} \times \mathfrak{O}\mathbf{C}$:

$$\begin{aligned} \mathbf{C}(A, B) &= (\text{dom} \times \text{cod})^{-1}((A, B)) \\ &= \text{dom}^{-1}(A) \cap \text{cod}^{-1}(B) \\ &= \{f \in \mathfrak{A}\mathbf{C} : \text{dom}(f) = A,\ \text{cod}(f) = B\}\ . \end{aligned} \tag{15}$$

And the collection $\mathfrak{A}\mathbf{C}$ of morphisms is the disjoint union

$$\mathfrak{A}\mathbf{C} = \bigcup_{A, B \in \mathfrak{O}\mathbf{C}} \mathbf{C}(A, B). \tag{16}$$

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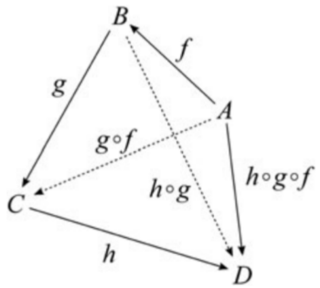
For other nuances [e.g., why there is no Axiom ($c1'$)] of the interplay between these two definitions of category and their consequences, see *RL*: 6.7–6.11.

0.3 Associativity Axioms ($c2$) and ($c2'$) imply parentheses are unnecessary in sequential compositions, and the composite in (5) and (13) may simply be denoted

(17) $$h \circ g \circ f : A \to D.$$

The equivalence is illustrated in the *commutative diagram*

(18)



which is a graphical representation that the four paths

(19) $$\begin{cases} A \xrightarrow{g \circ f} C \xrightarrow{h} D \\ A \xrightarrow{f} B \xrightarrow{h \circ g} D \\ A \xrightarrow{h \circ g \circ f} D \\ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \end{cases}$$

trace the same morphism in $\mathbf{C}(A, D)$.

0.4 Categorical Examples

Example i. Note that the only morphisms that are required to exist are the identities on the objects. When there are no objects, there are no identity morphisms. So trivially there is the *empty* category $\varnothing$, with no objects and no morphisms. The next trivial category $\mathbf{C}$ contains exactly one object A and the single identity morphism 1_A, i.e., $\mathfrak{O}\mathbf{C} = \{A\}$ and $\mathfrak{A}\mathbf{C} = \mathbf{C}(A, A) = \{1_A\}$.

Example ii. The correspondence $A \leftrightarrow 1_A$ is a bijection between $\mathfrak{O}\mathbf{C}$ and the subset of identity morphisms in $\mathfrak{A}\mathbf{C}$. The simplest nonempty category is one in

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which every morphism is an identity, where $\mathbf{C}(A,B)=\varnothing$ when $A\neq B$, and $\mathbf{C}(A,A)=\{\,1_A\,\}$. Such a category $\mathbf{C}$ is called *discrete*. Every set X is the set of objects of a discrete category $\mathbf{C}$, with $\mathfrak{O}\mathbf{C}=X$ and $\mathfrak{A}\mathbf{C}=\{\,1_x : x\in X\,\}$.

Example iii. A *monoid* is an algebraic structure with an associative binary operation and an identity element. For any category $\mathbf{C}$ and any $\mathbf{C}$-object X, the hom-set $\mathbf{C}(X,X)$ is a monoid (with the binary operation the composition of $\mathbf{C}$-morphisms, and the identity 1_X). Indeed, a monoid M is a category $\mathbf{C}$ with one object, such that $\mathfrak{O}\mathbf{C}=\{M\}$ and $\mathfrak{A}\mathbf{C}=M$.

Example iv. A *preorder* $\leq$ is a reflexive and transitive relation on a set X ($\leq\,\subset X\times X$; cf. *ML*:1.10). A *preordered set* $\langle X,\leq\rangle$ may be considered as a category, in which the objects are elements of X, and a hom-set $\mathbf{C}(x,y)$ for $x,y\in X$ has either a single element or is empty, according to whether $x\leq y$ or not. The identity $1_x\in\mathbf{C}(x,x)$ is reflexivity $x\leq x$, and the composition $\circ:\mathbf{C}(x,y)\times\mathbf{C}(y,z)\to\mathbf{C}(x,z)$ is transitivity that $x\leq y$ and $y\leq z$ imply $x\leq z$. In sum, $\mathfrak{O}\mathbf{C}=X$ and $\mathfrak{A}\mathbf{C}=\,\leq$. A preordered set is a category $\mathbf{C}$ in which the mapping dom $\times$ cod : $\mathfrak{A}\mathbf{C}\to\mathfrak{O}\mathbf{C}\times\mathfrak{O}\mathbf{C}$ ($f\mapsto(\mathrm{dom}(f),\ \mathrm{cod}(f))$ as in Definition 0.2ii′ above) is injective. This implies that each hom-set $\mathbf{C}(x,y)$ contains at most one morphism; a category with this property is called *thin*. Thus categories with larger hom-sets may be considered to 'generalize' preorders: each morphism defines a distinct preorder relation.

Preorders include *partial orders* (preorders with the additional antisymmetry axiom that $x\leq y$ and $y\leq x$ imply $x=y$; cf. *ML*: 1.20) and *total* (or *linear*) *orders* (partial orders such that, for all $x,y\in X$, either $x\leq y$ or $y\leq x$; cf. *ML*: 1.32). For a *partially ordered set* (*poset*) considered as a category $\mathbf{C}$, the antisymmetry means that if both $\mathbf{C}(x,y)$ and $\mathbf{C}(y,x)$ are nonempty then $x=y$; a category with this property is called *skeletal*. For a *totally ordered set* (*toset*) considered as a category $\mathbf{C}$, the total order means that for all $x,y\in X$, either $\mathbf{C}(x,y)$ or $\mathbf{C}(y,x)$ is nonempty (but if both are nonempty then $x=y$).

Example v. The category **Set** has its collection of objects the set of all sets (in a suitably naive universe of small sets), and its morphisms are mappings from one small set to another. Let me explain *en passant* the phrase 'a suitably naive universe of small sets'. One assumes the existence of a suitable universe U of sets, and then describe a set as a *small set* if it is a member of U. 'Suitable' simply means U has to be big enough for one's purpose, so that the set-theoretic constructions, used in contexts that occur naturally in mathematics, will exist, but U is not too big as to give rise to paradoxical contradictions. This is set theory from the "naive" point of view, and is the common approach of most mathematicians

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(other than, of course, those in mathematical logic and the foundations of mathematics). In other words, one (aspiring relational biologist included) acknowledges these paradoxes, and moves on.

In a category **C**, the **C**-objects are not necessary sets and the **C**-morphisms are not necessary mappings. But the category **Set** involves itself in an essential way in every category. This is because $\mathfrak{O}\mathbf{C}$ and $\mathfrak{A}\mathbf{C}$ themselves are (for most purposes) sets. Composition and identities are defined by mappings (from a set to a set; Definitions 0.2 iii′ & iv′). Above all, for each pair of **C**-objects A and B, the hom-*set* of **C**-morphisms $\mathbf{C}(A, B)$ is a set.

Example vi. The category **Mon** has its collection of objects the set of all monoids, and its morphisms are monoid homomorphisms from one monoid to another (that preserve the structure of the associative binary operation and the identity). The category **Pos** has as its collection of objects the set of all posets, and its morphisms are order-preserving (isotone) maps from one poset to another (cf. *ML*: 1.23).

Note the difference between the 'single-object-as-a-category' and the 'category of all objects-with-structure and structure-preserving morphisms' considered in the examples above. Contrast a single-set-as-a-category (i.e. a discrete category) with the category **Set** of all sets and mappings. Likewise, contrast a single-monoid-as-a-category (i.e., a single-object category) with **Mon**, and a skeletal category with **Pos**.

0.5 Isomorphism (*ML*: A.5) A morphism $f : A \to B$ is an *isomorphism* if there exists an *inverse morphism* $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. If such an inverse morphism exists, it is unique, and is denoted by f^{-1}.

An isomorphism with the same object A as domain and codomain is an *automorphism* on A. If there exists an isomorphism from A to B then A is *isomorphic* to B, and this relation is denoted by

$$A \cong B. \tag{20}$$

Isomorphic objects are considered abstractly (and often identified as) the same, and most constructions of category theory are 'unique up to isomorphism' (in the sense that two similarly constructed objects are isomorphic, if not necessarily identical). The isomorphism relation $\cong$ is an equivalence relation on the collection $\mathfrak{O}\mathbf{C}$ of objects in a category. So instead of "A is isomorphic to B" one may simply say "A and B are isomorphic" by symmetry.

In the category **Set** (of sets and single-valued mappings), isomorphism is the concept of *equipotence* (*RL*: 0.5, et seq. on cardinality); two sets are **Set**-isomorphic precisely when there exists a bijection between them.

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0.6 Subcategory (*ML*: A.7) Given categories **C** and **D**, one says that **C** is a *subcategory* of **D** if each **C**-object is a **D**-object, each **C**-morphism is a **D**-morphism, and compositions of morphisms are the same in the two categories. Thus $\mathfrak{O}\mathbf{C} \subset \mathfrak{O}\mathbf{D}$, and for any two **C**-objects A and B, $\mathbf{C}(A,B) \subset \mathbf{D}(A,B)$ (whence *a fortiori* $\mathfrak{A}\mathbf{C} \subset \mathfrak{A}\mathbf{D}$).

More formally, a *subcategory* **C** of a category **D** is given by

i. a subset $X \subset \mathfrak{O}\mathbf{D}$ of **D**-objects, and
ii. a subset $\Phi \subset \mathfrak{A}\mathbf{D}$ of **D**-morphisms,

such that

($s1$) for every $A \in X$, the identity morphism $1_A \in \Phi$;
($s2$) for every morphism $f : A \to B$ in Φ, both the domain A and the codomain B are in X; and
($s3$) for every pair of morphisms f and g in Φ, the composite $g \circ f$ is in Φ whenever it is defined.

These conditions ensure that **C** is a category in its own right: the collection of **C**-objects is $\mathfrak{O}\mathbf{C} = X$, the collection of **C**-morphisms is $\mathfrak{A}\mathbf{C} = \Phi$, and the identities and composition are as in **D**.

If $\mathbf{C}(A,B) = \mathbf{D}(A,B)$ holds for all **C**-objects A and B, **C** is a *full subcategory* of **D**. A full subcategory is one that includes *all* **D**-morphisms between objects of **C**. For any collection $X \subset \mathfrak{O}\mathbf{D}$ of **D**-objects, there is a unique full subcategory **C** of **D** with $X = \mathfrak{O}\mathbf{C}$.

Functor

> **functor** (noun): from Latin *functus*, past participle of the verb *fungi* "to perform" (not the same as the *fungi* meaning yeasts and molds). The Indo-European root is *bheug*- "to enjoy". … [-*or* "a male person or thing that does the indicated action".] A functor is a mapping from one category into another that is compatible with it; the Latin word means literally "performer".
>
> — Steven Schwartzman (1994)
> *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English*

A functor is a morphism of categories, a mapping from one category to another that preserves the structures and processes therein. A category is defined by the roles of its four cast members: objects, morphisms, composition, identities. A functor, in its performance, must therefore suitably relate these four roles.

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0.7 Definition A (*ML*: A.10) Let **C** and **D** be categories. A (*covariant*) *functor* F from **C** to **D**, $F : \mathbf{C} \to \mathbf{D}$, consists of a pair of mappings $\langle F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D},\ F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D} \rangle$ on the categorical 'components' of objects and morphisms, called respectively the *object mapping* and the *arrow mapping*, that assigns

i. to each **C**-object A a **D**-object FA,

$$F : A \mapsto FA, \tag{21}$$

and

ii. to each **C**-morphism $f : A \to B$ a **D**-morphism $Ff : FA \to FB$

$$F : [f : A \to B] \mapsto [Ff : FA \to FB]. \tag{22}$$

The object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ and the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ are related in such a way that

(*f*1) if $g \circ f$ is defined in **C**, then $Fg \circ Ff$ is defined in **D**, with

$$F(g \circ f) = Fg \circ Ff; \tag{23}$$

and

(*f*2) for each **C**-object A,

$$F 1_A = 1_{FA}. \tag{24}$$

Category theory is a formal image of the modelling process itself. It is, indeed, the *general* theory of modelling relations, and not just some *specific* way of making models of one thing in another. It thus generates mathematical counterparts of epistemologies, entirely within the formal realm. One may think of the functor $F : \mathbf{C} \to \mathbf{D}$ as providing, for the category **C**, a *model* $F(\mathbf{C})$ in another category **D**, of all the **C**-objects and **C**-morphisms.

The object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ maps material causes in **C** to material causes in **D**; the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ maps efficient causes in **C** to efficient causes in **D**. The pairwise functorial connection thus extends to the various manifestations; whence $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ maps structures to structures, material entailment to material entailment, and $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ maps functions to functions, repair to repair, etc.

0.8 Injection and Surjection The functor $F : \mathbf{C} \to \mathbf{D}$ is *injective on objects* if the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ is injective, and is *surjective on objects* if $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ is surjective. Similarly, $F : \mathbf{C} \to \mathbf{D}$ is *injective* (respectively,

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surjective) *on arrows* (or *on morphisms*) if the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ is injective (respectively, surjective).

In set theory, equality of sets is formulated as the Axiom of Extension (*ML* : 0.2): *Two sets are equal if and only if they have the same elements.* (Hence, a priori, two elements of a set are either equal or not.) The object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ is surjective if, by definition, for each **D**-object X there exists a **C**-object A such that $X = FA$. When the requirement of **D**-object-equality is relaxed to **D**-isomorphism, one generalizes the property of surjectivity on objects: a functor $F : \mathbf{C} \to \mathbf{D}$ is *essentially surjective on objects* if for each **D**-object X there exists a **C**-object A such that $X \cong FA$. And of course, if a functor is surjective on objects then it is essentially surjective on objects. 'Essential injectivity on objects', on the other hand, has finer nuances, and its various degrees shall, indeed, be important contributing characteristics towards invertibility.

Property (*f*2), that a functor $F : \mathbf{C} \to \mathbf{D}$ maps an identity morphism in **C** to an identity morphism in **D**, implies that the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ entails the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$. This is because, when the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ takes the value $F1_A = 1_X \in \mathbf{D}(X, X)$ at the **C**-morphism $1_A \in \mathbf{C}(A, A)$, with the correspondence $X \leftrightarrow 1_X$ one may uniquely define the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ to take the value $FA = X$ at the **C**-object A.

A functor, just like a category, may alternatively be defined in terms of arrows (without the redundant postulate i′ for the object mapping):

0.9 Definition B A (*covariant*) *functor* F from category **C** to category **D**, $F : \mathbf{C} \to \mathbf{D}$, is

ii′. a mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ of arrows that sends $f \in \mathfrak{A}\mathbf{C}$ to $Ff \in \mathfrak{A}\mathbf{D}$,

$$F : f \mapsto Ff, \tag{25}$$

carrying

(*f*1′) each composable pair of **C**-morphisms $(f, g) \in \mathfrak{A}\mathbf{C} \times_{\mathfrak{O}\mathbf{C}} \mathfrak{A}\mathbf{C}$ to a composable pair of **D**-morphisms $(Ff, Fg) \in \mathfrak{A}\mathbf{D} \times_{\mathfrak{O}\mathbf{D}} \mathfrak{A}\mathbf{D}$, with

$$F(g \circ f) = Fg \circ Ff; \tag{26}$$

and

(*f*2′) each identity morphism in $\mathfrak{A}\mathbf{C}$ to an identity morphism in $\mathfrak{A}\mathbf{D}$.

Often, for the sake of clarity, however, one explicitly specifies the action of a functor on both objects and arrows.

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0.10 Functorial Representation A functor $F : \mathbf{C} \to \mathbf{D}$ may be succinctly represented in

(27) $$F : \begin{cases} A \mapsto FA & (A \in \mathfrak{O}\mathbf{C}) \\ [f : A \to B] \mapsto [Ff : FA \to FB] & (f \in \mathfrak{A}\mathbf{C}) \end{cases},$$

the two lines denoting respectively the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ and the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$.

As denoted in (27), the general representation does not, of course, provide additional information about F. Its use lies in the specific forms that the final causes FA and $Ff : FA \to FB$ would take for specific functors under study. Then representation (27) provides a concise summary of the actions of the functor F.

0.11 Contravariant Functor Besides the covariant functors there is a dual kind of functors that reverses the direction of the processes and the order of composition. A *contravariant functor* F from $\mathbf{C}$ to $\mathbf{D}$ assigns

i. to each $\mathbf{C}$-object A a $\mathbf{D}$-object FA,

(28) $$F : A \mapsto FA,$$

and

ii $^{\text{op}}$. to each $\mathbf{C}$-morphism $f : A \to B$ a $\mathbf{D}$-morphism $Ff : FB \to FA$

(29) $$F : [f : A \to B] \mapsto [Ff : FB \to FA],$$

such that

($f1^{\text{op}}$) if $g \circ f$ is defined in $\mathbf{C}$, then $Ff \circ Fg$ is defined in $\mathbf{D}$, and

(30) $$F(g \circ f) = Ff \circ Fg.$$

and

($f2$) for each $\mathbf{C}$-object A,

(31) $$F1_A = 1_{FA}.$$

Its succinct representation is

(32) $$F : \begin{cases} A \mapsto FA & (A \in \mathfrak{O}\mathbf{C}) \\ [f : A \to B] \mapsto [Ff : FB \to FA] & (f \in \mathfrak{A}\mathbf{C}) \end{cases}.$$

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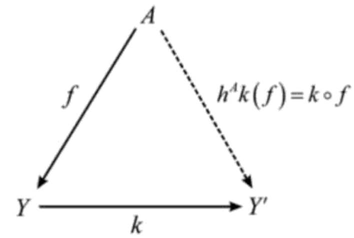
0.12 Hom-Functors (*ML*: A.13) For any category **C** and a **C**-object A, the *covariant hom-functor* $h^A = \mathbf{C}(A, \bullet)$ from **C** to **Set** assigns to each **C**-object Y the set $h^A Y = \mathbf{C}(A, Y)$, and to a **C**-morphism $k : Y \to Y'$ the mapping $h^A k : \mathbf{C}(A, Y) \to \mathbf{C}(A, Y')$ defined by

(33) $$h^A k : f \mapsto k \circ f \quad \text{for} \quad f : A \to Y;$$

i.e. via the diagram

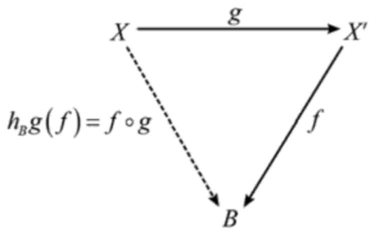
(34)


Note the action of $h^A k$ may be described as 'composition with k-on-the-left'.

Dually, for a category **C** and a **C**-object B, the *contravariant hom-functor* $h_B = \mathbf{C}(\bullet, B)$ assigns to each **C**-object X the set $h_B X = \mathbf{C}(X, B)$, and to a **C**-morphism $g : X \to X'$ the mapping $h_B g : \mathbf{C}(X', B) \to \mathbf{C}(X, B)$ defined by

(35) $$h_B g(f) = f \circ g \quad \text{for} \quad f : X' \to B;$$

i.e. via the diagram

(36)


Note the action of $h_B g$ may be described as 'composition with g-on-the-right'.

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0.13 The Category Cat (*ML*: A.15) The idea of category applied to categories and functors themselves yields the category **Cat**, with objects all categories (i.e. all *small* categories in a suitably naïve universe) and morphisms all functors between them.

Functors can be composed—given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, the maps $A \mapsto G(FA)$ and $f \mapsto G(Ff)$ on $\mathbf{C}$-objects A and $\mathbf{C}$-morphisms f define a functor $G \circ F : \mathbf{C} \to \mathbf{E}$. This composition is associative, since it is associative componentwise on objects and morphisms. For each category $\mathbf{C}$ there is an *identity functor* $I_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$, defined in the natural way as the identity map componentwise, sending each $\mathbf{C}$-object to itself and each $\mathbf{C}$-morphism to itself.

An *isomorphism* $F : \mathbf{C} \to \mathbf{D}$ of categories is a functor that is a bijection both on objects and on morphisms. This is equivalent to the existence of an 'inverse functor' $F^{-1} : \mathbf{D} \to \mathbf{C}$.

0.14 Faithful and Full Functors (*ML*: A.16) For each pair of $\mathbf{C}$-objects A and B, the functor $F : \mathbf{C} \to \mathbf{D}$ assigns to each $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$ a $\mathbf{D}$-morphism $Ff \in \mathbf{D}(FA, FB)$, and so defines a (single-valued) mapping

$$F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB) \tag{37}$$

with $F_{A,B}(f) = Ff$. The functor may alternatively be considered as the collection of these doubly-indexed mappings:

$$F = \{\, F_{A,B} : A, B \in \mathfrak{O}\mathbf{C} \,\}. \tag{38}$$

The functor F is *faithful* when each $F_{A,B}$ is injective, and *full* when each $F_{A,B}$ is surjective.

Faithfulness and fullness are functorial conditions on the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$, and each by itself does not impose limitations on the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$. So a faithful functor need not be injective on objects: two $\mathbf{C}$-objects may map to the same $\mathbf{D}$-object. Likewise, a full functor need not be surjective on objects: there may be $\mathbf{D}$-objects not of the form FA for some $A \in \mathfrak{O}\mathbf{C}$.

Injectivity on arrows is a stronger condition than faithfulness: if $F : \mathbf{C} \to \mathbf{D}$ is injective on arrows then it is faithful. But the converse implication is not true: a faithful functor need not be injective on arrows. The collection of $\mathbf{C}$-hom-sets $\{\, \mathbf{C}(A, B) : A, B \in \mathfrak{O}\mathbf{C} \,\}$ forms a partition of $\mathfrak{A}\mathbf{C}$ (cf. (16) above), and faithfulness only requires that the *restriction* of the arrow mapping to each block $\mathbf{C}(A, B)$, $F_{A,B} = F|_{\mathbf{C}(A,B)} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$, be injective, whereas injectivity on

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arrows requires $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ to be injective on the whole domain $\mathfrak{A}\mathbf{C}$. A functor $F : \mathbf{C} \to \mathbf{D}$ that is faithful may still map two $\mathbf{C}$-morphisms with different domains or codomains, (therefore belonging to different $\mathbf{C}$-hom-sets) to the same $\mathbf{D}$-morphism. Injectivity on arrows also implies injectivity on objects. This is because, if $F : \mathbf{C} \to \mathbf{D}$ is injective on arrows, then in particular, for $A, B \in \mathfrak{O}\mathbf{C}$ and $A \neq B$, F must map the distinct $1_A, 1_B \in \mathfrak{A}\mathbf{C}$ to distinct $1_{FA}, 1_{FB} \in \mathfrak{A}\mathbf{D}$, whence $FA \neq FB$ in $\mathfrak{O}\mathbf{D}$.

Similarly, surjectivity on arrows implies surjectivity on objects: if $F : \mathbf{C} \to \mathbf{D}$ is surjective on arrows, for each $X \in \mathfrak{O}\mathbf{D}$ there is an $f \in \mathfrak{A}\mathbf{C}$ that gets mapped by $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$ to $1_X \in \mathbf{D}(X, X) \subset \mathfrak{A}\mathbf{D}$, thence both $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathfrak{O}\mathbf{C}$ (which need not coincide) are mapped by $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ to X. Further, if a functor is surjective on arrows then it is full, hence contrapositively a functor that is not full cannot be surjective on arrows. Conversely, a full functor $F : \mathbf{C} \to \mathbf{D}$ need not be surjective on arrows: $\mathbf{D}$-morphisms between $\mathbf{D}$-objects that are not of the form FA for some $A \in \mathfrak{O}\mathbf{C}$ cannot come from $\mathbf{C}$-morphisms.

Even if $F : \mathbf{C} \to \mathbf{D}$ is both faithful and full, whence each mapping $F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$ is bijective, the collection $\{F_{A,B} : A, B \in \mathfrak{O}\mathbf{C}\}$ of **Set**-isomorphisms is still not sufficient to ensure that F is an isomorphism in the category **Cat**. As explicated above, the range $F(\mathbf{C})$ is not necessarily isomorphic to either $\mathbf{C}$ or $\mathbf{D}$. A faithful and full functor is, however, necessarily injective on objects *up to isomorphism*. When $F : \mathbf{C} \to \mathbf{D}$ is a faithful and full functor, one may readily verify, using the definition of isomorphism and the premise that all mapping $F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB)$ are then bijections, that $FA \cong FB$ implies $A \cong B$. This defines one version of 'essentially injective on objects'.

0.15 Inclusion Functor (*ML*: A.12(v)) If $\mathbf{C}$ is a subcategory of $\mathbf{D}$, there is a functor that takes objects and morphisms to themselves; i.e., both the object mapping and the arrow mapping are the corresponding inclusion maps. This is the *inclusion functor* (of $\mathbf{C}$ in $\mathbf{D}$), denoted $i : \mathbf{C} \to \mathbf{D}$.

The inclusion functor $i : \mathbf{C} \to \mathbf{D}$ is injective on objects, injective on arrows, and faithful. It is full if and only if $\mathbf{C}$ is a full subcategory of $\mathbf{D}$.

0.16 Concrete Category and Forgetful Functor A *concrete category* $\mathbf{C}$ is a category equipped with a faithful functor $F : \mathbf{C} \to \mathbf{Set}$. The faithfulness of F allows the (one-to-one) identification of a $\mathbf{C}$-morphism $f \in \mathfrak{A}\mathbf{C}$ with the mapping $Ff \in \mathfrak{A}\mathbf{Set}$. A concrete category may be described as a category $\mathbf{C}$ in which each $\mathbf{C}$-object A comes equipped with an 'underlying set' FA, each $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$ is an actual mapping $Ff : FA \to FB$, and the composition of $\mathbf{C}$-morphisms is a composition of mappings. Stated otherwise, the faithful functor $F : \mathbf{C} \to \mathbf{Set}$ allows the consideration of $\mathbf{C}$-objects as *sets with additional*

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structure, and of **C**-morphisms as *structure-preserving mappings*. The functor $F : \mathbf{C} \to \mathbf{Set}$ then, in essence, 'forgets' the additional structure of the objects and hence the structure-preserving aspect of the mappings; it is therefore called the *forgetful functor*.

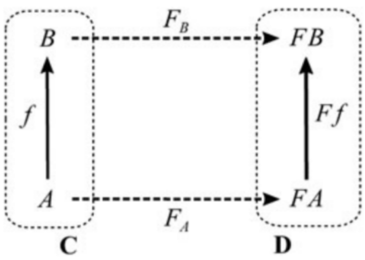
Many important categories have interpretations as concrete categories; for example, the category **Grp** of groups and homomorphisms, the category **Vct** of vector spaces and linear transformations, and the category **Top** of topological spaces and continuous mappings (*ML*: A.6).

The requirement for a concrete category **C** is that the functor $F : \mathbf{C} \to \mathbf{Set}$ be faithful, but not necessarily injective on arrows. This means that F must take different morphisms in $\mathbf{C}(A, B)$ to different mappings in $\mathbf{Set}(FA, FB)$, but it may take different **C**-objects to the same set, since injectivity on objects is not a requirement (say $A, B \in \mathfrak{O}\mathbf{C}$, $A \neq B$, but the sets $FA = FB$). If this occurs, it will also take corresponding **C**-morphisms in $\mathbf{C}(A, Y)$ and $\mathbf{C}(B, Y)$, for example, to the same mapping in $\mathbf{Set}(FA, FY) = \mathbf{Set}(FB, FY)$.

0.17 Membership and Element-Tracing In a concrete category **C**, one may speak of 'membership' $a \in A$ for a **C**-object $A \in \mathfrak{O}\mathbf{C}$, and 'element chase' $f : a \mapsto b = f(a)$ associated with a **C**-morphism $f : A \to B$ where $f \in \mathfrak{A}\mathbf{C}$. (For the element-trace notation $f : a \mapsto f(a)$ see *ML*: 1.5 and *RL*: 1.7; I shall also re-introduce it in *IL*: Chapter 2.)

When $F : \mathbf{C} \to \mathbf{D}$ is a functor between concrete categories, the object mapping $F : \mathfrak{O}\mathbf{C} \to \mathfrak{O}\mathbf{D}$ at $A \in \mathfrak{O}\mathbf{C}$, $F : A \mapsto FA$, hierarchically entails the element mapping $F_A : A \to FA$. The action of the arrow mapping $F : \mathfrak{A}\mathbf{C} \to \mathfrak{A}\mathbf{D}$, taking $f : A \to B$ to $Ff : FA \to FB$, may then be represented in the commutative diagram

(39)



which declares the equality of two sequential compositions

(40) $$F_B \circ f = Ff \circ F_A : A \to FB.$$

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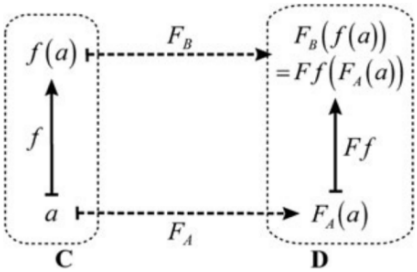
The compositional equality entails, for $a \in A$ and the traces of the paths

(41) $$\begin{cases} a \mapsto f(a) \mapsto F_B(f(a)) \\ a \mapsto F_A(a) \mapsto Ff(F_A(a)) \end{cases},$$

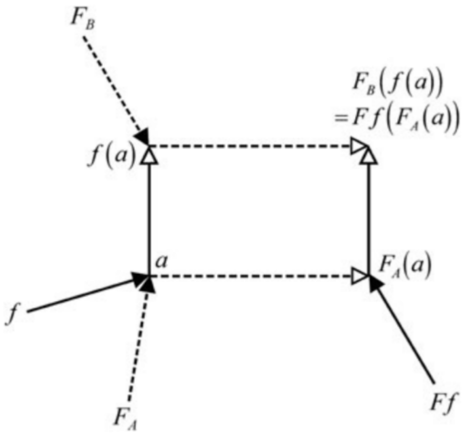
elemental equality of the final causes, resulting in

(42) $$F_B(f(a)) = Ff(F_A(a)) \in FB.$$

The corresponding element-trace diagram is

(43) 

In terms of the solid-headed and hollow-headed arrows of a *relational diagram in graph-theoretic form* (*ML*: 5.4–5.11; *RL*: E.6 & 3.1; and, in anticipation, *IL*: 2.2), the confluence of two sequential compositions (40) is represented thus:

(44) 

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Natural Transformation

> … "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".
>
> — Saunders Mac Lane (1997)
> *Category Theory for the Working Mathematician*
> § I.4

A natural transformation is a morphism of functors (*ML*: A.17). This is the vehicle with which one functor models another.

0.18 Definition Suppose

(45) $$\mathbf{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{D}$$

are two functors between the same two categories. A *natural transformation* τ from F to G, notated

(46) $$\tau : F \to G,$$

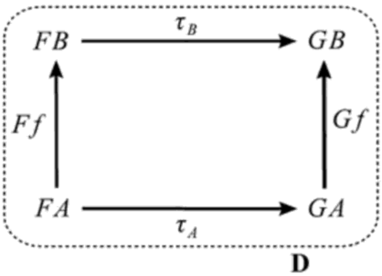
i. assigns to each $\mathbf{C}$-object A a $\mathbf{D}$-morphism $\tau_A \in \mathbf{D}(FA, GA)$,

such that,

($t1$) for each $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$, the $\mathbf{D}$-morphisms $Gf \in \mathbf{D}(GA, GB)$, $\tau_A \in \mathbf{D}(FA, GA)$, $\tau_B \in \mathbf{D}(FB, GB)$, and $Ff \in \mathbf{D}(FA, FB)$ commute:

(47) $$Gf \circ \tau_A = \tau_B \circ Ff.$$

Graphically, this is the commutative diagram

(48)


τ_A is called the *component of* τ at A.

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Note the antitone decrease in the numbers of requirements as one ascends the hierarchy: a category (Definition 0.1) has four assignments and three properties (*c*1)–(*c*3); a functor (Definition 0.7) has two assignments and two properties (*f*1)–(*f*2); a natural transformation has one assignment i and one property (*t*1).

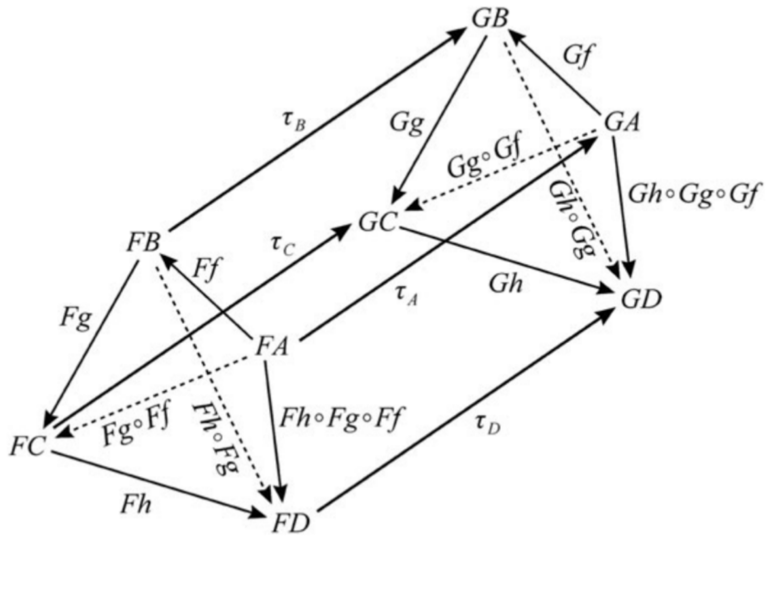
A natural transformation $\tau : F \to G$ may be considered to be determined by the collection of components

(49) $$\{\, \tau_A \in \mathbf{D}(FA, GA) : A \in \mathfrak{O}\mathbf{C} \}.$$

$\tau_A \in \mathbf{D}(FA, GA)$ is said to be *natural in A*, in the sense that when the **C**-object A is treated as a variable, the **D**-morphism $\tau_A \in \mathbf{D}(FA, GA)$ is 'defined in the same way for each A'. This is the standard terminology ("informal parlance") of a more proper "$\tau_{(\bullet)} : F(\bullet) \to G(\bullet)$ is natural in its variable".

Since a functor $F : \mathbf{C} \to \mathbf{D}$ gives a picture (or model) in **D** for any collection of objects and morphisms of **C**, one may consider a natural transformation $\tau : F \to G$ to be a translation (alternate description or model) of the picture F to the picture G. For example, picture (18), the commutative diagram of **C**-morphism associativity, has the following translation from F to G:

(50)



a digraph of $\mathbf{D}$-morphisms in which all paths are commutative (i.e., any two directed paths with the same initial and final vertices trace the same morphism).

0.19 Functor Category The *functor category* $\mathbf{D}^{\mathbf{C}}$ has as objects all (covariant) functors from $\mathbf{C}$ to $\mathbf{D}$, and as morphisms natural transformations, and to have composition and identities the 'pointwise' ones (*ML*: A.18).

0.20 Natural Isomorphism A natural transformation $\tau : F \to G$ is a *natural isomorphism*, denoted

$$\tau : F \cong G, \tag{51}$$

if and only if for each $\mathbf{C}$-object A, $\tau_A \in \mathbf{D}(FA, GA)$ is an isomorphism in $\mathbf{D}$.

Stated otherwise, a natural isomorphism is an isomorphism in the functor category $\mathbf{D}^{\mathbf{C}}$.

0.21 Category of Diagrams If $\mathbf{C}$ is a trivial category with only a single object A and only the single morphism 1_A in $\mathbf{C}(A, A)$ (Example 0.4i), then the functor category $\mathbf{D}^{\mathbf{C}}$ is a discrete category (Example 0.4ii), consisting of the objects of $\mathbf{D}$ together with their identity morphisms. That is, $\mathfrak{O}\mathbf{D}^{\mathbf{C}} \cong \mathfrak{O}\mathbf{D}$ and $\mathfrak{A}\mathbf{D}^{\mathbf{C}} \cong \{1_X : X \in \mathfrak{O}\mathbf{D}\} \cong \mathfrak{O}\mathbf{D}$.

Next, let $\mathbf{C}$ consist of a pair of objects A, B, and suppose that the morphisms in $\mathbf{C}$ consist only of 1_A, 1_B, and a single morphism $f : A \to B$. Then given any other category $\mathbf{D}$, the functor category $\mathbf{D}^{\mathbf{C}}$ may be regarded as consisting of all the morphisms in $\mathbf{D}$; i.e., $\mathfrak{O}\mathbf{D}^{\mathbf{C}} \cong \mathfrak{A}\mathbf{D}$.

A graphic interpretation is as follows: the category $\mathbf{C}$ may be regarded as being specified by the simple diagram

$$A \xrightarrow{f} B. \tag{52}$$

(The identity morphisms correspond to self-loops (*ML*: 6.3) on the objects, and may be omitted.) The functor category $\mathbf{D}^{\mathbf{C}}$ consists of all copies of this diagram in $\mathbf{D}$; i.e., all diagrams of the form

$$X \xrightarrow{g} Y, \tag{53}$$

where $X = FA$, $Y = FB$, $g = Ff$ for some covariant functor $F : \mathbf{C} \to \mathbf{D}$. More illustratively, when the category $\mathbf{C}$ is concrete and $f : A \to B$ is a mapping, the relational diagram in graph-theoretic form of (52) may be drawn as

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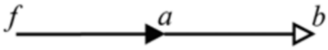
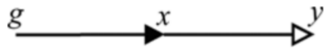
(54) 

with corresponding relational-diagrammatic representation

(55) 

in $\mathbf{D}$. Thus, if the category $\mathbf{C}$ is regarded as specifying the 'pattern' (54), the functor category $\mathbf{D}^{\mathbf{C}}$ consists of all copies of this pattern which may be formed in $\mathbf{D}$.

More generally, *any* diagram of $\mathbf{C}$-morphisms (i.e., a network) in a category $\mathbf{C}$ can be regarded as specifying a subcategory $\mathbf{C}'$ of $\mathbf{C}$ (with careful inclusion of composites); then the functor category $\mathbf{D}^{\mathbf{C}'}$ (which is a subcategory of $\mathbf{D}^{\mathbf{C}}$) may again be regarded as the collection of copies of this diagram that may be formed from the objects and morphisms of $\mathbf{D}$. Hence the larger functor category $\mathbf{D}^{\mathbf{C}}$ contains copies of all $\mathbf{C}$-diagrams, and is therefore also called the *category of diagrams* over $\mathbf{C}$.

0.22 Binary Operation Let $R : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor (Definition 0.16) that sends a group $G \in \mathfrak{O}\mathbf{Grp}$ to its underlying set $RG \in \mathfrak{O}\mathbf{Set}$ and a homomorphism $\varphi \in \mathbf{Grp}(G, H)$ to the mapping $R\varphi \in \mathbf{Set}(RG, RH)$. Let $S : \mathbf{Grp} \to \mathbf{Set}$ be the "Cartesian square functor", defined by

(56) $$S : \begin{cases} G \mapsto RG \times RG & (G \in \mathfrak{O}\mathbf{Grp}) \\ [\varphi : G \to H] \mapsto [R\varphi : RG \times RG \to RH \times RH] & (\varphi \in \mathfrak{A}\mathbf{Grp}) \end{cases},$$

where

(57) $$R\varphi(x, y) = (\varphi \times \varphi)(x, y) = (\varphi x, \varphi y) \quad (x, y \in G).$$

The binary operation $\cdot_G$ of a group $G \in \mathfrak{O}\mathbf{Grp}$ is a mapping

(58) $$\tau_G : RG \times RG \to RG,$$

i.e., $\tau_G \in \mathbf{Set}(RG \times RG, RG) = \mathbf{Set}(SG, RG)$, defined by

(59) $$\tau_G(x, y) = x \cdot_G y \quad (x, y \in G).$$

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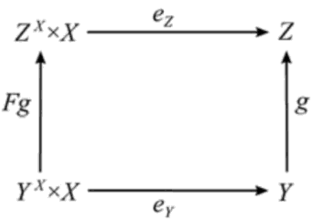
0.24 Evaluation Map For sets X and Y, the set $\mathbf{Set}(X, Y)$ of all mappings from X to Y is denoted Y^X. The *evaluation* mapping $e : Y^X \times X \to Y$, defined, for $f : X \to Y$ and $x \in X$, by $e(f, x) = f(x)$, may be interpreted as a natural transformation as follows. For a fixed X, the map $Y \mapsto Y^X \times X$ extends to a functor $F : \mathbf{Set} \to \mathbf{Set}$ with, for $g : Y \to Z$, $Fg : Y^X \times X \to Z^X \times X$ defined by $Fg : (f, x) \mapsto (g \circ f, x)$ for $f : X \to Y$ and $x \in X$. Then, for this fixed X, $e : F \to I_{\mathbf{Set}}$ is a natural transformation from the functor F to the identity functor $I_{\mathbf{Set}}$, i.e., the following square commutes for any mapping $g : Y \to Z$:

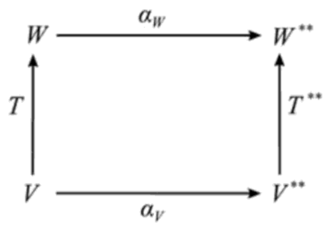
(70)



This reduces to the equation $g(e_Y(f, x)) = e_Z(g \circ f, x)$, which says simply that $g(f(x)) = (g \circ f)(x)$.

0.25 Dual Vector Spaces In the category **Vct** of vector spaces over a fixed field K, evaluation takes the following form. Each element $x \in V$ defines an *evaluation mapping* $\hat{x} : V^* \to K$ by $\hat{x}(f) = f(x)$ for every $f \in V^*$. $\hat{x}$ is a linear functional on V^*, hence it is a member of V^{**}, the second dual space of V. The mapping $\alpha_V : V \to V^{**}$ defined by $\alpha_V(x) = \hat{x}$ is an isomorphism (of vector-spaces) when V is finite dimensional. It is called the *natural isomorphism* between V and V^{**}. (Note this linear-algebraic terminology is part of the inspiration for its category-theoretic analogue.) For a linear transformation $T : V \to W$, one has $T^{**} \circ \alpha_X = \alpha_Y \circ T$, i.e., the diagram

(71)



commutes, which says precisely that $\alpha : I_{\mathbf{Vct}} \to (\bullet)^{**}$ is a natural transformation.

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0.26 Material and Functional Entailments A mapping of two variables $t : X \times Y \to Z$ may be considered as a mapping $\varphi t : X \to Z^Y$ of one variable (in X), and the values of which are mappings with domain in the second variable (in Y) and codomain in Z:

(72) $$[\varphi t(x)](y) = t(x, y) \quad \text{for} \quad x \in X \quad \text{and} \quad y \in Y.$$

Equality (72) describes φ as a bijection (i.e. an isomorphism in **Set**)

(73) $$\varphi : \mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}\left(X, Z^Y\right)$$

that is natural in X, Y, and Z. The isomorphism (73) may be written as

(74) $$\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, \mathbf{Set}(Y, Z))$$

or

(75) $$H(X \times Y, Z) \cong H(X, H(Y, Z)).$$

The last bijection (75), connecting material entailment (metabolism) on the left-hand side with functional entailment (repair) on the right-hand side, is of particular importance in (M,R)-systems. It has wonderful consequences in relational biology, from ontogenesis (*ML*: 13.25) to therapeutics (*RL*: 14.9–14.10). It also leads into the category-theoretic concept of adjunction, and will reappear many times as we proceed in *IL*.

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Part I
Potestas
The Power Set Functor

—Doxology of the *Pater Noster*

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Ascent

The *power set functor* is the most important functor in relational biology. It plays an indispensable role in the category-theoretic formulation of closure to efficient causation (*RL*: 9.3 & 9.4), the very characterization of life. It is (usually) defined as the covariant functor $\mathfrak{P} : \mathbf{Set} \to \mathbf{Set}$ that assigns to a set X its power set $\mathfrak{P}X$ and assigns to a mapping $f : X \to Y$ the mapping $\mathfrak{P}f : \mathfrak{P}X \to \mathfrak{P}Y$ that sends each subset $A \subset X$ to its image $(\mathfrak{P}f)(A) = f(A) \subset Y$, viz.

$$\mathfrak{P} : \begin{cases} X \mapsto \mathfrak{P}X & (X \in \mathfrak{O}\mathbf{Set}) \\ [f : x \mapsto f(x)] \mapsto [\mathfrak{P}f : A \mapsto f(A)] & (f \in \mathfrak{A}\mathbf{Set}) \end{cases}.$$

The power set functor $\mathfrak{P}$ is an essential tool in the analysis of impredicative systems through the reconciliation of two alternate descriptions of an impredicative system. Tersely, the entities '$\langle X, f \rangle$' and '$\langle \mathfrak{P}X, \mathfrak{P}f \rangle$' are alternate descriptions on different 'levels' of the same system 'X'. The mapping $f : X \to Y$ maps on the 'element level' (i.e. parts) while the mapping $\mathfrak{P}f : \mathfrak{P}X \to \mathfrak{P}Y$ maps on the 'set level' (i.e. whole). Thus the power set functor $\mathfrak{P}$ efficiently ascends hierarchical levels.

On our journey in relational biology, the power set functor $\mathfrak{P} : \mathbf{Set} \to \mathbf{Set}$ was first introduced as an example in *ML*: A.12(ii) and explicated in more detail in *RL*: 1.18 *et seq*. I shall presently formulate it alternatively in the category **Rel** of sets and relations.

I would like to share an anecdote. During the algebra session of my PhD comprehensive examination (the other two sessions being analysis and mathematical biology) in the spring of 1980, I was verily grilled by professors on everything I knew about the subject. But my supervisor Robert Rosen asked me exactly one question: 'What are the actions of the power set functor?'

It may therefore be said that thence planted was the intangible seed of the tangible manifestation of a model of the *arbor scientiae* that is this monograph *IL*.

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1
Prooemium
Relations

> It is idle to talk always of the alternative of reason and faith. Reason is itself a matter of faith. It is an act of faith to assert that our thoughts have any relation to reality at all.
>
> — G.K. Chesterton (1908)
> *Orthodoxy*
> Chapter III. "The Suicide of Thought"

Let me begin with a parody of a few passages from the Prologomenon of *RL*. Expository divergence is, however, imminent …

Sets

1.1 Subset and Superset If A and B are sets and if every element of A is an element of B, then A is a *subset* of B, and B is a *superset* of A, denoted

(1) $$A \subset B \quad (\text{equivalently,} \quad B \supset A).$$

Note that this symbolism of containment means *either* $A = B$ (which means the sets A and B have the same elements; Axiom of Extension, *ML*: 0.2) *or* A is a *proper subset* of B (which means that B contains at least one element that is not in A). Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$ (*ML*: 0.4).

1.2 Inclusion Map For $A \subset B$, the mapping $i : A \to B$ defined by $i(a) = a$ for all $a \in A$ is called the *inclusion map* (of A in B). If the sets involved need to be emphasized, one may use the notation $i_{A \subset B}$ for the inclusion map. The inclusion map of A in A is called the *identity map* on A, denoted 1_A $(= i_{A \subset A})$.

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$$|A \cup B| = |A| + |B| - |A \cap B|, \tag{5}$$

implies, in particular, the inequality $|A \cup B| \leq |A| + |B|$, with $|A \cup B| = |A| + |B|$ iff $|A \cap B| = |\varnothing| = 0$ (i.e., iff sets A and B are *disjoint*). The results generalize for finite sets $A_1, A_2, \ldots, A_n$ to

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| = {} & |A_1| + |A_2| + \cdots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\ & \qquad + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ & \vdots \\ & + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|, \end{aligned} \tag{6}$$

which may be succinctly written as

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \right). \tag{7}$$

Further, $\left| \bigcap_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$, with equality iff the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint.

1.12 Power Set If X is a set, the *power set* $\mathrm{P}X$ of X is the family of all subsets of X.

The inclusion relation $\subset$ is a *partial order* on the power set $\mathrm{P}X$; i.e., $\langle \mathrm{P}X, \subset \rangle$ is a *poset* (*ML*: 1.22). The least element of $\langle \mathrm{P}X, \subset \rangle$ is $\varnothing$, and the greatest element of $\langle \mathrm{P}X, \subset \rangle$ is X (*ML*: 1.28). Note that even when $X = \varnothing$, $\varnothing \in \mathrm{P}X$ (indeed, $\mathrm{P}X = \{\varnothing\}$) so $\mathrm{P}X \neq \varnothing$. $\langle \mathrm{P}X, \cup, \cap \rangle$ is a complete, complemented *lattice* (*ML*: 2.1, 2.12, 3.12). $\langle \mathrm{P}X, \cup, \cap, ^c \rangle$ is a Boolean algebra (*ML*: 3.19), called the *power set algebra* of X. A *field of sets* is a subalgebra of a power set algebra. The power set algebra is, indeed, the 'universal' Boolean algebra, in the sense that every Boolean algebra is isomorphic to a field of sets (Stone Representation Theorem, *ML*: 3.20).

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1.13 Characteristic Mapping A subset A of X may be identified with its *characteristic mapping*, a mapping χ_A from X to $2 = \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}. \tag{8}$$

When X is a finite set with n members, there are 2^n different mappings $\chi : X \to 2$, because for each element $x \in X$ there are precisely two choices for the value $\chi(x)$, either 0 or 1. If one defines $A = \chi^{-1}(1) \subset X$, then $\chi = \chi_A$.

1.14 Cardinality of the Power Set Thus if $|X| = n$, then $|\text{Ꝑ}X| = 2^n$, and the equality may be extended to all cardinal numbers n, finite and infinite. This gives an alternate notation of the power set ꝐX as 2^X. One may succinctly write

$$|\text{Ꝑ}X| = \left|2^X\right| = 2^{|X|}. \tag{9}$$

This is consistent even if $X = \varnothing$, when $|X| = 0$ and $|\text{Ꝑ}X| = 2^0 = 1$. Cantor's Theorem (*RL*: 0.8) states that, for all sets X, $|X| < 2^{|X|}$.

The equivalent notation $\text{Ꝑ}X = 2^X$ expressing the power set as a 'power' is, of course, the origin of its name.

> **power** (noun): from Old French *poeir*, from Vulgar Latin *potere*, a variant of Classical Latin *posse* "to be able". The Indo-European root is *poti-* "powerful; lord". If you are able to do many things, you are powerful. A powerful person typically has a large number of possessions (a word derived from *posse*) and a large amount of money. In algebra, when even a relatively small number like 2 is multiplied by itself a number of times the result gets large very quickly; metaphorically speaking, the result is powerful. … If the term *power* is used precisely, it refers to the result of multiplying a number by itself a certain number of times. Consider $2^3 = 8$, which says that the 3rd power of 2 is 8. The power is 8. In less precise usage, however, 3 is identified as the power, when it is actually the exponent.
>
> — Steven Schwartzman (1994)
> *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English*

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1.15 Product Given two sets X and Y, one denotes by $X \times Y$ the set of all *ordered pairs* of the form (x, y) where $x \in X$ and $y \in Y$. The set $X \times Y$ is called the *product* (or *Cartesian product*) of the sets X and Y. If either X or Y is empty, then $X \times Y = \varnothing$.

For all sets X and Y, the cardinality of the product set is the product of the cardinalities of the components:

(10) $$|X \times Y| = |X||Y|.$$

1.16 Projections The mappings

(11) $$\pi_1 : X \times Y \to X \quad \text{and} \quad \pi_2 : X \times Y \to Y,$$

defined, for $x \in X$ and $y \in Y$, by

(12) $$\pi_1(x, y) = x \quad \text{and} \quad \pi_2(x, y) = y,$$

are the *canonical projections* (of the product $X \times Y$ onto its components; cf. *ML*: A.22).

For $A \subset X$, the set $\pi_1^{-1}(A)$ of the *inverse image* of A is the subset of $X \times Y$ containing all ordered pairs (x, y) that are sent by π_1 into A:

(13) $$\pi_1^{-1}(A) = \{(x, y) \in X \times Y : \pi_1(x, y) = x \in A\} = A \times Y.$$

Similarly, for $B \subset Y$,

(14) $$\pi_2^{-1}(B) = \{(x, y) \in X \times Y : \pi_2(x, y) = y \in B\} = X \times B.$$

The product set $A \times B \subset X \times Y$ may be identified with the set $\pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ of intersection of inverse images, since

(15) $$\pi_1^{-1}(A) \cap \pi_2^{-1}(B) = (A \times Y) \cap (X \times B) = A \times B.$$

Relations

1.17 Definition A A *relation* R is an ordered triple (X, Y, Γ) where X and Y are sets and Γ is a subset of the Cartesian product $X \times Y$. The sets X and Y are respectively called the *domain* and *codomain* of the relation, and $\Gamma \subset X \times Y$ is called its *graph*.

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One may indicate the dependence of X, Y, and Γ on R with the notations $X = \mathrm{dom}(R)$, $Y = \mathrm{cod}(R)$, and $\Gamma(R)$.

According to the formal Definition 1.17, a relation uniquely determines its domain and codomain, so two relations with identical graphs but different domains or different codomains are considered different. [This is, indeed, the category-theoretic requirement that a morphism uniquely entails its domain and codomain; Definitions 0.1, 0.2, and cf. *ML*: A.1; *RL*: 6.7 et seq.] Consider the simple example $\Gamma = \{(2, \mathrm{A}), (1, \mathrm{C}), (2, \mathrm{B})\}$. The relations $R_1 = (\{1, 2, 3, 4, 5\}, \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}, \Gamma)$, $R_2 = (\mathbb{N}, \text{alphanumeric characters}, \Gamma)$, $R_3 = (\mathbb{Z}, \text{Latin alphabet}, \Gamma)$, and $R_4 = (\mathbb{R}, \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}, \Gamma)$ are all distinct.

A relation is often identified with its graph (hence the minor equivocation $R = \Gamma(R)$), so one also has the (more common but less rigorous)

1.18 Definition B A *relation* is a set R of ordered pairs; i.e. $R \subset X \times Y$ for some sets X and Y.

Equivalently, a relation R is an element of the power set $\mathcal{P}(X \times Y)$, i.e., $R \in \mathcal{P}(X \times Y)$. With domain X and codomain Y, the relation R is *from X to Y*. The collection of *all* relations from X to Y is thus the power set $\mathcal{P}(X \times Y)$, and, in view of (9) and (10) above, the cardinality of this collection is

$$|\mathcal{P}(X \times Y)| = 2^{|X \times Y|} = 2^{|X||Y|}. \tag{16}$$

If $(x, y) \in R$ (or more precisely $(x, y) \in \Gamma(R)$), then one may say that *x is R-related to y* (or simply *x is related to y* when the involved relation R is understood).

There is a chirality inherent in $(x, y) \in R \subset X \times Y$. When $X \neq Y$, the asymmetry between a relation from X to Y and a relation from Y to X are apparent. But even when $R \subset X \times X$ (whence $\mathrm{dom}(R) = \mathrm{cod}(R) = X$ and one says R is a *relation on X*), $(x, y) \in R$ and $(y, x) \in R$ (for $x, y \in X$) are independent statements. (See *ML*: 1.9 et seq. for an exposition of the epistemological consequences of relations on X.) To emphasize the chirality inherent in $(x, y) \in R$, one may also say that x is a *left R-relative* (*left relative*) of y, and that y is a *right R-relative* (*right relative*) of x.

1.19 External and Internal Entailments Note that even in the formulation 1.18, a relation still has to uniquely determine its domain and codomain, although

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