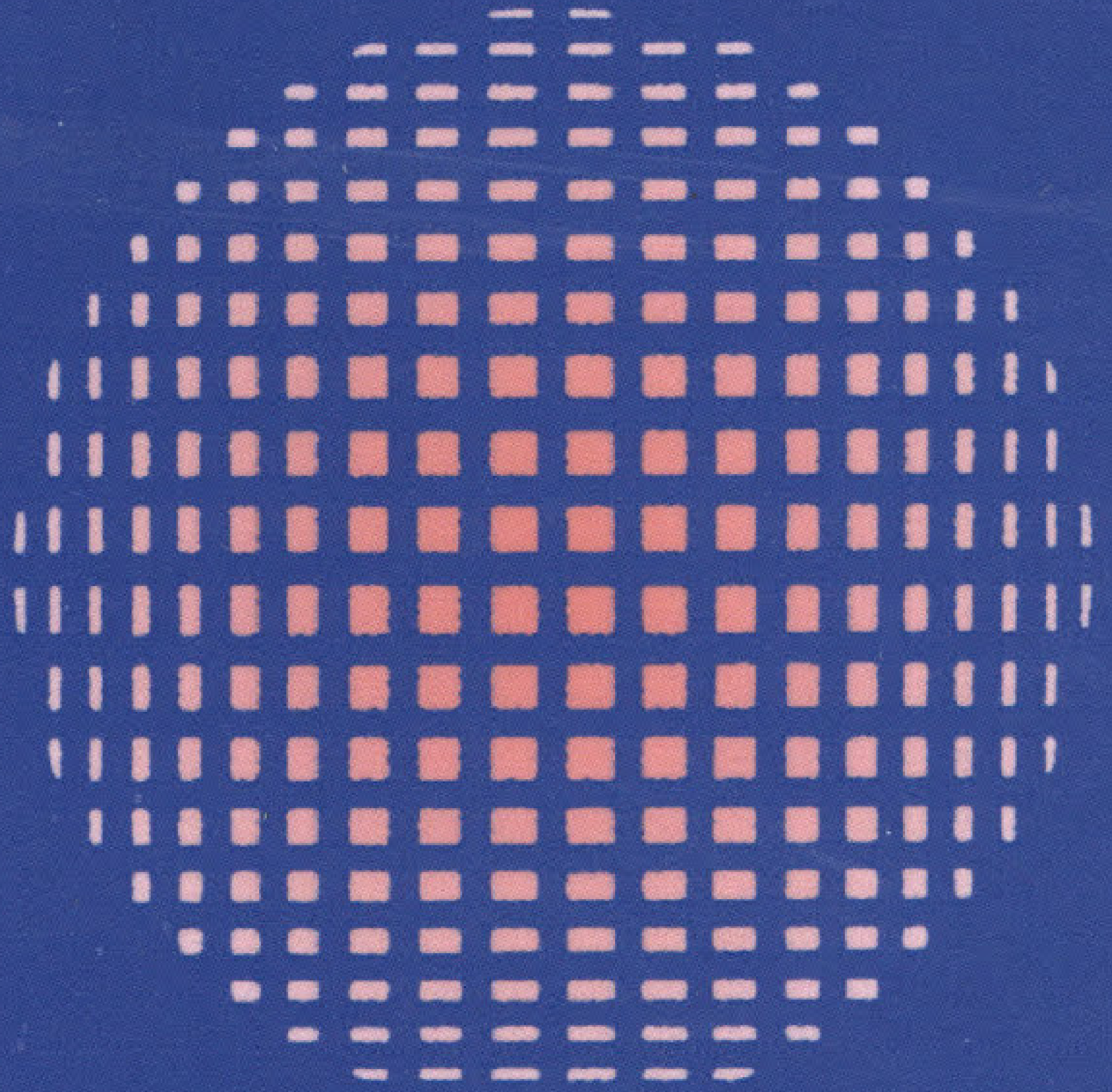
INTRODUCTION TO LOGIC

AND TO THE METHODOLOGY OF DEDUCTIVE SCIENCES



Alfred Tarski

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METHODOLOGY OF DEDUCTIVE SCIENCES

ALFRED TARSKI

TRANSLATED BY OLAF HELMER

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PREFACE

The present book is a partially modified and extended edition of my book *On Mathematical Logic and Deductive Method*, which appeared first in 1936 in Polish and then in 1937 in an exact German translation (under the title: *Einführung in die mathematische Logik und in die Methodologie der Mathematik*). In its original form it was intended as a popular scientific book; its aim was to present to the educated layman—in a manner which would combine scientific exactitude with the greatest possible intelligibility—a clear idea of that powerful trend of contemporary thought which is concentrated about modern logic. This trend arose originally from the somewhat limited task of stabilizing the foundations of mathematics. In its present phase, however, it has much wider aims. For it seeks to create a unified conceptual apparatus which would supply a common basis for the whole of human knowledge. Furthermore, it tends to perfect and sharpen the deductive method, which in some sciences is regarded as the sole permitted means of establishing truths, and indeed in every domain of intellectual activity is at least an indispensable auxiliary tool for deriving conclusions from accepted assumptions.

The response accorded to the Polish and German editions, and especially some suggestions made by reviewers, gave rise to the idea of making the new edition not merely a popular scientific book, but also a textbook upon which an elementary college course in logic and the methodology of deductive sciences could be based. The experiment seemed the more desirable in view of a certain lack of suitable elementary textbooks in this domain.

In order to carry out the experiment, it was necessary to make several changes in the book.

Some very fundamental questions and notions were entirely passed over or merely touched upon in the previous editions, either because of their more technical character, or in order to avoid points of a controversial nature. As examples may be cited such topics as the difference between the usage of certain logical notions in systematic developments of logic and in the language of everyday life, the general method of verifying the laws of the sentential calculus, the necessity of a sharp distinction between words and their names, the concepts of the universal class and the null class, the fundamental notions of the calculus of relations, and finally the conception of methodology as a general science of sciences. In the present edition all these topics are discussed (although not all in an equally thorough manner), since it seemed to me that to avoid them would constitute an essential gap in any textbook of modern logic. Consequently, the chapters of the first, general part of the book have been more or less extended; in particular, Chapter II, which is devoted to the sentential calculus, contains much new material. I have also added many new exercises to these chapters, and have increased the number of historical indications.

While in previous editions the use of special symbols was reduced to a minimum, I considered it necessary in the present edition to familiarize the reader with the elements of logical symbolism. Nevertheless, the use of this symbolism in practice remains very restricted, and is limited mostly to exercises.

In previous editions the principal domain from which examples were drawn for illustrating general and abstract considerations was high-school mathematics; for it was, and still is, my opinion that elementary mathematics, and especially algebra, because of the simplicity of its concepts and the uniformity of its methods of inference, is peculiarly appropriate for exemplifying various fundamental phenomena of a logical and methodological nature. Nevertheless, in the present edition, particularly in the newly added passages, I draw examples more frequently from other domains, especially from everyday life.

Independent of these additions, I have rewritten certain sections whose mastery by students had been found somewhat difficult.

The essential features of the book remain unchanged. The preface to the original edition, the major part of which is reprinted in the next few pages, will give the reader an idea of the general character of the book. Perhaps, however, it is desirable to point out explicitly at this place what he should not expect to find in it.

First, the book contains no systematic and strictly deductive presentation of logic; such a presentation would obviously not lie within the framework of an elementary textbook. It was originally my intention to include, in the present edition, an additional chapter entitled *Logic as a Deductive Science*, which—as an illustration of the general methodological remarks contained in Chapter VI—would outline a systematic development of some elementary parts of logic. For a number of reasons this intention could not be realized; but I hope that several new exercises on this subject included in Chapter VI will to some extent compensate for the omission.

Secondly, apart from two rather short passages, the book gives no information about the traditional Aristotelian logic, and contains no material drawn from it. But I believe that the space here devoted to traditional logic corresponds well enough to the small role to which this logic has been reduced in modern science; and I also believe that this opinion will be shared by most contemporary logicians.

And, finally, the book is not concerned with any problems belonging to the so-called logic and methodology of empirical sciences. I must say that I am inclined to doubt whether any special "logic of empirical sciences," as opposed to logic in general or the "logic of deductive sciences," exists at all (at least so long as the word "logic" is used as in the present book—that is to say, as the name of a discipline which analyzes the meaning of the concepts common to all the sciences, and establishes the general laws governing the concepts). But this is rather a terminological, than a factual, problem. At any rate the methodology of empirical sciences constitutes an important domain of scientific research. The knowledge of logic is of course valuable in the study of this methodology, as it is in the case of any other discipline. It must be admitted, however, that logical concepts and methods have not, up to the present, found any specific or fertile applications in this domain. And it is at least possible that this situation is not merely a consequence of the present stage of methodological researches. It arises, perhaps, from the circumstance that, for the purpose of an adequate methodological treatment, an empirical science may have to be considered, not merely as a scientific theory —that is, as a system of asserted statements arranged according to certain rules—, but rather as a complex consisting partly of such statements and partly of human activities. It should be added that, in striking opposition to the high development of the empirical sciences themselves, the methodology of these sciences can hardly boast of comparably definite achievements—despite the great efforts that have been made. Even the preliminary task of clarifying the concepts involved in this domain has not yet been carried out in a satisfactory way. Consequently, a course in the methodology of empirical sciences must have a quite different character from one in logic and must be largely confined to evaluations and criticisms of tentative gropings and unsuccessful efforts. For these and other reasons, I see little rational justification for combining the discussion of logic and the methodology of empirical sciences in the same college course.

A few remarks concerning the arrangement of the book and its use as a college text.

The book is divided into two parts The first gives a general introduction to logic and the methodology of deductive sciences; the second shows, by means of a concrete example, the sort of applications which logic and methodology find in the construction of mathematical theories, and thus affords an opportunity to assimilate and deepen the knowledge acquired in the first part. Each chapter is followed by appropriate exercises. Brief historical indications are contained in footnotes.

Passages, and even whole sections, which are set off by asterisks "*" both at the beginning and at the end, contain more difficult material, or presuppose familiarity with other passages containing such material; they can be omitted without jeopardizing the intelligibility of subsequent parts of the book. This also applies to the exercises whose numbers are preceded by asterisks.

I feel that the book contains sufficient material for a full-year course. Its arrangement, however, makes it feasible to use it in half-year courses as well. If used as a text in half-year logic courses in a department of philosophy, I suggest the thorough study of its first part, including the more difficult portions, with the entire omission of the second part. If the book is used in a half-year course in a mathematics department—for instance, in the foundations of mathematics—, I suggest the study of both parts of the book, with the omission of the more difficult passages.

In any case, I should like to emphasize the importance of working out the exercises carefully and thoroughly; for they not only facilitate the assimilation of the concepts and principles discussed, but also touch upon many problems for the discussion of which the text provided no opportunity.

I shall be very happy if this book contributes to the wider diffusion of logical knowledge. The course of historical events has assembled in this country the most eminent representatives of contemporary logic, and has thus created here especially favorable conditions for the development of logical thought. These favorable conditions can, of course, be easily overbalanced by other and more powerful factors. It is obvious that the future of logic, as well as of all theoretical science, depends essentially upon normalizing the political and social relations of mankind, and thus upon a factor which is beyond the control of professional scholars. I have no illusions that the development of logical thought, in particular, will have a very essential effect upon the process of the normalization of human relationships; but I do believe that the wider diffusion of the knowledge of logic may contribute positively to the acceleration of this process. For, on the one hand, by making the meaning of concepts precise and uniform in its own field and by stressing the necessity of such a precision and uniformization in any other domain, logic leads to the possibility of better understanding between those who have the will to do so. And, on the other hand, by perfecting and sharpening the tools of thought, it makes men more critical—and thus makes less likely their being misled by all the pseudo-reasonings to which they are in various parts of the world incessantly exposed today.

I gratefully acknowledge my indebtedness to Dr. O. Helmer, who performed the translation of the German edition into English.

I want to express my warmest gratitude to Dr. A. Hofstadter, Mr. L. K. Krader, Professor E. Nagel, Professor W. V. Quine, Mr. M. G. White and especially Dr. J. C. C. McKinsey and Dr. P. P. Wiener, who were unsparing in their advice and assistance while I was preparing the English edition. I also owe many thanks to Mr. K. J. Arrow for his help in reading proofs.

Alfred Tarski

Harvard University September 1940

The present book is a photographic reprint of the first English edition, and no large-scale changes could be introduced in it. Misprints, however, have been corrected, and a number of improvements in detail have been made. I wish to thank readers and reviewers for their helpful suggestions, and I am especially indebted to Miss Louise H. Chin for her assistance in preparing the present edition for publication.

A. T.

University of California,
Berkeley, August 1945

FROM THE PREFACE TO THE ORIGINAL EDITION

In the opinion of many laymen mathematics is today already a dead science: after having reached an unusually high degree of development, it has become petrified in rigid perfection. This is an entirely erroneous view of the situation; there are but few domains of scientific research which are passing through a phase of such intensive development at present as mathematics. Moreover, this development is extraordinarily manifold: mathematics is expanding its domain in all possible directions, it is growing in height, in width, and in depth. It is growing in height, since, on the soil of its old theories which look back upon hundreds if not thousands of years of development, new problems appear again and again, and ever more perfeet results are being achieved. It is growing in width, since its methods permeate other branches of sciences, while its domain of investigation embraces increasingly more comprehensive ranges of phenomena and ever new theories are being included in the large circle of mathematical disciplines. And finally it is growing in depth, since its foundations become more and more firmly established, its methods perfected, and its principles stabilized.

It has been my intention in this book to give those readers who are interested in contemporary mathematics, without being actively concerned with it, at least a very general idea of that third line of mathematical development, i.e. its growth in depth. My aim has been to acquaint the reader with the most important concepts of a discipline which is known as mathematical logic, and which has been created for the purpose of a firmer and more profound establishment of the foundations of mathematics; this discipline, in spite of its brief existence of barely a century, has already attained a high degree of perfection and plays today a role in the totality of our knowledge that far transcends its originally intended boundaries. It has been my intention to show that the concepts of logic permeate the whole of mathematics, that they comprehend all specifically mathematical concepts as special cases, and that logical laws are constantly applied—be it consciously or unconsciously—in mathematical reasonings. Finally, I have tried to present the most important principles in the construction of mathematical theories—principles which form the subject matter of still another discipline, the methodology of mathematics—and to show how one sets about using those principles in practice.

It has not been easy to carry this whole plan through within the framework of a relatively small book without presupposing on the part of the reader any specialized mathematical knowledge or any specific training in reasonings of an abstract character. Throughout the book a combination of the greatest possible intelligibility with the necessary conciseness had to be attempted, with a constant care for avoiding errors or cruder inexactitudes from the scientific standpoint. A language had to be used which deviates as little as possible from the language of everyday life. The employment of a special logical symbolism had to be given up, although this symbolism is an invaluable tool which permits us to combine conciseness with precision, removes to a large degree the possibility of ambiguities and misunderstandings, and is thereby of essential service in all subtler considerations. The idea of a systematic treatment had to be abandoned from the beginning. Of the abundance of questions which present themselves only a few could be discussed in detail, others could only be touched upon superficially, while still others had to be passed over entirely, with the consciousness that the selection of topics discussed would inevitably exhibit a more or less arbitrary character. In those cases in which contemporary science has as yet not taken any definite stand and offers a number of possible and equally correct solutions, it was out of the question to present objectively all known views. A decision in favor of a definite point of view had to be made. When making such a decision I have taken care, not primarily to have it conform to my personal inclinations, but rather to choose a method of solution which would be as simple as possible and which would lend itself to a popular mode of presentation.

I do not have the illusion that I have entirely succeeded in overcoming these and other difficulties.

FIRST PART

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ELEMENTS OF LOGIC

DEDUCTIVE METHOD

· I ·

ON THE USE OF VARIABLES

1. Constants and variables

Every scientific theory is a system of sentences which are accepted as true and which may be called LAWS or ASSERTED STATEMENTS or, for short, simply STATEMENTS. In mathematics, these statements follow one another in a definite order according to certain principles which will be discussed in detail in Chapter VI, and they are, as a rule, accompanied by considerations intended to establish their validity. Considerations of this kind are referred to as PROOFS, and the statements established by them are called THEOREMS.

Among the terms and symbols occurring in mathematical theorems and proofs we distinguish CONSTANTS and VARIABLES.

In arithmetic, for instance, we encounter such constants as "*number*", "*zero*" ("0"), "*one*" ("1"), "*sum*" ("+"), and many others.[1] Each of these terms has a well-determined meaning which remains unchanged throughout the course of the considerations.

As variables we employ, as a rule, single letters, e.g. in arithmetic the small letters of the English alphabet: "*a*", "*b*", "*c*",

[1] By "arithmetic" we shall here understand that part of mathematics which is concerned with the investigation of the general properties of numbers, relations between numbers and operations on numbers. In place of the word "arithmetic" the term "algebra" is frequently used, particularly in high-school mathematics. We have given preference to the term "arithmetic" because, in higher mathematics, the term "algebra" is reserved for the much more special theory of algebraic equations. (In recent years the term "algebra" has obtained a wider meaning, which is, however, still different from that of "arithmetic".)—The term "number" will here always be used with that meaning which is normally attached to the term "real number" in mathematics; that is to say, it will cover integers and fractions, rational and irrational, positive and negative numbers, but not imaginary or complex numbers.

. . ., "x", "y", "z". As opposed to the constants, the variables do not possess any meaning by themselves. Thus, the question:

does zero have such and such a property?

e.g.:

is zero an integer?

can be answered in the affirmative or in the negative; the answer may be true or false, but at any rate it is meaningful. A question concerning x, on the other hand, for example the question:

is x an integer?

cannot be answered meaningfully.

In some textbooks of elementary mathematics, particularly the less recent ones, one does occasionally come across formulations which convey the impression that it is possible to attribute an independent meaning to variables. Thus it is said that the symbols "x", "y", . . . also denote certain numbers or quantities, not "constant numbers" however (which are denoted by constants like "0", "1", . . .), but the so-called "variable numbers" or rather "variable quantities". Statements of this kind have their source in a gross misunderstanding. The "variable number" x could not possibly have any specified property, for instance, it could be neither positive nor negative nor equal to zero; or rather, the properties of such a number would change from case to case, that is to say, the number would sometimes be positive, sometimes negative, and sometimes equal to zero. But entities of such a kind we do not find in our world at all; their existence would contradict the fundamental laws of thought. The classification of the symbols into constants and variables, therefore, does not have any analogue in the form of a similar classification of the numbers.

2. Expressions containing variables—sentential and designatory functions

In view of the fact that variables do not have a meaning by themselves, such phrases as:

x is an integer

are not sentences, although they have the grammatical form of sentences; they do not express a definite assertion and can be neither confirmed nor refuted. From the expression:

x is an integer

we only obtain a sentence when we replace "x" in it by a constant denoting a definite number; thus, for instance, if "x" is replaced by the symbol "1", the result is a true sentence, whereas a false sentence arises on replacing "x" by "$\frac{1}{2}$". An expression of this kind, which contains variables and, on replacement of these variables by constants, becomes a sentence, is called a SENTENTIAL FUNCTION. But mathematicians, by the way, are not very fond of this expression, because they use the term "function" with a different meaning. More often the word "CONDITION" is employed in this sense; and sentential functions and sentences which are composed entirely of mathematical symbols (and not of words of everyday language), such as:

$$x + y = 5,$$

are usually referred to by mathematicians as FORMULAS. In place of "sentential function" we shall sometimes simply say "sentence" —but only in cases where there is no danger of any misunderstanding.

The role of the variables in a sentential function has sometimes been compared very adequately with that of the blanks left in a questionnaire; just as the questionnaire acquires a definite content only after the blanks have been filled in, a sentential function becomes a sentence only after constants have been inserted in place of the variables. The result of the replacement of the variables in a sentential function by constants—equal constants taking the place of equal variables—may lead to a true sentence; in that case, the things denoted by those constants are said to SATISFY the given sentential function. For example, the numbers 1, 2 and $2\frac{1}{2}$ satisfy the sentential function:

$$x < 3,$$

but the numbers 3, 4 and $4\frac{1}{2}$ do not.

Besides the sentential functions there are some further expressions containing variables that merit our attention, namely, the so-called DESIGNATORY OF DESCRIPTIVE FUNCTIONS. They are expressions which, on replacement of the variables by constants, turn into designations ("descriptions") of things. For example, the expression:

$$2x + 1$$

is a designatory function, because we obtain the designation of a certain number (e.g., the number 5), if in it we replace the variable "x" by an arbitrary numerical constant, that is, by a constant denoting a number (e.g., "2").

Among the designatory functions occurring in arithmetic, we have, in particular, all the so-called algebraic expressions which are composed of variables, numerical constants and symbols of the four fundamental arithmetical operations, such as:

$$x - y, \qquad \frac{x + 1}{y + 2}, \qquad 2 \cdot (x + y - z).$$

Algebraic equations, on the other hand, that is to say, formulas consisting of two algebraic expressions connected by the symbol "$=$", are sentential functions. As far as equations are concerned, a special terminology has become customary in mathematics; thus, the variables occurring in an equation are referred to as the unknowns, and the numbers satisfying the equation are called the roots of the equation. E.g., in the equation:

$$x^2 + 6 = 5x$$

the variable "x" is the unknown, while the numbers 2 and 3 are roots of the equation.

Of the variables "x", "y", $\cdots$ employed in arithmetic it is said that they STAND FOR DESIGNATIONS OF NUMBERS or that numbers are VALUES of these variables. Thereby approximately the following is meant: a sentential function containing the symbols "x", "y", $\cdots$ becomes a sentence, if these symbols are replaced by such constants as designate numbers (and not by expressions designating operations on numbers, relations between numbers or even things outside the field of arithmetic like geometrical configurations, animals, plants, etc.). Likewise, the variables occurring in geometry stand for designations of points and geometrical figures. The designatory functions which we meet in arithmetic can also be said to stand for designations of numbers. Sometimes it is simply said that the symbols "x", "y", $\cdots$ themselves, as well as the designatory functions made up out of them, denote numbers or are designations of numbers, but this is then a merely abbreviative terminology.

3. Formation of sentences by means of variables—universal and existential sentences

Apart from the replacement of variables by constants there is still another way in which sentences can be obtained from sentential functions. Let us consider the formula:

$$x + y = y + x.$$

It is a sentential function containing the two variables "x" and "y" that is satisfied by any arbitrary pair of numbers; if we put any numerical constants in place of "x" and "y", we always obtain a true formula. We express this fact briefly in the following manner:

$$\textit{for any numbers } x \textit{ and } y, \quad x + y = y + x.$$

The expression just obtained is already a genuine sentence and, moreover, a true sentence; we recognize in it one of the fundamental laws of arithmetic, the so-called commutative law of addition. The most important theorems of mathematics are formulated similarly, namely, all so-called UNIVERSAL SENTENCES, or SENTENCES OF A UNIVERSAL CHARACTER, which assert that arbitrary things of a certain category (e.g., in the case of arithmetic, arbitrary numbers) have such and such a property. It has to be noticed that in the formulation of universal sentences the phrase "*for any things* (or *numbers*) x, y, $\cdots$ " is often omitted and has to be inserted mentally; thus, for instance, the commutative law of addition may simply be given in the following form:

$$x + y = y + x.$$

This has become a well accepted usage, to which we shall generally adhere in the course of our further considerations.

Let us now consider the sentential function:

$$x > y + 1.$$

This formula fails to be satisfied by every pair of numbers; if, for instance, "3" is put in place of "x" and "4" in place of "y", the false sentence:

$$3 > 4 + 1$$

is obtained. Therefore, if one says:

for any numbers x and y, $x > y + 1,$

one does undoubtedly state a meaningful, though obviously false, sentence. There are, on the other hand, pairs of numbers which satisfy the sentential function under consideration; if, for example, "x" and "y" are replaced by "4" and "2", respectively, the result is the true formula:

$$4 > 2 + 1.$$

This situation is expressed briefly by the following phrase:

for some numbers x and y, $x > y + 1,$

or, using a more frequently employed form:

there are numbers x and y such that $x > y + 1.$

The expressions just given are true sentences; they are examples of EXISTENTIAL SENTENCES, OR SENTENCES OF AN EXISTENTIAL CHARACTER, stating the existence of things (e.g., numbers) with a certain property.

With the help of the methods just described we can obtain sentences from any given sentential function; but it depends on the content of the sentential function whether we arrive at a true or a false sentence. The following example may serve as a further illustration. The formula:

$$x = x + 1$$

is satisfied by no number; hence, no matter whether the words "*for any number x*" or "*there is a number x such that*" are prefixed, the resulting sentence will be false.

In contradistinction to sentences of a universal or existential character we may denote sentences not containing any variables, such as:

$$3 + 2 = 2 + 3,$$

as SINGULAR SENTENCES. This classification is not at all exhaustive, since there are many sentences which cannot be counted among any of the three categories mentioned. An example is represented by the following statement:

for any numbers x and y there is a number z such that
$$x = y + z.$$

Sentences of this type are sometimes called CONDITIONALLY EXISTENTIAL SENTENCES (as opposed to the existential sentences considered before, which may also be called ABSOLUTELY EXISTENTIAL SENTENCES); they state the existence of numbers having a certain property, but on condition that certain other numbers exist.

4. Universal and existential quantifiers; free and bound variables

Phrases like:

for any x, y, ···

and

there are x, y, ··· such that

are called QUANTIFIERS; the former is said to be a UNIVERSAL, the latter an EXISTENTIAL QUANTIFIER. Quantifiers are also known as OPERATORS; there are, however, expressions counted likewise among operators, which are different from quantifiers. In the preceding section we tried to explain the meaning of both quantifiers. In order to emphasize their significance it may be pointed out that, only by the explicit or implicit employment of operators, can an expression containing variables occur as a sentence, that is, as the statement of a well-determined assertion.

Without the help of operators, the usage of variables in the formulation of mathematical theorems would be excluded.

In everyday language it is not customary (though quite possible) to use variables, and quantifiers are also, for this reason, not in use. There are, however, certain words in general usage which exhibit a very close connection with quantifiers, namely, such words as "*every*", "*all*", "*a certain*", "*some*". The connection becomes obvious when we observe that expressions like:

all men are mortal

or

some men are wise

have about the same meaning as the following sentences, formulated with the help of quantifiers:

for any x, if x is a man, then x is mortal

and

there is an x, such that x is both a man and wise,

respectively.

· For the sake of brevity, the quantifiers are sometimes replaced by symbolic expressions. We can, for instance, agree to write in place of:

for any things (or *numbers*) $x, y, \cdots$

and

there exist things (or *numbers*) $x, y, \cdots$ *such that*

the following symbolic expressions:

$$\underset{x,y,\cdots}{\mathsf{A}} \quad \text{and} \quad \underset{x,y,\cdots}{\mathsf{E}}$$

respectively (with the understanding that the sentential functions following the quantifiers are put in parentheses). According to this agreement, the statement which was given at the end of the preceding section as an example of a conditionally existential sentence, for instance, assumes the following form:

(I) $$\underset{x,y}{\mathsf{A}}\,[\underset{z}{\mathsf{E}}\,(x = y + z)]$$

A sentential function in which the variables "x", "y", "z", ... occur automatically becomes a sentence as soon as one prefixes to it one or several operators containing all those variables. If, however, some of the variables do not occur in the operators, the expression in question remains a sentential function, without becoming a sentence. For example, the formula:

$$x = y + z$$

changes into a sentence if preceded by one of the phrases:

for any numbers x, y and z;

there are numbers x, y and z such that;

for any numbers x and y, there is a number z such that;

and so on. But if we merely prefix the quantifier:

there is a number z such that or $\underset{z}{\mathsf{E}}$

we do not yet arrive at a sentence; the expression obtained, namely:

(II) $$\underset{z}{\mathsf{E}}(x = y + z)$$

is, however, undoubtedly a sentential function, for it immediately becomes a sentence when we substitute some constants in the place of "x" and "y" and leave "z" unaltered, or else, when we prefix another suitable quantifier, e.g.:

for any numbers x and y or $\underset{x,y}{\mathsf{A}}$

It is seen from this that, among the variables which may occur in a sentential function, two different kinds can be distinguished. The occurrence of variables of the first kind—they will be called FREE OR REAL VARIABLES—is the decisive factor in determining that the expression under consideration is a sentential function and not a sentence; in order to effect the change from a sentential function to a sentence it is necessary to replace these variables by constants or else to put operators in front of the sentential function that contain those free variables. The remaining, so-called BOUND OR APPARENT VARIABLES, however, are not to be changed in such a transformation. In the above sentential function (II), for instance, "x" and "y" are free variables, and the symbol "z" occurs twice as a bound variable; on the other hand, the expression (I) is a sentence, and thus contains bound variables only.

*It depends entirely upon the structure of the sentential function, namely, upon the presence and position of the operators, whether any particular variable occurring in it is free or bound. This may be best seen by means of a concrete example. Let us, for instance, consider the following sentential function:

(III) *for any number x, if* $x = 0$ *or* $y \neq 0$, *then there exists a number z such that* $x = y \cdot z$.

This function begins with a universal quantifier containing the variable "x", and therefore the latter, which occurs three times in this function, occurs at all these places as a bound variable; at the first place it makes up part of the quantifier, while at the other two places it is, as we say, BOUND BY THE QUANTIFIER. The situation is similar with respect to the variable "z". For, although the initial quantifier of (III) does not contain this variable, we can, nevertheless, recognize a certain sentential function forming a part of (III) which opens with an existential quantifier containing the variable "z"; this is the function:

(IV) *there exists a number z such that* $x = y \cdot z$.

Both places at which the variable "z" occurs in (III) belong to the partial function (IV) just stated. It is for this reason that we say that "z" occurs everywhere in (III) as a bound variable; at the first place it makes up part of the existential quantifier, and at the second place it is bound by that quantifier. As for the variable "y" also occurring in (III), we see that there is no quantifier in (III) containing this variable, and therefore it occurs in (III) twice as a free variable.

The fact that quantifiers bind variables—that is, that they change free into bound variables in the sentential functions which follow them—constitutes a very essential property of quantifiers. Several other expressions are known which have an analogous property; with some of them we shall become acquainted later (in Sections 20 and 22), while some others—such as, for instance, the integral sign—play an important role in higher mathematics. The term "operator" is the general term used to denote all expressions having this property.*

5. The importance of variables in mathematics

As we have seen in Section 3 variables play a leading role in the formulation of mathematical theorems. From what has been said it does not follow, however, that it would be impossible in principle to formulate the latter without the use of variables. But in practice it would scarcely be feasible to do without them, since even comparatively simple sentences would assume a complicated and obscure form. As an illustration let us consider the following theorem of arithmetic:

for any numbers x and y, $x^3 - y^3 = (x - y) \cdot (x^2 + xy + y^2)$.

Without the use of variables, this theorem would look as follows:

> *the difference of the third powers of any two numbers is equal to the product of the difference of these numbers and a sum of three terms, the first of which is the square of the first number, the second the product of the two numbers, and the third the square of the second number.*

An even more essential significance, from the standpoint of the economy of thought, attaches to variables as far as mathematical proofs are concerned. This fact will be readily confirmed by the reader if he attempts to eliminate the variables in any of the proofs which he will meet in the course of our further considerations. And it should be pointed out that these proofs are much simpler than the average considerations to be found in the various fields of higher mathematics; attempts at carrying the latter through without the help of variables would meet with very considerable difficulties. It may be added that it is to the introduction of variables that we are indebted for the development of so fertile a method for the solution of mathematical problems as the method of equations. Without exaggeration it can be said that the invention of variables constitutes a turning point in the history of mathematics; with these symbols man acquired a tool that prepared the way for the tremendous development of the mathematical science and for the solidification of its logical foundations.[2]

Exercises

1. Which among the following expressions are sentential functions, and which are designatory functions:

(a) *x is divisible by* 3,

(b) *the sum of the numbers x and* 2,

(c) $y^2 - z^2$,

(d) $y^2 = z^2$,

(e) $x + 2 < y + 3$,

(f) $(x + 3) - (y + 5)$,

(g) *the mother of x and z*,

(h) *x is the mother of z* ?

2. Give examples of sentential and designatory functions from the field of geometry.

3. The sentential functions which are encountered in arithmetic and which contain only one variable (which may, however, occur at several different places in the given sentential function) can be divided into three categories: (i) functions satisfied by every number; (ii) functions not satisfied by any number; (iii) functions satisfied by some numbers, and not satisfied by others.

[2] Variables were already used in ancient times by Greek mathematicians and logicians,—though only in special circumstances and in rare cases. At the beginning of the 17th century, mainly under the influence of the work of the French mathematician F. Vieta (1540–1603), people began to work systematically with variables and to employ them consistently in mathematical considerations. Only at the end of the 19th century, however, due to the introduction of the notion of a quantifier, was the role of variables in scientific language and especially in the formulation of mathematical theorems fully recognized; this was largely the merit of the outstanding American logician and philosopher Ch. S. Peirce (1839–1914).

To which of these categories do the following sentential functions belong:

(a) $x + 2 = 5 + x$,

(b) $x^2 = 49$,

(c) $(y + 2)\cdot(y - 2) < y^2$,

(d) $y + 24 > 36$,

(e) $z = 0$ *or* $z < 0$ *or* $z > 0$,

(f) $z + 24 > z + 36$?

4. Give examples of universal, absolutely existential and conditionally existential theorems from the fields of arithmetic and geometry.

5. By writing quantifiers containing the variables "x" and "y" in front of the sentential function:

$$x > y$$

it is possible to obtain various sentences from it, for instance:

for any numbers x and y, $x > y$;

for any number x, there exists a number y such that $x > y$;

there is a number y such that, for any number x, $x > y$.

Formulate them all (there are six altogether) and determine which of them are true.

6. Do the same as in Exercise 5 for the following sentential functions:

$$x + y^2 > 1$$

and

$$x \textit{ is the father of } y$$

(assuming that the variables "x" and "y" in the latter stand for names of human beings).

7. State a sentence of everyday language that has the same meaning as:

for every x, if x is a dog, then x has a good sense of smell

and that contains no quantifier or variables.

8. Replace the sentence:

some snakes are poisonous

by one which has the same meaning but is formulated with the help of quantifiers and variables.

9. Differentiate, in the following expressions, between the free and bound variables:

(a) *x is divisible by y;*

(b) *for any x,* $x - y = x + (-y)$,

(c) *if* $x < y$, *then there is a number z such that* $x < y$ *and* $y < z$;

(d) *for any number y, if* $y > 0$, *then there is a number z such that* $x = y \cdot z$;

(e) *if* $x = y^2$ *and* $y > 0$, *then, for any number z,* $x > -z^2$;

(f) *if there exists a number y such that* $x > y^2$, *then, for any number z,* $x > -z^2$.

Formulate the above expressions by replacing the quantifiers by the symbols introduced in Section 4.

*10. If, in the sentential function (e) of the preceding exercise, we replace the variable "z" in both places by "y", we obtain an expression in which "y" occurs in some places as a free and in others as a bound variable; in what places and why?

(In view of some difficulties in operating with expressions in which the same variable occurs both bound and free, some logicians prefer to avoid the use of such expressions altogether and not to treat them as sentential functions.)

*11. Try to state quite generally under which conditions a variable occurs at a certain place of a given sentential function as a free or as a bound variable.

12. Which numbers satisfy the sentential function:

there is a number y such that $x = y^2$,

and which satisfy:

there is a number y such that $x \cdot y = 1$?

· II ·

ON THE SENTENTIAL CALCULUS

6. Logical constants; the old logic and the new logic

The constants with which we have to deal in every scientific theory may be divided into two large groups. The first group consists of terms which are specific for a given theory. In the case of arithmetic, for instance, they are terms denoting either individual numbers or whole classes of numbers, relations between numbers, operations on numbers, etc.; the constants which we used in Section 1 as examples belong here among others. On the other hand, there are terms of a much more general character occurring in most of the statements of arithmetic, terms which are met constantly both in considerations of everyday life and in every possible field of science, and which represent an indispensable means for conveying human thoughts and for carrying out inferences in any field whatsoever; such words as "*not*", "*and*", "*or*", "*is*", "*every*", "*some*" and many others belong here. There is a special discipline, namely LOGIC, considered the basis for all the other sciences, whose concern it is to establish the precise meaning of such terms and to lay down the most general laws in which these terms are involved.

Logic developed into an independent science long ago, earlier even than arithmetic and geometry. And yet it has only been recently—after a long period of almost complete stagnation—that this discipline has begun an intensive development, in the course of which it has undergone a complete transformation with the effect of assuming a character similar to that of the mathematical disciplines; in this new form it is known as MATHEMATICAL or DEDUCTIVE or SYMBOLIC LOGIC, and sometimes it is also called LOGISTIC. The new logic surpasses the old in many respects,—not only because of the solidity of its foundations and the perfection of the methods employed in its development, but mainly on account of the wealth of concepts and theorems that have been established. Fundamentally, the old traditional logic forms only a fragment of the new, a fragment moreover which, from the point of view of the requirements of other sciences, and of mathematics in particular, is entirely insignificant. Thus, in regard to the aim which we here have, there will in this whole book be but very little opportunity to draw the material for our considerations from traditional logic.[1]

7. Sentential calculus; negation of a sentence, conjunction and disjunction of sentences

Among the terms of a logical character there is a small distinguished group, consisting of such words as "*not*", "*and*", "*or*", "*if* ... , *then* ...". All these words are well-known to us from everyday language, and serve to build up compound sentences from simpler ones. In grammar, they are counted among the so-called sentential conjunctions. If only for this reason, the presence of these terms does not represent a specific property of any particular science. To establish the meaning and usage of these terms is the task of the most elementary and fundamental part of logic, which is called SENTENTIAL CALCULUS, or sometimes PROPOSITIONAL CALCULUS or (less happily) THEORY OF DEDUCTION.[2]

[1] Logic was created by ARISTOTLE, the great Greek thinker of the 4th century B.C. (384–322); his logical writings are collected in the work *Organon*. As the creator of mathematical logic we have to look upon the great German philosopher and mathematician of the 17th century G. W. LEIBNIZ (1646–1716). However, the logical works of LEIBNIZ failed to have a great influence upon the further development of logical investigations; there was even a period in which they sank into oblivion. A continuous development of mathematical logic began only towards the middle of the 19th century, namely at the time when the logical system of the English mathematician G. BOOLE was published (1815–1864; principal work: *An Investigation of the Laws of Thought*, London 1854). So far the new logic has found its most perfect expression in the epochal work of the great contemporary English logicians A. N. WHITEHEAD and B. RUSSELL: *Principia Mathematica* (Cambridge, 1910-1913).

[2] The historically first system of sentential calculus is contained in the work *Begriffsschrift* (Halle 1879) of the German logician G. FREGE (1848–1925) who, without doubt, was the greatest logician of the 19th century. The eminent contemporary Polish logician and historian of logic J. ŁUKASIEWICZ succeeded in giving sentential calculus a particularly simple and precise form and caused extensive investigations concerning this calculus.

We will now discuss the meaning of the most important terms of sentential calculus.

With the help of the word *"not"* one forms the NEGATION of any sentence; two sentences, of which the first is a negation of the second, are called CONTRADICTORY SENTENCES. In sentential calculus, the word *"not"* is put in front of the whole sentence, whereas in everyday language it is customary to place it with the verb; or should it be desirable to have it at the beginning of the sentence, it must be replaced by the phrase *"it is not the case that"*. Thus, for example, the negation of the sentence:

1 *is a positive integer*

reads as follows:

1 *is not a positive integer,*

or else:

it is not the case that 1 *is a positive integer.*

Whenever we utter the negation of a sentence, we intend to express the idea that the sentence is false; if the sentence is actually false, its negation is true, while otherwise its negation is false.

The joining of two sentences (or more) by the word *"and"* results in their so-called CONJUNCTION OR LOGICAL PRODUCT; the sentences joined in this manner are called the MEMBERS OF THE CONJUNCTION or the FACTORS OF THE LOGICAL PRODUCT. If, for instance, the sentences:

2 *is a positive integer*

and

$$2 < 3$$

are joined in this way, we obtain the conjunction:

2 *is a positive integer and* $2 < 3$.

The stating of the conjunction of two sentences is tantamount to stating that both sentences of which the conjunction is formed are true. If this is actually the case, then the conjunction is true, but if at least one of its members is false, then the whole conjunction is false.

By joining sentences by means of the word *"or"* one obtains the DISJUNCTION of those sentences, which is also called the LOGICAL SUM; the sentences forming the disjunction are called the MEMBERS OF THE DISJUNCTION or the SUMMANDS OF THE LOGICAL SUM. The word *"or"*, in everyday language, possesses at least two different meanings. Taken in the so-called NON-EXCLUSIVE meaning, the disjunction of two sentences merely expresses that at least one of these sentences is true, without saying anything as to whether or not both sentences may be true; taken in another meaning, the so-called EXCLUSIVE one, the disjunction of two sentences asserts that one of the sentences is true but that the other is false. Suppose we see the following notice put up in a bookstore:

Customers who are teachers or college students are entitled to a special reduction.

Here the word *"or"* is undoubtedly used in the first sense, since it is not intended to refuse the reduction to a teacher who is at the same time a college student. If, on the other hand, a child has asked to be taken on a hike in the morning and to a theater in the afternoon, and we reply:

no, we are going on a hike or we are going to the theater,

then our usage of the word *"or"* is obviously of the second kind since we intend to comply with only one of the two requests. In logic and mathematics, the word *"or"* is always used in the first, non-exclusive meaning; the disjunction of two sentences is considered true if both or at least one of its members are true, and otherwise false. Thus, for instance, it may be asserted:

every number is positive or less than 3,

although it is known that there are numbers which are both positive and less than 3. In order to avoid misunderstandings, it would be expedient, in everyday as well as in scientific language, to use the word *"or"* by itself only in the first meaning, and to replace it by the compound expression "*either ⋯ or ⋯*" whenever the second meaning is intended.

* Even if we confine ourselves to those cases in which the word "*or*" occurs in its first meaning, we find quite noticeable differences between the usages of it in everyday language and in logic. In common language, two sentences are joined by the word "*or*" only when they are in some way connected in form and content. (The same applies, though perhaps to a lesser degree, to the usage of the word "*and*".) The nature of this connection is not quite clear, and a detailed analysis and description of it would meet with considerable difficulties. At any rate, anybody unfamiliar with the language of contemporary logic would presumably be little inclined to consider such a phrase as:

$$2 \cdot 2 = 5 \quad \textit{or} \quad \textit{New York is a large city}$$

as a meaningful expression, and even less so to accept it as a true sentence. Moreover, the usage of the word "*or*" in everyday English is influenced by certain factors of a psychological character. Usually we affirm a disjunction of two sentences only if we believe that one of them is true but wonder which one. If, for example, we look upon a lawn in normal light, it will not enter our mind to say that the lawn is green or blue, since we are able to affirm something simpler and, at the same time, stronger, namely that the lawn is green. Sometimes even, we take the utterance of a disjunction as an admission by the speaker that he does not know which of the members of the disjunction is true. And if we later arrive at the conviction that he knew at the time that one—and, specifically, which—of the members was false, we are inclined to look upon the whole disjunction as a false sentence, even should the other member be undoubtedly true. Let us imagine, for instance, that a friend of ours, upon being asked when he is leaving town, answers that he is going to do so today, tomorrow or the day after. Should we then later ascertain that, at that time, he had already decided to leave the same day, we shall probably get the impression that we were deliberately misled and that he told us a lie.

The creators of contemporary logic, when introducing the word "*or*" into their considerations, desired, perhaps unconsciously, to simplify its meaning and to render the latter clearer and independent of all psychological factors, especially of the presence or absence of knowledge. Consequently, they extended the usage of the word "*or*", and decided to consider the disjunction of any two sentences as a meaningful whole, even should no connection between their contents or forms exist; and they also decided to make the truth of a disjunction—like that of a negation or conjunction—dependent only and exclusively upon the truth of its members. Therefore, a man using the word "*or*" in the meaning of contemporary logic will consider the expression given above:

$$2 \cdot 2 = 5 \quad or \quad New\ York\ is\ a\ large\ city$$

as a meaningful and even a true sentence, since its second part is surely true. Similarly, if we assume that our friend, who was asked about the date of his departure, used the word "*or*" in its strict logical meaning, we shall be compelled to consider his answer as true, independent of our opinion as to his intentions.*

8. Implication or conditional sentence; implication in material meaning

If we combine two sentences by the words "*if* ···, *then* ···", we obtain a compound sentence which is denoted as an IMPLICATION or a CONDITIONAL SENTENCE. The subordinate clause to which the word "*if*" is prefixed is called ANTECEDENT, and the principal clause introduced by the word "*then*" is called CONSEQUENT. By asserting an implication one asserts that it does not occur that the antecedent is true and the consequent is false. An implication is thus true in any one of the following three cases: (i) both antecedent and consequent are true, (ii) the antecedent is false and the consequent is true, (iii) both antecedent and consequent are false; and only in the fourth possible case, when the antecedent is true and the consequent is false, the whole implication is false. It follows that, whoever accepts an implication as true, and at the same time accepts its antecedent as true, cannot but accept its consequent; and whoever accepts an implication as true and rejects its consequent as false, must also reject its antecedent.

* As in the case of disjunction, considerable differences between the usages of implication in logic and everyday language manifest themselves. Again, in ordinary language, we tend to join two sentences by the words "*if* ..., *then* ..." only when there is some connection between their forms and contents. This connection is hard to characterize in a general way, and only sometimes is its nature relatively clear. We often associate with this connection the conviction that the consequent follows necessarily from the antecedent, that is to say, that if we assume the antecedent to be true we are compelled to assume the consequent, too, to be true (and that possibly we can even deduce the consequent from the antecedent on the basis of some general laws which we might not always be able to quote explicitly). Here again, an additional psychological factor manifests itself; usually we formulate and assert an implication only if we have no exact knowledge as to whether or not the antecedent and consequent are true. Otherwise the use of an implication seems unnatural and its sense and truth may raise some doubt.

The following example may serve as an illustration. Let us consider the law of physics:

every metal is malleable,

and let us put it in the form of an implication containing variables:

if x is a metal, then x is malleable.

If we believe in the truth of this universal law, we believe also in the truth of all its particular cases, that is, of all implications obtainable by replacing "x" by names of arbitrary materials such as iron, clay or wood. And, indeed, it turns out that all sentences obtained in this way satisfy the conditions given above for a true implication; it never happens that the antecedent is true while the consequent is false. We notice, further, that in any of these implications there exists a close connection between the antecedent and the consequent, which finds its formal expression in the coincidence of their subjects. We are also convinced that, assuming the antecedent of any of these implications, for instance, *"iron is a metal"*, as true, we can deduce from it its consequent *"iron is malleable"*, for we can refer to the general law that every metal is malleable.

Nevertheless, some of the sentences discussed just now seem artificial and doubtful from the point of view of common language. No doubt is raised by the universal implication given above, or by any of its particular cases obtained by replacing "x" by the name of a material of which we do not know whether it is a metal or whether it is malleable. But if we replace "x" by *"iron"*, we are confronted with a case in which the antecedent and consequent are undoubtedly true; and we shall then prefer to use, instead of an implication, an expression such as:

since iron is a metal, it is malleable.

Similarly, if for "x" we substitute *"clay"*, we obtain an implication with a false antecedent and a true consequent, and we shall be inclined to replace it by the expression:

although clay is not a metal, it is malleable.

And finally, the replacement of "x" by *"wood"* results in an implication with a false antecedent and a false consequent; if, in this case, we want to retain the form of an implication, we should have to alter the grammatical form of the verbs:

if wood were a metal, then it would be malleable.

The logicians, with due regard for the needs of scientific languages, adopted the same procedure with respect to the phrase *"if ..., then ..."* as they had done in the case of the word *"or"*. They decided to simplify and clarify the meaning of this phrase, and to free it from psychological factors. For this purpose they extended the usage of this phrase, considering an implication as a meaningful sentence even if no connection whatsoever exists between its two members, and they made the truth or falsity of an implication dependent exclusively upon the truth or falsity of the antecedent and consequent. To characterize this situation briefly, we say that contemporary logic uses IMPLICATIONS IN MATERIAL MEANING, or simply, MATERIAL IMPLICATIONS; this is opposed to the usage of IMPLICATION IN FORMAL MEANING

OR FORMAL IMPLICATION, in which case the presence of a certain formal connection between antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication. The concept of formal implication is not, perhaps, quite clear, but, at any rate, it is narrower than that of material implication; every meaningful and true formal implication is at the same time a meaningful and true material implication, but not vice versa.

In order to illustrate the foregoing remarks, let us consider the following four sentences:

if $2 \cdot 2 = 4$, *then* *New York is a large city;*

if $2 \cdot 2 = 5$, *then* *New York is a large city;*

if $2 \cdot 2 = 4$, *then* *New York is a small city;*

if $2 \cdot 2 = 5$, *then* *New York is a small city.*

In everyday language, these sentences would hardly be considered as meaningful, and even less as true. From the point of view of mathematical logic, on the other hand, they are all meaningful, the third sentence being false, while the remaining three are true. Thereby it is, of course, not asserted that sentences like these are particularly relevant from any viewpoint whatever, or that we apply them as premisses in our arguments.

It would be a mistake to think that the difference between everyday language and the language of logic, which has been brought to light here, is of an absolute character, and that the rules, outlined above, of the usage of the words "*if* ..., *then* ..." in common language admit of no exceptions. Actually, the usage of these words fluctuates more or less, and if we look around, we can find cases in which this usage does not comply with our rules. Let us imagine that a friend of ours is confronted with a very difficult problem and that we do not believe that he will ever solve it. We can then express our disbelief in a jocular form by saying:

if you solve this problem, I shall eat my hat.

The tendency of this utterance is quite clear. We affirm here an implication whose consequent is undoubtedly false; therefore, since we affirm the truth of the whole implication, we thereby, at the same time, affirm the falsity of the antecedent; that is to say, we express our conviction that our friend will fail to solve the problem in which he is interested. But it is also quite clear that the antecedent and the consequent of our implication are in no way connected, so that we have a typical case of a material and not of a formal implication.

The divergency in the usage of the phrase "*if* ···, *then* ···" in ordinary language and mathematical logic has been at the root of lengthy and even passionate discussions,—in which, by the way, professional logicians took only a minor part.[3] (Curiously enough, considerably less attention was paid to the analogous divergency in the case of the word "*or*".) It has been objected that the logicians, on account of their employment of the material implication, arrived at paradoxes and even plain nonsense. This has resulted in an outcry for a reform of logic to the effect of bringing about a far-reaching rapprochement between logic and ordinary language regarding the use of implication.

It would be hard to grant that these criticisms are well-founded. There is no phrase in ordinary language that has a precisely determined sense. It would scarcely be possible to find two people who would use every word with exactly the same meaning, and even in the language of a single person the meaning of the same word varies from one period of his life to another. Moreover, the meaning of words of everyday language is usually very complicated; it depends not only on the external form of the word but also on the circumstances in which it is uttered and sometimes even on subjective psychological factors. If a scientist wants to

[3] It is interesting to notice that the beginning of this discussion dates back to antiquity. It was the Greek philosopher Philo of Megara (in the 4th century B.C.) who presumably was the first in the history of logic to propagate the usage of material implication; this was in opposition to the views of his master, Diodorus Cronus, who proposed to use implication in a narrower sense, rather related to what is called here the formal meaning. Somewhat later (in the 3d century B.C.)—and probably under the influence of Philo—various possible conceptions of implication were discussed by the Greek philosophers and logicians of the Stoic School (in whose writings the first beginnings of sentential calculus are to be found).

introduce a concept from everyday life into a science and to establish general laws concerning this concept, he must always make its content clearer, more precise and simpler, and free it from inessential attributes; it does not matter here whether he is a logician concerned with the phrase "*if* ..., *then* ..." or, for instance, a physicist establishing the exact meaning of the word "*metal*". In whatever way the scientist realizes his task, the usage of the term as it is established by him will deviate more or less from the practice of everyday language. If, however, he states explicitly in what meaning he decides to use the term, and if he consistently acts in conformity to this decision, nobody will be in a position to object that his procedure leads to nonsensical results.

Nevertheless, in connection with the discussions that have taken place, some logicians have undertaken attempts to reform the theory of implication. They do not, generally, deny material implication a place in logic, but they are anxious to find also a place for another concept of implication, for instance, of such a kind that the possibility of deducing the consequent from the antecedent constitutes a necessary condition for the truth of an implication; they even desire, so it seems, to place the new concept in the foreground. These attempts are of a relatively recent date, and it is too early to pass a final judgment as to their value.[4] But it appears today almost certain that the theory of material implication will surpass all other theories in simplicity, and, in any case, it must not be forgotten that logic, founded upon this simple concept, turned out to be a satisfactory basis for the most complicated and subtle mathematical reasonings.*

9. The use of implication in mathematics

The phrase "*if* ..., *then* ..." belongs to those expressions of logic which are used most frequently in other sciences and, especially, in mathematics. Mathematical theorems, particularly those of a universal character, tend to have the form of implications; the antecedent is called in mathematics the HYPOTHESIS, and the consequent is called the CONCLUSION.

[4] The first attempt of this kind was made by the contemporary American philosopher and logician C. I. LEWIS.

As a simple example of a theorem of arithmetic, having the form of an implication, we may quote the following sentence:

if x is a positive number, then $2x$ is a positive number

in which "*x is a positive number*" is the hypothesis, while "*$2x$ is a positive number*" is the conclusion.

Apart from this, so to speak, classical form of mathematical theorems, there are, occasionally, different formulations, in which hypothesis and conclusion are connected in some other way than by the phrase "*if* ..., *then* ...". The theorem just mentioned, for instance, can be paraphrased in any of the following forms:

from: x is a positive number, it follows: $2x$ is a positive number;

the hypothesis: x is a positive number, implies (or *has as a consequence*) *the conclusion: $2x$ is a positive number;*

the condition: x is a positive number, is sufficient for $2x$ to be a positive number;

for $2x$ to be a positive number it is sufficient that x be a positive number;

the condition: $2x$ is a positive number, is necessary for x to be a positive number;

for x to be a positive number it is necessary that $2x$ be a positive number.

Therefore, instead of asserting a conditional sentence, one might usually just as well say that the hypothesis IMPLIES the conclusion or HAS it AS A CONSEQUENCE, or that it is a SUFFICIENT CONDITION for the conclusion; or one can express it by saying that the conclusion FOLLOWS from the hypothesis, or that it is a NECESSARY CONDITION for the latter. A logician may raise various objections against some of the formulations given above, but they are in general use in mathematics.

* The objections which might be raised here concern those of the above formulations in which any of the words "*hypothesis*", "*conclusion*", "*consequence*", "*follows*", "*implies*" occur.

In order to understand the essential points in these objections, we observe first that those formulations differ in content from the ones originally given. While in the original formulation we talk only about numbers, properties of numbers, operations upon numbers and so on—hence, about things with which mathematics is concerned—, in the formulations now under discussion we talk about hypotheses, conclusions, conditions, that is about sentences or sentential functions occurring in mathematics. It might be noted on this occasion that, in general, people do not distinguish clearly enough the terms which denote things dealt with in a given science from those which denote various kinds of expressions occurring within it. This can be observed, in particular, in the domain of mathematics, especially on the elementary level. Presumably only few are aware of the fact that such terms as "equation", "inequality", "polynomial" or "algebraic fraction", which are met at every turn in textbooks of elementary algebra, do not, strictly speaking, belong to the domain of mathematics or logic, since they do not denote things considered in this domain; equations and inequalities are certain special sentential functions, while polynomials and algebraic fractions—especially as they are treated in elementary textbooks—are particular instances of designatory functions (cf. Section 2). The confusion on this point is brought about by the fact that terms of this kind are frequently used in the formulation of mathematical theorems. This has become a very common usage, and perhaps it is not worth our while to put up a stand against it, since it does not present any particular danger; but it might be worth our while to get to recognize that, for every theorem formulated with the help of such terms, there is another formulation, logically more correct, in which those terms do not occur at all. For instance, the theorem:

the equation: $x^2 + ax + b = 0$ *has at most two roots*

can be expressed in a more correct manner as follows:

there are at most two numbers x such that $x^2 + ax + b = 0$.

Returning to the questionable formulations of an implication, we must emphasize one still more important point. In these formulations we assert that one sentence, namely the antecedent of the implication, has another—the consequent of the implication—as a consequence, or that the second follows from the first. But ordinarily when we express ourselves in this way, we have in mind that the assumption that the first sentence is true leads us, so to speak, necessarily to the same assumption concerning the second sentence (and that possibly we are even able to derive the second sentence from the first). As we already know from Section 8, however, the meaning of an implication, as it was established in contemporary logic, does not depend on whether its consequent has any such connection with its antecedent. Anyone shocked by the fact that the expression:

if $2 \cdot 2 = 4$, *then New York is a large city*

is considered in logic as a meaningful and even true sentence will find it still harder to reconcile himself with such a transformation of this phrase as:

the hypothesis that $2 \cdot 2 = 4$ *has as a consequence that New York is a large city.*

We see, thus, that the manners discussed here of formulating or transforming a conditional sentence lead to paradoxical sounding utterances and make yet more profound the discrepancies between common language and mathematical logic. It is for this reason that they repeatedly brought about various misunderstandings and have been one of the causes of those passionate and frequently sterile discussions which we mentioned above.

From the purely logical point of view we can obviously avoid all objections raised here by stating explicitly once and for all that, in using the formulations in question, we shall disregard their usual meaning and attribute to them exactly the same content as to the ordinary conditional sentence. But this would be inconvenient in another respect; for there are situations—though not in logic itself, but in a field closely related to it, namely, the methodology of deductive sciences (cf. Chapter VI)—in which we talk about sentences and the relation of consequence between them, and in which we use such terms as "*implies*" and "*follows*" in a different meaning more closely akin to the ordinary one.

It would, therefore, be better to avoid those formulations altogether, all the more since we have several formulations at our disposal which are not open to any of these objections.*

10. Equivalence of sentences

We shall consider one more expression from the field of sentential calculus. It is one which is comparatively rarely met in everyday language, namely, the phrase "*if, and only if*". If any two sentences are joined up by this phrase, the result is a compound sentence called an EQUIVALENCE. The two sentences connected in this way are referred to as the LEFT and RIGHT SIDE OF THE EQUIVALENCE. By asserting the equivalence of two sentences, it is intended to exclude the possibility that one is true and the other false; an equivalence, therefore, is true if its left and right sides are either both true or both false, and otherwise the equivalence is false.

The sense of an equivalence can also be characterized in still another way. If, in a conditional sentence, we interchange antecedent and consequent, we obtain a new sentence which, in its relation to the original sentence, is called the CONVERSE SENTENCE (or the CONVERSE OF THE GIVEN SENTENCE). Let us take, for instance, as the original sentence the implication:

(I) *if x is a positive number, then $2x$ is a positive number;*

the converse of this sentence will then be:

(II) *if $2x$ is a positive number, then x is a positive number.*

As is shown by this example, it occurs that the converse of a true sentence is true. In order to see, on the other hand, that this is not a general rule, it is sufficient to replace "$2x$" by "x^2" in (I) and (II); the sentence (I) will remain true, while the sentence (II) becomes false. If, now, it happens that two conditional sentences, of which one is the converse of the other, are both true, then the fact of their simultaneous truth can also be expressed by joining the antecedent and consequent of any one of the two sentences by the words "*if, and only if*". Thus, the above two implications—the original sentence (I) and the converse sentence (II)—may be replaced by a single sentence:

x is a positive number if, and only if, $2x$ is a positive number

(in which the two sides of the equivalence may yet be interchanged).

There are, incidentally, still a few more possible formulations which may serve to express the same idea, e.g.:

from: x is a positive number, it follows: $2x$ is a positive number, and conversely;

the conditions that x is a positive number and that $2x$ is a positive number are equivalent with each other;

the condition that x is a positive number is both necessary and sufficient for $2x$ to be a positive number;

for x to be a positive number it is necessary and sufficient that $2x$ be a positive number.

Instead of joining two sentences by the phrase "*if, and only if*", it is therefore, in general, also possible to say that the RELATION OF CONSEQUENCE holds between these two sentences IN BOTH DIRECTIONS, or that the two sentences are EQUIVALENT, or, finally, that each of the two sentences represents a NECESSARY AND SUFFICIENT CONDITION for the other.

11. The formulation of definitions and its rules

The phrase "*if, and only if*" is very frequently used in laying down DEFINITIONS, that is, conventions stipulating what meaning is to be attributed to an expression which thus far has not occurred in a certain discipline, and which may not be immediately comprehensible. Imagine, for instance, that in arithmetic the symbol "$\leqq$" has not as yet been employed but that one wants to introduce it now into the considerations (looking upon it, as usual, as an abbreviation of the expression "*is less than or equal to*"). For this purpose it is necessary to define this symbol first, that is, to explain exactly its meaning in terms of expressions which are already known and whose meanings are beyond doubt. To achieve this, we lay down the following definition,—assuming that "$>$" belongs to the symbols already known:

we say that $x \leqq y$ *if, and only if, it is not the case that* $x > y$.

The definition just formulated states the equivalence of the two sentential functions:

$$x \leqq y$$

and

it is not the case that $x > y$;

it may be said, therefore, that it permits the transformation of the formula "$x \leqq y$" into an equivalent expression which no longer contains the symbol "$\leqq$" but is formulated entirely in terms already comprehensible to us. The same holds for any formula obtained from "$x \leqq y$" by replacing "x" and "y" by arbitrary symbols or expressions designating numbers. The formula:

$$3 + 2 \leqq 5,$$

for instance, is equivalent with the sentence:

it is not the case that $3 + 2 > 5$;

since the latter is a true assertion, so is the former. Similarly, the formula:

$$4 \leqq 2 + 1$$

is equivalent with the sentence:

it is not the case that $4 > 2 + 1$,

both being false assertions. This remark applies also to more complicated sentences and sentential functions; by transforming, for instance, the sentence:

if $x \leqq y$ *and* $y \leqq z$, *then* $x \leqq z$,

we obtain:

if it is not the case that $x > y$ *and if it is not the case that* $y > z$, *then it is not the case that* $x > z$.

In short, by virtue of the definition given above, we are in a position to transform any simple or compound sentence containing the symbol "$\leqq$" into an equivalent one no longer containing it; in other words, so to speak, to translate it into a language in which the symbol "$\leqq$" does not occur. And it is this very fact which constitutes the role which definitions play within the mathematical disciplines.

If a definition is to fulfil its proper task well, certain precautionary measures have to be observed in its formulation. To this effect special rules are laid down, the so-called RULES OF DEFINITION, which specify how definitions should be constructed correctly. Since we shall not here go into an exact formulation of these rules, it may merely be remarked that, on their basis, every definition may assume the form of an equivalence; the first member of that equivalence, the DEFINIENDUM, should be a short, grammatically simple sentential function containing the constant to be defined; the second member, the DEFINIENS, may be a sentential function of an arbitrary structure, containing, however, only constants whose meaning either is immediately obvious or has been explained previously. In particular, the constant to be defined, or any expression previously defined with its help, must not occur in the definiens; otherwise the definition is incorrect, it contains an error known as a VICIOUS CIRCLE IN THE DEFINITION (just as one speaks of a VICIOUS CIRCLE IN THE PROOF, if the argument meant to establish a certain theorem is based upon that theorem itself, or upon some other theorem previously proved with its help). In order to emphasize the conventional character of a definition and to distinguish it from other statements which have the form of an equivalence, it is expedient to prefix it by words such as "*we say that*". It is easy to verify that the above definition of the symbol "$\leqq$" satisfies all these conditions; it has the definiendum:

$$x \leqq y,$$

whereas the definiens reads:

it is not the case that $x > y$.

It is worth noticing that mathematicians, in laying down definitions, prefer the words "*if*" or "*in case that*" to the phrase "*if, and only if*". If, for example, they had to formulate the definition of the symbol "$\leq$", they would, presumably, give it the following form:

we say that $x \leq y$, *if it is not the case that* $x > y$.

It looks as if such a definition merely states that the definiendum follows from the definiens, without emphasizing that the relation of consequence also holds in the opposite direction, and thus fails to express the equivalence of definiendum and definiens. But what we actually have here is a tacit convention to the effect that "*if*" or "*in case that*", if used to join definiendum and definiens, are to mean the same as the phrase "*if, and only if*" ordinarily does.—It may be added that the form of an equivalence is not the only form in which definitions may be laid down.

12. Laws of sentential calculus

After having come to the end of our discussion of the most important expressions of sentential calculus, we shall now try to clarify the character of the laws of this calculus.

Let us consider the following sentence:

if 1 *is a positive number and* $1 < 2$, *then* 1 *is a positive number*.

This sentence is obviously true, it contains exclusively constants belonging to the field of logic and arithmetic, and yet the idea of listing this sentence as a special theorem in a textbook of mathematics would not occur to anybody. If one reflects why this is so, one comes to the conclusion that this sentence is completely uninteresting from the standpoint of arithmetic; it fails to enrich in any way our knowledge about numbers, its truth does not at all depend upon the content of the arithmetical terms occurring in it, but merely upon the sense of the words "*and*", "*if*", "*then*". In order to make sure that this is so, let us replace in the sentence under consideration the components:

1 *is a positive number*

and

$$1 < 2$$

by any other sentences from an arbitrary field; the result is a series of sentences, each of which, like the original sentence, is true; for example:

> *if the given figure is a rhombus and if the same figure is a rectangle, then the given figure is a rhombus;*
>
> *if today is Sunday and the sun is shining, then today is Sunday.*

In order to express this fact in a more general form, we shall introduce the variables "p" and "q", stipulating that these symbols are not designations of numbers or any other things, but that they stand for whole sentences; variables of this kind are denoted as SENTENTIAL VARIABLES. Further, we shall replace in the sentence under consideration the phrase:

1 *is a positive number*

by "p" and the formula:

$$1 < 2$$

by "q"; in this manner we arrive at the sentential function:

if p and q, then p.

This sentential function has the property that only true sentences are obtained if arbitrary sentences are substituted for "p" and "q". This observation may be given the form of a universal statement:

For any p and q, if p and q, then p.

We have here obtained a first example of a law of sentential calculus, which will be referred to as the LAW OF SIMPLIFICATION for logical multiplication. The sentence considered above was merely a special instance of this universal law—just as, for instance, the formula:

$$2 \cdot 3 = 3 \cdot 2$$

is merely a special instance of the universal arithmetical theorem:

for arbitrary numbers x and y, $x \cdot y = y \cdot x$.

In a similar way, other laws of sentential calculus can be obtained. We give here a few examples of such laws; in their formulation we omit the universal quantifier "*for any p, q, . . .*"— in accordance with the usage mentioned in Section 3, which becomes almost a rule throughout sentential calculus.

If p, then p.

If p, then q or p.

If p implies q and q implies p, then p if, and only if, q.

If p implies q and q implies r, then p implies r.

The first of these four statements is known as the LAW OF IDENTITY, the second as the LAW OF SIMPLIFICATION for logical addition, and the fourth as the LAW OF THE HYPOTHETICAL SYLLOGISM.

Just as the arithmetical theorems of a universal character state something about the properties of arbitrary numbers, the laws of sentential calculus assert something, so one may say, about the properties of arbitrary sentences. The fact that in these laws only such variables occur as stand for quite arbitrary sentences is characteristic of sentential calculus and decisive for its great generality and the scope of its applicability.

13. Symbolism of sentential calculus; truth functions and truth tables

There exists a certain simple and general method, called METHOD OF TRUTH TABLES OR MATRICES, which enables us, in any particular case, to recognize whether a given sentence from the domain of the sentential calculus is true, and whether, therefore, it can be counted among the laws of this calculus.[5]

[5] This method originates with PEIRCE (who has already been cited at an earlier occasion; cf. footnote 2 on p. 14).

In describing this method it is convenient to apply a special symbolism. We shall replace the expressions:

not; and; or; if ..., then ...; if, and only if

by the symbols:

$\sim$; $\wedge$; $\vee$; $\rightarrow$; $\leftrightarrow$

respectively. The first of these symbols is to be placed in front of the expression whose negation one wants to obtain; the remaining symbols are always placed between two expressions ("$\rightarrow$" stands therefore in the place of the word "*then*", while the word "*if*" is simply omitted). From one or two simpler expressions we are, in this way, led to a more complicated expression; and if we want to use the latter for the construction of further still more complicated expressions, we enclose it in parentheses.

With the help of variables, parentheses and the constant symbols listed above (and sometimes also additional constants of a similar character which will not be discussed here), we are able to write down all sentences and sentential functions belonging to the domain of sentential calculus. Apart from the individual sentential variables the simplest sentential functions are the expressions:

$$\sim p, \quad p \wedge q, \quad p \vee q, \quad p \rightarrow q, \quad p \leftrightarrow q$$

(and other similar expressions which differ from these merely in the shape of the variables used). As an example of a compound sentential function let us consider the expression:

$$(p \vee q) \rightarrow (p \wedge r),$$

which we read, translating symbols into common language:

if p or q, then p and r.

A still more complicated expression is the law of the hypothetical syllogism given above, which now assumes the form:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r).$$

We can easily make sure that every sentential function occurring in our calculus is a so-called TRUTH FUNCTION. This means to say that the truth or falsehood of any sentence obtained from that function by substituting whole sentences for variables depends exclusively upon the truth or falsehood of the sentences which have been substituted. As for the simplest sentential functions "$\sim p$", "$p \wedge q$", and so on, this follows immediately from the remarks made in Sections 7, 8 and 10 concerning the meaning attributed in logic to the words "*not*", "*and*", and so on. But the same applies, likewise, to compound functions. Let us consider, for instance, the function "$(p \vee q) \rightarrow (p \wedge r)$". A sentence obtained from it by substitution is an implication, and, therefore, its truth depends on the truth of its antecedent and consequent only; the truth of the antecedent, which is a disjunction obtained from "$p \vee q$", depends only on the truth of the sentences substituted for "p" and "q", and similarly the truth of the consequent depends only on the truth of the sentences substituted for "p" and "r". Thus, finally, the truth of the whole sentence obtained from the sentential function under consideration depends exclusively on the truth of the sentences substituted for "p", "q" and "r".

In order to see quite exactly how the truth or falsity of a sentence obtained by substitution from a given sentential function depends upon the truth or falsity of the sentences substituted for variables, we construct what is called the TRUTH TABLE or MATRIX for this function. We shall begin by giving such a table for the function "$\sim p$":

p	$\sim p$
T	F
F	T

And here is the joint truth table for the other elementary functions "$p \wedge q$", "$p \vee q$", and so on:

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
F	T	F	T	T	F
T	F	F	T	F	F
F	F	F	F	T	T

The meaning of these tables becomes at once comprehensible if we take the letters "T" and "F" to be abbreviations of "true sentence" and "false sentence", respectively. In the second table, for instance, we find, in the second line below the headings "p", "q" and "$p \rightarrow q$", the letters "F", "T" and "T", respectively. We gather from that that a sentence obtained from the implication "$p \rightarrow q$" is true if we substitute any false sentence for "p" and any true sentence for "q"; this, obviously, is entirely consistent with the remarks made in Section 8.—The variables "p" and "q" occurring in the tables can, of course, be replaced by any other variables.

With the help of the two above tables, called **FUNDAMENTAL TRUTH TABLES**, we can construct **DERIVATIVE TRUTH TABLES** for any compound sentential function. The table for the function "$(p \vee q) \rightarrow (p \wedge r)$", for instance, looks as follows:

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \rightarrow (p \wedge r)$
T	T	T	T	T	T
F	T	T	T	F	F
T	F	T	T	T	T
F	F	T	F	F	T
T	T	F	T	F	F
F	T	F	T	F	F
T	F	F	T	F	F
F	F	F	F	F	T

In order to explain the construction of this table, let us concentrate, say, on its fifth horizontal line (below the headings). We substitute true sentences for "p" and "q" and a false sentence for "r". According to the second fundamental table, we then obtain from "$p \vee q$" a true sentence and from "$p \wedge r$" a false sentence. From the whole function "$(p \vee q) \rightarrow (p \wedge r)$" we obtain then an implication with a true antecedent and a false consequent; hence, again with the help of the second fundamental table (in which we think of "p" and "q" being for the moment replaced by "$p \vee q$" and "$p \wedge r$"), we conclude that this implication is a false sentence.

The horizontal lines of a table that consist of symbols "T" and "F" are called **ROWS** of the table, and the vertical lines are called **COLUMNS**. Each row or, rather, that part of each row which is on the left of the vertical bar represents a certain substitution of true or false sentences for the variables. When constructing the matrix of a given function, we take care to exhaust all possible ways in which a combination of symbols "T" and "F" could be correlated to the variables; and, of course, we never write in a table two rows which do not differ either in the number or in the order of the symbols "T" and "F". It can then be seen very easily that the number of rows in a table depends in a simple way on the number of different variables occurring in the function; if a function contains 1, 2, 3, ··· variables of different shape, its matrix consists of $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, ··· rows. As for the number of columns, it is equal to the number of partial sentential functions of different form contained in the given function (where the whole function is also counted among its partial functions).

We are now in a position to say how it may be decided whether or not a sentence of sentential calculus is true. As we know, in sentential calculus, there is no external difference between sentences and sentential functions; the only difference consisting in the fact that the expressions considered to be sentences are always completed mentally by the universal quantifier. In order to recognize whether the given sentence is true, we treat it, for the time being, as a sentential function, and construct the truth table for it. If, in the last column of this table, no symbol "F" occurs, then every sentence obtainable from the function in question by substitution will be true, and therefore our original universal sentence (obtained from the sentential function by mentally prefixing the universal quantifier) is also true. If, however, the last column contains at least one symbol "F", our sentence is false.

Thus, for instance, we have seen that in the matrix constructed for the function "$(p \lor q) \to (p \land r)$" the symbol "F" occurs four times in the last column. If, therefore, we considered this expression as a sentence (that is, if we prefixed to it the words "*for any p, q and r*"), we would have a false sentence. On the other hand,. it can be easily verified with the help of the method of truth tables that all the laws of sentential calculus stated in Section 12, that is, the laws of simplification, identity, and so on, are true sentences. The table for the law of simplification:

$$(p \land q) \to p,$$

for instance, is as follows:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

We give here a number of other important laws of sentential calculus whose truth can be ascertained in a similar way:

$\sim[p \wedge (\sim p)]$, $p \vee (\sim p)$,

$(p \wedge p) \leftrightarrow p$, $(p \vee p) \leftrightarrow p$,

$(p \wedge q) \leftrightarrow (q \wedge p)$, $(p \vee q) \leftrightarrow (q \vee p)$,

$[p \wedge (q \wedge r)] \leftrightarrow [(p \wedge q) \wedge r]$, $[p \vee (q \vee r)] \leftrightarrow [(p \vee q) \vee r]$.

The two laws in the first line are called the LAW OF CONTRADICTION and the LAW OF EXCLUDED MIDDLE; we next have the two LAWS OF TAUTOLOGY (for logical multiplication and addition); we then have the two COMMUTATIVE LAWS, and finally the two ASSOCIATIVE LAWS. It can easily be seen how obscure the meaning of these last two laws becomes if we try to express them in ordinary language. This exhibits very clearly the value of logical symbolism as a precise instrument for expressing more complicated thoughts.

*It occurs that the method of matrices leads us to accept sentences as true whose truth seemed to be far from obvious before the application of this method. Here are some examples of sentences of this kind:

$$p \rightarrow (q \rightarrow p),$$

$$(\sim p) \rightarrow (p \rightarrow q),$$

$$(p \rightarrow q) \vee (q \rightarrow p).$$

That these sentences are not immediately obvious is due mainly to the fact that they are a manifestation of the specific usage of implication characteristic of modern logic, namely, the usage of implication in material meaning.