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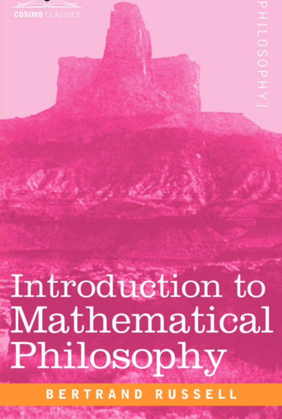

Introduction to Mathematical Philosophy

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Introduction to Mathematical Philosophy

CHAPTER I

THE SERIES OF NATURAL NUMBERS

Mathematics is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analysing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathematical philosophy as opposed to ordinary mathematics. But it should be understood that the distinction is one, not in the subject matter, but in the state of mind of the investigator. Early Greek geometers, passing from the empirical rules of Egyptian land-surveying to the general propositions by which those rules were found to be justifiable, and thence to Euclid's axioms and postulates, were engaged in mathematical philosophy, according to the above definition; but when once the axioms and postulates had been reached, their deductive employment, as we find it in Euclid, belonged to mathematics in the ordinary sense. The distinction between mathematics and mathematical philosophy is one which depends upon the interest inspiring the research, and upon the stage which the research has reached; not upon the propositions with which the research is concerned.

We may state the same distinction in another way. The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using "simple" in a *logical* sense). And as we need two sorts of instruments, the telescope and the microscope, for the enlargement of our visual powers, so we need two sorts of instruments for the enlargement of our logical powers, one to take us forward to the higher mathematics, the other to take us backward to the logical foundations of the things that we are inclined to take for granted in mathematics. We shall find that by analysing our ordinary mathematical notions we acquire fresh insight, new powers, and the means of reaching whole new mathematical subjects by adopting fresh lines of advance after our backward journey. It is the purpose of this book to explain mathematical philosophy simply and untechnically, without enlarging upon those portions which are so doubtful or difficult that an elementary treatment is scarcely possible. A full treatment will be found in *Principia Mathematica*;[1] the treatment in the present volume is intended merely as an introduction.

To the average educated person of the present day, the obvious starting-point of mathematics would be the series of whole numbers,

$$1, 2, 3, 4, \ldots \text{etc.}$$

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[1] Cambridge University Press, vol. i., 1910; vol. ii., 1911; vol. iii., 1913. By Whitehead and Russell.

Probably only a person with some mathematical knowledge would think of beginning with 0 instead of with 1, but we will presume this degree of knowledge; we will take as our starting-point the series:

$$0, 1, 2, 3, \ldots n, n+1, \ldots$$

and it is this series that we shall mean when we speak of the "series of natural numbers."

It is only at a high stage of civilisation that we could take this series as our starting-point. It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2: the degree of abstraction involved is far from easy. And the discovery that 1 is a number must have been difficult. As for 0, it is a very recent addition; the Greeks and Romans had no such digit. If we had been embarking upon mathematical philosophy in earlier days, we should have had to start with something less abstract than the series of natural numbers, which we should reach as a stage on our backward journey. When the logical foundations of mathematics have grown more familiar, we shall be able to start further back, at what is now a late stage in our analysis. But for the moment the natural numbers seem to represent what is easiest and most familiar in mathematics.

But though familiar, they are not understood. Very few people are prepared with a definition of what is meant by "number," or "0," or "1." It is not very difficult to see that, starting from 0, any other of the natural numbers can be reached by repeated additions of 1, but we shall have to define what we mean by "adding 1," and what we mean by "repeated." These questions are by no means easy. It was believed until recently that some, at least, of these first notions of arithmetic must be accepted as too simple and primitive to be defined. Since all terms that are defined are defined by means of other terms, it is clear that human knowledge must always be content to accept some terms as intelligible without definition, in order to have a starting-point for its definitions. It is not clear that there must be terms which are *incapable* of definition: it is possible that, however far back we go in defining, we always *might* go further still. On the other hand, it is also possible that, when analysis has been pushed far enough, we can reach terms that really are simple, and therefore logically incapable of the sort of definition that consists in analysing. This is a question which it is not necessary for us to decide; for our purposes it is sufficient to observe that, since human powers are finite, the definitions known to us must always begin somewhere, with terms undefined for the moment, though perhaps not permanently.

All traditional pure mathematics, including analytical geometry, may be regarded as consisting wholly of propositions about the natural numbers. That is to say, the terms which occur can be defined by means of the natural numbers, and the propositions can be deduced from the properties of the natural numbers—with the addition, in each case, of the ideas and propositions of pure logic.

That all traditional pure mathematics can be derived from the natural numbers is a fairly recent discovery, though it had long been suspected. Pythagoras, who believed that not only mathematics, but everything else could be deduced from numbers, was the discoverer of the most serious obstacle in the way of what is called the "arithmetising" of mathematics. It was Pythagoras who discovered the existence of incommensurables, and, in particular, the incommensurability of the side of a square and the diagonal. If the length of the side is 1 inch, the number of inches in the diagonal is the square root of 2, which appeared not to be a number at all. The problem thus raised was solved only in our own day, and was only solved *completely* by the help of the reduction of arithmetic to logic, which will be explained in following chapters. For the present, we shall take for granted the arithmetisation of mathematics, though this was a feat of the very greatest importance.

Having reduced all traditional pure mathematics to the theory of the natural numbers, the next step in logical analysis was to reduce this theory itself to the smallest set of premisses and undefined terms from which it could be derived. This work was accomplished by Peano. He showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of pure logic. These three ideas and five propositions thus became, as it were, hostages for the whole of traditional pure mathematics. If they could be defined and proved in terms of others, so could all pure mathematics. Their logical "weight," if one may use such an expression, is equal to that of the whole series of sciences that have been deduced from the theory of the natural numbers; the truth of this whole series is assured if the truth of the five primitive propositions is guaranteed, provided, of course, that there is nothing erroneous in the purely logical apparatus which is also involved. The work of analysing mathematics is extraordinarily facilitated by this work of Peano's.

The three primitive ideas in Peano's arithmetic are:

0, number, successor.

By "successor" he means the next number in the natural order. That is to say, the successor of 0 is 1, the successor of 1 is 2, and so on. By "number" he means, in this connection, the class of the natural numbers.[1] He is not assuming that we know all the members of this class, but only that we know what we mean when we say that this or that is a number, just as we know what we mean when we say "Jones is a man," though we do not know all men individually.

The five primitive propositions which Peano assumes are:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.

[1] We shall use "number" in this sense in the present chapter. Afterwards the word will be used in a more general sense.

(4) 0 is not the successor of any number.

(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

The last of these is the principle of mathematical induction. We shall have much to say concerning mathematical induction in the sequel; for the present, we are concerned with it only as it occurs in Peano's analysis of arithmetic.

Let us consider briefly the kind of way in which the theory of the natural numbers results from these three ideas and five propositions. To begin with, we define 1 as "the successor of 0," 2 as "the successor of 1," and so on. We can obviously go on as long as we like with these definitions, since, in virtue of (2), every number that we reach will have a successor, and, in virtue of (3), this cannot be any of the numbers already defined, because, if it were, two different numbers would have the same successor; and in virtue of (4) none of the numbers we reach in the series of successors can be 0. Thus the series of successors gives us an endless series of continually new numbers. In virtue of (5) all numbers come in this series, which begins with 0 and travels on through successive successors: for (*a*) 0 belongs to this series, and (*b*) if a number n belongs to it, so does its successor, whence, by mathematical induction, every number belongs to the series.

Suppose we wish to define the sum of two numbers. Taking any number m, we define $m+0$ as m, and $m+(n+1)$ as the successor of $m+n$. In virtue of (5) this gives a definition of the sum of m and n, whatever number n may be. Similarly we can define the product of any two numbers. The reader can easily convince himself that any ordinary elementary proposition of arithmetic can be proved by means of our five premisses, and if he has any difficulty he can find the proof in Peano.

It is time now to turn to the considerations which make it necessary to advance beyond the standpoint of Peano, who represents the last perfection of the "arithmetisation" of mathematics, to that of Frege, who first succeeded in "logicising" mathematics, *i.e.* in reducing to logic the arithmetical notions which his predecessors had shown to be sufficient for mathematics. We shall not, in this chapter, actually give Frege's definition of number and of particular numbers, but we shall give some of the reasons why Peano's treatment is less final than it appears to be.

In the first place, Peano's three primitive ideas—namely, "0," "number," and "successor"—are capable of an infinite number of different interpretations, all of which will satisfy the five primitive propositions. We will give some examples.

(1) Let "0" be taken to mean 100, and let "number" be taken to mean the numbers from 100 onward in the series of natural numbers. Then all our primitive propositions are satisfied, even the fourth, for, though 100 is the successor of 99, 99 is not a "number" in the sense which we are now giving to the word "number." It is obvious that any number may be substituted for 100 in this example.

(2) Let "0" have its usual meaning, but let "number" mean what we usually call "even numbers," and let the "successor" of a number be what results from adding two to it. Then "1" will stand for the number two, "2" will stand for the number four, and so on; the series of "numbers" now will be

0, two, four, six, eight . . .

All Peano's five premisses are satisfied still.

(3) Let "0" mean the number one, let "number" mean the set

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots$$

and let "successor" mean "half." Then all Peano's five axioms will be true of this set.

It is clear that such examples might be multiplied indefinitely. In fact, given any series

$$x_0, x_1, x_2, x_3, \ldots x_n, \ldots$$

which is endless, contains no repetitions, has a beginning, and has no terms that cannot be reached from the beginning in a finite number of steps, we have a set of terms verifying Peano's axioms. This is easily seen, though the formal proof is somewhat long. Let "o" mean x_0, let "number" mean the whole set of terms, and let the "successor" of x_n mean x_{n+1}. Then

(1) "o is a number," *i.e.* x_0 is a member of the set.

(2) "The successor of any number is a number," *i.e.* taking any term x_n in the set, x_{n+1} is also in the set.

(3) "No two numbers have the same successor," *i.e.* if x_m and x_n are two different members of the set, x_{m+1} and x_{n+1} are different; this results from the fact that (by hypothesis) there are no repetitions in the set.

(4) "o is not the successor of any number," *i.e.* no term in the set comes before x_0.

(5) This becomes: Any property which belongs to x_0, and belongs to x_{n+1} provided it belongs to x_n, belongs to all the x's.

This follows from the corresponding property for numbers.

A series of the form

$$x_0, x_1, x_2, \ldots x_n, \ldots$$

in which there is a first term, a successor to each term (so that there is no last term), no repetitions, and every term can be reached from the start in a finite number of steps, is called a *progression*. Progressions are of great importance in the principles of mathematics. As we have just seen, every progression verifies Peano's five axioms. It can be proved, conversely, that every series which verifies Peano's five axioms is a progression. Hence these five axioms may be used to define the class of progressions: "progressions" are "those series which verify these five axioms." Any progression may be taken as the basis of pure mathematics: we may give the name "o" to its first term, the name "number" to the whole set of its terms, and the name "successor" to the next in the progression. The progression need not be composed of numbers: it may be composed of points in space, or moments of time, or any other terms of which there is an infinite supply. Each different progression will give rise to a different interpretation of all the propositions of traditional pure mathematics; all these possible interpretations will be equally true.

In Peano's system there is nothing to enable us to distinguish between these different interpretations of his primitive ideas. It is assumed that we know what is meant by "0," and that we shall not suppose that this symbol means 100 or Cleopatra's Needle or any of the other things that it might mean.

This point, that "0" and "number" and "successor" cannot be defined by means of Peano's five axioms, but must be independently understood, is important. We want our numbers not merely to verify mathematical formulæ, but to apply in the right way to common objects. We want to have ten fingers and two eyes and one nose. A system in which "1" meant 100, and "2" meant 101, and so on, might be all right for pure mathematics, but would not suit daily life. We want "0" and "number" and "successor" to have meanings which will give us the right allowance of fingers and eyes and noses. We have already some knowledge (though not sufficiently articulate or analytic) of what we mean by "1" and "2" and so on, and our use of numbers in arithmetic must conform to this knowledge. We cannot secure that this shall be the case by Peano's method; all that we can do, if we adopt his method, is to say "we know what we mean by '0' and 'number' and 'successor,' though we cannot explain what we mean in terms of other simpler concepts." It is quite legitimate to say this when we must, and at *some* point we all must; but it is the object of mathematical philosophy to put off saying it as long as possible. By the logical theory of arithmetic we are able to put it off for a very long time.

It might be suggested that, instead of setting up "0" and "number" and "successor" as terms of which we know the meaning although we cannot define them, we might let them stand for *any* three terms that verify Peano's five axioms. They will then no longer be terms which have a meaning that is definite though undefined: they will be "variables," terms concerning which we make certain hypotheses, namely, those stated in the five axioms, but which are otherwise undetermined. If we adopt this plan, our theorems will not be proved concerning an ascertained set of terms called "the natural numbers," but concerning all sets of terms having certain properties. Such a procedure is not fallacious; indeed for certain purposes it represents a valuable generalisation. But from two points of view it fails to give an adequate basis for arithmetic. In the first place, it does not enable us to know whether there are any sets of terms verifying Peano's axioms; it does not even give the faintest suggestion of any way of discovering whether there are such sets. In the second place, as already observed, we want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. This definite meaning is defined by the logical theory of arithmetic.

CHAPTER II

DEFINITION OF NUMBER

THE question "What is a number?" is one which has been often asked, but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his *Grundlagen der Arithmetik*.[1] Although this book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author in 1901.

In seeking a definition of number, the first thing to be clear about is what we may call the grammar of our inquiry. Many philosophers, when attempting to define number, are really setting to work to define plurality, which is quite a different thing. *Number* is what is characteristic of numbers, as *man* is what is characteristic of men. A plurality is not an instance of number, but of some particular number. A trio of men, for example, is an instance of the number 3, and the number 3 is an instance of number; but the trio is not an instance of number. This point may seem elementary and scarcely worth mentioning; yet it has proved too subtle for the philosophers, with few exceptions.

A particular number is not identical with any collection of terms having that number: the number 3 is not identical with

[1] The same answer is given more fully and with more development in his *Grundgesetze der Arithmetik*, vol. i., 1893.

the trio consisting of Brown, Jones, and Robinson. The number 3 is something which all trios have in common, and which distinguishes them from other collections. A number is something that characterises certain collections, namely, those that have that number.

Instead of speaking of a "collection," we shall as a rule speak of a "class," or sometimes a "set." Other words used in mathematics for the same thing are "aggregate" and "manifold." We shall have much to say later on about classes. For the present, we will say as little as possible. But there are some remarks that must be made immediately.

A class or collection may be defined in two ways that at first sight seem quite distinct. We may enumerate its members, as when we say, "The collection I mean is Brown, Jones, and Robinson." Or we may mention a defining property, as when we speak of "mankind" or "the inhabitants of London." The definition which enumerates is called a definition by "extension," and the one which mentions a defining property is called a definition by "intension." Of these two kinds of definition, the one by intension is logically more fundamental. This is shown by two considerations: (1) that the extensional definition can always be reduced to an intensional one; (2) that the intensional one often cannot even theoretically be reduced to the extensional one. Each of these points needs a word of explanation.

(1) Brown, Jones, and Robinson all of them possess a certain property which is possessed by nothing else in the whole universe, namely, the property of being either Brown or Jones or Robinson. This property can be used to give a definition by intension of the class consisting of Brown and Jones and Robinson. Consider such a formula as "x is Brown or x is Jones or x is Robinson." This formula will be true for just three x's, namely, Brown and Jones and Robinson. In this respect it resembles a cubic equation with its three roots. It may be taken as assigning a property common to the members of the class consisting of these three

men, and peculiar to them. A similar treatment can obviously be applied to any other class given in extension.

(2) It is obvious that in practice we can often know a great deal about a class without being able to enumerate its members. No one man could actually enumerate all men, or even all the inhabitants of London, yet a great deal is known about each of these classes. This is enough to show that definition by extension is not *necessary* to knowledge about a class. But when we come to consider infinite classes, we find that enumeration is not even theoretically possible for beings who only live for a finite time. We cannot enumerate all the natural numbers: they are 0, 1, 2, 3, *and so on*. At some point we must content ourselves with "and so on." We cannot enumerate all fractions or all irrational numbers, or all of any other infinite collection. Thus our knowledge in regard to all such collections can only be derived from a definition by intension.

These remarks are relevant, when we are seeking the definition of number, in three different ways. In the first place, numbers themselves form an infinite collection, and cannot therefore be defined by enumeration. In the second place, the collections having a given number of terms themselves presumably form an infinite collection: it is to be presumed, for example, that there are an infinite collection of trios in the world, for if this were not the case the total number of things in the world would be finite, which, though possible, seems unlikely. In the third place, we wish to define "number" in such a way that infinite numbers may be possible; thus we must be able to speak of the number of terms in an infinite collection, and such a collection must be defined by intension, *i.e.* by a property common to all its members and peculiar to them.

For many purposes, a class and a defining characteristic of it are practically interchangeable. The vital difference between the two consists in the fact that there is only one class having a given set of members, whereas there are always many different characteristics by which a given class may be defined. Men may be defined as featherless bipeds, or as rational animals, or (more correctly) by the traits by which Swift delineates the Yahoos. It is this fact that a defining characteristic is never unique which makes classes useful; otherwise we could be content with the properties common and peculiar to their members.[1] Any one of these properties can be used in place of the class whenever uniqueness is not important.

Returning now to the definition of number, it is clear that number is a way of bringing together certain collections, namely, those that have a given number of terms. We can suppose all couples in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections, each bundle consisting of all the collections that have a certain number of terms. Each bundle is a class whose members are collections, *i.e.* classes; thus each is a class of classes. The bundle consisting of all couples, for example, is a class of classes: each couple is a class with two members, and the whole bundle of couples is a class with an infinite number of members, each of which is a class of two members.

How shall we decide whether two collections are to belong to the same bundle? The answer that suggests itself is: "Find out how many members each has, and put them in the same bundle if they have the same number of members." But this presupposes that we have defined numbers, and that we know how to discover how many terms a collection has. We are so used to the operation of counting that such a presupposition might easily pass unnoticed. In fact, however, counting, though familiar, is logically a very complex operation; moreover it is only available, as a means of discovering how many terms a collection has, when the collection is finite. Our definition of number must not assume in advance that all numbers are finite; and we cannot in any case, without a vicious circle,

[1] As will be explained later, classes may be regarded as logical fictions, manufactured out of defining characteristics. But for the present it will simplify our exposition to treat classes as if they were real.

use counting to define numbers, because numbers are used in counting. We need, therefore, some other method of deciding when two collections have the same number of terms.

In actual fact, it is simpler logically to find out whether two collections have the same number of terms than it is to define what that number is. An illustration will make this clear. If there were no polygamy or polyandry anywhere in the world, it is clear that the number of husbands living at any moment would be exactly the same as the number of wives. We do not need a census to assure us of this, nor do we need to know what is the actual number of husbands and of wives. We know the number must be the same in both collections, because each husband has one wife and each wife has one husband. The relation of husband and wife is what is called "one-one."

A relation is said to be "one-one" when, if x has the relation in question to y, no other term x' has the same relation to y, and x does not have the same relation to any term y' other than y. When only the first of these two conditions is fulfilled, the relation is called "one-many"; when only the second is fulfilled, it is called "many-one." It should be observed that the number 1 is not used in these definitions.

In Christian countries, the relation of husband to wife is one-one; in Mahometan countries it is one-many; in Tibet it is many-one. The relation of father to son is one-many; that of son to father is many-one, but that of eldest son to father is one-one. If n is any number, the relation of n to $n+1$ is one-one; so is the relation of n to $2n$ or to $3n$. When we are considering only positive numbers, the relation of n to n^2 is one-one; but when negative numbers are admitted, it becomes two-one, since n and $-n$ have the same square. These instances should suffice to make clear the notions of one-one, one-many, and many-one relations, which play a great part in the principles of mathematics, not only in relation to the definition of numbers, but in many other connections.

Two classes are said to be "similar" when there is a one-one relation which correlates the terms of the one class each with one term of the other class, in the same manner in which the relation of marriage correlates husbands with wives. A few preliminary definitions will help us to state this definition more precisely. The class of those terms that have a given relation to something or other is called the *domain* of that relation: thus fathers are the domain of the relation of father to child, husbands are the domain of the relation of husband to wife, wives are the domain of the relation of wife to husband, and husbands and wives together are the domain of the relation of marriage. The relation of wife to husband is called the *converse* of the relation of husband to wife. Similarly *less* is the converse of *greater*, *later* is the converse of *earlier*, and so on. Generally, the converse of a given relation is that relation which holds between y and x whenever the given relation holds between x and y. The *converse domain* of a relation is the domain of its converse: thus the class of wives is the converse domain of the relation of husband to wife. We may now state our definition of similarity as follows:—

One class is said to be "similar" to another when there is a one-one relation of which the one class is the domain, while the other is the converse domain.

It is easy to prove (1) that every class is similar to itself, (2) that if a class α is similar to a class β, then β is similar to α, (3) that if α is similar to β and β to γ, then α is similar to γ. A relation is said to be *reflexive* when it possesses the first of these properties, *symmetrical* when it possesses the second, and *transitive* when it possesses the third. It is obvious that a relation which is symmetrical and transitive must be reflexive throughout its domain. Relations which possess these properties are an important kind, and it is worth while to note that similarity is one of this kind of relations.

It is obvious to common sense that two finite classes have the same number of terms if they are similar, but not otherwise. The act of counting consists in establishing a one-one correlation between the set of objects counted and the natural numbers (excluding 0) that are used up in the process. Accordingly common sense concludes that there are as many objects in the set to be counted as there are numbers up to the last number used in the counting. And we also know that, so long as we confine ourselves to finite numbers, there are just n numbers from 1 up to n. Hence it follows that the last number used in counting a collection is the number of terms in the collection, provided the collection is finite. But this result, besides being only applicable to finite collections, depends upon and assumes the fact that two classes which are similar have the same number of terms; for what we do when we count (say) 10 objects is to show that the set of these objects is similar to the set of numbers 1 to 10. The notion of similarity is logically presupposed in the operation of counting, and is logically simpler though less familiar. In counting, it is necessary to take the objects counted in a certain order, as first, second, third, etc., but order is not of the essence of number: it is an irrelevant addition, an unnecessary complication from the logical point of view. The notion of similarity does not demand an order: for example, we saw that the number of husbands is the same as the number of wives, without having to establish an order of precedence among them. The notion of similarity also does not require that the classes which are similar should be finite. Take, for example, the natural numbers (excluding 0) on the one hand, and the fractions which have 1 for their numerator on the other hand: it is obvious that we can correlate 2 with $\frac{1}{2}$, 3 with $\frac{1}{3}$, and so on, thus proving that the two classes are similar.

We may thus use the notion of "similarity" to decide when two collections are to belong to the same bundle, in the sense in which we were asking this question earlier in this chapter. We want to make one bundle containing the class that has no members: this will be for the number 0. Then we want a bundle of all the classes that have one member: this will be for the number 1. Then, for the number 2, we want a bundle consisting of all couples; then one of all trios; and so on. Given any collection, we can define the bundle it is to belong to as being the class of all those collections that are "similar" to it. It is very easy to see that if (for example) a collection has three members, the class of all those collections that are similar to it will be the class of trios. And whatever number of terms a collection may have, those collections that are "similar" to it will have the same number of terms. We may take this as a *definition* of "having the same number of terms." It is obvious that it gives results conformable to usage so long as we confine ourselves to finite collections.

So far we have not suggested anything in the slightest degree paradoxical. But when we come to the actual definition of numbers we cannot avoid what must at first sight seem a paradox, though this impression will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive. Accordingly we set up the following definition :—

The number of a class is the class of all those classes that are similar to it.

Thus the number of a couple will be the class of all couples. In fact, the class of all couples will *be* the number 2, according to our definition. At the expense of a little oddity, this definition secures definiteness and indubitableness; and it is not difficult to prove that numbers so defined have all the properties that we expect numbers to have.

We may now go on to define numbers in general as any one of the bundles into which similarity collects classes. A number will be a set of classes such as that any two are similar to each other, and none outside the set are similar to any inside the set. In other words, a number (in general) is any collection which is the number of one of its members; or, more simply still:

A number is anything which is the number of some class.

Such a definition has a verbal appearance of being circular, but in fact it is not. We define "the number of a given class" without using the notion of number in general; therefore we may define number in general in terms of "the number of a given class" without committing any logical error.

Definitions of this sort are in fact very common. The class of fathers, for example, would have to be defined by first defining what it is to be the father of somebody; then the class of fathers will be all those who are somebody's father. Similarly if we want to define square numbers (say), we must first define what we mean by saying that one number is the square of another, and then define square numbers as those that are the squares of other numbers. This kind of procedure is very common, and it is important to realise that it is legitimate and even often necessary.

We have now given a definition of numbers which will serve for finite collections. It remains to be seen how it will serve for infinite collections. But first we must decide what we mean by "finite" and "infinite," which cannot be done within the limits of the present chapter.

CHAPTER III

FINITUDE AND MATHEMATICAL INDUCTION

THE series of natural numbers, as we saw in Chapter I., can all be defined if we know what we mean by the three terms "0," "number," and "successor." But we may go a step farther: we can define all the natural numbers if we know what we mean by "0" and "successor." It will help us to understand the difference between finite and infinite to see how this can be done, and why the method by which it is done cannot be extended beyond the finite. We will not yet consider how "0" and "successor" are to be defined: we will for the moment assume that we know what these terms mean, and show how thence all other natural numbers can be obtained.

It is easy to see that we can reach any assigned number, say 30,000. We first define "1" as "the successor of 0," then we define "2" as "the successor of 1," and so on. In the case of an assigned number, such as 30,000, the proof that we can reach it by proceeding step by step in this fashion may be made, if we have the patience, by actual experiment: we can go on until we actually arrive at 30,000. But although the method of experiment is available for each particular natural number, it is not available for proving the general proposition that *all* such numbers can be reached in this way, *i.e.* by proceeding from 0 step by step from each number to its successor. Is there any other way by which this can be proved?

Let us consider the question the other way round. What are the numbers that can be reached, given the terms "0" and "successor"? Is there any way by which we can define the whole class of such numbers? We reach 1, as the successor of 0; 2, as the successor of 1; 3, as the successor of 2; and so on. It is this "and so on" that we wish to replace by something less vague and indefinite. We might be tempted to say that "and so on" means that the process of proceeding to the successor may be repeated *any finite number* of times; but the problem upon which we are engaged is the problem of defining "finite number," and therefore we must not use this notion in our definition. Our definition must not assume that we know what a finite number is.

The key to our problem lies in *mathematical induction*. It will be remembered that, in Chapter I., this was the fifth of the five primitive propositions which we laid down about the natural numbers. It stated that any property which belongs to 0, and to the successor of any number which has the property, belongs to all the natural numbers. This was then presented as a principle, but we shall now adopt it as a definition. It is not difficult to see that the terms obeying it are the same as the numbers that can be reached from 0 by successive steps from next to next, but as the point is important we will set forth the matter in some detail.

We shall do well to begin with some definitions, which will be useful in other connections also.

A property is said to be "hereditary" in the natural-number series if, whenever it belongs to a number n, it also belongs to $n+1$, the successor of n. Similarly a class is said to be "hereditary" if, whenever n is a member of the class, so is $n+1$. It is easy to *see*, though we are not yet supposed to know, that to say a property is hereditary is equivalent to saying that it belongs to all the natural numbers not less than some one of them, *e.g.* it must belong to all that are not less than 100, or all that are less than 1000, or it may be that it belongs to all that are not less than 0, *i.e.* to all without exception.

A property is said to be "inductive" when it is a hereditary property which belongs to 0. Similarly a class is "inductive" when it is a hereditary class of which 0 is a member.

Given a hereditary class of which 0 is a member, it follows that 1 is a member of it, because a hereditary class contains the successors of its members, and 1 is the successor of 0. Similarly, given a hereditary class of which 1 is a member, it follows that 2 is a member of it; and so on. Thus we can prove by a step-by-step procedure that any assigned natural number, say 30,000, is a member of every inductive class.

We will define the "posterity" of a given natural number with respect to the relation "immediate predecessor" (which is the converse of "successor") as all those terms that belong to every hereditary class to which the given number belongs. It is again easy to *see* that the posterity of a natural number consists of itself and all greater natural numbers; but this also we do not yet officially know.

By the above definitions, the posterity of 0 will consist of those terms which belong to every inductive class.

It is now not difficult to make it obvious that the posterity of 0 is the same set as those terms that can be reached from 0 by successive steps from next to next. For, in the first place, 0 belongs to both these sets (in the sense in which we have defined our terms); in the second place, if n belongs to both sets, so does $n+1$. It is to be observed that we are dealing here with the kind of matter that does not admit of precise proof, namely, the comparison of a relatively vague idea with a relatively precise one. The notion of "those terms that can be reached from 0 by successive steps from next to next" is vague, though it *seems* as if it conveyed a definite meaning; on the other hand, "the posterity of 0" is precise and explicit just where the other idea is hazy. It may be taken as giving what we *meant* to mean when we spoke of the terms that can be reached from 0 by successive steps.

We now lay down the following definition:—

The "natural numbers" are the posterity of 0 *with respect to the* *relation "immediate predecessor"* (which is the converse of "successor").

We have thus arrived at a definition of one of Peano's three primitive ideas in terms of the other two. As a result of this definition, two of his primitive propositions—namely, the one asserting that 0 is a number and the one asserting mathematical induction—become unnecessary, since they result from the definition. The one asserting that the successor of a natural number is a natural number is only needed in the weakened form "every natural number has a successor."

We can, of course, easily define "0" and "successor" by means of the definition of number in general which we arrived at in Chapter II. The number 0 is the number of terms in a class which has no members, *i.e.* in the class which is called the "null-class." By the general definition of number, the number of terms in the null-class is the set of all classes similar to the null-class, *i.e.* (as is easily proved) the set consisting of the null-class all alone, *i.e.* the class whose only member is the null-class. (This is not identical with the null-class: it has one member, namely, the null-class, whereas the null-class itself has no members. A class which has one member is never identical with that one member, as we shall explain when we come to the theory of classes.) Thus we have the following purely logical definition:—

0 *is the class whose only member is the null-class.*

It remains to define "successor." Given any number n, let α be a class which has n members, and let x be a term which is not a member of α. Then the class consisting of α with x added on will have $n+1$ members. Thus we have the following definition:—

The successor of the number of terms in the class α *is the number of terms in the class consisting of* α *together with* x, *where* x *is any term not belonging to the class.*

Certain niceties are required to make this definition perfect, but they need not concern us.[1] It will be remembered that we

[1] See *Principia Mathematica*, vol. ii. ∗ 110.

have already given (in Chapter II.) a logical definition of the number of terms in a class, namely, we defined it as the set of all classes that are similar to the given class.

We have thus reduced Peano's three primitive ideas to ideas of logic: we have given definitions of them which make them definite, no longer capable of an infinity of different meanings, as they were when they were only determinate to the extent of obeying Peano's five axioms. We have removed them from the fundamental apparatus of terms that must be merely apprehended, and have thus increased the deductive articulation of mathematics.

As regards the five primitive propositions, we have already succeeded in making two of them demonstrable by our definition of "natural number." How stands it with the remaining three? It is very easy to prove that 0 is not the successor of any number, and that the successor of any number is a number. But there is a difficulty about the remaining primitive proposition, namely, "no two numbers have the same successor." The difficulty does not arise unless the total number of individuals in the universe is finite; for given two numbers m and n, neither of which is the total number of individuals in the universe, it is easy to prove that we cannot have $m+1=n+1$ unless we have $m=n$. But let us suppose that the total number of individuals in the universe were (say) 10; then there would be no class of 11 individuals, and the number 11 would be the null-class. So would the number 12. Thus we should have $11=12$; therefore the successor of 10 would be the same as the successor of 11, although 10 would not be the same as 11. Thus we should have two different numbers with the same successor. This failure of the third axiom cannot arise, however, if the number of individuals in the world is not finite. We shall return to this topic at a later stage.[1]

Assuming that the number of individuals in the universe is not finite, we have now succeeded not only in defining Peano's

[1] See Chapter XIII.

three primitive ideas, but in seeing how to prove his five primitive propositions, by means of primitive ideas and propositions belonging to logic. It follows that all pure mathematics, in so far as it is deducible from the theory of the natural numbers, is only a prolongation of logic. The extension of this result to those modern branches of mathematics which are not deducible from the theory of the natural numbers offers no difficulty of principle, as we have shown elsewhere.[1]

The process of mathematical induction, by means of which we defined the natural numbers, is capable of generalisation. We defined the natural numbers as the "posterity" of 0 with respect to the relation of a number to its immediate successor. If we call this relation N, any number m will have this relation to $m+1$. A property is "hereditary with respect to N," or simply "N-hereditary," if, whenever the property belongs to a number m, it also belongs to $m+1$, *i.e.* to the number to which m has the relation N. And a number n will be said to belong to the "posterity" of m with respect to the relation N if n has every N-hereditary property belonging to m. These definitions can all be applied to any other relation just as well as to N. Thus if R is any relation whatever, we can lay down the following definitions:[2]—

A property is called "R-hereditary" when, if it belongs to a term x, and x has the relation R to y, then it belongs to y.

A class is R-hereditary when its defining property is R-hereditary.

A term x is said to be an "R-ancestor" of the term y if y has every R-hereditary property that x has, provided x is a term which has the relation R to something or to which something has the relation R. (This is only to exclude trivial cases.)

[1] For geometry, in so far as it is not purely analytical, see *Principles of Mathematics*, part vi.; for rational dynamics, *ibid.*, part vii.

[2] These definitions, and the generalised theory of induction, are due to Frege, and were published so long ago as 1879 in his *Begriffsschrift*. In spite of the great value of this work, I was, I believe, the first person who ever read it—more than twenty years after its publication.

The "R-posterity" of x is all the terms of which x is an R-ancestor.

We have framed the above definitions so that if a term is the ancestor of anything it is its own ancestor and belongs to its own posterity. This is merely for convenience.

It will be observed that if we take for R the relation "parent," "ancestor" and "posterity" will have the usual meanings, except that a person will be included among his own ancestors and posterity. It is, of course, obvious at once that "ancestor" must be capable of definition in terms of "parent," but until Frege developed his generalised theory of induction, no one could have defined "ancestor" precisely in terms of "parent." A brief consideration of this point will serve to show the importance of the theory. A person confronted for the first time with the problem of defining "ancestor" in terms of "parent" would naturally say that A is an ancestor of Z if, between A and Z, there are a certain number of people, B, C, . . ., of whom B is a child of A, each is a parent of the next, until the last, who is a parent of Z. But this definition is not adequate unless we add that the number of intermediate terms is to be finite. Take, for example, such a series as the following:—

$$-1, -\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{8}, \ldots \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, 1.$$

Here we have first a series of negative fractions with no end, and then a series of positive fractions with no beginning. Shall we say that, in this series, $-\frac{1}{8}$ is an ancestor of $\frac{1}{8}$? It will be so according to the beginner's definition suggested above, but it will not be so according to any definition which will give the kind of idea that we wish to define. For this purpose, it is essential that the number of intermediaries should be finite. But, as we saw, "finite" is to be defined by means of mathematical induction, and it is simpler to define the ancestral relation generally at once than to define it first only for the case of the relation of n to $n+1$, and then extend it to other cases. Here, as constantly elsewhere, generality from the first, though it may require more thought at the start, will be found in the long run to economise thought and increase logical power.

The use of mathematical induction in demonstrations was, in the past, something of a mystery. There seemed no reasonable doubt that it was a valid method of proof, but no one quite knew why it was valid. Some believed it to be really a case of induction, in the sense in which that word is used in logic. Poincaré[1] considered it to be a principle of the utmost importance, by means of which an infinite number of syllogisms could be condensed into one argument. We now know that all such views are mistaken, and that mathematical induction is a definition, not a principle. There are some numbers to which it can be applied, and there are others (as we shall see in Chapter VIII.) to which it cannot be applied. We *define* the "natural numbers" as those to which proofs by mathematical induction can be applied, *i.e.* as those that possess all inductive properties. It follows that such proofs can be applied to the natural numbers, not in virtue of any mysterious intuition or axiom or principle, but as a purely verbal proposition. If "quadrupeds" are defined as animals having four legs, it will follow that animals that have four legs are quadrupeds; and the case of numbers that obey mathematical induction is exactly similar.

We shall use the phrase "inductive numbers" to mean the same set as we have hitherto spoken of as the "natural numbers." The phrase "inductive numbers" is preferable as affording a reminder that the definition of this set of numbers is obtained from mathematical induction.

Mathematical induction affords, more than anything else, the essential characteristic by which the finite is distinguished from the infinite. The principle of mathematical induction might be stated popularly in some such form as "what can be inferred from next to next can be inferred from first to last." This is true when the number of intermediate steps between first and last is finite, not otherwise. Anyone who has ever

[1] *Science and Method*, chap. iv.

watched a goods train beginning to move will have noticed how the impulse is communicated with a jerk from each truck to the next, until at last even the hindmost truck is in motion. When the train is very long, it is a very long time before the last truck moves. If the train were infinitely long, there would be an infinite succession of jerks, and the time would never come when the whole train would be in motion. Nevertheless, if there were a series of trucks no longer than the series of inductive numbers (which, as we shall see, is an instance of the smallest of infinites), every truck would begin to move sooner or later if the engine persevered, though there would always be other trucks further back which had not yet begun to move. This image will help to elucidate the argument from next to next, and its connection with finitude. When we come to infinite numbers, where arguments from mathematical induction will be no longer valid, the properties of such numbers will help to make clear, by contrast, the almost unconscious use that is made of mathematical induction where finite numbers are concerned.

CHAPTER IV

THE DEFINITION OF ORDER

WE have now carried our analysis of the series of natural numbers to the point where we have obtained logical definitions of the members of this series, of the whole class of its members, and of the relation of a number to its immediate successor. We must now consider the *serial* character of the natural numbers in the order 0, 1, 2, 3, . . . We ordinarily think of the numbers as in this *order*, and it is an essential part of the work of analysing our data to seek a definition of " order " or " series " in logical terms.

The notion of order is one which has enormous importance in mathematics. Not only the integers, but also rational fractions and all real numbers have an order of magnitude, and this is essential to most of their mathematical properties. The order of points on a line is essential to geometry; so is the slightly more complicated order of lines through a point in a plane, or of planes through a line. Dimensions, in geometry, are a development of order. The conception of a *limit*, which underlies all higher mathematics, is a serial conception. There are parts of mathematics which do not depend upon the notion of order, but they are very few in comparison with the parts in which this notion is involved.

In seeking a definition of order, the first thing to realise is that no set of terms has just *one* order to the exclusion of others. A set of terms has all the orders of which it is capable. Sometimes one order is so much more familiar and natural to our

our second. A relation having our second property is called *transitive.*

(3) Given any two terms of the class which is to be ordered, there must be one which precedes and the other which follows. For example, of any two integers, or fractions, or real numbers, one is smaller and the other greater; but of any two complex numbers this is not true. Of any two moments in time, one must be earlier than the other; but of events, which may be simultaneous, this cannot be said. Of two points on a line, one must be to the left of the other. A relation having this third property is called *connected.*

When a relation possesses these three properties, it is of the sort to give rise to an order among the terms between which it holds; and wherever an order exists, some relation having these three properties can be found generating it.

Before illustrating this thesis, we will introduce a few definitions.

(1) A relation is said to be an aliorelative,[1] or to *be contained in* or *imply diversity*, if no term has this relation to itself. Thus, for example, "greater," "different in size," "brother," "husband," "father" are aliorelatives; but "equal," "born of the same parents," "dear friend" are not.

(2) The *square* of a relation is that relation which holds between two terms x and z when there is an intermediate term y such that the given relation holds between x and y and between y and z. Thus "paternal grandfather" is the square of "father," "greater by 2" is the square of "greater by 1," and so on.

(3) The *domain* of a relation consists of all those terms that have the relation to something or other, and the *converse domain* consists of all those terms to which something or other has the relation. These words have been already defined, but are recalled here for the sake of the following definition:—

(4) The *field* of a relation consists of its domain and converse domain together.

[1] This term is due to C. S. Peirce.

(5) One relation is said to *contain* or *be implied by* another if it holds whenever the other holds.

It will be seen that an *asymmetrical* relation is the same thing as a relation whose square is an aliorelative. It often happens that a relation is an aliorelative without being asymmetrical, though an asymmetrical relation is always an aliorelative. For example, "spouse" is an aliorelative, but is symmetrical, since if x is the spouse of y, y is the spouse of x. But among *transitive* relations, all aliorelatives are asymmetrical as well as *vice versa*.

From the definitions it will be seen that a *transitive* relation is one which is implied by its square, or, as we also say, "contains" its square. Thus "ancestor" is transitive, because an ancestor's ancestor is an ancestor; but "father" is not transitive, because a father's father is not a father. A transitive aliorelative is one which contains its square and is contained in diversity; or, what comes to the same thing, one whose square implies both it and diversity—because, when a relation is transitive, asymmetry is equivalent to being an aliorelative.

A relation is *connected* when, given any two different terms of its field, the relation holds between the first and the second or between the second and the first (not excluding the possibility that both may happen, though both cannot happen if the relation is asymmetrical).

It will be seen that the relation "ancestor," for example, is an aliorelative and transitive, but not connected; it is because it is not connected that it does not suffice to arrange the human race in a series.

The relation "less than or equal to," among numbers, is transitive and connected, but not asymmetrical or an aliorelative.

The relation "greater or less" among numbers is an aliorelative and is connected, but is not transitive, for if x is greater or less than y, and y is greater or less than z, it may happen that x and z are the same number.

Thus the three properties of being (1) an aliorelative, (2) twelve steps bring us back to our starting-point. Thus in such a case, though the relation "proper R-ancestor" is connected, and though R itself is an aliorelative, we do not get a series because "proper R-ancestor" is not an aliorelative. It is for this reason that we cannot say that one person comes before another with respect to the relation "right of" or to its ancestral derivative.

The above was an instance in which the ancestral relation was connected but not contained in diversity. An instance where it is contained in diversity but not connected is derived from the ordinary sense of the word "ancestor." If x is a proper ancestor of y, x and y cannot be the same person; but it is not true that of any two persons one must be an ancestor of the other.

The question of the circumstances under which series can be generated by ancestral relations derived from relations of consecutiveness is often important. Some of the most important cases are the following: Let R be a many-one relation, and let us confine our attention to the posterity of some term x. When so confined, the relation "proper R-ancestor" must be connected; therefore all that remains to ensure its being serial is that it shall be contained in diversity. This is a generalisation of the instance of the dinner-table. Another generalisation consists in taking R to be a one-one relation, and including the ancestry of x as well as the posterity. Here again, the one condition required to secure the generation of a series is that the relation "proper R-ancestor" shall be contained in diversity.

The generation of order by means of relations of consecutiveness, though important in its own sphere, is less general than the method which uses a transitive relation to define the order. It often happens in a series that there are an infinite number of intermediate terms between any two that may be selected, however near together these may be. Take, for instance, fractions in order of magnitude. Between any two fractions there are others—for example, the arithmetic mean of the two. Consequently there is no such thing as a pair of consecutive fractions. If we depended

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