Lectures on the Philosophy of Mathematics

Joel David Hamkins

The MIT Press
Cambridge, Massachusetts
London, England

Library of Congress Cataloging-in-Publication Data

Names: Hamkins, Joel David, author.
Title: Lectures on the philosophy of mathematics / Joel David Hamkins.
Description: Cambridge, Massachusetts: The MIT Press, [2020] | Includes bibliographical references and index.
Identifiers: LCCN 2020009876 | ISBN 9780262542234 (paperback)
Subjects: LCSH: Mathematics–Philosophy.
Classification: LCC QA8.6.H35 2020 | DDC 510.1–dc23
LC record available at https://lccn.loc.gov/2020009876

d_r0

Contents

Preface
About the Author

1 Numbers
1.1 Numbers versus numerals
1.2 Number systems
Natural numbers
Integers
Rational numbers
1.3 Incommensurable numbers
An alternative geometric argument
1.4 Platonism
Plenitudinous platonism
1.5 Logicism
Equinumerosity
The Cantor-Hume principle
The Julius Caesar problem
Numbers as equinumerosity classes
Neologicism
1.6 Interpreting arithmetic
Numbers as equinumerosity classes
Numbers as sets
Numbers as primitives
Numbers as morphisms
Numbers as games
Junk theorems
Interpretation of theories
1.7 What numbers could not be
The epistemological problem
1.8 Dedekind arithmetic
Arithmetic categoricity
1.9 Mathematical induction
Fundamental theorem of arithmetic
Infinitude of primes
1.10 Structuralism
Definability versus Leibnizian structure
Role of identity in the formal language

Isomorphism orbit
Categoricity
Structuralism in mathematical practice
Eliminative structuralism
Abstract structuralism
1.11 What is a real number?
Dedekind cuts
Theft and honest toil
Cauchy real numbers
Real numbers as geometric continuum
Categoricity for the real numbers
Categoricity for the real continuum
1.12 Transcendental numbers
The transcendence game
1.13 Complex numbers
Platonism for complex numbers
Categoricity for the complex field
A complex challenge for structuralism?
Structure as reduct of rigid structure
1.14 Contemporary type theory
1.15 More numbers
1.16 What is a philosophy for?
1.17 Finally, what is a number?
Questions for further thought
Further reading
Credits

2 Rigor
2.1 Continuity
Informal account of continuity
The definition of continuity
The continuity game
Estimation in analysis
Limits
2.2 Instantaneous change
Infinitesimals
Modern definition of the derivative
2.3 An enlarged vocabulary of concepts
2.4 The least-upper-bound principle
Consequences of completeness
Continuous induction
2.5 Indispensability of mathematics
Science without numbers
Fictionalism
The theory/metatheory distinction
2.6 Abstraction in the function concept
The Devil's staircase
Space-filling curves

Conway base-13 function
2.7 Infinitesimals revisited
Nonstandard analysis and the hyperreal numbers
Calculus in nonstandard analysis
Classical model-construction perspective
Axiomatic approach
"The" hyperreal numbers?
Radical nonstandardness perspective
Translating between nonstandard and classical perspectives
Criticism of nonstandard analysis
Questions for further thought
Further reading
Credits

3 Infinity
3.1 Hilbert's Grand Hotel
Hilbert's bus
Hilbert's train
3.2 Countable sets
3.3 Equinumerosity
3.4 Hilbert's half-marathon
Cantor's cruise ship
3.5 Uncountability
Cantor's original argument
Mathematical cranks
3.6 Cantor on transcendental numbers
Constructive versus nonconstructive arguments
3.7 On the number of subsets of a set
On the number of infinities
Russell on the number of propositions
On the number of possible committees
The diary of Tristram Shandy
The cartographer's paradox
The Library of Babel
On the number of possible books
3.8 Beyond equinumerosity to the comparative size principle
Place focus on reflexive preorders
3.9 What is Cantor's continuum hypothesis?
3.10 Transfinite cardinals—the alephs and the beths
Lewis on the number of objects and properties
3.11 Zeno's paradox
Actual versus potential infinity
3.12 How to count
Questions for further thought
Further reading
Credits

4 Geometry
4.1 Geometric constructions

Contemporary approach via symmetries
Collapsible compasses
Constructible points and the constructible plane
Constructible numbers and the number line
4.2 Nonconstructible numbers
Doubling the cube
Trisecting the angle
Squaring the circle
Circle-squarers and angle-trisectors
4.3 Alternative tool sets
Compass-only constructibility
Straightedge-only constructibility
Construction with a marked ruler
Origami constructibility
Spirograph constructibility
4.4 The ontology of geometry
4.5 The role of diagrams and figures
Kant
Hume on arithmetic reasoning over geometry
Manders on diagrammatic proof
Contemporary tools
How to lie with figures
Error and approximation in geometric construction
Constructing a perspective chessboard
4.6 Non-Euclidean geometry
Spherical geometry
Elliptical geometry
Hyperbolic geometry
Curvature of space
4.7 Errors in Euclid?
Implicit continuity assumptions
The missing concept of "between"
Hilbert's geometry
Tarski's geometry
4.8 Geometry and physical space
4.9 Poincaré on the nature of geometry
4.10 Tarski on the decidability of geometry
Questions for further thought
Further reading
Credits

5 Proof
5.1 Syntax-semantics distinction
Use/mention
5.2 What is proof?
Proof as dialogue
Wittgenstein
Thurston

Formalization and mathematical error
Formalization as a sharpening of mathematical ideas
Mathematics does not take place in a formal language
Voevodsky
Proofs without words
How to lie with figures
Hard arguments versus soft
Moral mathematical truth
5.3 Formal proof and proof theory
Soundness
Completeness
Compactness
Verifiability
Sound and verifiable, yet incomplete
Complete and verifiable, yet unsound
Sound and complete, yet unverifiable
The empty structure
Formal deduction examples
The value of formal deduction
5.4 Automated theorem proving and proof verification
Four-color theorem
Choice of formal system
5.5 Completeness theorem
5.6 Nonclassical logics
Classical versus intuitionistic validity
Informal versus formal use of “constructive”
Epistemological intrusion into ontology
No unbridgeable chasm
Logical pluralism
Classical and intuitionistic realms
5.7 Conclusion
Questions for further thought
Further reading
Credits

6 Computability
6.1 Primitive recursion
Implementing logic in primitive recursion
Diagonalizing out of primitive recursion
The Ackermann function
6.2 Turing on computability
Turing machines
Partiality is inherent in computability
Examples of Turing-machine programs
Decidability versus enumerability
Universal computer
“Stronger” Turing machines
Other models of computatibility

6.3 Computational power: Hierarchy or threshold?
The hierarchy vision
The threshold vision
Which vision is correct?
6.4 Church-Turing thesis
Computation in the physical universe
6.5 Undecidability
The halting problem
Other undecidable problems
The tiling problem
Computable decidability versus enumerability
6.6 Computable numbers
6.7 Oracle computation and the Turing degrees
6.8 Complexity theory
Feasibility as polynomial-time computation
Worst-case versus average-case complexity
The black-hole phenomenon
Decidability versus verifiability
Nondeterministic computation
P versus NP
Computational resiliency
Questions for further thought
Further reading

7 Incompleteness
7.1 The Hilbert program
Formalism
Life in the world imagined by Hilbert
The alternative
Which vision is correct?
7.2 The first incompleteness theorem
The first incompleteness theorem, via computability
The Entscheidungsproblem
Incompleteness, via diophantine equations
Arithmetization
First incompleteness theorem, via Gödel sentence
7.3 Second incompleteness theorem
Löb proof conditions
Provability logic
7.4 Gödel-Rosser incompleteness theorem
7.5 Tarski's theorem on the nondefinability of truth
7.6 Feferman theories
7.7 Ubiquity of independence
Tower of consistency strength
7.8 Reverse mathematics
7.9 Goodstein's theorem
7.10 Löb's theorem
7.11 Two kinds of undecidability

image
not
available

image
not
available

Preface

Philosophical conundrums pervade mathematics, from fundamental questions of mathematical ontology—What is a number? What is infinity?—to questions about the relations among truth, proof, and meaning. What is the role of figures in geometric argument? Do mathematical objects exist that we cannot construct? Can every mathematical question be solved in principle by computation? Is every truth of mathematics true for a reason? Can every mathematical truth be proved?

This book is an introduction to the philosophy of mathematics, in which we shall consider all these questions and more. I come to the subject from mathematics, and I have strived in this book for what I hope will be a fresh approach to the philosophy of mathematics—one grounded in mathematics, motivated by mathematical inquiry or mathematical practice. I have strived to treat philosophical issues as they arise organically in mathematics. Therefore, I have organized the book by mathematical themes, such as number, infinity, geometry, and computability, and I have included some mathematical arguments and elementary proofs when they bring philosophical issues to light.

The philosophical positions of platonism, realism, logicism, structuralism, formalism, intuitionism, type theorism, and others arise naturally in various mathematical contexts. The mathematical progression from ancient Pythagorean incommensurability and the irrationality of $\sqrt{2}$, for example, through to Liouville's construction of transcendental numbers, paralleling the discovery of nonconstructible numbers in geometry, is an opportunity to contrast platonism with structuralism and other accounts of what numbers and mathematical objects are. Structuralism finds its origin in Dedekind's arithmetic categoricity theorem, gaining strength with categorical accounts of the real numbers and our other familiar mathematical structures. The rise of rigor in the calculus is a natural setting to discuss whether the indispensability of mathematics in science offers grounds for mathematical truth. Zeno's paradoxes of motion and Galileo's paradoxes of infinity lead to the Cantor-Hume principle, and then to both Frege's number concept and Cantor's work on the transfinite. Thus, mathematical themes traverse millennia, giving rise again and again to philosophical considerations.

I therefore aim to present a mathematics-oriented philosophy of mathematics. Years ago, Penelope Maddy (1991) criticized parts of the philosophy of mathematics at that time as amounting to

> an intramural squabble between metaphysicians, and a squabble in which it is not clear what, if anything, is really at stake. (p. 158)

She sought to refocus the philosophy of mathematics on philosophical issues closer to mathematics:

> What I'm recommending is a hands-on sort of philosophy of mathematics, a sort relevant to actual practice, a sort sensitive to the problems, procedures, and concerns of mathematicians themselves. (p. 159)

I find that inspiring, and part of what I have aimed to do in this book is follow that advice—to present an introduction to the philosophy of mathematics that both mathematicians and philosophers might find relevant. Whether or not you agree with Maddy's harsh criticism, there are many truly compelling issues in the philosophy of mathematics, which I hope to share with you in this book. I hope that you will enjoy them.

Another aim I have with the book is to try to help develop a little the mathematical sophistication of the reader on mathematical topics important in the philosophy of mathematics, such as the foundations of number theory, non-Euclidean geometry, nonstandard analysis, Gödel's incompleteness theorems, and uncountability. Readers surely come to this subject from diverse mathematical backgrounds, ranging from novice to expert, and so I have tried to provide something useful for everyone, always beginning gently but still reaching deep waters. The allegory of Hilbert's Grand Hotel, for example, is an accessible entryway to the discussion of Cantor's results on countable and uncountable infinities, and ultimately to the topic of large cardinals. I have aimed high on several mathematical topics, but I have also strived to treat them with a light touch, without getting bogged down in difficult details.

This book served as the basis for the lecture series I gave on the philosophy of mathematics at the University of Oxford for Michaelmas terms in 2018, and again in 2019 and 2020. I am grateful to the Oxford philosophy of mathematics community for wide-ranging discussions that have helped me to improve this book. Special thanks to Daniel Isaacson, Alex Paseau, Beau Mount, Timothy Williamson, Volker Halbach, and especially Robin Solberg, who gave me extensive comments on earlier drafts. Thanks also to Justin Clarke-Doane of Columbia University in New York for comments. And thanks to Theresa Carcaldi for extensive help with editing.

This book was typeset using LaTeX. Except for the image on page 89, which is in the public domain, I created all the other images in this book using TikZ in LaTeX, specifically for this book, and in several instances also for my book *Proof and the Art of Mathematics* (2020), available from MIT Press.

About the Author

I am both a mathematician and a philosopher, undertaking research in the area of mathematical and philosophical logic, with a focus on the mathematics and philosophy of the infinite, especially set theory and the philosophy of set theory and the mathematics and philosophy of potentialism. My new book on proof-writing, *Proof and the Art of Mathematics* (2020), is available from MIT Press.

I have recently taken up a position in Oxford, after a longstanding appointment at the City University of New York. I have also held visiting positions over the years at New York University, Carnegie Mellon University, Kobe University, University of Amsterdam, University of Muenster, University of California at Berkeley, and elsewhere. My 1994 Ph.D. in mathematics was from the University of California at Berkeley, after an undergraduate degree in mathematics at the California Institute of Technology.

Joel David Hamkins
Professor of Logic and Sir Peter Strawson Fellow in Philosophy
Oxford University, University College
High Street, Oxford OX1 4BH

joeldavid.hamkins@philosophy.ox.ac.uk
joeldavid.hamkins@maths.ox.ac.uk
Blog: http://jdh.hamkins.org
MathOverflow: http://mathoverflow.net/users/1946/joel-david-hamkins
Twitter: @JDHamkins

1
Numbers

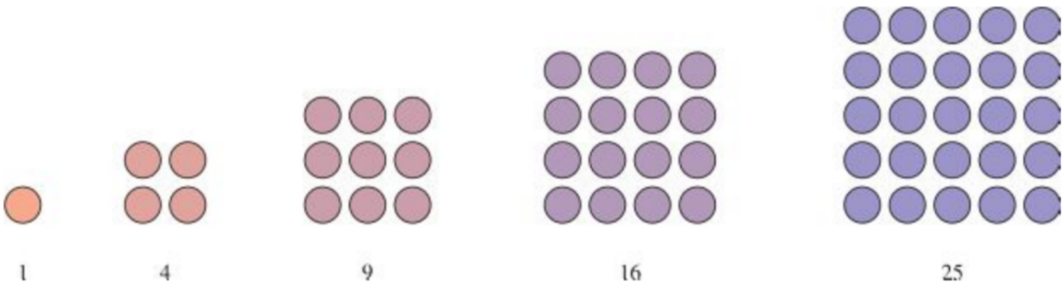
Abstract. Numbers are perhaps the essential mathematical idea, but what are numbers? There are many kinds of numbers—natural numbers, integers, rational numbers, real numbers, complex numbers, hyperreal numbers, surreal numbers, ordinal numbers, and more—and these number systems provide a fruitful background for classical arguments on incommensurability and transcendentality, while setting the stage for discussions of platonism, logicism, the nature of abstraction, the significance of categoricity, and structuralism.

1.1 Numbers versus numerals

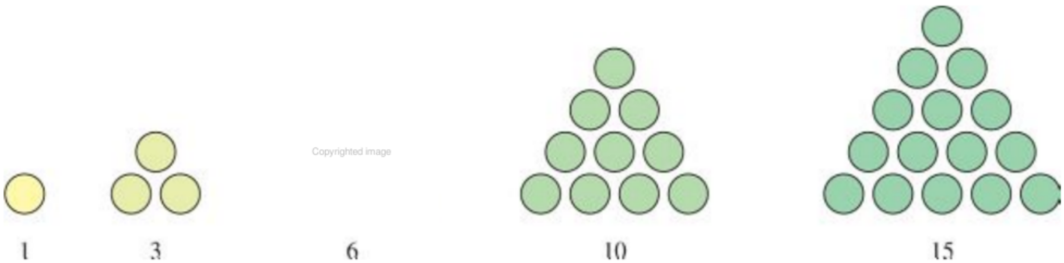
What is a number? Consider the number 57. What is it? We distinguish between the number and the numerals used to represent it. The notation 57—I mean literally the symbol 5 followed by the symbol 7—is a description of how to build the number: take five tens and add seven. The number 57 is represented in binary as 111001, which is a different recipe: start with a fresh thirty-two, fold in sixteen and eight, and garnish with one on top, chill and serve. The Romans would have written LVII, which is the following recipe: start with fifty, add five, and then two more.

In the delightful children's novel, *The Phantom Tollbooth*, Juster and Feiffer (1961), numbers come from the number mine in Digitopolis, and they found there *the largest number*! It was...*ahem*...a gigantic number 3—over 4 meters tall—made of stone. Broken numbers from the mine were used for fractions, like 5/3, when the number 5 has broken into three pieces. But of course, this confuses the number with the numeral, the object with its description. We would not confuse Hypatia, the person, with *Hypatia*, the string of seven letters forming her name; or the pecan pie (delicious!) with the written instructions (chewy, like cardboard) for how to prepare it.

There are many diverse kinds of natural numbers. The *square* numbers, for example, are those that can be arranged in the form of a square:



The *triangular* numbers, in contrast, are those that can be arranged in the form of a triangle:



One proceeds to the *hexagonal* numbers, and so forth. The number zero is a degenerate instance in each case.

The *palindromic* numbers are the numbers, such as 121 or 523323325, whose digits read the same forward and backward, like the palindromic phrase, "I prefer pi." Whereas squareness and triangularity are properties of numbers, however, the question of whether a number is palindromic depends on the base in which it is presented (and for this reason, mathematicians sometimes find the notion unnatural or amateurish). For example, 27 is not a palindrome, but in binary, it is represented as 11011, which is a palindrome. Every number is a palindrome in any sufficiently large base, for it thereby becomes a single digit, a palindrome. Thus, palindromicity is not a property of numbers, but of number descriptions.

1.2 Number systems

Let us lay our various number systems on the table.

Natural numbers

The *natural numbers* are commonly taken to be the numbers:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$$

The set of natural numbers is often denoted by $\mathbb{N}$. The introduction of zero, historically, was a difficult conceptual step. The idea that one might need a number to represent *nothing* or an absence is deeper than one might expect, given our current familiarity with this concept. Zero as a number was first fully explicit in the fifth century in India, although it was used earlier as a placeholder, but not with Roman numerals, which have no place values. Even today, some mathematicians prefer to start the natural numbers with 1; some computer programming languages start their indices with 0, and others with 1. And consider the cultural difference between Europe and the US in the manner of counting floors in a building, or the Chinese method of counting a person's age, where a baby is "one" at birth and during

image
not
available

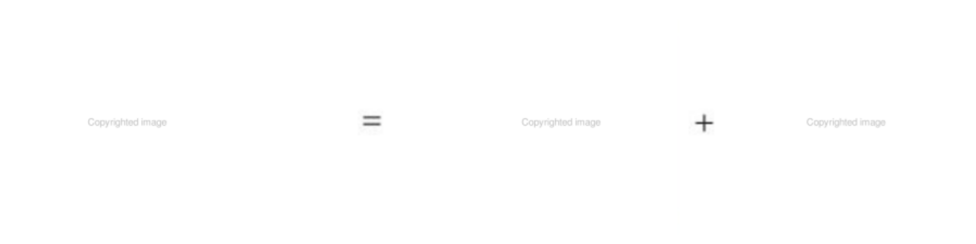
image
not
available

be even as well. So both p and q are even, which contradicts our assumption that the fraction p/q was in lowest terms. So $\sqrt{2}$ cannot be represented as a fraction and therefore is irrational. □

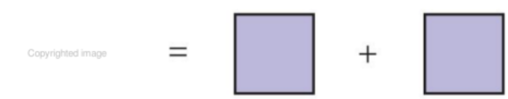
An alternative geometric argument

Here is another argument—one with a more geometric flavor, due to the logician Stanley Tennenbaum from New York City.

Proof. If $\sqrt{2} = p/q$ is rational, then let us represent the resulting equation $p^2 = q^2 + q^2$ geometrically, with a large, blue integer square having the same area as two medium, red integer squares:



Let us consider the smallest possible integer instance of this relationship. Place the two red squares overlapping inside the blue square, as shown here. The two blue corners are uncovered, peeking through, while the red square of overlap in the center is covered twice. Since the original blue square had the same area as the two red squares, it follows that the area of the double-counted central square is exactly balanced by the two small, uncovered blue squares in the corners. Let us pull these smaller squares out of the figure and consider them separately. We have said that the red, central square has the same area as the two small blue squares in the corners.



Furthermore, these smaller squares also each have integer sides, since they arise as differences from the previous squares. Thus, we have found a strictly smaller integer square as the sum of two copies of another integer square. This contradicts our assumption that we had begun with the smallest instance. So there can be no such instance at all, and therefore $\sqrt{2}$ is irrational. □

1.4 Platonism

Truly, what is a number? Let us begin to survey some possible answers. According to the philosophical position known as *platonism*, numbers and other mathematical objects exist as abstract objects. Plato held them to exist in a realm of ideal forms. A particular line or circle that you might draw on paper is flawed and imperfect; in the platonic realm, there are perfect lines and circles—and numbers. From this view, for a mathematician to say, "There is a natural number with such-and-such property," means that in the platonic realm, there are such numbers to be found. Contemporary approaches to platonism assert that abstract objects exist—this is the core issue—but are less connected with Plato's idea of an ideal form or the platonic realm, a place where they are all gathered together.

What does it mean to say, "There is a function *f* that is continuous but not differentiable," "There is a solution to this differential equation," or "There is a topological space that is sequentially compact, but not compact"? This is not physical existence; we cannot hold these "objects" in our hands, as we might briefly hold a hot potato. What kind of existence is this?

According to platonism, mathematical objects are abstract but enjoy a real existence. For the platonist, ordinary talk in mathematics about the existence of mathematical objects can be taken literally—the objects do exist, but abstractly rather than physically. According to this perspective, the nature of mathematical existence is similar to the nature of existence for other abstractions, such as beauty or happiness. Does beauty exist? I believe so. Do parallel lines exist? According to platonism, the answers are similar. But what are abstract objects? What is the nature of this existence?

Consider a piece of writing: Henrik Ibsen's play *A Doll's House*. This exists, surely, but what is it specifically that exists here? I could offer you a printed manuscript, saying, "This is *A Doll's House*." But that would not be fully true, for if that particular manuscript were damaged, we would not say that the play itself was damaged; we would not say that the play had been taken in my back pocket on a motorcycle ride. I could see a performance of the play on Broadway, but no particular performance would seem to be the play itself. We do not say that Ibsen's play existed only in 1879 at its premiere, or that the play comes into and out of existence with each performance. The play is an abstraction, an idealization of its various imperfect instantiations in manuscripts and performances.

Like the play, the number 57 similarly exists in various imperfect instantiations: 57 apples in the bushel and 57 cards in the cheater's deck. The existence of abstract objects is mediated somehow through the existence of their various instantiations. Is the existence of the number 57 similar to the existence of a play, a novel, or a song? The play, as with other pieces of art, was created: Ibsen wrote *A Doll's House*. And while some mathematicians describe their work as an act of creation, doubtless no mathematician would claim to have created the number 57 in that sense. Is mathematics discovered or created? Part of the contemporary platonist view is that numbers and other mathematical objects have an *independent* existence; like the proverbial tree falling in the forest, the next number would exist anyway, even if nobody ever happened to count that high.

Plenitudinous platonism

The position known as *plenitudinous platonism*, defended by Mark Balaguer (1998), is a generous form of platonism, generous in its metaphysical commitments; it overflows with them. According to plenitudinous platonism, every coherent mathematical theory is realized in a corresponding mathematical structure in the platonic realm. The theory is true in an ideal mathematical structure, instantiating the subject matter that the theory is about.

According to plenitudinous platonism, every conceivable coherent mathematical theory is true in an actual mathematical structure, and so this form of platonism offers us a rich mathematical ontology.

1.5 Logicism

Pursuing the philosophical program known as *logicism*, Gottlob Frege, and later Bertrand Russell and others at the dawn of the twentieth century, aimed to reduce all mathematics, including the concept of number, to logic. Frege begins by analyzing what it means to say that there are a certain number of things of a certain kind. There are exactly two things with property P, for example, when there is a thing x with that property and there is another distinct thing y with that property, and furthermore, anything with the property is either x or y. In logical notation, therefore, "There are exactly two Ps" can be expressed like this:

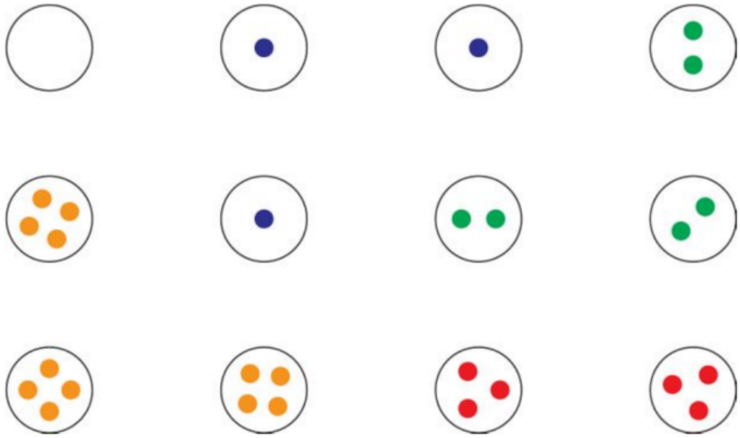
$$\exists x, y\,(Px \wedge Py \wedge x \neq y \wedge \forall z(Pz \rightarrow z = x \vee z = y)).$$

The quantifier symbol $\exists$ is read as "There exists" and $\forall$ as "For all," while $\wedge$ and $\vee$ mean "and" and "or," and $\rightarrow$ means "implies." In this way, Frege has expressed the concept of the number 2 in purely logical terms. You can have two sheep or two apples or two hands, and the thing that is common between these situations is what the number 2 is.

Equinumerosity

Frege's approach to cardinal numbers via logic has the effect that classes placed in a one-to-one correspondence with each other will fulfill exactly the same Fregean number assertions, because the details of the truth assertion transfer through the correspondence from one class to the other. Frege's approach, therefore, is deeply connected with the *equinumerosity* relation, a concept aiming to express the idea that two sets or classes have the same cardinal size. Equinumerosity also lies at the core of Georg Cantor's analysis of cardinality, particularly the infinite cardinalities discussed in chapter 3. Specifically, two classes of objects (or as Frege would say: two concepts) are *equinumerous*—they have the same cardinal size—when they can be placed into a one-to-one correspondence. Each object in the first class is associated with a unique object in the second class and conversely, like the shepherd counting his sheep off on his fingers.

So let us consider the equinumerosity relation on the collection of all sets. Amongst the sets pictured here, for example, equinumerosity is indicated by color: all the green sets are equinumerous, with two elements, and all the red sets are equinumerous, and the orange sets, and so on.



In this way, equinumerosity enables us systematically to compare any two sets and determine whether they have the same cardinal size.

The Cantor-Hume principle

At the center of Frege's treatment of cardinal numbers, therefore, is the following criterion for number identity:

Cantor-Hume principle. *Two concepts have the same number if and only if those concepts can be placed in a one-to-one correspondence.*

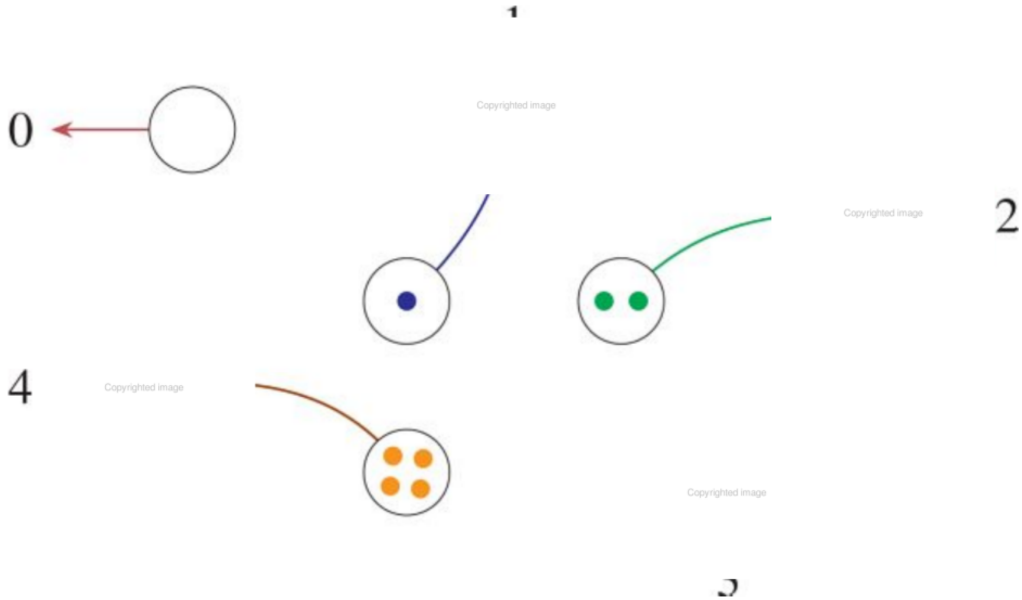
In other words, the number of objects with property P is the same as the number of objects with property Q exactly when there is a one-to-one correspondence between the class $\{ x \mid P(x) \}$ and the class $\{ x \mid Q(x) \}$. Expressed in symbols, the principle asserts

$$\#P = \#Q \qquad \text{if and only if} \qquad \{ x \mid P(x) \} \quad \simeq \quad \{ x \mid Q(x) \},$$

where $\#P$ and $\#Q$ denote the number of objects with property P or Q, respectively, and the symbol $\simeq$ denotes the equinumerosity relation.

The Cantor-Hume principle is also widely known simply as *Hume's principle*, in light of Hume's brief statement of it in *A Treatise of Human Nature* (1739, I.III.I), which Frege mentions (the relevant Hume quotation appears in section 4.5 of this book). Much earlier, Galileo mounted an extended discussion of equinumerosity in his *Dialogues Concerning Two New Sciences* (1638), considering it as a criterion of size identity, particularly in the confounding case of infinite collections, including the paradoxical observation that line segments of different lengths and circles of different radii are nevertheless equinumerous as collections of points. Meanwhile, to my way of thinking, the principle is chiefly to be associated with Cantor, who takes it as the core motivation underlying his foundational development of cardinality, perhaps the most successful and influential, and the first finally to be clear on the nature of countable and uncountable cardinalities (see chapter 3). Cantor treats equinumerosity in his seminal set-theoretic article, Cantor (1874), and states a version of the Cantor-Hume principle in the opening sentence of Cantor (1878). In an 1877 letter to Richard Dedekind, he proved the equinumerosity of the unit interval with the square, the cube, and indeed the unit hypercube in any finite dimension, saying of the discovery, "I see it, but I don't believe it!" (Dauben, 2004, p. 941) We shall return to this example in section 3.8, page 106.

The Cantor-Hume principle provides a criterion of number identity, a criterion for determining when two concepts have the same number. Yet it expresses on its face merely a necessary feature of the number concept, rather than identifying fully what numbers are. Namely, the principle tells us that numbers are classification invariants of the equinumerosity relation. A classification *invariant* of an equivalence relation is a labeling of the objects in the domain of the relation, such that equivalent objects get the same label and inequivalent objects get different labels. For example, if we affix labels to all the apples we have picked, with a different color for each day of picking, then the color of the label will be an invariant for the picked-on-the-same-day-as relation on these apples. But there are many other invariants; we could have written the date on the labels, encoded it in a bar code, or we could simply have placed each day's apples into a different bushel.



The Cantor-Hume principle tells us that numbers—whatever they are—are assigned to every class in such a way that equinumerous classes get the same number and nonequinumerous classes get different numbers. And this is precisely what it means for numbers to be a classification invariant of the equinumerosity relation. But ultimately, what are these "number" objects that get assigned to the sets? The Cantor-Hume principle does not say.

The Julius Caesar problem

Frege had sought in his logicist program an *eliminative* definition of number, for which numbers would be defined in terms of other specific concepts. Since the Cantor-Hume principle does not tell us what numbers are, he ultimately found it unsatisfactory to base a number concept solely upon it. Putting the issue boldly, he proclaimed

> we can never—to take a crude example—decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not. (Frege, 1968 [1884], §57)

The objection is that although the Cantor-Hume principle provides a number identity criterion for identities of the form $\#P = \#Q$, comparing the numbers of two classes, it does not