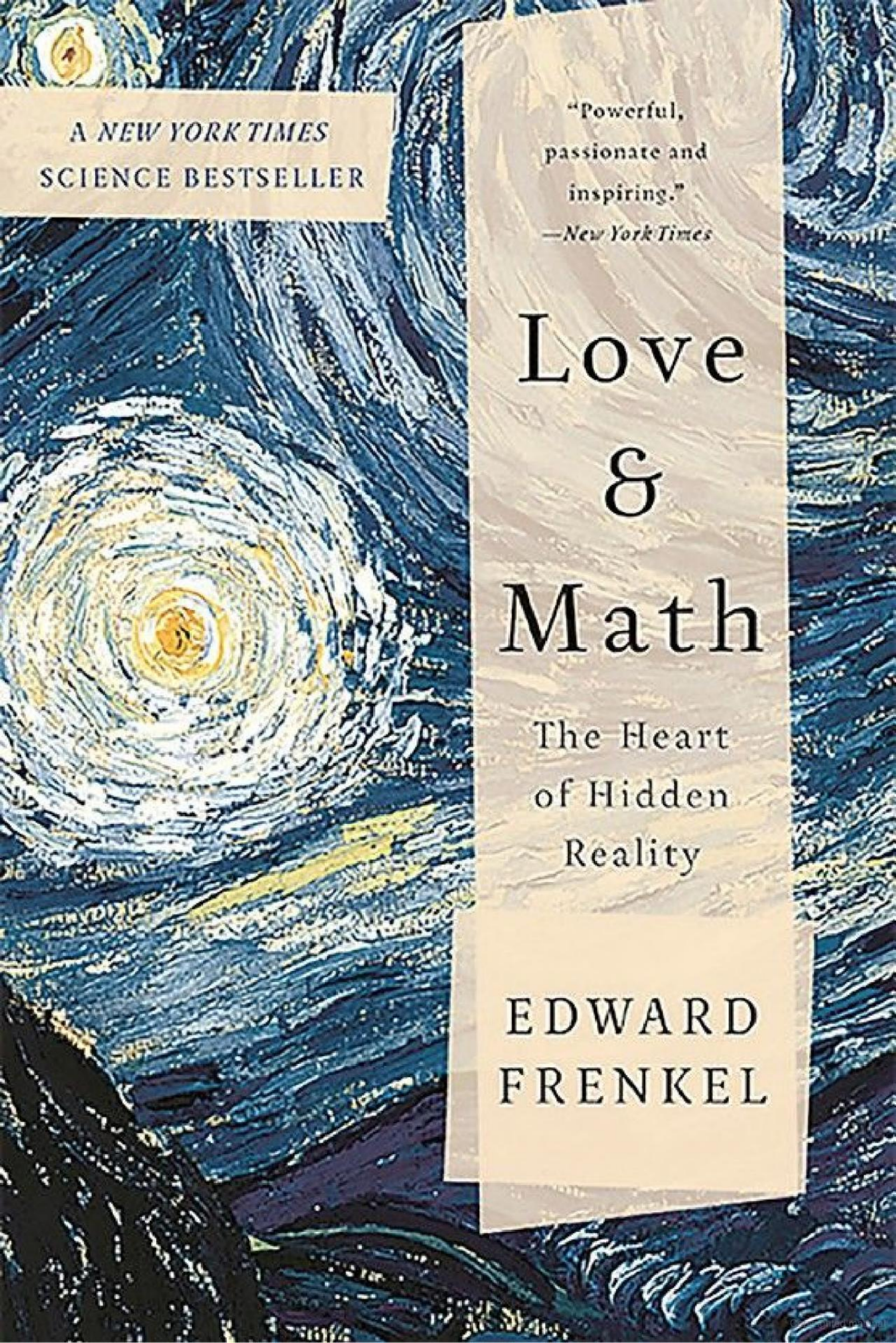

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Love & Math

The Heart of Hidden Reality

EDWARD FRENKEL

—The New York Times Book Review

"The words love and math aren't usually uttered in the same breath. But mathematician Edward Frenkel is on a mission to change that... [in his] book, 'Love and Math' [in which] the tenured professor at the University of California at Berkeley argues that the boring way that math is traditionally taught in schools has led to a widespread ignorance that may have even been responsible for the recession... [the] book tells his personal story and goes on to describe his research in the Langlands program, as well as recent mathematical discoveries that aren't regularly taught in classrooms."

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—Wilmott

"Part ode, part autobiography, *Love and Math* is an admirable attempt to

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Contents

Preface

There's a secret world out there. A hidden parallel universe of beauty and elegance, intricately intertwined with ours. It's the world of mathematics. And it's invisible to most of us. This book is an invitation to discover this world.

Consider this paradox: On the one hand, mathematics is woven in the very fabric of our daily lives. Every time we make an online purchase, send a text message, do a search on the Internet, or use a GPS device, mathematical formulas and algorithms are at play. On the other hand, most people are daunted by math. It has become, in the words of poet Hans Magnus Enzensberger, "a blind spot in our culture – alien territory, in which only the elite, the initiated few have managed to entrench themselves." It's rare, he says, that we "encounter a person who asserts vehemently that the mere thought of reading a novel, or looking at a picture, or seeing a movie causes him insufferable torment," but "sensible, educated people" often say "with a remarkable blend of defiance and pride" that math is "pure torture" or a "nightmare" that "turns them off."

How is this anomaly possible? I see two main reasons. First, mathematics is more abstract than other subjects, hence not as accessible. Second, what we study in school is only a tiny part of math, much of it established more than a millennium ago. Mathematics has advanced tremendously since then, but the treasures of modern math have been kept hidden from most of us.

What if at school you had to take an "art class" in which you were only taught how to paint a fence? What if you were never shown the paintings of Leonardo da Vinci and Picasso? Would that make you appreciate art? Would you want to learn more about it? I doubt it. You would probably say something like this: "Learning art at school was a waste of my time. If I ever need to have my fence painted, I'll just hire people to do this for me." Of course, this sounds ridiculous, but this is how math is taught, and so in the eyes of most of us it becomes the equivalent of watching paint dry. While the paintings of the great masters are readily available, the math of the great masters is locked away.

However, it's not just the aesthetic beauty of math that's captivating. As Galileo famously said, "The laws of Nature are written in the language of mathematics." Math is a way to describe reality and figure out how the world works, a universal language that has become the gold standard of truth. In our world, increasingly driven by science and technology, mathematics is becoming, ever more, the source of power, wealth, and progress. Hence those who are fluent in this new language will be on the cutting edge of progress.

One of the common misconceptions about mathematics is that it can only be used as a "toolkit": a biologist, say, would do some field work, collect data, and then try to build a mathematical model fitting these data (perhaps, with some help from a mathematician). While this is an important mode of operation, math offers us *a lot more*: it enables us to make groundbreaking, paradigm-shifting leaps that we couldn't make otherwise. For example, Albert Einstein was not trying to fit any data into equations when he understood that gravity causes our space to curve. In fact, there was no such data. No one could even imagine at the time that our space is curved; everyone "knew" that our world was flat! But Einstein understood that this was the only way to generalize his special relativity theory to non-inertial systems, coupled with his insight that gravity and acceleration have the same effect. This was a high-level intellectual exercise within the realm of math, one in which Einstein relied on the work of a mathematician, Bernhard Riemann, completed fifty years earlier. The human brain is wired in such a way that we simply cannot imagine curved spaces of dimension greater than two; we can only access them through mathematics. And guess what, Einstein was right – our universe *is* curved, and furthermore, it's expanding. That's the power of mathematics I am talking about!

Many examples like this may be found, and not only in physics, but in other areas of science (we will discuss some of them below). History shows that science and technology are transformed by mathematical ideas at an accelerated pace; even mathematical theories that are initially viewed as abstract and esoteric later become indispensable for applications. Charles Darwin, whose work at first did not rely on math, later wrote in his autobiography: “I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense.” I take it as prescient advice to the next generations to capitalize on mathematics’ immense potential.

When I was growing up, I wasn’t aware of the hidden world of mathematics. Like most people, I thought math was a stale, boring subject. But I was lucky: in my last year of high school I met a professional mathematician who opened the magical world of math to me. I learned that mathematics is full of infinite possibilities as well as elegance and beauty, just like poetry, art, and music. I fell in love with math.

Dear reader, with this book I want to do for you what my teachers and mentors did for me: unlock the power and beauty of mathematics, and enable *you* to enter this magical world the way I did, even if you are the sort of person who has never used the words “math” and “love” in the same sentence. Mathematics will get under your skin just like it did under mine, and your worldview will never be the same.

* * *

Mathematical knowledge is unlike any other knowledge. While our perception of the physical world can always be distorted, our perception of mathematical truths can’t be. They are objective, persistent, necessary truths. A mathematical formula or theorem means the same thing to anyone anywhere – no matter what gender, religion, or skin color; it will mean the same thing to anyone a thousand years from now. And what’s also amazing is that we own all of them. No one can patent a mathematical formula, it’s ours to share. There is nothing in this world that is so deep and exquisite and yet so readily available to all. That such a reservoir of knowledge really exists is nearly unbelievable. It’s too precious to be given away to the “initiated few.” It belongs to all of us.

One of the key functions of mathematics is the ordering of information. This is what distinguishes the brush strokes of Van Gogh from a mere blob of paint. With the advent of 3D printing, the reality we are used to is undergoing a radical transformation: everything is migrating from the sphere of physical objects to the sphere of information and data. We will soon be able to convert information into matter on demand by using 3D printers just as easily as we now convert a PDF file into a book or an MP3 file into a piece of music. In this brave new world, the role of mathematics will become even more central: as the way to organize and order information, and as the means to facilitate the conversion of information into physical reality.

In this book, I will describe one of the biggest ideas to come out of mathematics in the last fifty years: the Langlands Program, considered by many as the Grand Unified Theory of mathematics. It's a fascinating theory that weaves a web of tantalizing connections between mathematical fields that at first glance seem to be light years apart: algebra, geometry, number theory, analysis, and quantum physics. If we think of those fields as continents in the hidden world of mathematics, then the Langlands Program is the ultimate teleportation device, capable of getting us instantly from one of them to another, and back.

Launched in the late 1960s by Robert Langlands, the mathematician who currently occupies Albert Einstein's office at the Institute for Advanced Study in Princeton, the Langlands Program had its roots in a groundbreaking mathematical theory of symmetry. Its foundations were laid two centuries ago by a French prodigy, just before he was killed in a duel, at age twenty. It was subsequently enriched by another stunning discovery, which not only led to the proof of Fermat's Last Theorem, but revolutionized the way we think about numbers and equations. Yet another penetrating insight was that mathematics has its own Rosetta stone and is full of mysterious analogies and metaphors. Following these analogies as creeks in the enchanted land of math, the ideas of the Langlands Program spilled into the realms of geometry and quantum physics, creating order and harmony out of seeming chaos.

I want to tell you about all this to expose the sides of mathematics we rarely get to see: inspiration, profound ideas, startling revelations. Mathematics is a way to break the barriers of the conventional, an expression of unbounded imagination in the search for truth. Georg Cantor, creator of the theory of infinity, wrote: "The essence of mathematics lies in its freedom." Mathematics teaches us to rigorously analyze reality, study the facts, follow them wherever they lead. It liberates us from dogmas and prejudice, nurtures the capacity for innovation. It thus provides tools that transcend the subject itself.

These tools can be used for good and for ill, forcing us to reckon with math's real-world effects. For example, the global economic crisis was caused to a large extent by the widespread use of inadequate mathematical models in the financial markets. Many of the decision makers didn't fully understand these models due to their mathematical illiteracy, but were arrogantly using them anyway – driven by greed – until this practice almost wrecked the entire system. They were taking unfair advantage of the asymmetric access to information and hoping that no one would call their bluff because others weren't inclined to ask how these mathematical models worked either. Perhaps, if more people understood how these models functioned, how the system really worked, we wouldn't have been fooled for so long.

As another example, consider this: in 1996, a commission appointed by the U.S. government gathered in secret and altered a formula for the Consumer Price Index, the measure of inflation that determines the tax brackets, Social Security, Medicare, and other indexed payments. Tens of millions of Americans were affected, but there was little public discussion of the new formula and its consequences. And recently there was another attempt to exploit this arcane formula as a backdoor on the U.S. economy.[1]

Far fewer of these sorts of backroom deals could be made in a mathematically literate society. Mathematics equals rigor plus intellectual integrity times reliance on facts. We should all have access to the mathematical knowledge and tools needed to protect us from arbitrary decisions made by the powerful few in an increasingly math-driven world. Where there is no mathematics, there is no freedom.

***

Mathematics is as much part of our cultural heritage as art, literature, and music. As humans, we have a hunger to discover something new, reach new meaning, understand better the universe and our place in it. Alas, we can't discover a new continent like Columbus or be the first to set foot on the Moon. But what if I told you that you don't have to sail across an ocean or fly into space to discover the wonders of the world? They are right here, intertwined with our present reality. In a sense, within us. Mathematics directs the flow of the universe, lurks behind its shapes and curves, holds the reins of everything from tiny atoms to the biggest stars.

This book is an invitation to this rich and dazzling world. I wrote it for readers without any background in mathematics. If you think that math is hard, that you won't get it, if you are terrified by math, but at the same time curious whether there is something there worth knowing – then this book is for you.

There is a common fallacy that one has to study mathematics for years to appreciate it. Some even think that most people have an innate learning disability when it comes to math. I disagree: most of us have heard of and have at least a rudimentary understanding of such concepts as the solar system, atoms and elementary particles, the double helix of DNA, and much more, without taking courses in physics and biology. And nobody is surprised that these sophisticated ideas are part of our culture, our collective consciousness. Likewise, everybody can grasp key mathematical concepts and ideas, if they are explained in the right way. To do this, it is not necessary to study math for years; in many cases, we can cut right to the point and jump over tedious steps.

The problem is: while the world at large is always talking about planets, atoms, and DNA, chances are no one has ever talked to you about the fascinating ideas of modern math, such as symmetry groups, novel numerical systems in which 2 and 2 isn't always 4, and beautiful geometric shapes like Riemann surfaces. It's like they keep showing you a little cat and telling you that this is what a tiger looks like. But actually the tiger is an entirely different animal. I'll show it to you in all of its splendor, and you'll be able to appreciate its "fearful symmetry," as William Blake eloquently said.

Don't get me wrong: reading this book won't by itself make you a mathematician. Nor am I advocating that everyone should become a mathematician. Think about it this way: learning a small number of chords will enable you to play quite a few songs on a guitar. It won't make you the world's best guitar player, but it will enrich your life. In this book I will show you the chords of modern math, which have been hidden from you. And I promise that this will enrich your life.

One of my teachers, the great Israel Gelfand, used to say: "People think they don't understand math, but it's all about how you explain it to them. If you ask a drunkard what number is larger, 2/3 or 3/5, he won't be able to tell you. But if you rephrase the question: what is better, 2 bottles of vodka for 3 people or 3 bottles of vodka for 5 people, he will tell you right away: 2 bottles for 3 people, of course."

My goal is to explain this stuff to you in terms that you will understand.

I will also talk about my experience of growing up in the former Soviet Union, where mathematics became an outpost of freedom in the face of an oppressive regime. I was denied entrance to Moscow State University because of the discriminatory policies of the Soviet Union. The doors were slammed shut in front of me. I was an outcast. But I didn't give up. I would sneak into the University to attend lectures and seminars. I would read math books on my own, sometimes late at night. And in the end, I was able to hack the system. They didn't let me in through the front door; I flew in through a window. When you are in love, who can stop you?

Two brilliant mathematicians took me under their wings and became my mentors. With their guidance, I started doing mathematical research. I was still a college student, but I was already pushing the boundaries of the unknown. This was the most exciting time of my life, and I did it even though I was sure that the discriminatory policies would never allow me to have a job as a mathematician in the Soviet Union.

But there was a surprise in store: my first mathematical papers were smuggled abroad and became known, and I got invited to Harvard University as a Visiting Professor at age twenty-one. Miraculously, at exactly the same time *perestroika* in the Soviet Union lifted the iron curtain, and citizens were allowed to travel abroad. So there I was, a Harvard professor without a Ph.D., hacking the system once again. I continued on my academic path, which led me to research on the frontiers of the Langlands Program and enabled me to participate in some of the major advances in this area during the last twenty years. In what follows, I will describe spectacular results obtained by brilliant scientists as well as what happened behind the scenes.

This book is also about love. Once, I had a vision of a mathematician discovering the "formula of love," and this became the premise of a film *Rites of Love and Math*, which I will talk about later in the book. Whenever I show the film, someone always asks: "Does a formula of love really exist?"

My response: "Every formula we discover is a formula of love."

Mathematics is the source of timeless profound knowledge, which goes to the heart of all matter and unites us across cultures, continents, and centuries. My dream is that all of us will be able to see, appreciate, and marvel at the magic beauty and exquisite harmony of these ideas, formulas, and equations, for this will give so much more meaning to our love for this world and for each other.

A Guide for the Reader

I have made every effort to present mathematical concepts in this book in the most elementary and intuitive way. However, I realize that some parts of the book are somewhat heavier on math (particularly, some parts of Chapters 8, 14, 15, and 17). It is *perfectly fine to skip* those parts that look confusing or tedious at the first reading (this is what I often do myself). Coming back to those parts later, equipped with newly gained knowledge, you might find the material easier to follow. But that is usually not necessary in order to be able to follow what comes next.

Perhaps, a bigger point is that *it is perfectly OK if something is unclear.* That's how I feel 90 percent of the time when I do mathematics, so welcome to my world! The feeling of confusion (even frustration, sometimes) is an essential part of being a mathematician. But look at the bright side: how boring would life be if everything in it could be understood with little effort! What makes doing mathematics so exciting is our desire to overcome this confusion; to understand; to lift the veil on the unknown. And the feeling of personal triumph when we do understand something makes it all worthwhile.

My focus in this book is on the big picture and the logical connections between different concepts and different branches of math, not technical details. A more in-depth discussion is often relegated to the endnotes, which also contain references and suggestions for further reading. However, although endnotes may enhance your understanding, they may be safely skipped (at least, at the first reading).

I have tried to minimize the use of formulas – opting, whenever possible, for verbal explanations. Feel free to skip the few formulas that do appear.

A word of warning on mathematical terminology: while writing this book, I discovered, to my surprise, that certain terms that mathematicians use in a specific way actually mean something entirely different to non-mathematicians. Terms like correspondence, representation, composition, loop, manifold, and theory. Whenever I detected this issue, I included an explanation. Also, whenever possible, I changed obscure mathematical terms to terms with more transparent meaning (for example, I would write

“Langlands relation” instead of “Langlands correspondence”). You might find it useful to consult the Glossary and the Index whenever there is a word that seems unclear.

Please check out my website http://edwardfrenkel.com for updates and supporting materials, and send me an e-mail to share your thoughts about the book (my e-mail address can be found on the website). Your feedback will be much appreciated.

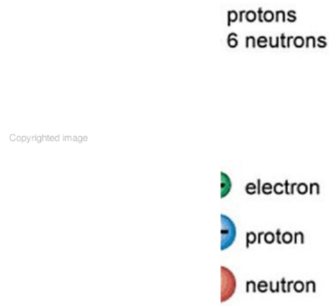
Chapter 1

A Mysterious Beast

How does one become a mathematician? There are many ways that this can happen. Let me tell you how it happened to me.

It might surprise you, but I hated math when I was at school. Well, "hated" is perhaps too strong a word. Let's just say I didn't like it. I thought it was boring. I could do my work, sure, but I didn't understand why I was doing it. The material we discussed in class seemed pointless, irrelevant. What really excited me was physics – especially quantum physics. I devoured every popular book on the subject that I could get my hands on. I grew up in Russia, where such books were easy to find.

I was fascinated with the quantum world. Ever since ancient times, scientists and philosophers had dreamed about describing the fundamental nature of the universe – some even hypothesized that all matter consists of tiny pieces called atoms. Atoms were proved to exist at the beginning of the twentieth century, but at around the same time, scientists discovered that each atom could be divided further. Each atom, it turned out, consists of a nucleus in the middle and electrons orbiting it. The nucleus, in turn, consists of protons and neutrons, as shown on the diagram below.[1]



Carbon atom

And what about protons and neutrons? The popular books that I was reading told me that they are built of the elementary particles called "quarks."

I liked the name quarks, and I especially liked how this name came about. The physicist who invented these particles, Murray Gell-Mann, borrowed this name from James Joyce's book *Finnegans Wake*, where there is a mock poem that goes like this:

> Three quarks for Muster Mark!
> Sure he hasn't got much of a bark
> And sure any he has it's all beside the mark.

I thought it was really cool that a physicist would name a particle after a novel. Especially such a complex and non-trivial one as *Finnegans Wake.* I must have been around thirteen, but I already knew by then that scientists were supposed to be these reclusive and unworldly creatures who were so deeply involved in their work that they had no interest whatsoever in other aspects of life such as Art and Humanities. I wasn't like this. I had many friends, liked to read, and was interested in many things besides science. I liked to play soccer and spent endless hours chasing the ball with my friends. I discovered Impressionist paintings around the same time (it started with a big volume about Impressionism, which I found in my parents' library). Van Gogh was my favorite. Enchanted by his works, I even tried to paint myself. All of these interests had actually made me doubt whether I was really cut out to be a scientist. So when I read that Gell-Mann, a great physicist, Nobel Prize–winner, had such diverse interests (not only literature, but also linguistics, archaeology, and more), I was very happy.

According to Gell-Mann, there are two different types of quarks, "up" and "down," and different mixtures of them give neutrons and protons their characteristics. A neutron is made of two down and one up quarks, and a proton is made of two up and one down quarks, as shown on the pictures.[2]

Neutron

Proton

That was clear enough. But how physicists guessed that protons and neutrons were not indivisible particles but rather were built from smaller blocks was murky.

The story goes that by the late 1950s, a large number of apparently elementary particles, called hadrons, was discovered. Neutrons and protons are both hadrons, and of course they play major roles in everyday life as the building blocks of matter. As for the rest of hadrons – well, no one had any idea what they existed for (or "who ordered them," as one researcher put it). There were so many of them that the influential physicist Wolfgang Pauli joked that physics was turning into botany. Physicists desperately needed to rein in the hadrons, to find the underlying principles that govern their behavior and would explain their maddening proliferation.

Gell-Mann, and independently Yuval Ne'eman, proposed a novel classification scheme. They both showed that hadrons can be naturally split into small families, each consisting of eight or ten particles. They called them octets and decuplets. Particles within each of the families had similar properties.

In the popular books I was reading at the time, I would find octet diagrams like this:

Here the proton is marked as *p*, the neutron is marked as *n*, and there are six other particles with strange names expressed by Greek letters.

But why 8 and 10, and not 7 and 11, say? I couldn't find a coherent explanation in the books I was reading. They would mention a mysterious idea of Gell-Mann called the "eightfold way" (referencing the "Noble Eightfold Path" of Buddha). But they never attempted to explain what this was all about.

This lack of explanation left me deeply unsatisfied. The key parts of the story remained hidden. I wanted to unravel this mystery but did not know how.

As luck would have it, I got help from a family friend. I grew up in a small industrial town called Kolomna, population 150,000, which was about seventy miles away from Moscow, or just over two hours by train. My parents worked as engineers at a large company making heavy machinery. Kolomna is an old town on the intersection of two rivers that was founded in 1177 (only thirty years after the founding of Moscow). There are still a few pretty churches and the city wall to attest to Kolomna's storied past. But it's not exactly an educational or intellectual center. There was only one small college there, which prepared schoolteachers. One of the professors there, a mathematician named Evgeny Evgenievich Petrov, however, was an old friend of my parents. And one day my mother met him on the street after a long time, and they started talking. My mom liked to tell her friends about me, so I came up in conversation. Hearing that I was interested in science, Evgeny Evgenievich said, "I must meet him. I will try to convert him to math."

"Oh no," my mom said, "he doesn't like math. He thinks it's boring. He wants to do quantum physics."

"No worries," replied Evgeny Evgenievich, "I think I know how to change his mind."

A meeting was arranged. I wasn't particularly enthusiastic about it, but I went to see Evgeny Evgenievich at his office anyway.

I was just about to turn fifteen, and I was finishing the ninth grade, the penultimate year of high school. (I was a year younger than my classmates because I had skipped the sixth grade.) Then in his early forties, Evgeny Evgenievich was friendly and unassuming. Bespectacled, with a beard stubble, he was just what I imagined a mathematician would look like, and yet there was something captivating in the probing gaze of his big eyes. They exuded unbounded curiosity about everything.

It turned out that Evgeny Evgenievich indeed had a clever plan how to convert me to math. As soon as I came to his office, he asked me, "So, I hear you like quantum physics. Have you heard about Gell-Mann's eightfold way and the quark model?"

"Yes, I've read about this in several popular books."

"But do you know what was the basis for this model? How did he come up with these ideas?"

"Well..."

"Have you heard about the group $SU(3)$?"

"*SU* what?"

"How can you possibly understand the quark model if you don't know what the group $SU(3)$ is?"

He pulled out a couple of books from his bookshelf, opened them, and showed me pages of formulas. I could see the familiar octet diagrams, such as the one shown above, but these diagrams weren't just pretty pictures; they were part of what looked like a coherent and detailed explanation.

Though I could make neither head nor tail of these formulas, it became clear to me right away that they contained the answers I had been searching for. This was a moment of epiphany. I was mesmerized by what I was seeing and hearing; touched by something I had never experienced before; unable to express it in words but feeling the energy, the excitement one feels from hearing a piece of music or seeing a painting that makes an unforgettable impression. All I could think was "Wow!"

"You probably thought that mathematics is what they teach you in school," Evgeny Evgenievich said. He shook his head, "No, this" – he pointed at the formulas in the book – "is what mathematics is about. And if you really want to understand quantum physics, this is where you need to start. Gell-Mann predicted quarks using a beautiful mathematical theory. It was in fact a mathematical discovery."

"But how do I even begin to understand this stuff?"

It looked kind of scary.

"No worries. The first thing you need to learn is the concept of a symmetry group. That's the main idea. A large part of mathematics, as well as theoretical physics, is based on it. Here are some books I want to give you. Start reading them and mark the sentences that you don't understand. We can meet here every week and talk about this."

He gave me a book about symmetry groups and also a couple of others on different topics: about the so-called *p*-adic numbers (a number system radically different from the numbers we are used to) and about topology (the study of the most fundamental properties of geometric shapes). Evgeny Evgenievich had impeccable taste: he found a perfect combination of topics that would allow me to see this mysterious beast – *Mathematics* – from different sides and get excited about it.

At school we studied things like quadratic equations, a bit of calculus, some basic Euclidean geometry, and trigonometry. I had assumed that all mathematics somehow revolved around these subjects, that perhaps problems became more complicated but stayed within the same general framework I was familiar with. But the books Evgeny Evgenievich gave me contained glimpses of an entirely different world, whose existence I couldn't even imagine.

I was instantly converted.

Chapter 2

The Essence of Symmetry

In the minds of most people, mathematics is all about numbers. They imagine mathematicians as people who spend their days crunching numbers: big numbers, and even bigger numbers, all having exotic names. I had thought so too – at least, until Evgeny Evgenievich introduced me to the concepts and ideas of modern math. One of them turned out to be the key to the discovery of quarks: the concept of symmetry.

What is symmetry? All of us have an intuitive understanding of it – we know it when we see it. When I ask people to give me an example of a symmetric object, they point to butterflies, snowflakes, or the human body.

Photo by K.G. Libbrecht

But if I ask them what we mean when we say that an object is symmetrical, they hesitate.

Here is how Evgeny Evgenievich explained it to me. "Let's look at this round table and this square table," he pointed at the two tables in his office.

“Which one is more symmetrical?”

“Of course, the round table, isn’t it obvious?”

“But why? Being a mathematician means that you don’t take ‘obvious’ things for granted but try to reason. Very often you’ll be surprised that the most obvious answer is actually wrong.”

Noticing confusion on my face, Evgeny Evgenievich gave me a hint: “What is the property of the round table that makes it more symmetrical?”

I thought about this for a while, and then it hit me: “I guess the symmetry of an object has to do with it keeping its shape and position unchanged even when we apply changes to it.”

Evgeny Evgenievich nodded.

“Indeed. Let’s look at all possible transformations of the two tables which preserve their shape and position,” he said. “In the case of the round table...”

I interrupted him: “Any rotation around the center point will do. We will get back the same table positioned in the same way. But if we apply an arbitrary rotation to a square table, we will typically get a table positioned differently. Only rotations by 90 degrees and its multiples will preserve it.”

“Exactly! If you leave my office for a minute, and I turn the round table by any angle, you won’t notice the difference. But if I do the same to the square table, you will, unless I turn it by 90, 180, or 270 degrees.”

Rotation of a round table by any angle does not change its position, but rotation of a square table by an angle that is not a multiple of 90 degrees does change its position (both are viewed here from above)

He continued: “Such transformations are called symmetries. So you see that the square table has only four symmetries, whereas the round table has many more of them – it actually has infinitely many symmetries. That's why we say that the round table is more symmetrical."

This made a lot of sense.

"This is a fairly straightforward observation," continued Evgeny Evgenievich. "You don't have to be a mathematician to see this. But if you are a mathematician, you ask the next question: what are *all* possible symmetries of a given object?"

Let's look at the square table. Its symmetries[1] are these four rotations around the center of the table: by 90 degrees, 180 degrees, 270 degrees, and 360 degrees, counterclockwise.[2] A mathematician would say that the *set* of symmetries of the square table consists of four elements, corresponding to the angles 90, 180, 270, and 360 degrees. Each rotation takes a fixed corner (marked with a balloon on the picture below) to one of the four corners.

90°

180°

360° = 0°

270°

One of these rotations is special; namely, rotation by 360 degrees is the same as rotation by 0 degrees, that is, no rotation at all. This is a special symmetry because it actually does nothing to our object: each point of the table ends up in exactly the same position as it was before. We call it the *identical symmetry*, or just the *identity*.[3]

Note that rotation by any angle greater than 360 degrees is equivalent to rotation by an angle between 0 and 360 degrees. For example, rotation by 450 degrees is the same as rotation by 90 degrees, because 450 = 360 + 90. That's why we will only consider rotations by angles between 0 and 360 degrees.

Here comes the crucial observation: if we apply two rotations from the list {90°, 180°, 270°, 360°} one after another, we obtain another rotation from the same list. We call this new symmetry the *composition* of the two.

Of course, this is obvious: each of the two symmetries preserves the table. Hence the composition of the two symmetries also preserves it. Therefore this composition has to be a symmetry as well. For example, if we rotate the table by 90 degrees and then again by 180 degrees, the net result is the rotation by 270 degrees.

Let's see what happens with the table under these symmetries. Under the counterclockwise rotation by 90 degrees, the right corner of the table (the one marked with a balloon on the previous picture) will go to the upper corner. Next, we apply the rotation by 180 degrees, so the upper corner will go to the down corner. The net result will be that the right corner will go to the down corner. This is the result of the counterclockwise rotation by 270 degrees.

Here is one more example:

$$90^\circ + 270^\circ = 0^\circ.$$

By rotating by 90 degrees and then by 270 degrees, we get the rotation by 360 degrees. But the effect of the rotation by 360 degrees is the same as that of the rotation by 0 degrees, as we have discussed above – this is the "identity symmetry."

In other words, the second rotation by 270 degrees undoes the initial rotation by 90 degrees. This is in fact an important property: any symmetry can be *undone*; that is, for any symmetry S there exists another symmetry S' such that their composition is the identity symmetry. This S' is called the *inverse* of symmetry S. So we see that rotation by 270 degrees is the inverse of the rotation by 90 degrees. Likewise, the inverse of the rotation by 180 degrees is the same rotation by 180 degrees.

We now see that what looks like a very simple collection of symmetries of the square table – the four rotations {90°, 180°, 270°, 0°} – actually has a lot of inner structure, or rules for how the members of the set can interact.

First of all, we can compose any two symmetries (that is, apply them one after another).

Second, there is a special symmetry, the identity. In our example, this is the rotation by 0 degrees. If we compose it with any other symmetry, we get back the same symmetry. For example,

$$90^\circ + 0^\circ = 90^\circ, \qquad 180^\circ + 0^\circ = 180^\circ, \qquad \text{etc.}$$

Third, for any symmetry S, there is the inverse symmetry S' such that the composition of S and S' is the identity.

And now we come to the main point: the set of rotations along with these three structures comprise an example of what mathematicians call a *group*.

The symmetries of any other object also constitute a group, which in general has more elements – possibly, infinitely many.[4]

Let's see how this works in the case of a round table. Now that we have gained some experience, we can see right away that the set of all symmetries of the round table is just the set of all possible rotations (not just by multiples of 90 degrees), and we can visualize it as the set of all points of a circle.

Each point on this circle corresponds to an angle between 0 and 360 degrees, representing the rotation of the round table by this angle in the counterclockwise direction. In particular, there is a special point corresponding to rotation by 0 degrees. It is marked on the picture below, together with another point corresponding to rotation by 30 degrees.

We should not think of the points of this circle as points of the round table, though. Rather, each point of the circle represents a particular rotation of the round table. Note that the round table does not have a preferred point, but our circle does; namely, the one corresponding to rotation by 0 degrees.[5]

Now let’s see if the above three structures can be applied to the set of points of the circle.

First, the composition of two rotations, by $_1$ and $_2$ degrees, is the rotation by $_1$ + $_2$ degrees. If $_1$ + $_2$ is greater than 360, we simply subtract 360 from it. In mathematics, this is called *addition modulo* 360. For example, if $_1$ = 195 and $_2$ = 250, then the sum of the two angles is 445, and the rotation by 445 degrees is the same as the rotation by 85 degrees. So, in the group of rotations of the round table we have

$$195^\circ + 250^\circ = 85^\circ.$$

Second, there is a special point on the circle corresponding to the rotation by 0 degrees. This is the identity element of our group.

Third, the inverse of the counterclockwise rotation by degrees is the counterclockwise rotation by (360–) degrees, or equivalently, clockwise rotation by degrees (see the drawing).

Thus, we have described the group of rotations of the round table. We will call it the *circle group*. Unlike the group of symmetries of the square table, which has four elements, this group has infinitely many elements because there are infinitely many angles between 0 and 360 degrees.

We have now put our intuitive understanding of symmetry on firm theoretical ground – indeed, we’ve turned it into a mathematical concept. First, we postulated that a symmetry of a given object is a transformation that preserves it and its properties. Then we made a decisive step: we focused on the set of all symmetries of a given object. In the case of a square table, this set consists of four elements (rotations by multiples of 90 degrees); in the case of a round table, it is an infinite set (of all points on the circle). Finally, we described the neat structures that this set of symmetries always possesses: any two symmetries can be composed to produce another symmetry, there exists the identical symmetry, and for each symmetry there exists its inverse. (The composition of symmetries also satisfies the associativity property described in endnote 4.) Thus, we came to the mathematical concept of a group.

A group of symmetries is an abstract object that is quite different from the concrete object we started with. We cannot touch or hold the set of symmetries of a table (unlike the table itself), but we can imagine it, draw its elements, study it, talk about it. Each element of this abstract set has a concrete meaning, though: it represents a particular transformation of a concrete object, its symmetry.

Mathematics is about the study of such abstract objects and concepts.

Experience shows that symmetry is an essential guiding principle for the laws of nature. For example, a snowflake forms a perfect hexagonal shape because that turns out to be the lowest energy state into which crystallized water molecules are forced. The symmetries of the snowflake are rotations by multiples of 60 degrees; that is, 60, 120, 180, 240, 300, and 360 (which is the same as 0 degrees). In addition, we can "flip" the snowflake along each of the six axes corresponding to those angles. All of these rotations and flips preserve the shape and position of the snowflake, and hence they are its symmetries.*

In the case of a butterfly, flipping it turns it upside down. Since it has legs on one side, the flip is not, strictly speaking, a symmetry of the butterfly. When we say that a butterfly is symmetrical, we are talking about an idealized version of it, where its front and back are exactly the same (unlike those of an actual butterfly). Then the flip exchanging the left and the right wings becomes a symmetry. (Alternatively, we can imagine exchanging the wings without turning the butterfly upside down.)

This brings up an important point: there are many objects in nature whose symmetries are approximate. A real-life table is not perfectly round or perfectly square, a live butterfly has an asymmetry between its front and back, and a human body is not fully symmetrical. However, even in this case it turns out to be useful to consider their abstract, idealized versions, or models – a perfectly round table or an image of the butterfly in which we don't distinguish between the front and the back. We then explore the symmetries of these idealized objects and adjust whatever inferences we can make from this analysis to account for the difference between a real-life object and its model.

This is not to say that we do not appreciate asymmetry; we do, and we often find beauty in it. But the main point of the mathematical theory of symmetry is not aesthetic. It is to formulate the concept of symmetry in the most general, and hence inevitably most abstract, terms so that it could be applied in a unified fashion in different domains, such as geometry, number theory, physics, chemistry, biology, and so on. Once we develop such a theory, we can also talk about the mechanisms of symmetry breaking – viewing asymmetry as emergent, if you will. For example, elementary particles acquire masses because the so-called gauge symmetry they obey (which will be discussed in Chapter 16) gets broken. This is facilitated by the Higgs boson, an elusive particle recently discovered at the Large Hadron Collider under the city of Geneva.[6] The study of such mechanisms of symmetry breaking yields invaluable insights into the behavior of the fundamental blocks of nature.

I'd like to point out some of the basic qualities of the abstract theory of symmetry because this is a good illustration of why mathematics is important.

The first is *universality*. The circle group is not only the group of symmetries of a round table, but also of all other round objects, like a glass, a bottle, a column, and so forth. In fact, to say that a given object is round is the same as to say that its group of symmetries is the circle group. This is a powerful statement: we realize that we can describe an important attribute of an object ("being round") by describing its symmetry group (the circle). Likewise, "being square" means that the group of symmetries is the group of four elements described above. In other words, the same abstract mathematical object (such as the circle group) serves many different concrete objects, and it points to universal properties that they all have in common (such as roundness).[7]

The second is *objectivity*. The concept of a group, for example, is independent of our interpretation. It means the same thing to anyone who learns it. Of course, in order to understand it, one has to know the language in which it is expressed, that is, mathematical language. But anyone can learn this language. Likewise, if you want to understand the meaning of René Descartes' sentence "*Je pense, donc je suis*," you need to know French (at least, those words that are used in this sentence) – but anyone can learn it. However, in the case of the latter sentence, once we understand it, different interpretations of it are possible. Also, different people may agree or disagree on whether a particular interpretation of this sentence is true or false. In contrast, the meaning of a logically consistent mathematical statement is not subject to interpretation.[8] Furthermore, its truth is also objective. (In general, the truth of a particular statement may depend on the system of axioms within which it is considered. However, even then, this dependence on the axioms is also objective.) For example, the statement "the group of symmetries of a round table is a circle" is true to anyone, anywhere, at any time. In other words, mathematical truths are the necessary truths. We will talk more about this in Chapter 18.

The third, closely related, quality is *endurance*. There is little doubt that the Pythagorean theorem meant the same thing to the ancient Greeks as it does to us today, and there is every reason to expect that it will mean the same thing to anyone in the future. Likewise, all true mathematical statements we talk about in this book will remain true forever.

The fact that such objective and enduring knowledge exists (and moreover, belongs to all of us) is nothing short of a miracle. It suggests that mathematical concepts exist in a world separate from the physical and mental worlds – which is sometimes referred to as the Platonic world of mathematics (we will talk more about that in the closing chapter). We still don't fully understand what it is and what drives mathematical discovery. But it's clear that this hidden reality is bound to play a larger and larger role in our lives, especially with the advent of new computer technologies and 3D printing.

The fourth quality is *relevance* of mathematics to the physical world. For example, a lot of progress has been made in quantum physics in the past fifty years because of the application of the concept of symmetry to elementary particles and interactions between them. From this point of view, a particle, such as an electron or a quark, is like a round table or a snowflake, and its behavior is very much determined by its symmetries. (Some of these symmetries are exact, and some are approximate.)

The discovery of quarks is a perfect example of how this works. Reading the books Evgeny Evgenievich gave me, I learned that at the root of the Gell-Mann and Ne'eman classification of hadrons that we talked about in the previous chapter is a *symmetry group*. This group had been previously studied by mathematicians – who did not anticipate any connections to subatomic particles whatsoever. The mathematical name for it is $SU(3)$. Here S and U stand for "special unitary." This group is very similar in its properties to the group of symmetries of the sphere, which we will talk about in detail in Chapter 10.

Mathematicians had previously described the representations of the group $SU(3)$, that is, different ways that the group $SU(3)$ can be realized as a symmetry group. Gell-Mann and Ne'eman noticed the similarity between the structure of these representations and the patterns of hadrons that they had found. They used this information to classify hadrons.

The word "representation" is used in mathematics in a particular way, which is different from its more common usage. So let me pause and explain what this word means in the present context. Perhaps, it would help if I first give an example. Recall the group of rotations of a round table discussed above, the circle group. Now imagine extending the tabletop infinitely far in all directions. This way we obtain an abstract mathematical object: a plane. Each rotation of the tabletop, around its center, gives rise to a rotation of this plane around the same point. Thus, we obtain a rule that assigns a symmetry of this plane (a rotation) to every element of the circle group. In other words, each element of the circle group may be represented by a symmetry of the plane. For this reason mathematicians refer to this process as a *representation* of the circle group.

Now, the plane is two-dimensional because it has two coordinate axes and hence each point has two coordinates:

Therefore, we say that we have constructed a "two-dimensional representation" of the group of rotations. It simply means that each element of the group of rotations is realized as a symmetry of a plane.[9]

There are also spaces of dimension greater than two. For example, the space around us is three-dimensional. That is to say, it has three coordinate axes, and so in order to specify a position of a point, we need to specify its three coordinates (x, y, z) as shown on this picture:

We cannot imagine a four-dimensional space, but mathematics gives us a universal language that allows us to talk about spaces of any dimension. Namely, we represent points of the four-dimensional space by quadruples of numbers (x, y, z, t), just like points of the three-dimensional space are represented by triples of numbers (x, y, z). In the same way, we represent points of an n-dimensional space, for any natural number n, by n-tuples of numbers. If you have used a spreadsheet program, then you have encountered such n-tuples: they appear as rows in a spreadsheet, each of the n numbers corresponding to a particular attribute of the stored data. Thus, every row in a spreadsheet refers to a point in an n-dimensional space. (We will talk more about spaces of various dimensions in Chapter 10.)

If each element of a group can be realized, in a consistent manner,[10] as a symmetry of an n-dimensional space, then we say that the group has an "n-dimensional representation."

It turns out that a given group can have representations of different dimensions. The reason elementary particles can be assembled in families of 8 and 10 particles is that the group $SU(3)$ is known to have an 8-dimensional and a 10-dimensional representation. The 8 particles of each octet constructed by Gell-Mann and Ne'eman (like the one shown on the diagram in the previous chapter) are in one-to-one correspondence with the 8 coordinate axes of an 8-dimensional space which is a representation of $SU(3)$. The same goes for the decuplet of particles. (But particles cannot be assembled in families of, say, 7 or 11 particles because mathematicians have proved that the group $SU(3)$ has no 7- or 11-dimensional representations.)

At first, this was just a convenient way to combine the particles with similar properties. But then Gell-Mann went further. He postulated that there was a deep reason behind this classification scheme. He basically said that this scheme works so well because hadrons consist of smaller particles – sometimes two and sometimes three of them – the quarks. A similar proposal was made independently by physicist George Zweig (who called the particles "aces").

This was a stunning proposal. Not only did it go against the popular belief at the time that protons and neutrons as well as other hadrons were indivisible elementary particles, these new particles were supposed to have electric charges that were fractions of the charge of the electron. This was a startling prediction because no one had seen such particles before. Yet, quarks were soon found experimentally, and as predicted, they had fractional electric charges!

What motivated Gell-Mann and Zweig to predict the existence of quarks? Mathematical theory of representations of the group $SU(3)$. Specifically, the fact that the group $SU(3)$ has two different 3-dimensional representations. (Actually, that's the reason there is a "3" in this group's name.) Gell-Mann and Zweig suggested that these two representations should describe two families of fundamental particles: 3 quarks and 3 anti-quarks. It turns out that the 8- and 10-dimensional representations of $SU(3)$ can be built from the 3-dimensional ones. And this gives us a precise blueprint for how to construct hadrons from quarks – just like in Lego.

Gell-Mann named the 3 quarks "up," "down," and "strange."[11] A proton consists of two up quarks and one down quark, whereas a neutron consists of two down quarks and one up quark, as we saw on the pictures in the previous chapter. Both of these particles belong to the octet shown on the diagram in the previous chapter. Other particles from this octet involve the strange quark as well as the up and down quarks. There are also octets that consist of particles that are composites of one quark and one anti-quark.

The discovery of quarks is a good example of the paramount role played by mathematics in science that we discussed in the Preface. These particles were predicted not on the basis of empirical data, but on the basis of mathematical symmetry patterns. This was a purely theoretical prediction, made within the framework of a sophisticated mathematical theory of representations of the group $SU(3)$. It took physicists years to master this theory (and in fact there was some resistance to it at first), but it is now the bread and butter of elementary particle physics. Not only did it provide a classification of hadrons, it also led to the discovery of quarks, which forever changed our understanding of physical reality.

Imagine: a seemingly esoteric mathematical theory empowered us to get to the heart of the building blocks of nature. How can we not be enthralled by the magic harmony of these tiny blobs of matter, not marvel at the capacity of mathematics to reveal the inner workings of the universe?

As the story goes, Albert Einstein's wife Elsa remarked, upon hearing that a telescope at the Mount Wilson Observatory was needed to determine the shape of space-time: "Oh, my husband does this on the back of an old envelope."

Physicists do need expensive and sophisticated machines such as the Large Hadron Collider in Geneva, but the amazing fact is that scientists like Einstein and Gell-Mann have used what looks like the purest and most abstract mathematical knowledge to unlock the deepest secrets of the world around us.

Regardless of who we are and what we believe in, we all share this knowledge. It brings us closer together and gives a new meaning to our love for the universe.

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*Note that flipping a table is not a symmetry: this would turn it upside down – let's not forget that a table has legs. If we were to consider a square or a circle (no legs attached), then flips would be bona fide symmetries. We would have to include them in the corresponding symmetry groups.

Chapter 3

The Fifth Problem

Evgeny Evgenievich's plan worked perfectly: I was "converted" to math. I was learning quickly, and the deeper I delved into math, the more my fascination grew, the more I wanted to know. This is what happens when you fall in love.

I started meeting with Evgeny Evgenievich on a regular basis. He would give me books to read, and I would meet him once a week at the pedagogical college where he taught to discuss what I had read. Evgeny Evgenievich played soccer, ice hockey, and volleyball on a regular basis, but like many men in the Soviet Union in those days, he was a chain smoker. For a long time afterward, the smell of cigarettes was associated in my mind with doing mathematics.

Sometimes our conversations would last well into the night. Once, the auditorium we were in was locked by the custodian who couldn't fathom that there would be someone inside at such a late hour. And we must have been so deep into our conversation that we didn't hear the turning of the key. Fortunately, the auditorium was on the ground floor, and we managed to escape through a window.

The year was 1984, my senior year at high school. I had to decide which university to apply to. Moscow had many schools, but there was only one place to study pure math: Moscow State University, known by its Russian abbreviation MGU, for *Moskovskiy Gosudarstvenny Universitet*. Its famous *Mekh-Mat*, the Department of Mechanics and Mathematics, was the flagship mathematics program of the USSR.

Entrance exams to colleges in Russia are not like the SAT that American students take. At *Mekh-Mat* there were four: written math, oral math, literature essay composition, and oral physics. Those who, like me, graduated from high school with highest honors (in the Soviet Union we were then given a gold medal) would be automatically accepted after getting a 5, the highest grade, at the first exam.

I had by then progressed far beyond high school math, and so it looked like I would sail through the exams at MGU.

But I was too optimistic. The first warning shot came in the form of a letter I received from a correspondence school with which I had studied. This school had been organized some years earlier by, among others, Israel Gelfand, the famous Soviet mathematician (we will talk much more about him later). The school intended to help those students who, like me, lived outside of major cities and did not have access to special mathematical schools. Every month, participating students would receive a brochure elucidating the material that was studied in school and going a little beyond. It also contained some problems, more difficult than those discussed at school, which a student was supposed to solve and mail back. Graders (usually undergrads of Moscow University) read those solutions and returned them, marked, to the students. I was enrolled in this school for three years, as well as in another school, which was more physics-oriented. It was a helpful resource for me, though the material was pretty close to what was studied at school (unlike the stuff I was studying privately with Evgeny Evgenievich).

The letter I received from this correspondence school was short: "If you would like to apply to Moscow University, stop by our office, and we will be happy to give you advice," and it gave the address on the campus of MGU and the visiting hours. Shortly after receiving the letter, I took the two-hour train ride to Moscow. The school's office was a big room with a bunch of desks and a number of people working, typing, and correcting papers. I introduced myself, produced my little letter, and was immediately led to a diminutive young woman, in her early thirties.

"What's your name?" she said by way of greeting.

"Eduard Frenkel." (I used the Russian version of Edward in those days.)

"And you want to apply to MGU?"

"Yes."

"Which department?"

“*Mekh-Mat*.”

“I see.” She lowered her eyes and asked:

“And what’s your nationality?”

I said, “Russian.”

“Really? And what are your parents’ nationalities?”

“Well… My mother is Russian.”

“And your father?”

“My father is Jewish.”

She nodded.

This dialogue might sound surreal to you, and as I am writing it now, it sounds surreal to me too. But in the Soviet Union circa 1984 – remember Orwell?* – it was not considered bizarre to ask someone what his or her “nationality” was. In the interior passport that all Soviet citizens had to carry with them, there was in fact a special line for “nationality.” It came after (1) first name, (2) patronymic name, (3) last name, and (4) the date of birth. For this reason, it was called *pyataya grafa*, “the fifth line.” Nationality was also recorded in one’s birth certificate, as were the nationalities of the parents. If their nationalities were different, as in my case, then the parents had a choice of which nationality to give to their child.

For all intents and purposes, the fifth line was a code for asking whether one was Jewish or not. (People of other nationalities, such as Tatars and Armenians, against whom there were prejudices and persecution – though not nearly at the same scale as against the Jews – were also picked up this way.) My fifth line said that I was Russian, but my last name – which was my father’s last name, and clearly sounded Jewish – gave me away.

It is important to note that my family was not religious at all. My father was not brought up in a religious tradition, and neither was I. Religion in the Soviet Union was in fact all but non-existent in those days. Most Christian Orthodox churches were destroyed or closed. In the few existing churches, one could typically only find a few old *babushkas* (grandmothers), such as my maternal grandmother. She occasionally attended service at the only active church in my hometown. There were even fewer synagogues. There were none in my hometown; in Moscow, whose population was close to 10 million, officially there was only one synagogue.[1] Going to a service in a church or a synagogue was dangerous: one could be spotted by special plain-clothed agents and would then get in a lot of trouble. So when someone was referred to as being Jewish, it was meant not in the sense of religion but rather in the sense of ethnicity, of "blood."

Even if I hadn't been using my father's last name, my Jewish origin would be picked up by the admissions committee anyway, because the application form specifically asked for the full names of both parents. Those full names included patronymic names; that is, the first names of the grandparents of the applicant. My father's patronymic name is Joseph, which sounded unmistakably Jewish in the Soviet Union of that era, so this was another way to find out (if his last name hadn't given me away). The system was set up in such a way that it would flag those who were at least one-quarter Jewish.

Having established that by this definition I was a Jew, the woman said, "Do you know that Jews are not accepted to Moscow University?"

"What do you mean?"

"What I mean is that you shouldn't even bother to apply. Don't waste your time. They won't let you in."

I didn't know what to say.

"Is that why you sent me this letter?"

"Yes. I'm just trying to help you."

I looked around. It was clear that everyone in the office was aware of what this conversation was about, even if they weren't listening closely. This must have already happened dozens of times, and everybody seemed used to it. They all averted their eyes, as if I were a terminally ill patient. My heart sank.

I had encountered anti-Semitism before, but at a personal, not institutional, level. When I was in fifth grade, some of my classmates took to taunting me with *evrey, evrey* ("Jew, Jew"). I don't think they had any idea what this meant (which was clear from the fact that some of them confused the word *evrey*, which meant "Jew," with *evropeyets*, which meant "European") – they must have heard anti-Semitic remarks from their parents or other adults. (Unfortunately, anti-Semitism was deeply rooted in the Russian culture.) I was strong enough and lucky enough to have a couple of true friends who stood by me, so I was never actually beaten up by these bullies, but this was an unpleasant experience. I was too proud to tell the teachers or my parents, but one day a teacher overheard and intervened. As a result, those kids were immediately called to the principal, and the taunting stopped.

My parents had heard of the discrimination against Jews in entrance exams to universities, but somehow they did not pay much attention to this. In my hometown, there weren't many Jews to begin with, and all the purported discrimination cases my parents had heard of concerned programs in physics. A typical argument went that Jews weren't accepted there because the studies in such a program were related to nuclear research and hence to national defense and state secrets; the government didn't want Jews in those areas because Jews could emigrate to Israel or somewhere else. By this logic, there shouldn't have been a reason to care about those who studied pure math. Well, apparently, someone did.

Everything about my conversation at MGU was strange. And I am not just talking about the Kafkaesque aspect of it. It is possible to conclude that the woman I talked to simply tried to help me and other students by warning us of what's going to happen. But could this really be the case? Remember, we are talking about 1984, when the Communist Party and the KGB still tightly controlled all aspects of life in the Soviet Union. The official policy of the state was that all nationalities were equal, and publicly suggesting otherwise would put one in danger. Yet, this woman calmly talked about this to me, a stranger she had just met, and she didn't seem to be worried about being overheard by her colleagues.

Besides, the exams at MGU were always scheduled one month ahead of all other schools. Therefore, students who were failed at MGU would still have a chance to apply elsewhere. Why would someone try to convince them not even to try? It sounded like some powerful forces were trying to scare me and other Jewish students away.

But I would not be deterred. After talking about all this at great length, my parents and I felt that I had nothing to lose. We decided that I would apply to MGU anyway and just hope for the best.

The first exam, at the beginning of July, was a written test in mathematics. It always consisted of five problems. The fifth problem was considered deadly and unsolvable. It was like the fifth element of the exam. But I was able to solve all problems, including the fifth. Aware as I was of the strong likelihood that whoever graded my exam could be biased against me and would try to find gaps in my solutions, I wrote everything out in excruciating detail. I then checked and double-checked all my arguments and calculations to make sure that I hadn't made any mistakes. Everything looked perfect! I was in an upbeat mood on the train ride home. The next day I told Evgeny Evgenievich my solutions, and he confirmed that everything was correct. It seemed like I was off to a good start.

My next exam was oral math. It was scheduled for July 13, which happened to be a Friday.

I remember very clearly many details about that day. The exam was scheduled for the early afternoon, and I took the train from home with my mother that morning. I entered the room at MGU a few minutes before the exam. It was a regular classroom, and there were probably between fifteen and twenty students there and four or five examiners. At the start of the exam each of us had to draw a piece of paper from a big pile on the desk at the front of the room. Each paper had two questions written on it, and it was turned blank side up. It was like drawing a lottery ticket, so we called this piece of paper *bilet*, ticket. There were perhaps one hundred questions altogether, all known in advance. I didn't really care which ticket I would draw as I knew this material inside-out. After drawing the ticket, each student had to sit down at one of the desks and prepare the answer, using only the provided blank sheets of paper.

The questions on my ticket were: (1) a circle inscribed in a triangle and the formula for the area of the triangle using its radius; and (2) derivative of the ratio of two functions (the formula only). I was so ready for these questions, I could have answered them in my sleep.

I sat down, wrote down a few formulas on a sheet of paper, and collected my thoughts. This must have taken me about two minutes. There was no need to prepare more; I was ready. I raised my hand. There were several examiners present in the room, and they were all waiting for the students to raise their hands, but, bizarrely, they ignored me, as if I did not exist. They looked right through me. I was sitting with my hand raised for a while: no response.

Then, after ten minutes or so, a couple of other kids raised their hands, and as soon as they did, the examiners rushed to them. An examiner would take a seat next to a student and listen to him or her answer the questions. They were quite close to me, so I could hear them. The examiners were very polite and were mostly nodding their heads, only occasionally asking follow-up questions. Nothing out of the ordinary. When a student finished answering the questions on the ticket (after ten minutes or so), the examiner would give him or her one additional problem to solve. Those problems seemed rather simple, and most students solved them right away. And that was it!

The first couple of students were already happily gone, having obviously earned a 5, the highest grade, and I was still sitting there. Finally, I grabbed one of the examiners passing by, a young fellow who seemed like he was a fresh Ph.D., and asked him pointedly: "Why aren't you talking to me?" He looked away and said quietly: "Sorry, we are not allowed to talk to you."

An hour or so into the exam, two middle-aged men entered the room. They briskly walked up to the table at the front of the room and presented themselves to the guy who was sitting there. He nodded and pointed at me. It became clear that these were the people I'd been waiting for: my inquisitors.

They came up to my desk and introduced themselves. One was lean and quick, the other slightly overweight and with a big mustache.

"OK," the lean man said – he did most of the talking – "what have we got here? What's the first question?"

"The circle inscribed in a triangle and..."

He interrupted me: "What is the definition of a circle?"

He was quite aggressive, which was in sharp contrast to how other examiners treated students. Besides, the other examiners never asked anything before the student had a chance to fully present their answer to the question on the ticket.

I said, "A circle is the set of points on the plane equidistant from a given point."

This was the standard definition.

"Wrong!" declared the man cheerfully.

How could this possibly be wrong? He waited for a few seconds and then said, "It's the set of *all* points on the plane equidistant from a given point."

That sounded like excessive parsing of words – the first sign of trouble ahead.

"OK," the man said, "What is the definition of a triangle?"