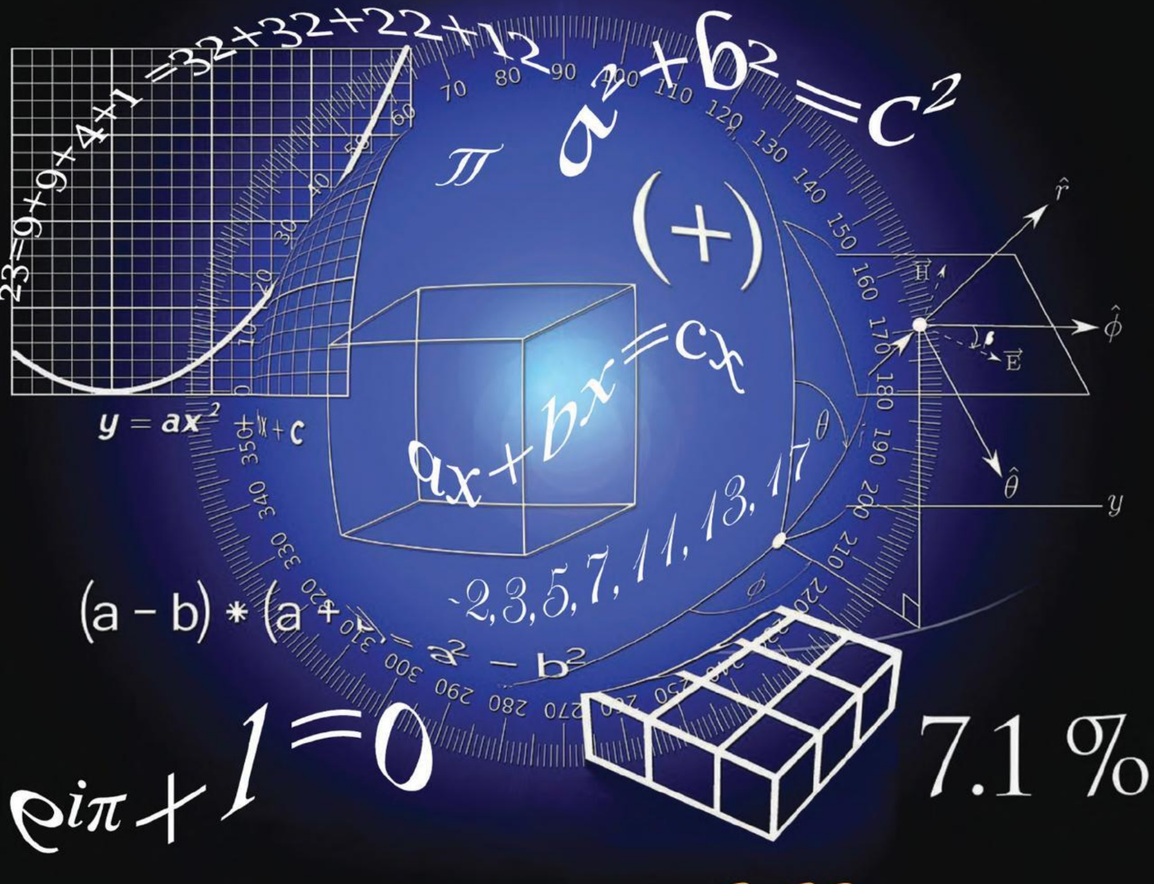
Mathematical Elegance

An Approachable Guide to Understanding Basic Concepts



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Introduction

I was never very good at math, but I have always loved it for the way it all fits together so nicely. Fortunately, it is a peculiarity of human beings that we can love that, perhaps *only* that, which we do not fully understand. And in this you and I are perhaps more like than unlike the very greatest of mathematicians, men whose thoughts represent some of the highest and most profound we human beings have had.

All of us face a universe in which mathematics has forever unfolded in directions we—and even the greatest mathematicians among us—cannot begin to visualize. Thus, our humility is maintained by the knowledge that, possibly as opposed to the physical universe, the universe of numbers goes on everywhere forever in every possible direction and dimension. These numbers contain unimaginably more mathematical truths than we now know, more even than our universe, even if it has eternity at its disposal, can ever know.

In the mathematical universe, any knowledge of truth we will ever have is insignificant compared to all that is true. Facing such a universe, we realize that the difference between us and the very greatest mathematicians virtually evaporates—sort of. These fellows were a *tad* better than we are at finding mathematical truths. And it is the glory of mathematics that many of the deepest truths can be discovered by looking through the tiny window at the tiny portion of the mathematical universe provided by our little world. It is the mathematicians who look through the window and tell us what they see.

This Is All You Have to Know

There are many different sorts of numbers, most of which are primarily of interest to mathematicians and too arcane to concern us in this book. However, a few types of numbers—most of which are already familiar to you—are the subject of this book. They are the following:

Natural Numbers
Natural numbers are what you first think of when you think of numbers: 1, 2, 3 . . . (". . ." means "goes on forever").

Integers or Whole Numbers
Integers and whole numbers are natural numbers, their negative equivalents (–1, –2, –3 . . .), and 0.

A mathematician named Leopold Kronecker once said that "God made the integers; all else was the work of man." It has always seemed to me that God made the primes and the rest of the numbers are merely a mopping-up operation. But who am I to argue with a guy whose moniker was Kronecker?

Prime Numbers
A prime number is an integer greater than 1 that is only evenly divisible by 1 and itself. Two is a prime number (indeed, it is the only even prime number because every other even number is divisible by 1, itself, and 2). Three, 5, and 7 are prime numbers, but 9 (which is divisible by 1, 9, and 3) is not. Primes are the crème de la crème of numbers because they are the mathematical atoms, the building blocks of all other numbers.

Euclid proved that there is an infinite number of prime numbers (i.e., there is always a larger prime number than the one you thought must be the largest). Some people consider this proof to be the greatest intellectual achievement our species has yet accomplished. Other people do not.

If you can figure out an easy way of determining whether a large number is prime (or an easy formula for generating the primes and only the primes), you will become very, very rich.

Composite Numbers
A composite number is any integer that is not a prime number. Four is a composite number because it is evenly divisible not only

0.123456789 . . .
0.987654321 . . .
0.555555555 . . .
0.273403949 . . .
etc.

Now, draw a diagonal line, beginning with the first digit of the first number, the second digit of the second number, the third digit of the third number, ad infinitum. The unending diagonal number we get will begin 7074553 . . . .

Add 1 to each digit of our diagonal number. The new number will begin 8185664 . . . . Notice that the diagonal number we end up with cannot be the same as the first number because its first digit is an 8, not a 7. It cannot be the same as the second number because its second digit is a 7, not a 6. It cannot be the same as the third number . . . well, you get the idea. The infinity of listed numbers does not include the diagonal number. Therefore, the diagonal number is a member of an infinity that cannot be counted (i.e., an infinity larger than the countable infinities). This demonstrates that some infinities are larger than others. Moreover, because the same sort of argument can be made against *any* infinity claiming to be the largest, there is no more a largest infinity than there is a largest number.

The mathematician and science author John L. Casti, has a marvelous way of demonstrating the diagonal proof that is easier for some people to see:

Consider these six names:

Twain
fUrman
beRry
sprIng
lockNer
herzoG

Create a word using the letter after the first letter of the first word, the letter after the second letter of the second word, and so on. You will get "turing." This must be different from every word on the list because it will differ from the first word by a different first letter, the second word by a different second letter, and so on.

I hope you now have some slight feeling for what elegance is. This book is a compendium of examples of elegance, with some other stuff thrown in just because it is interesting. The point is not to explain in any detail, but to give the reader a taste that will, I hope, lead to exploration of one of the many paths hinted at in these pages.

I cannot stress this final point enough: This book will not teach you mathematics. I am far, far from qualified to do this, and there are many, many mathematicians—some of them, unfortunately, unemployed—who are eminently qualified to do so.

My purpose is simply to provide a sort of tasting menu to whet your appetite. I attempt to serve up fascinating and beautiful findings of mathematics, in the hope of persuading you to follow up on those you find most appealing by searching out the works of mathematicians who have described those findings in detail.

A Million

Picture a die, like one used for gambling only much smaller, just one-tenth of an inch on each side. A million of these lined up horizontally will stretch out about one and two-thirds miles. Ignoring the height and width of the dice, this is a one-dimensional arrangement.

Now, let's arrange the dice flat in a square, a thousand dice long and a thousand dice wide. The square holding the million dice will be a bit over eight feet (a hundred inches—one-tenth inch times one thousand) on each side. Ignoring the third dimension, the height of the die, this is a two-dimensional arrangement.

Finally, let's make a square of a hundred dice by a hundred dice (i.e., $100 \times 100 = 10{,}000$ dice). Now make ninety-nine more such squares and pile each on top of the previous one. You now have a cube, $100 \times 100 \times 100 = 1{,}000{,}000$. This cube will be only ten inches by ten inches by ten inches. This is a three-dimensional arrangement, probably much smaller than you would have guessed.

This Kid Shows Promise

When Carl Gauss, the greatest of all mathematicians in some people's view, was seven years old, his teacher did not feel like lecturing to his second graders one day. So he gave the boys a math problem that would easily take them the entire class: add the numbers from 1 to 100. Seconds later, young Carl raised his hand and told the teacher that the sum was 5,050. Both the teacher and the other students were dumbfounded.

Carl had realized that he could add pairs from the two ends: that is, (1 + 100) + (2 + 99) + (3 + 98) . . . . Each pair of numbers adds up to 101. Because there are fifty pairs of numbers, the total is 50 × 101, or 5,050.

A Mathematical Fact

Mathematics is full of facts that you would never guess are facts. For example, Joseph Louis Lagrange proved that every natural number (i.e., positive integer) is equal to the sum of four or fewer square numbers: for example, $23 = 9 + 9 + 4 + 1 = 3^2 + 3^2 + 2^2 + 1^2$. Trial and error establishes that not all natural numbers can be represented as the sum of *fewer* than four squares, though some can: for example, $18 = 3^2 + 3^2$.

You Will Never Get Rich

Most people know that the more often interest is compounded, the more money you end up with. Banks know that people know this and compete by offering ever-more-frequent compounding. What banks also know is that the difference between semiannual compounding and daily, or even second-by-second, compounding is insignificant.

Let's say you have $100 in a savings account and the bank compounds the interest *annually* (i.e., *once* a year). Your money will double (i.e., grow to $200) in, for example, ten years. Now, if the bank compounds the interest *semiannually*, your $100 will grow to about $269, rather than $200, in the same amount of time.

This might lead you to the conclusion that ever-more-frequent compounding will engender similar increases. Banks occasionally exploit this belief by compounding ever-more-frequently.

But that conclusion is not correct. In fact, even if the $100 dollars is compounded second-by-second, it will grow to only about $272 (actually a tad less) in the same time that semiannual compounding would make it grow to $269.

This is because compounding approaches the limit of the "*e*" (2.7182 . . .). Why? You might well ask *why* does *e* equal 2.7182 . . . but you will not be satisfied with the answer, which is "because it does." The *e* is an irrational number—a number not expressible as a fraction composed of two integers—that is, as we have seen was the case with the square root of 2, just as legitimate a number as 6 or 3/4. The *e* is one of those numbers, like π, that is a mathematical constant that keeps popping up in divergent mathematical areas for no obvious reason.

Each increase in the number of times your savings is compounded *does* increase the total; daily compounding is better than weekly