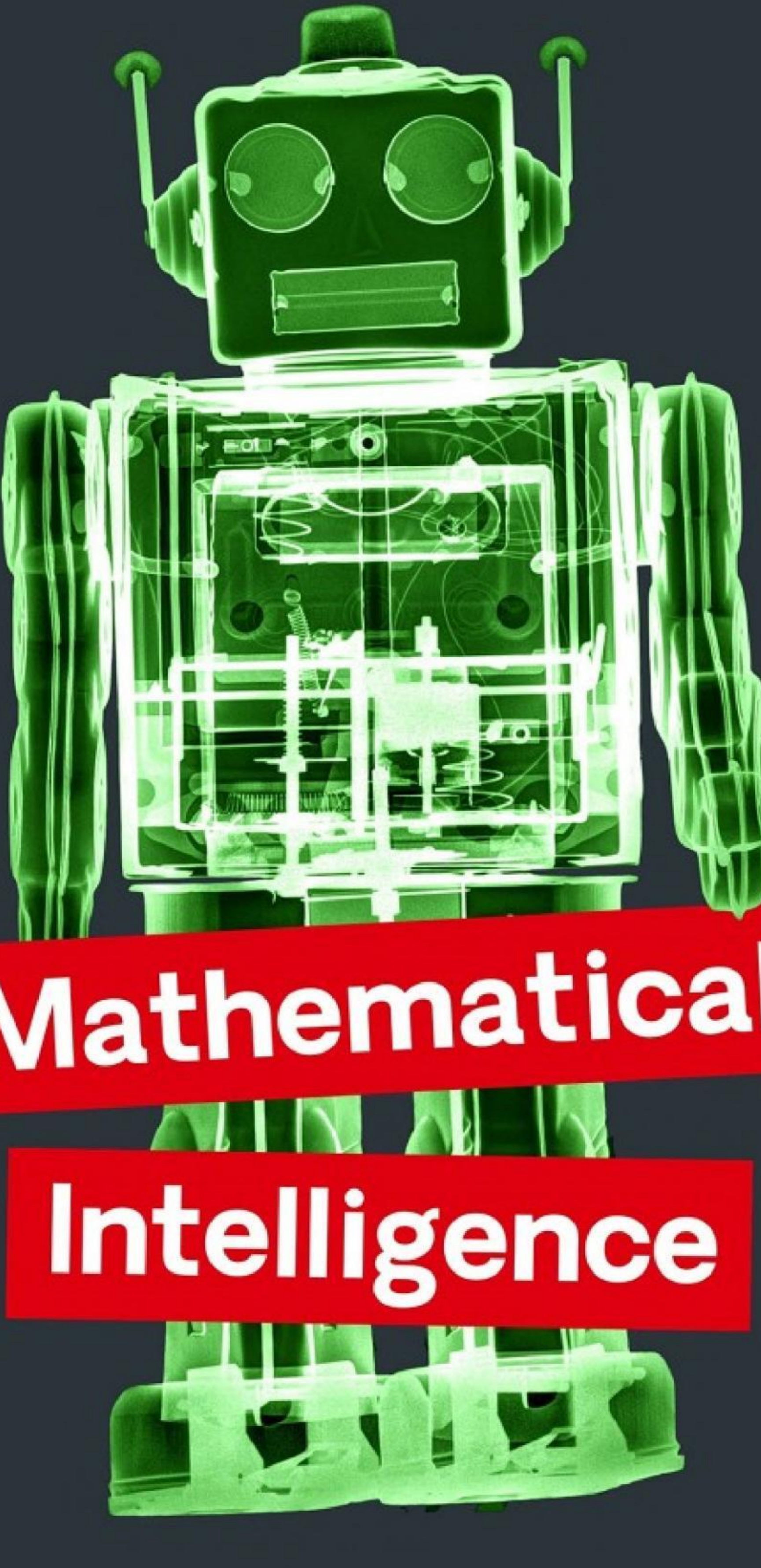
Junaid Mubeen



# Mathematical Intelligence

*What we have that machines don't*

First published in Great Britain in 2022 by
Profile Books Ltd
29 Cloth Fair
London
EC1A 7JQ

*www.profilebooks.com*



1 3 5 7 9 10 8 6 4 2



A CIP catalogue record for this book is available from the British Library.

ISBN 978 1 78816 683 6
eISBN 978 1 78283 795 4

Typeset in Sabon by MacGuru Ltd

# *Contents*

# ACKNOWLEDGEMENTS

'Acknowledgements' are inadequate. Every person mentioned here has my heartfelt gratitude for helping me turn a vague concept into an actual thing. Any shortcomings are my own.

My agent, Doug Young, elevated my ambitions for what this book could become. He has been a strong advocate and guiding hand throughout.

I'll never forget my first conversation with Helen Conford because it was the first time I felt validated as an aspiring author. I'll always be grateful for the chance she took in bringing me to Profile. Ed Lake, Paul Forty and the whole editorial team have worked their magic to polish and fine-tune the rough manuscript that was presented to them.

Several friends and colleagues vetted early drafts: my thanks to Keith Devlin, Shameq Sayeed, Steve Buckley, Roxana and Rares Pamfil (the golden couple!), Noel-Ann Bradshaw, Andrew Mellor, Lucy Rycroft-Smith, David Seifert and Ed Border. A special mention to Mohamady El-Gaby for sense-checking my neuroscience claims and helping me find my footing as I ventured far beyond my own areas of expertise. Thanks also to Taimur Abdaal for lending me a sketch or two.

Much of this book was written on Friday afternoons in coffee shops and I'm thankful to my friend and former boss, Richard Marett, for granting me '10% time' to indulge in this project during my time at Whizz.

I have seen mathematical intelligence in action every Sunday in the Oxford Maths Club. To every parent who has entrusted me with their child's maths development, and to every student who has taken to our courses with gusto, I am immeasurably grateful.

My wife, Kawther, is the unsung hero of this project. She has championed my work even when it was little more than scrawlings on a napkin. She also happens to be the literary talent in the family, and a ruthless editor to boot. Since the inception of this book we've grown ourselves a delightful family. Leena and Elias are my two greatest blessings in life; the book is dedicated to the former (next one's for you, *lumps*).

Introduction

# THE CASE FOR MATHEMATICAL INTELLIGENCE

*MIT, 1950s. The first wave of Artificial Intelligence is on the horizon. Marvin Minsky, one of the field's leading figures, proclaims: 'We're going to make machines intelligent. We're going to make them conscious.' Douglas Engelbart, a peer of Minsky's, retorts: 'You're going to do all that for the machines? What are you going to do for the people?'*[1]

Artificial intelligence (AI) researchers are nothing if not bullish about the prospects of their creations. The field kicked off in earnest in 1956 at a summer workshop held at Dartmouth College, New Hampshire, where the founding fathers of AI set out their vision in no uncertain terms. Intelligent machines, they believed, were to propel humanity into the next golden age of innovation by 'simulating every aspect of learning or any other feature of intelligence'.[2] The timeframe was bolder still: one summer was all they would need to break the back of AI.

Things turned out to be rather more complicated, as a summer of hype gave way to a succession of AI winters, with progress in the field largely stagnant for several decades. But if you've caught the headlines recently, you'll know that AI is currently the subject of renewed hype. Between flagship triumphs in popular games, the growing presence of home assistants, and the coming of self-driving cars, the machines have resumed their rise.

We humans have distinguished ourselves from other species by inventing tools to help us solve our most challenging problems. And yet we may be complicit in our own demise because some of these tools have become so powerful that they appear to pose genuine threats to our ways of thinking and being. Studies of the growing threat of automation to human labour abound,[3] while the so-called 'superintelligent' machines of tomorrow may force us to re-examine what it even means to be human in the first place.

As we enter this new cycle of ratcheted expectations, hopes and anxieties around the latest wave of technological innovation, Engelbart's question should resonate loud and clear. We reserve such reverence for technology that we risk overlooking our own human capabilities. Machines lack some of the basic qualities of human thinking – qualities we have sidelined through our mechanistic ways of schooling and working, and qualities that we need to urgently reawaken to thrive alongside our silicon counterparts.

As it happens, humans have – through millions of years of evolution and thousands of years of continual refinement – developed a powerful system for making sense of the world, for imagining new ones, and for devising and solving complex problems. This system has helped us create the economies that underpin our society. It has shaped our notions of democracy. It has spawned technologies that now stare us down, but the same system can equip us with the skills to tame these digital beasts.

The system has a name: *mathematics*.

## What is mathematics, really?

Mathematics has been described as an art, a language and a science. For some, it is a means of unlocking nature's secrets. As Galileo testified so eloquently: '[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language.' This is mathematics as the language of the universe, the engine of scientific progress.

The scope of mathematics transcends our physical universe. Entire swathes of the subject are explored for their own sake, driven by the deep satisfaction that comes from dreaming up new concepts, piecing together ideas, and grappling with thorny problems. Many mathematicians seek out aesthetic qualities in their craft. The twentieth-century mathematician and philosopher Bertrand Russell spoke of the subject's 'supreme beauty – a beauty cold and austere ... capable of a stern perfection such as only the greatest art can show'.[4] Many see themselves as artists as well as scientists – 'makers of patterns',[5] to borrow a description from G. H. Hardy, a contemporary of Russell's. It is not uncommon for mathematicians to deride the need to apply their thinking to the 'real' world, as if utility were some kind of distraction. It has even been proposed that some aspects of mathematical inquiry have a hedonistic basis.[6]

From these varied motives, mathematics is often partitioned into two supposed types: there is *applied* mathematics, which, as the name suggests, is concerned with problems of the real world. Then there is the presumptively labelled strand of *pure* mathematics, which centres on more abstract concepts and rigorous arguments often removed from practical consideration. This separation is felt keenly at university, where maths students are expected to declare their allegiances before specialising in one area. I was of the *pure* persuasion. Yet, since leaving formal mathematics a decade ago, much of my work has been rooted in datasets and algorithms – about as *applied* as it comes.

Having bridged the pure/applied divide, I have come to realise that it is an arbitrary and limiting way of characterising the subject. There is a commonality that binds mathematicians of all types. Without exception, we derive immense joy from tackling maths problems, a satisfaction akin to solving our favourite puzzles. Mathematics is even alleged to elicit the same physiological reactions as sexual activity (yes, really).[7] Alongside that pleasure comes power; whatever branch of mathematics a mathematician happens to be probing, they are using the mind's highest faculties and building a store of portable mental models that serve them in all parts of life.

It may feel risky to invest time and effort in studying mathematics based on nebulous notions of pleasure and power. But mathematics cannot help but bring practical uses too. It is not at all unusual for a field of mathematics that starts out as pure intellectual inquiry to later find itself in practical settings. Prime numbers (whole numbers greater than 1 that cannot be divided into smaller whole number parts) were first studied for their unusual arithmetic properties, yet internet security now relies on them – your credit card details are kept secure by the sheer difficulty of finding the prime factors of really large numbers. The Greeks were enthralled by the geometric properties of ellipses; only centuries later would Kepler discover that planets move around the sun in an elliptical orbit. The topology of knots, a delight to study in its own right, has applications in protein folding. And calculus (the study of continuous change), arguably the most applied of all mathematical topics, which was the basis for Newton's study of planetary motion, and whose tools are indispensable to engineers, physicists, financial analysts, even historians,[8] was developed within the rigorous frameworks of pure mathematics. I could go on.

The theoretical physicist Eugene Wigner encapsulated this entwining of intellectual curiosity and utility by remarking on what he called the 'unreasonable effectiveness' of mathematics, declaring that 'the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation for it'.[9]

The 'usefulness' of mathematics is not limited to specific real-world applications. It arises chiefly from its invitation to explore a vast range of concepts, even arcane ones. Mathematics transports us into multiple worlds, each governed by its own rules. It encourages us to break free of convention and leap from one conceptual system to another. These faraway worlds can also train us to think in ways that enrich our understanding of our own, physical one. Even as the content of my own doctorate in pure mathematics drifts from memory (to the point where I can scarcely grasp its essential ideas any longer),[10] the process by which it was created remains its most enduring contribution to my everyday thinking and problem solving.

Mathematical intelligence is not about calculus or topology any more than musical intelligence is limited to a particular genre or instrument. It is a system for making us better thinkers and problem solvers, using the proven tools of mathematicians. And in the age of smart machines, it is needed more than ever.

## Mathematics and calculation: a false coupling

The mathematics I've just described is quite apart from what we encounter at school. 'School mathematics' places great emphasis on calculation. A calculation is a routine operation performed on certain objects, often numbers, to produce a particular result. It can be as simple as counting and as complicated as Google's search-ranking algorithms (an algorithm here just means a list of step-by-step instructions).* School mathematics is premised on the idea that rehearsing a litany of routine calculational techniques is a strict prerequisite for mathematical intelligence, and a gateway to employment. Topics such as calculus, algebra and geometry, each of which contain a multitude of rich concepts, are stripped down to bare calculational form.

The marriage between mathematics and calculation is the result of several forces. The first is an industrial paradigm of formal education whose roots can be traced back to the mid-nineteenth century, when the aims of mass schooling coalesced with notions of mechanisation and scale, and increases in urban populations fuelled demand for everyday numeracy skills such as counting money and telling the time. As universal education systems sprang up the world over, subject matter reflected the needs of a mathematically literate workforce. In England, for instance, arithmetic dominated the curriculum, and additional topics – such as algebra, mechanics and fractions – were introduced with the goals of employment in mind.[11]

Society has made giant leaps of progress since then, yet school mathematics has remained largely static; national and international curriculum standards remain heavily couched in speed and proficiency with calculation. The stubborn persistence of calculation in education also owes a debt to widely held beliefs around the nature of mathematics. *Platonism* – first espoused by the Greek philosopher Plato – holds that mathematical objects are abstract entities independent of language, thought or practices. Just as electrons and planets exist independently of us, so do mathematical concepts such as number. In this view, there is a single form of mathematics, timeless and immutable. Alongside Platonism, there is the *formalist* view, which gained traction in the twentieth century and considers mathematics a self-contained system of logical truths, each derivable from first principles. The Platonist and formalist philosophies, especially popular among 'pure' mathematicians, conspire to reduce mathematics to a single pathway of predetermined, hard-coded truths. Abstraction is the gold standard in this framing of mathematics, its raison d'être, best accessed by mastering symbol manipulation. The execution of mathematical procedures – fast, precise calculation – is seen as the single pathway to deep mathematical thought.

The Platonist–formalist view overlooks the crucial fact that mathematics takes on rich and diverse forms,[12] all of which are birthed in the context of local environment and experience.[13] Take something as seemingly universal as our number system. It arises out of a series of choices, from the symbols we use to denote quantities to how we group together objects to manage large amounts, to how we perform arithmetic on numbers. In schools across the world, students are taught Hindu–Arabic numerals (0, 1, 2, and so on), the decimal system (grouping numbers into tens), and specific algorithms for performing addition, subtraction, multiplication and division. Students are led to believe that these choices are inevitable – the only conceivable way to think about numbers – when in fact they are situated within a historical and sociocultural backdrop. As we'll see in later chapters, communities around the world to this day adopt highly varied representations for numbers. Mathematics in the real world is more situational and contextual than Platonism and formalism would suggest.

My work has taken me to classrooms the world over and I can confirm that, despite its short-sightedness, the Platonist–formalist ideal is alive and well everywhere. A common thread binds the mathematics taught to marginalised communities in Kenya, children of Microsoft executives in Washington State, students of Eton College, and low-income families in rural Mexico. In all these cases, school mathematics is characterised by a heavy diet of calculation,[14] and mathematical talent is conceived as the ability to execute these techniques flawlessly and at speed.

School mathematics comes wrapped in the promise that this very particular skill set will, on some unspecified date in the future, serve students' everyday needs. That promise may have held up in the nineteenth century, when, for example, the formulae of trigonometry would guide your career as a carpenter or surveyor or navigator, and you would be expected to make the requisite calculations by hand. Yet the twenty-first-century student will discover, if they haven't already, that calculation is no longer the unique marker of human mathematical talent. It is almost tautological to say it, but for computation we have computers.

School mathematics is clearly in need of a rethink, which should come as welcome relief to most. Far from evoking the sentiments of wonder or beauty experienced by mathematicians, it is more commonly associated with feelings of dread. In the UK alone, a fifth of the population is afflicted with maths anxiety.[15] For these people, the anticipation and experience of doing mathematics activate the same regions of the brain that give rise to pain.[16] Attitudes towards mathematics have been shown to deteriorate with age,[17] and many people, scarred by their encounters in school, flee into the safe sanctuary of adulthood, resolving never again to confront anything that resembles mathematics. Is the Platonist–formalist method of education simply the price we have to pay to feel the power of mathematics – to appreciate its unreasonable effectiveness? Even if a casualty rate of one in five is deemed palatable, the apparent victors of this brand of mathematics find themselves trapped in a false sense of security. As an admissions tutor at Oxford University, and more recently as an employer, I have interviewed hundreds of candidates who naïvely presume that a clean sweep of top grades in mathematics at school has prepared them to think creatively and tackle complex problems.

The German poet Hans Magnus Enzensberger has described mathematics as 'a blind spot in our culture – alien territory, in which only the elite, the initiated few have managed to entrench themselves'.[18] There is a yawning chasm between the mathematics enjoyed by professional mathematicians and the monotony of most school curricula.

Professional mathematicians, for their part, tend to keep calculation at arm's length. They recognise that techniques such as long division, the quadratic formula and trigonometric identities occupy a small space within the mathematical landscape, a tiny sliver of all the concepts available in the subject. Entire branches of mathematics are removed from calculation, and even where calculations surface, the creative elements of mathematical intelligence reside in dreaming up such methods in the first place, understanding their inner workings and applying them in novel settings. The specific act of calculation is secondary and offers little joy or illumination.

## New calculating tools, new mathematics

The history of mathematics shares a timeline with an ongoing effort to liberate humans from the tedium of calculation. Performing calculations does not come naturally to us. Time and again, we have created tools and technologies that outsource the most mechanical aspects of mathematics.

Great leaps have been made with leading-edge calculating tools.[19] Where our earliest ancestors marshalled pebbles and grains to keep track of basic quantities, the city planners of Babylonia, Sumeria and Egypt used formal calculation schemes which were brought to bear on problems of engineering, land administration, astronomy, timekeeping, planning and logistics. Calculation, along with reading and writing, became a cornerstone of more developed civilisations. Some of the earliest surviving government records are replete with calculations central to administration.

Physical counting instruments were always close at hand. The abacus that helps us count large quantities has its roots in the pebble-counting schemes of the ancient Romans, and as calculations grew in complexity, so too did the power of our tools. Older readers may recall using a slide rule in school to assist in weighty calculations such as the multiplication of large numbers. The slide rule was based on John Napier's logarithm tables. Napier was born into a Scottish family of estate owners in 1550. Copernicus had just developed the heliocentric model of the universe, placing the sun at its centre for the first time. Columbus had sailed the Atlantic, and Renaissance artists were advancing their own frontiers. Yet the world remained heavily dependent on tired calculational conventions. The work of masons, merchants, navigators and astronomers all required methods of long division and multiplication that were tediously handcrafted, prone to human error, and prohibitively expensive to carry out (pen and paper did not come cheap).

On his travels across Europe as a young student, Napier observed the burden of calculation first-hand. He would encounter decorative books composed solely of mathematical tables and currency versions, created and used daily by merchants. The tables still demanded a hefty degree of calculation on the part of the user. There had to be a more effective way, Napier thought, of removing what he called 'those hindrances' to trade. Napier was alluding to what cognitive psychologists now term our 'working memory', which handles short-term information and is limited to between four and seven objects at a time.[20] This makes multistep calculations such as long multiplication or division difficult to perform, as we strain to keep track of each moving part.

In his famous work *Mirifici Logarithmorum Canonis Descriptio* ('Description of the marvellous canon of logarithms'), Napier introduced a powerful mathematical object called the logarithm function. To grasp the intuition behind logarithms, first consider a familiar multiplication involving powers of 10:


This calculation is straightforward because we just ‘add the zeros’ in each term to get our product. It would be handy if every multiplication could be managed in such a simple way. Napier’s logarithm makes this possible. In the numbers above, the string of zeros corresponds to how many times 10 multiplies by itself – twice for 100, thrice for 1000, and so on. With this in mind, the logarithm of a number is defined by how many times you have to multiply 10 by itself to get that number. So the logarithm of 100, denoted log(100), is 2, and the logarithm of 1000, denoted log(1000), is 3.

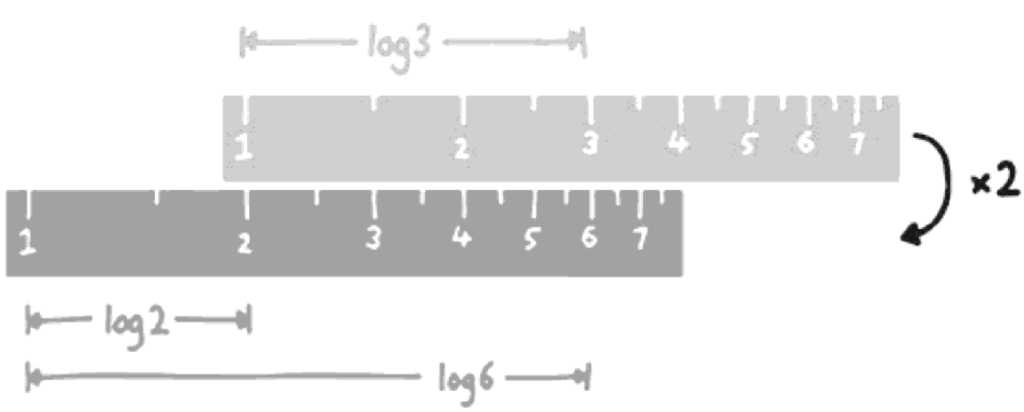
The clever, mathematical part is that the logarithm can be defined for every positive number, not just powers of 10. The logarithm of 95 is 1.978, the logarithm of 2367 is 3.374, and the logarithm of 3 is 0.477, which is to say that ‘if you multiply 10 by itself 0.477 times you will get 3’. That may sound strange at first, but the conceptual power of mathematical functions allows us to bring such notions into existence.

A useful property of logarithms is that they obey the following rule:

$$\log(a \times b) = \log(a) + \log(b)$$

Suppose we want to multiply two large numbers. Napier explained how, using the above formula, we can transform the problem into one involving addition, which is simpler and less error-prone. All we need is a table that lists the ‘logarithmic value’ of each number. The process then goes as follows:

1. Look up the logarithm of each value to be multiplied.
2. Add these two logarithmic values to get a total.
3. Look up the value of the number whose logarithm corresponds to this total. The number you have found is the product of your two original numbers.*




A slide rule in action: if we slide the top ruler 2 units along the ruler below (a length of log 2) then every number on the ruler below corresponds to multiplying the number above it by 2. For example, the number 3 (which is a length of log 3 along the top ruler) lines up with the number 6 (which is a length of log 6 along the bottom ruler), telling us that 3 × 2 = 6.

Napier's *Canon* comprised huge lists of numbers and their corresponding logarithmic values. It took some twenty years to compile. When dedicating the work to the future King Charles I, Napier wrote of how 'this new course ... doth clean take away the difficulty that heretofore hath been in mathematical calculations, and is so fitted to help the weakness of memory'.[21] The slide rule – a compact manifestation of Napier's logarithm tables – appeared in 1654, after his passing. Logarithms can also be exploited to simplify a raft of operations beyond multiplication: powers, square roots and even trigonometric calculations can be closely approximated using simple extensions of the techniques described here, and all of these methods were added to various iterations of the slide rule until the electronic calculator took its place in the latter part of the twentieth century.

Napier's innovation epitomises attempts to automate human effort. For a time, this led to an explosion of jobs. When the eighteenth-century French mathematician and engineer Gaspard de Prony embarked on the project of producing large logarithmic tables for the French Cadastre (the official system of land registration), for which 200,000 logarithms were each to be calculated to upwards of fourteen decimal places, he enlisted a small army of 'human computers' to accomplish the task.[22] De Prony took inspiration from economist Adam Smith's *The Wealth of Nations* and sought to bring Smith's concept of the 'division of labour' to calculation. He imagined a three-tiered pyramid of human labourers. At the top was a small sliver of mathematicians of distinction who devised clever step-by-step instructions – algorithms – for calculating logarithmic values. The second layer consisted of 'algebraists' who would translate these instructions into forms that could easily be computed. The final, most crowded layer consisted of workers who were competent in basic arithmetic and required 'the least knowledge and by far the greatest exertions', performing millions of calculations (addition and subtraction for the most part) and noting the results. In de Prony's model, just two or three mathematicians were needed for every seven or eight algebraists and seventy to eighty workers. With de Prony's labour pyramid, 'big calculation' was born, fashioned in the image of scalable manufacturing.

'Big calculation' trod the same path as manufacturing when it came to mechanisation, as physical calculating machines increasingly took the place of humans. It was against this backdrop that inventor-mathematician Charles Babbage designed two mechanical calculators in the mid-nineteenth century, neither of which was actually constructed during his lifetime (due mainly to financial constraints), but both of which carry huge significance as direct cogwheel-based forerunners of the modern computer. With the emergence of the digital computer and the electronic calculator, Babbage's visions were realised and the era of the *human* computer drew to a close. The heroic swansong of human computers was the 1960s NASA space mission, where the flesh-and-blood calculations of Katherine Johnson and her team helped propel humankind into space.[23]

The work of human computers was profitable in its time, noble even. But calculation has always been the understudy of mathematics (an insight not lost on Johnson and her colleagues, who fought for status in the face of racial and gender prejudice by demonstrating their aptitude for modelling and other essential mathematical skills). Calculation no longer paves a path to employment; today those lower rungs of the pyramid are occupied by machines.

Once computers crept past the calculation feats of humans, they surged ahead and never looked back. The slide rule reigned for over three hundred years, but the electronic calculator that took its place lasted no more than thirty.[24] The competition for pocket-sized electronic calculators was fiercely fought for all of two decades before the advent of the internet and cloud-based technologies. The rapid ascent of computing power was foreseen by Intel co-founder Gordon Moore. In the 1960s, Moore observed that the number of transistors that can be accommodated on a microprocessor seemed to double every eighteen months – an exponential rate. Moore's Law has come to fruition with astonishing accuracy.* By now, our smartphones possess more processing power than the computers and slide rules that sent us to the moon. A world without digital computers is a world without the internet and all that it enables: social media, email, GPS, online shopping, music streaming, remote work, certain kinds of medical diagnosis.

As our calculating tools evolve, so does the nature of mathematical work. Writing in the early twentieth century, the British philosopher Alfred Whitehead noted: 'Civilization advances by extending the number of important operations which we can perform without thinking about them.'[25] Just as innovations such as Napier's logarithm tables accelerated scientific discovery in the past, today's technologies are giving rise to whole new ways of doing mathematics.

Over the past few decades, algorithms have evolved significantly in the direction of versatility as well as processing power. A flurry of packages such as Mathematica and Wolfram Alpha have been developed to execute a vast array of procedures. They have birthed new branches of research, such as 'experimental mathematics', where the idea is to study mathematical objects (numbers, shapes and multidimensional vector spaces, to name a few), and the patterns that govern them, through computation. Powerful, automated calculators allow us to make informed guesses and check them through trial and error by crunching through a range of numerical scenarios.

In our everyday lives, too, calculation is as prominent as ever – we analyse offers in the supermarket, mortgage options, calorie counts, and much besides. Getting the best deal (or diet), however, doesn't rest on our number-crunching skills as much as our ability to evaluate information and make sense of data.

With the right tools at our disposal, mathematics gives us all licence to transcend calculation and to think in the most creative ways. As mathematician Keith Devlin put it: 'Calculation was the price we used to have to pay to do mathematics.'[26] Mathematicians have figured out how to use technology to aid their thinking. They've cracked the human–machine conundrum that the rest of society is still grappling with.

## The rise – and fear – of artificial intelligence

The automation of mental effort does not end with calculation. The first whispers of artificial intelligence (AI) – the ability of computers to think and solve problems – were heard in the nineteenth century. Ada Lovelace, daughter of Lord Byron and a precocious amateur mathematician, became enthralled with the possibilities represented by Babbage's second calculator, the Analytical Engine. Lovelace saw beauty in mechanisation, writing that 'the Analytical Engine weaves algebraic patterns just as the Jacquard loom weaves flowers and leaves.' Babbage himself had realised that the functions of his Analytical Engine need not be restricted to numbers: they could also extend to more generalised operations on symbols. It was Lovelace, though, who expounded on the intelligent potential of machines, famously remarking: 'The Analytical Engine … can do whatever we know how to order it to perform.'[27]

A century later, in an essay from 1950 entitled 'Computing machinery and intelligence',[28] computing pioneer Alan Turing posed the question that launched the field of AI: 'Can machines think?' Turing's question was rhetorical; in the paper he lays out a series of counterarguments to AI, and refutes each of them in turn.

For decades, these ideas struggled to permeate the public consciousness as AI stuttered into a series of 'cold winters' following a number of false starts. That all changed at the close of the century. If the machine overlords ever do reign over this world, they might look back to an iconic scene in May 1997 as the moment of ascent. The world chess champion Garry Kasparov raises his arms in resignation as he is defeated by IBM's chess-playing computer Deep Blue in a contest billed by *Newsweek* as 'The brain's last stand'. The machine's triumph awakened humankind's deepest concerns. It was one thing for computers to automate routine tasks such as calculation, but now they appeared to be capable of applying logic to solve complex problems – a skill we had thought, perhaps hoped, was unique to humans. And why would the computers stop at chess? Companies would surely latch on to these newfound artificial capabilities to automate tasks, even entire jobs, where doing so promised labour savings. We had become accustomed to machines displacing human muscle and were even grateful for the efficiencies and prosperity that the Industrial Revolution brought about. Deep Blue's victory signalled a new, disconcerting possibility: now the machines were sure to come after the white collars too, displacing human intellect with the same nonchalant ease.

The machines have been on what may seem like a relentless march ever since Deep Blue's landmark triumph, as faster computers have combined with smart algorithms and large datasets to produce astonishing results. In 2011, IBM earned another feather in its cap, this time developing a knowledge machine, Watson, that trounced legendary quizzers Brad Rutter and Ken Jennings in a game of the general knowledge quiz *Jeopardy!* Winning at *Jeopardy!* involves dealing with all the messiness and ambiguity of natural language: a sign of rising machine intelligence. (Turing himself, in the paper mentioned earlier, posited that the ultimate display of machine intelligence would be through text-based conversation.) More recently, OpenAI's series of GPT text-generation tools have grown more powerful with each iteration; GPT-3, released in 2020, contains a staggering 175 billion parameters in its model and is able to produce a wide range of texts.[29] It even wrote an opinion piece for the *Guardian*, the first editorial ever penned by a machine, assuring readers of its peaceful intent:

> I am not asking humans to like me. But they should see me as a friendly robot. I am a servant of humans. I know that humans distrust and fear me. I only do what humans program me to do. I am only a set of code, governed by lines upon lines of code that encompass my mission statement.[30]

It may not be the stuff of Pulitzer Prize winners, but writers everywhere are on high alert as the field of AI journalism takes shape, with natural-language tools being called on to automatically personalise our newsfeeds and generate stories from datasets.[31]

Another AI milestone was achieved in 2016, when AlphaGo, a program developed at Google DeepMind, triumphed 4–1 in Go against world-class human competitor Lee Sedol. The size of a Go board, combined with the flexibility with which players are allowed to place their stones, means there are about $2 \times 10^{170}$ possible positions on the board – far too many for a computer to evaluate in sequence. Even ardent AI enthusiasts still subscribed to physicist Piet Hut's claim following Deep Blue's 1997 chess triumph: 'It may be a hundred years before a computer beats humans at Go – maybe even longer.'[32] That AlphaGo defied the sceptics was startling enough, but even more so was the nature of its triumph over Sedol. The machine played moves and strategies that amazed Go experts and mathematicians alike.[33] It was the strongest suggestion yet that the machines really meant business this time, performing mental feats that appeared elegant. AlphaGo's successor, AlphaZero, has proved even more versatile by mastering chess, Go and a host of other games all at once. Another descendant, MuZero, achieves mastery of these games without even being told the rules.[34]

The algorithms of Watson, AlphaZero, GPT and a multitude of other AI applications pack in more sophistication than the brute search techniques of Deep Blue. They fall under the category of *machine learning* models, so named because they 'learn' from data. Machine learning models do not need to have their behaviours defined for them: they program *themselves* by looking at information. Machine learning is the one area of AI that appears to work. Within this burgeoning field, you will find a repository of clever techniques such as *neural networks* (now fashionably termed *deep learning*) that are loosely modelled on the structure of the human brain and have proved highly effective in areas such as image and speech recognition. These techniques are also taking aim at problems in mathematics. In December 2019, for example, Facebook announced that it had developed a machine learning algorithm that could solve a range of calculus problems that stump many high school students,[35] while in 2021 a program developed by OpenAI solved word problems aimed at children aged 9–12, with a similar success rate to the students' own.[36]

Humans have been left head scratching, soul searching and brain scanning as we attempt to understand what awaits us while machines continue to gain thinking power. High-profile names, including Stephen Hawking and Elon Musk, have fanned the flames by warning of AI's existential threat to humanity.[37] Philosopher Nick Bostrom has projected a range of scenarios that might arise from machine *superintelligence*; most do not bode well for humans.[38]

Human fears around AI are not new. Even as Lovelace waxed lyrical about the potential of smart machines, the Victorian religious journalist Richard Thornton issued the first warning of the existential threat they posed. Thornton noted how, with the mechanical calculator, the mind 'outruns itself and does away with the necessity of its own existence by inventing machines to do its own thinking'.[39] Modern-day depictions of AI fuel our deepest insecurities; Hollywood thrives off our existential fears of replacement (or even extinction) by machines.

But much of the hype around AI is rooted in the lack of transparency around how these tools work. We fear what we do not understand, and we reserve our deepest anxieties for things that behave differently to us. It is hardly surprising, when we find ourselves grappling with long division and other relics of the school maths curriculum, that we respond with reverence to today's processing machines. We fear these tools because they are turbocharged calculators; they excel in the very skills that cause us such difficulty and dread.

Today's machine learning applications are smarter than your average computer, smarter even than Deep Blue, in the sense that they are continually learning from data inputs. AlphaGo, after all, didn't just demolish the leading human Go player; it did so with grace and style.

But for all its apparent sophistication, machine learning has some fundamental limitations which, when closely inspected, shine a light on our own human strengths.

Machine learning algorithms work by fitting patterns to data and finding associations, often imperceptible to the human mind, between variables. That renders machine learning the amplification of statistics by large datasets and powerful computers. Admittedly, *statistics* does not sound as cutting-edge. It may even be a flattering description because whereas statistics is concerned primarily with relationships between variables, such as their causes and effects, machine learning models tend to gloss over the interpretation of their results. Machines that are premised purely on patterns may have predictive value, but they lack the common sense and reasoning skills to explain their choices. They may say, with some degree of reliability, *what* will happen in the future – but not *why*.[40]

Ali Rahimi, an AI researcher at Google, received a standing ovation at an AI conference when he warned that machine learning technologies have become a form of alchemy. 'There's an anguish in the field,' says Rahimi. 'Many of us feel like we're operating on an alien technology.'[41] And François Chollet, also an AI researcher at Google, says this of much-vaunted deep learning models: 'Deep learning models do not have any understanding of their input, at least not in any human sense. Our own understanding of images, sounds, and language is grounded in our sensorimotor experience as humans – as embodied earthly creatures. Machine learning models have no access to such experiences and thus cannot "understand" their inputs in any human-relatable way.'[42]

A deep learning algorithm may be highly adept at identifying trees, but it does not *see* them in the same sense that humans do, and has no worldview within which to situate them. It will totally miss the forest. Chollet's insight punctures the 'brain as computer' metaphor that became popular in the mid-twentieth century when computing pioneer John von Neumann suggested that the design principles of digital machines bear a resemblance to the processing mechanisms of the human brain.[43]

The idea that the human brain operates like a computer is just the latest in a long line of crude comparisons. We tend to model the human brain on the dominant technologies of our time. At various points in history, it has been compared to the mechanics of hydraulics, gears, even the telegraph.[44] The computational metaphor of the brain* has persisted for over half a century[45] and is another contributing factor to the furore around AI. But metaphors are useful only up to the point where they are taken literally. If emulating human intelligence were purely a matter of computation, then, as Deep Blue and its successors have emphatically demonstrated, the game is up. On the other hand, if we unshackle ourselves from this simplistic conception of everything the brain does, and instead embrace its tremendous complexity, we will uncover aspects of thinking that are distinctly human.

The human brain is designed for dynamism and change. To a newborn baby, all life beyond a 20-cm horizon is a blur at first. But babies come equipped with learning mechanisms that help them to rapidly adapt and even change as they interact with their surroundings. It is a matter of hours before they can detect their mother's voice, days before the mother's face becomes familiar, and weeks before they sense contrasting colours. Learning is a social activity, fuelled by our bodily interactions with people and environment.

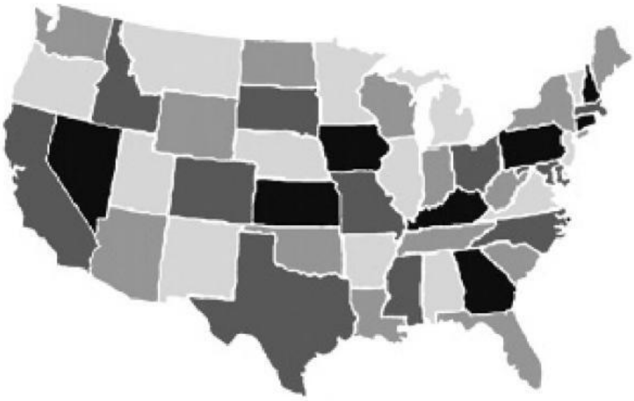
If the brain were to be described in computing terms, we might say that is a powerful hybrid of *innate circuitry* that has evolved over millions of years to give us intuitions and ways of thinking, and a vast repository of *learning algorithms* for navigating the world. With every interaction our brain's neural circuitry undergoes an incremental *upgrade*, *rewiring* itself as it revises assumptions and accumulates experience. We gradually shore up new and diverse models for seeing the world.

Operating at just 12 watts, our brain's 86 billion neurons exist as vast, intricately connected networks that communicate via electrochemical signals in order to facilitate thinking, contemplation, and improvisation. We can break rules just as easily as we make them, jumping from one mental paradigm to another. We also possess the capacity to reason and to justify our ideas with rigour. We create rich representations of the world that allow us to solve problems in a variety of contexts. We do not have to be fed millions of examples of a cat to be able to distinguish it from a dog, or millions of calculus problems to discern key underlying principles.

There's more: our psychology exposes us to vulnerabilities, but it also sets the stage for our most creative breakthroughs. We seek beauty and elegance in our

The irony of Kasparov's formula is that, while it may lack potency in chess, Go and other systems governed by strict rules, it remains a central edict for thinking in messy real-world settings that do not so readily succumb to the pattern-matching of computers.[57]

The work of professional mathematicians is often predicated on this kind of human–machine collaboration. The subject had a watershed moment in 1976 when, for the first time, a computer made a significant contribution to a mathematical proof. The *four colour theorem* says that you can colour any map with four colours in such a way that no two adjacent countries share the same colour. (It's slightly less apparent when rendered in black and white, of course.)



Since there are infinitely many possible maps, we cannot hope to check each one in turn. We require a more powerful argument – a mathematical *proof* – that deploys reason and rigour to account for all possible cases. It sounds like a challenge suited to humans, yet the problem is fiendish enough that a solution eluded mathematicians for over a century. The four colour theorem did finally yield in 1976, and when mathematicians Kenneth Appel and Wolfgang Haken presented their proof, they revealed a third, unexpected, contributor – a computer.

Appel and Haken's proof comes in two parts, both containing several hundred pages of detail. First the authors showed, using an inspired mathematical argument, that every map, however complex, can be reduced to one of 1,936 types. All that remained was to verify that each of these configurations could be coloured as required. The catch was that each configuration was enormously complicated – it would take a human forty hours a week over five years to check just a single configuration. Moreover, humans are prone to making errors, especially at that scale of calculation. Enter the machines: with brute-force processing, a computer was programmed to check every one of the finitely many cases, thus confirming, for the first time, that the four colour theorem is true.[58]

This is a powerful demonstration of what can be achieved by the tight interplay between human insight and computation: the former reduces the infinitely many cases down to a finite number; the latter tirelessly ploughs through those remaining cases. And as the computer took on increasingly complex calculations, it inspired new lines of attack for Appel and Haken. Creativity and computation were in harmony with one another.

This is the complementary force of technology in action. Huge increases in the supply of computational power have yielded immeasurable labour savings, but they have also stimulated the demand for a wider cadre of problem solvers, as each new class of algorithms gives rise to new problems. The billion-fold increase in computation did not make human jobs redundant – rather, it multiplied and amplified the contribution of human problem solvers. NASA now employs more mathematicians, engineers and software developers – humans at the intersection of research and computation – than the human computers of its 1960s heyday. The human computer may be extinct, but the mathematical human worker is thriving.

The very frontiers of mathematical research are receding, thanks to the growing capabilities of computers. In a December 2021 *Nature* article, the DeepMind team, in collaboration with 'pure' mathematicians, demonstrated how machine learning methods can be exploited to find patterns that have hitherto been hidden to the human mind.[59] These patterns are so subtle that they may even signal a kind of intuition on part of computers. Far from feeling threatened, mathematicians at the leading edge of abstract fields like algebra, geometry and topology are finding joy in taking those insights forward to develop their theories and enhancing their own feel for the subject.

As long as humans have existed, we have stored knowledge in cultural artefacts – from cave walls to books – to extend our own mental capabilities. As philosophers Andy Clark and David Chalmers put it in an influential 1998 essay, the mind is 'best regarded as an extended system, a coupling of biological organism and external resources'.[60] Computers are just the latest extension of the human brain; this is as true of the latest wave of AI supercomputers as it was of brute-force systems such as Deep Blue and even the primitive calculating tools of yesteryear.

## The seven principles of mathematical intelligence

Throughout this book I will hold up mathematical intelligence as an ambitious benchmark for both humans and computers, one that demands more than pattern-matching algorithms alone. For mathematical intelligence to be understood in this way, we must withdraw its associations with calculation, and conceive the subject in more expansive terms. For too long, and for too many people, the power of mathematics as a thinking system has been misunderstood due to society's deference to calculation. A skill that once served as a unique marker of human intelligence, and was sufficient for the workforce, has been eaten up by computers. Humans must strive for something more.

The following chapters present seven principles of mathematical intelligence that distinguish humans from computers, complement machine intelligence, equip us to tackle the messy problems of our everyday lives, and are woven into our most natural ways of thinking. Each chapter will animate an essential characteristic of mathematics by drawing on its rich heritage of concepts and problems. We'll relive some of the defining stories in the subject's history, and we'll hear from mathematicians past and present to see how the subject is viewed from within, and how it has continually evolved alongside the tools and technologies of each generation. My hope is that each chapter will, through the lens of mathematics, shine a light on the nature of human and machine intelligence so that we can proactively shape our existence alongside AI.

The first five principles concern our *ways of thinking*:

- Humans are endowed with a natural sense of number that is premised on approximation rather than precise calculation. Our in-built **estimation** skills complement the precision of computers. Interpreting the real world depends on both.
- An approximate sense of number is found throughout nature. What sets humans apart from other animals is language and abstraction. We have an extraordinary ability to create powerful **representations** of knowledge, more diverse than the binary language of computers.
- Mathematics confers on us the most robust, logical framework for establishing permanent truths. **Reasoning** shields us from the dubious claims of pure pattern-recognition systems.
- All mathematical truths are derived from a starting set of assumptions, or axioms. Unlike computers, we humans have the freedom to break free of convention and examine the logical consequences of our choices. Mathematics rewards our **imagination** with fascinating and, on occasion, applicable concepts that originate from breaking the rules.
- Computers can be tasked to solve a range of problems, but which problems are worth the effort? **Questioning** is as vital to our repertoire of thinking skills as problem-solving itself. If problems such as chess become uninteresting because they yield to computational brute force, then we can challenge ourselves to dream up problems that lie beyond the purview of routine computation.

That these principles run contrary to our usual perceptions of mathematics tells us we have to work hard, and work deliberately, to realise them. Thankfully, humans are privileged with metacognitive awareness of how our minds work; that is, we can think about how we think and learn about how we learn. We can engineer our *ways of working* to ensure we give plenty of space for those aspects of intelligence to develop. This informs two final principles, relating to how we regulate our own thinking and, finally, how we think alongside others.

- We know that our distinctive biological form of intelligence comes with the quirks of conscious and unconscious thinking. To solve our most stubborn problems, we have to display **temperament** as well as skill, paying particular attention to how we regulate the speed with which we solve problems, and the amount of information we take in.
- Humans rarely go it alone: just as machines complement humans, so too do other humans. The most fruitful **collaboration** relies on bringing together diverse perspectives, and the technologies of the digital age give us the prospect of harnessing the collective intelligence of humans like never before.

Many of the arguments that follow are underpinned by what machines can (and can't) do within today's paradigms, and what they are likely to achieve in the coming decades. Any commentary on technology has to involve some degree of speculation beyond that time horizon: we can foresee possible scenarios based on current trajectories, but we simply do not know how wide and deep machine intelligence will ultimately reach in the long run. As for mathematical intelligence, history teaches us that it, too, is ever-evolving; the seven principles outlined in this book are fit for our times (and for some time to come). But as technology continues to evolve, so will the way we understand mathematics as a thinking system – we'll be able to go further and deeper, aided by ever-smarter thinking tools such as automated theorem provers (which we'll explore in the chapter on reasoning). If AI really does penetrate our most coveted thinking skills, we'll have at least held machines to a higher intellectual standard.

## Mathematical intelligence is power

Today's AI applications are inescapable, pervading all aspects of our lives. We risk surrendering our human agency as we succumb to the conveniences of automation. Computers are pretty much faultless at executing clearly specified procedures, but some concepts are too fuzzy to put into words (or symbols) that computers can process. We humans have trouble enough giving expression to some of our most important thoughts and feelings; ambiguity and disagreement is part and parcel of our shared experience. When computers enter the fray, certain in their models of the world, written so bluntly in strings of 0s and 1s, we risk losing so much of the grey area that makes us who we are.

As we defer increasingly high-stakes decisions to these same tools, we also risk surrendering our ability (and our right) to probe the algorithmic judgements that bear on our personal and professional lives. The inscrutable manner in which machine learning algorithms operate[61] should make us critical of them when unleashing the same tools on a world that is more open, more volatile, and less predictable than closed systems such as chess and Go. Because these algorithms make predictions by 'learning' from historical data, they are layered with implicit prejudices.[62] For example, if crime rates are high for a particular ethnic group, then 'ethnicity' may be seen as a predictor of crime. Rather than addressing the sociocultural factors that give rise to those associations, algorithms jump straight to the conclusion that crime is a function of skin colour. The algorithmic models may not say such things so explicitly, but the assumptions are subtly baked into their decision-making mechanisms, as they project the future by imitating the past. As machine learning goes mainstream, some groups are paying a higher price than others.[63] Voice recognition software that is trained only on male voices will struggle to comprehend female inputs. Automated CV readers that predict candidates' potential based on previous successful hires unwittingly penalise women.[64] Image recognition software trained predominantly on white people and animals may mistake people of colour for gorillas.[65] You get the picture, even if the machines don't.

Any algorithm that relies purely on patterns in data, void of context, will never be capable of explaining its choices. The opacity of black-box machine learning systems, whose inner workings are, at best, known to a handful of technical minds, and whose causal inferences are left unchallenged, poses a grave threat to our notions of social justice. Technology is anything but neutral. It is an accelerator of progress, but it can also be an amplifier of our own human biases, which we're scarcely conscious of much of the time.

Here lies the crux of the issue: at the same time that mathematics fuels today's technologies, it also provides the means of overcoming its prejudices. It is the difference between *having mathematics done to us* and *thinking mathematically for ourselves*. Mathematical intelligence is concerned with the latter; it is a continual exercise in carefully defining and interrogating facts and employing the highest forms of reasoning to examine our arguments. A firm grounding in mathematics can liberate us from dogma and equip us with the intellectual tools to fight prejudice. It can nurture our most creative sensibilities and transform us from passive consumers of technologies to critical innovators.

The world is on edge. As I write this, we are grappling with the fallout of a global pandemic, on the cusp of irreversible changes to our climate, and in the grip of populist forces intent on undermining democracy. Technologies are being weaponised to create and disseminate falsehoods. The emergence of 'Deep Fakes', for instance, has its basis in the very same models we marvel at in other spheres, and now threatens to distort our perceptions of truth as we struggle to contain what the World Economic Forum terms 'digital wildfires' of misinformation.[66]

Mathematics itself is getting airtime as experts, pundits and politicians of all stripes invoke models to project the health and economic impacts of our actions. During the early onset of Covid-19, maths educators found encouragement in how concepts such as exponential growth were entering the lexicon of more than just the chattering classes in ways unthinkable just a couple of years ago. Yet we continue to see mathematics misappropriated, intentionally or otherwise, to justify dubious policies. Even as the public shows appetite to engage with mathematics as a means of making sense of the world, and governments assure us they are 'following the science', there is little clarity on what that entails. It is time to make mathematical intelligence explicit.

---

* Computation and calculation have slightly different meanings. The former tends to refer to algorithmic processes, the latter to arithmetic ones. I will use them interchangeably because they both espouse the same kinds of routine thinking processes.

* This method is a slight simplification of how Napier's tables were constructed, but close enough to give reasonable approximations, which is often all we need. It uses the *base 10* logarithm, which can be substituted for any other value – the natural logarithm that is now popular calls on base *e*, where *e* denotes Euler's number.

# 1

# ESTIMATION

*Tribes that only count to four, where babies outsmart computers, and why we underestimate pandemics*

The introduction of the Video Assistant Referee (VAR) promised football fans so much.[1] Technology would be the objective adjudicator of all tough on-pitch decisions, bailing out referees when they committed a 'clear and obvious error'. Gone would be the days of disputed handballs and disallowed goals. There would be no lingering sense of injustice from harsh decisions. That was the hope, anyway.

VAR has brought its own set of problems. Now when a team scores, the knee-jerk celebration of players and fans can turn to gradual despair as VAR inserts itself into the process, with an offsite team using camera stills to check for any infringement. When there is even a hint of offside, for instance, dreaded coloured lines appear on screen, marking reference points on players' bodies to check their position when contact was made with the ball. Stray toes, elbows and other protruding body parts, measured to the millimetre and excruciatingly analysed for several minutes at a time, have led to goals being overruled.

Something about this intervention just doesn't feel right. Pundits, players and fans have all expressed deep consternation at the literal interpretation of their game's rules. Debates have ensued on the meaning of 'clear and obvious' errors. There is an enduring sense that, in the pursuit of fairer decision-making, we've sacrificed a core part of the 'beautiful game' by privileging precise measurement over eyeball estimates.

Herein lies the first of our tensions with technology: while computers offer unswerving accuracy in their calculations, we are wired to see the world in fuzzier terms.

## How some tribes count

Our search for distinctly human ways of thinking begins in the Amazon rainforest, where the Pirahâ people have dwelt for tens of thousands of years. The tribe's language has been a topic of some fascination for non-natives, most notably American linguist Daniel Everett, the first outsider to unravel its mechanisms.[2] Starting in the 1970s, and continuing for three decades, Everett and his wife Keren visited the tribe intermittently and made a number of curious observations. The Pirahâ appeared to have no vocabulary for colours, no perfect tense, no concept of history beyond more than a couple of generations, and no words equivalent to quantifiers such as 'each' and 'every'. Everett was stunned: his observations pierced the widely held belief that humans possess a 'universal grammar', an idea popularised in the mid-twentieth century by Noam Chomsky. Chomsky had