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S U M S

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Mathematical Writing

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Contents

How to Use this Book

Students

This book is appropriate for self-study at undergraduate or master level. It will be of interest not only to dissertation-writing students looking for specific advice, but also to students at earlier stages of a mathematics programme, and for various purposes.

For example, a student may wish to raise the quality of their written output in coursework and exams, or may need to expand their vocabulary to reflect a growing mathematical maturity. Others may be interested in writing about a particular construct, or may seek to improve the form and style of their proofs.

One year of university mathematics should provide sufficient background for reading this book. Parts of it may be read earlier, even at the start of university if one has a good sense of conceptual accuracy. There is a large supply of exercises; solutions and hints are provided to facilitate self-assessment in absence of a teacher.

We explain succinctly the purpose of each chapter:

1. Getting into the spirit of the course.
2. Writing about sets and functions.
3. Writing about sequences and equations.
4. The logic underpinning complex sentences.
5. Advanced writing on functions.
6. Improving various aspects of a mathematical text.
7. Structuring a mathematical proof.
8. Using induction arguments.
9. Writing definitions.
10. Writing a thesis.

Chapter 1 should be read by everyone. Chapters 2 and 3 are fundamental but introductory; an advanced student should be familiar with much of their content, and will consult them only if necessary. Chapter 4 develops the basic logic needed, say, for a first analysis course. This material is a pre-requisite for Chaps. 5, 7, 9, while Chaps. 6, 10, and most of Chap. 8 may be read without it. Chapter 10 requires the mathematical maturity of a final year student. With this in mind, we suggest some possible paths through this book:

- To improve the use of basic terms: Chaps. 2–5.
- To improve conceptual accuracy: Chaps. 4, 6, 7, 9.
- To write better proofs: Chaps. 7, 8.
- To write a dissertation: Chaps. 6, 10.

Teachers

This book has been used for several years as the textbook for a second-year course in mathematical writing at the University of London, in classes of 50–70 students. The syllabus covers Chaps. 1–9, excluding the last section of Chaps. 2–4, which contain specialised material not needed elsewhere. Chapter 1 is assigned as a self-study exercise at the beginning of the course; Chap. 5 is given as a reading assignment during a pause in the teaching mid-way through the course.

The students are given weekly exercises taken from the book. In the early part of the course, the writing is limited to short phrases and sentences. Once the students have consolidated their mathematical vocabulary, more complex assignments are introduced. In a typical one, the students are given a two-page excerpt from a standard first-year textbook introducing a mainstream topic (the logarithm, Euclid's algorithm, complex numbers, etc.). They are asked to write a short document comprising a title, a few concise key points, and a short summary (150–200 words) *without using any mathematical symbol*. This form of writing is demanding but short, hence manageable in large classes. The requirement that no symbols be used has several educational virtues, and the added bonus of making plagiarism more difficult. An example of this assignment is given at the beginning of Sect. 6.6, in the form of a summary of a section of the present book. Related exercises are found at the end of Chap. 6.

The students submit their weekly work electronically, as a single pdf file. They are given complete freedom in the choice of the electronic medium used to generate their files (anything from scanning hand-written pages to LaTeX). This policy works well, and it also encourages independence and sense of responsibility. Each assignment requires only a limited use of symbols, to minimise the lure and distraction of electronic typesetting. Invariably, the best students are keen to learn LaTeX, and they do so without any supervision.

The coursework is marked by postgraduate students, who receive specialised training and are provided with detailed marking schemes. Lecturer and markers meet weekly to fine-tune the marking and resolve unusual cases. Postgraduate students tend to find this experience more instructive than marking conventional exercises.

The coursework constitutes 20–30 % of the assessment for the course, the rest being the final exam. In a small class, it may be desirable to adjust the assessment balance, increasing the weight of coursework and adding a mini-project or a short presentation at the end of the course.

Finally, this book could be used as supplementary material for various courses and programmes: a first-year introduction to mathematical structures, a course in analysis or in logic, a workshop on writing dissertations.

Chapter 1
Some Writing Tips

The following short mathematical sentences are poorly formed in one way or another. Can you identify all the errors and would you know how to fix them? Compare your answers with those given at the end of the book.

Exercise 1.1 Improve the writing.

1. a is positive.
2. Two is the only even prime.
3. If $x > 0\, g(x) \neq 0$.
4. We minus the equation.
5. $x^2 + 1$ has no real solution.
6. When you times it by negative x, $\leqslant$ becomes $\geqslant$.
7. The set of solutions are all odd.
8. $\sin(\pi x) = 0 \Rightarrow x$ is integer.
9. An invertible matrics is when the determinant is non-zero.
10. This infinete sequence has less negative terms.

In this chapter you will learn how to recognise and correct common mistakes, the first step towards writing mathematics well. By the time you reach the exercises at the end of the chapter you should already feel a sense of progress. You should return to this chapter repeatedly, to monitor the assimilation of good practice.

1.1 Grammar

You are advised to use an English dictionary, e.g.,[1] [34], and to recall the basic terminology of grammar (adjective, adverb, noun, pronoun, verb, etc.[2])—see, for instance, [37, pp. 89–95].

[1] Abbreviation for the Latin *exempli gratia*, meaning ‘for the sake of example’.

[2] Abbreviation for the Latin *et cetera*, meaning ‘and the rest’.

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- Write in complete sentences. Every sentence should begin with a capital letter, end with a full stop, and contain a subject and a verb. The expression 'A cubic polynomial' is not a sentence because it doesn't have a verb. It would be appropriate as a caption, or a title, but you can't simply insert it in the middle of a paragraph.
- Make sure that the nouns match the verbs grammatically.

  BAD: The set of primes are infinite.
  GOOD: The set of primes is infinite.

  (The verb refers to 'the set', which is singular.)
- Make a pronoun agree with its antecedent.

  BAD: Each function is greater than their minimum.
  GOOD: Each function is greater than its minimum.

  (The pronoun 'its' refers to 'function', which is singular.)
- If possible, do not split infinitives.

  BAD: We have to thoroughly examine this proof.
  GOOD: We have to examine this proof thoroughly.
  BAD: I was taught to always simplify fractions.
  GOOD: I was taught always to simplify fractions.

  (The infinitives are 'to examine' and 'to simplify'.) In some cases a split infinitive may be acceptable, even desirable.

  GOOD: This is a sure way to more than double the length of the manuscript.

- Check the spelling. No point in crafting a document carefully, if you then spoil it with spelling mistakes. If you use a word processor, take advantage of a spell checker. These are some frequent spelling mistakes:

  BAD: auxillary, catagory, consistant, correspondance, impliment, indispensible, ocurrence, preceeding, refering, seperate.

  These are misspelled mathematical words that I have found in mathematics examination papers:

  BAD: arithmatic, arithmatric, derivitive, divisable, falls (false), infinaty, matrics, orthoganal, orthoginal, othogonal, reciprical, scalor, theorom.

- Be careful about distinctions in meaning.
  Do not confuse *it's* (abbreviation for *it is*) with *its* (possessive pronoun).

  BAD: Its an equilateral triangle: it's sides all have the same length.

  Do not confuse the noun *principle* (general law, primary element) with the adjective *principal* (main, first in rank of importance).

  BAD: the principal of induction
  BAD: the principle branch of the logarithm

Do not use *less* (of smaller amount, quantity) when you should be using *fewer* (not as many as).

BAD: There are less primes between 100 and 200 than between 1 and 100.

- Do not use *where* inappropriately. As a relative adverb, *where* stands for *in which* or *to which*; it does not stand for *of which*.

BAD: We consider the logarithmic function, where the derivative is positive.
GOOD: We consider the logarithmic function, whose derivative is positive.

The adverb *when* is subject to similar misuse.

BAD: A prime number is when there are no proper divisors.
GOOD: A prime number is an integer with no proper divisors.

- Do not use *which* when you should be using *that*. Even when both words are correct, they have different meanings. The pronoun *that* is defining, it is used to identify an object uniquely, while *which* is non-defining, it adds information to an object already identified.

> The argument that was used above is based on induction.
> [*Specifies which argument.*]
> The following argument, which will be used in subsequent proofs, is based on induction.
> [*Adds a fact about the argument in question.*]

A simple rule is to use *which* only when it is preceded by a comma or by a preposition, or when it is used interrogatively.

- In presence of parentheses, the punctuation follows strict rules. The punctuation outside parentheses should be correct if the statement in parentheses is removed; the punctuation within parentheses should be correct independently of the outside.

BAD: This is bad. (Superficially, it looks good).
GOOD: This is good. (Superficially, it looks like the BAD one.)
BAD: This is bad, (on two accounts.)
GOOD: This is good (as you would expect).

1.2 Numbers and Symbols

Effectively combining numbers, symbols, and words is a main theme in this course. We begin to look at some basic conventions.

- A sentence containing numbers and symbols must still be a correct English sentence, including punctuation.

BAD: $a < b \;\; a \neq 0$
GOOD: We have $a < b$ and $a \neq 0$.
GOOD: We find that $a < b$ and $a \neq 0$.

GOOD: Let $a < b$, with $a \neq 0$.
BAD: $x^2 - 7^2 = 0$. $x = \pm 7$.
GOOD: Let $x^2 - 7^2 = 0$; then $x = \pm 7$.
GOOD: The equation $x^2 - 7^2 = 0$ has two solutions: $x = \pm 7$.

- Omit unnecessary symbols.

BAD: Every differentiable real function f is continuous.
GOOD: Every differentiable real function is continuous.

- If you use small numbers for counting, write them out in full; if you refer to specific numbers, use numerals.

BAD: The equation has 4 solutions.
GOOD: The equation has four solutions.
GOOD: The equation has 127 solutions.
BAD: Both three and five are prime numbers.
GOOD: Both 3 and 5 are prime numbers.

- If at all possible, do not begin a sentence with a numeral or a symbol.

BAD: ρ is a rational number with odd denominator.
GOOD: The rational number ρ has odd denominator.

- Do not combine operators ($+$, $\neq$, $<$, etc.) with words.

BAD: The difference $b - a$ is < 0
GOOD: The difference $b - a$ is negative.

- Do not misuse the implication operator $\Rightarrow$ or the symbol $\therefore$. The former is employed only in symbolic sentences (Sect. 4.2); the latter is not used in higher mathematics.

BAD: a is an integer $\Rightarrow$ a is a rational number.
GOOD: If a is an integer, then a is a rational number.
BAD: $\Rightarrow \quad x = 3$.
BAD: $\therefore \quad x = 3$.
GOOD: hence $x = 3$.
GOOD: and therefore $x = 3$.

- Within a sentence, adjacent formulae or symbols must be separated by words.

BAD: Consider A_n, $n < 5$.
GOOD: Consider A_n, where $n < 5$.
BAD: Add p k times to c.
BAD: Add p to c k times.
GOOD: Add p to c, repeating this process k times.

For displayed equations the rules are a bit different, because the spacing between symbols becomes a syntactic element. Thus an expression of the type

$$A_n = B_n, \quad n < 5$$

is quite acceptable (see Sect. 6.3).

1.3 Style

A good sentence needs a lot more than grammatical correctness.

- Give priority to clarity over fancy language. Avoid long and involved sentences; break long sentences into shorter ones.

  BAD: We note the fact that the polynomial $2x^2 - x - 1$ has the coefficient of the x^2 term positive.
  GOOD: The leading coefficient of the polynomial $2x^2 - x - 1$ is positive.
  BAD: The inverse of the matrix A requires the determinant of A to be non-zero in order to exist, but the matrix A has zero determinant, and so its inverse does not exist.
  GOOD: The matrix A has zero determinant, hence it has no inverse.

- Place important words in a prominent position within the sentence. Suppose you are introducing the logarithm:

  BAD: An important example of a transcendental function is the logarithm.

  In this classic bad opening '*an example of something is something else*', the focus of attention is the transcendental functions, not the logarithm.

  GOOD: The logarithm is an important example of a transcendental function.
  GOOD: Let us now define a key transcendental function: the logarithm.

  Suppose you wish to emphasise the scalar product:

  BAD: A commonly used method to check the orthogonality of two vectors is to verify that their scalar product is zero.
  GOOD: If the scalar product of two vectors is zero, then the vectors are orthogonal.

- Prefer the active to the passive voice.

  BAD: The convergence of the above series will now be established.
  GOOD: We establish the convergence of the above series.

- Vary the choice of words to avoid repetition and monotony.[3]

[3] If necessary, consult a thesaurus.

BAD: The function defined above is a function of both x and y.
GOOD: The function defined above depends on both x and y.

- Do not use unfamiliar words unless you know their exact meaning.

BAD: A simplistic argument shows that our polynomial is irreducible.
GOOD: A simple argument shows that our polynomial is irreducible.

- Do not use vague, general statements to lend credibility to your writing. Avoid emphatic statements.

BAD: Differential equations are extremely important in modern mathematics.
BAD: The proof is very easy, as it makes an elementary use of the triangle inequality.
GOOD: The proof uses the triangle inequality.

- Do not use jargon, or informal abbreviations: it looks immature rather than 'cool'.

BAD: Spse U subs x into T eq. Wot R T soltns?

- Enclose side remarks within commas, which is very effective, or parentheses (getting them out of the way). To isolate a phrase, use hyphenation—it really sticks out—or, if you have a word processor, *change font* (**but** `don't` overdo *it*).
- Take punctuation seriously. To improve it, read [40] or [41].

1.4 Preparation and Structure

There are things one must keep in mind when preparing any document.

- Begin by writing your document in draft form, or at least write down a list of key points. Few people are able to produce good writing at the first attempt.
- Consider the background of your readers; are they familiar with the meaning of the words you use? It's easy to write a mathematics text that's too difficult; it's almost impossible to write one that's too easy.
- Form each sentence in your head before writing it down. Then read carefully what you have written. Read it aloud: how does it sound? Have you written what you intended to write? Is it clear? Don't hesitate to rewrite.
- Split the text into paragraphs. Each paragraph should be about one 'idea', and it should be clear how you are moving from one idea to the next. Be prepared to re-arrange paragraphs. The first idea you thought of may not have been the best one; the sequence of arguments you have chosen may not be optimal.
- When you finish writing, consider the opening and closing sentences of your document. The former should motivate the readers to keep reading, the latter should mark a resting place, like the final bars in a piece of music.

- Word processing has changed the way we write, and often a document is the end-product of several successive approximations. After prolonged editing, one stops seeing things. If you have time, leave your document to rest for a day or two, and then read it again.

Exercise 1.2 Improve the writing, following the guidelines given in this chapter.

1. There are 3 special cases.
2. X is a finite set.
3. It does not tend to infinaty.
4. It follows $x - 1 = y^4$.
5. $\therefore c^{-1}$ is undefined.
6. The product of 2 negatives is positive.
7. We square the equation.
8. We have less solutions than we had before.
9. $x^2 = y^2$ are two othogonal lines.
10. Let us device a strategy for a proof.
11. This set of matrixs are all invertible.
12. If the integral $= 0$ the function is undefined.
13. Purely imaginary is when the real part is zero.
14. Construct the set of vertex of triangles.
15. From the fact that $x = 0$, I can't divide by x.
16. A circle is when major and minor axis are the same.
17. The function f is not discontinuous.
18. Plug-in that expression in the other equation.
19. I found less solutions than I expected.
20. When the discriminant is < 0, you get complex.
21. We prove Euler theorom.
22. The definate integral is where you don't have integration limits.
23. The asyntotes of this hiperbola are othogonal.
24. A quadratic function has 1 stationery point.
25. The solution is not independent of s.
26. a is negative $\therefore \sqrt{a}$ is complex.
27. Thus $x = a$. (We assume that a is positive).
28. Each value is greater than their reciprical.
29. Remember to always check the sign.
30. Differentiate f n times.

Chapter 2
Essential Dictionary I

In writing mathematics we use words and symbols to describe facts. We need to explain the meanings of words and symbols, and to state and prove the facts.

We'll be concerned with facts later. In this chapter and the next we list mathematical words with accompanying notation. This is our essential mathematical dictionary. It contains some 200 entries, organised around few fundamental terms: **set**, **function**, **sequence**, **equation**. As we introduce new words, we use them in short phrases and sentences.

Dictionaries are not meant to be read through, so don't be surprised if you find the exposition demanding. Take it in small doses. The last section of this chapter deals with advanced terminology and may be skipped on first reading.

2.1 Sets

A **set** is a collection of *well-defined*, *unordered*, *distinct* objects. (This is the so-called 'naive definition' of a set, due to Cantor.[1]) These objects are called the **elements** of a set, and a set is determined by its elements. We may write

The set of all odd integers
The set of vertices of a pentagon
The set of differentiable real functions

In simple cases, a set can be defined by listing its elements, separated by commas, enclosed within curly brackets. The expression

$$\{1, 2, 3\}$$

[1] Georg Cantor (German: 1845–1918).

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denotes the set whose elements are the integers 1, 2 and 3. Two sets are equal if they have the same elements:

$$\{1, 2, 3\} = \{3, 2, 1\}.$$

(By definition, the order in which the elements of a set are listed is irrelevant.)

It is customary to ignore repeated set elements: $\{2, 1, 3, 1, 3\} = \{2, 1, 3\}$. This convention, adopted by computer algebra systems, simplifies the definition of sets. If repeated elements are allowed and not collapsed, then we speak of a **multiset**: $\{2, 1, 3, 1, 3\}$. The **multiplicity** of an element of a multiset is the number of times the element occurs. Reference to multiplicity usually signals that there is a multiset in the background:

Every quadratic equation has two complex solutions, counting multiplicities.

Multisets are not as common as sets.

The set {} with no elements is called the **empty set**, denoted by the symbol ∅. The empty set is distinct from 'nothing', it is more like an empty container. For example, the statements

This equation has no solutions.
The solution set of this equation is empty.

have the same meaning.

To assign a symbol to a mathematical object, we use an **assignment statement** (or **definition**), which has the following syntax:

$$A := \{1, 2, 3\}. \tag{2.1}$$

This expression assigns the symbolic name A to the set $\{1, 2, 3\}$, and now we may use the former in place of the latter. The symbol ':=' denotes the **assignment operator**. It reads '*becomes*', or '*is defined to be*', rather than '*is equal to*', to underline the difference between assignment and equality (in computer algebra, the symbols = and **:=** are not interchangeable at all!). So we can't write $\{1, 2, 3\} := A$, because the left operand of an assignment operator must be a symbol or a symbolic expression.

The right-hand side of an assignment statement such as (2.1) is a collection of symbols or words that pick out a unique thing, which logicians call the *definiens* (Latin for 'thing that defines'). The left-hand side is a symbol that will be used to stand for this unique thing, which is called the *definiendum* (Latin for 'thing to be defined'). These terms are rather heavy, but they are widely used [36, Chap. 8]. The definiendum may also be a symbolic expression—see below.

While it's very common to use the equal sign '=' also for an assignment, the specialised notation := improves clarity. There are other symbols for the assignment operator, namely

$$\stackrel{\text{def}}{=} \qquad \stackrel{\nabla}{=}, \tag{2.2}$$

which make an even stronger point.

To indicate that x is an element of a set A, we write

$$x \in A \qquad x \text{ is an element of } A \qquad x \text{ belongs to } A.$$

The symbol $\notin$ is used to negate membership. Thus

$$\{7, 5\} \in \{5, \{5, 7\}\} \qquad 7 \notin \{5, \{5, 7\}\}.$$

(Think about it.)

A **subset** B of a set A is a set whose elements all belong to A. We write

$$B \subset A \qquad B \text{ is a subset of } A \qquad B \text{ is contained in } A$$

and we use $\not\subset$ to negate set inclusion. For example

$$\{3, 1\} \subset \{1, 2, 3\} \qquad \emptyset \subset \{1\} \qquad \{2, 3\} \not\subset \{2, \{2, 3\}\}.$$

Every set has at least two subsets: itself and the empty set. Sometimes these are referred to as the **trivial** subsets. Every other subset—if any—is called a **proper subset**. Motivated by an analogy with $\leqslant$ and $<$, some authors write $\subseteq$ in place of $\subset$, reserving the latter for proper inclusion: $\mathbb{R} \subseteq \mathbb{R}$, $\mathbb{Q} \subset \mathbb{R}$. Proper inclusion is occasionally expressed with the pedantic notation $\underset{\neq}{\subset}$.

The **cardinality** of a set is the number of its elements, denoted by the prefix #:

$$\#\{7, -1, 0\} = 3 \qquad \#A = n.$$

The absolute value symbol $|\cdot|$ is also used to denote cardinality: $|A| = n$. Common sense will tell when this choice is sensible. A set is **finite** if its cardinality is an integer, and **infinite** otherwise. To indicate that the set A is finite, without disclosing its cardinality, we write

$$\#A < \infty. \tag{2.3}$$

A more rigorous account of cardinality will be given in Sect. 2.3.3.

Next we consider the words associated with operations between sets. We write $A \cap B$ for the **intersection** of the sets A and B: this is the set comprising elements that belong to both A and B. If $A \cap B = \emptyset$, we say that A and B are **disjoint**, or have **empty intersection**. The sets $A_1, A_2, \ldots$ are **pairwise disjoint** if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

We write $A \cup B$ for the **union** of A and B, which is the set comprising elements that belong to A or to B (or to both A and B).

We write $A \smallsetminus B$ for the (**set**) **difference** of A and B, which is the collection of the elements of A that do not belong to B. The **symmetric difference** of A and B, denoted by $A \mathbin{\Delta} B$, is defined as

$$A \mathbin{\Delta} B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

The assignment operator '$\stackrel{\text{def}}{=}$' [cf. (2.2)] makes it clear that this is a definition. This notation establishes the meaning of $A \mathbin{\Delta} B$, which is a symbolic expression rather than an individual symbol. The following examples illustrate the action of set operators:

$$\begin{aligned}
\{1, 2, 3\} \cap \{3, 4, 5\} &= \{3\} \\
\{1, 2, 3\} \cup \{3, 4, 5\} &= \{1, 2, 3, 4, 5\} \\
\{1, 2, 3\} \setminus \{3, 4, 5\} &= \{1, 2\} \\
\{1, 2, 3\} \mathbin{\Delta} \{3, 4, 5\} &= \{1, 2, 4, 5\}.
\end{aligned}$$

The above **set operators** are **binary**; they have two sets as **operands**. The identities

$$A \cap B = B \cap A \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

express the **commutative** and **associative** properties of the intersection operator. Union and symmetric difference enjoy the same properties, but set difference doesn't.

Let A be a subset of a set X. The **complement** of A (in X) is the set $X \setminus A$, denoted by A' or by A^c. The complement of a set is defined with respect to an **ambient set** X. Reference to the ambient set may be omitted if there is no ambiguity. So we write

The odd integers is the complement of the even integers

since it's clear that the ambient set is the integers.

With set operators we can construct new sets from old ones, although, in a sense, we are recycling things we already have. To create genuinely new sets, we introduce the notion of **ordered pair**. This is an expression of the type (a, b), with a and b arbitrary quantities. Ordered pairs are defined by the property

$$(a, b) = (c, d) \qquad \text{if} \qquad a = c \quad \text{and} \quad b = d. \tag{2.4}$$

The ordered pair (a, b) should not be confused with the set $\{a, b\}$, since for pairs order is essential and repetition is allowed. (Ordered pairs may be defined solely in terms of sets—see Exercise 2.14.) Let A and B be sets. We consider the set of all ordered pairs (a, b), with a in A and b in B. This set is called the **cartesian product** of A and B, and is written as

$$A \times B.$$

Note that A and B need not be distinct; one may write A^2 for $A \times A$, A^3 for $A \times A \times A$, etc. Because the cartesian product is **associative**, the product of more than two sets is defined unambiguously. Also note that the explicit presence of the multiplication operator '$\times$' is needed here, because the expression AB has a different meaning [see Eq. (2.21), Sect. 2.3].

2.1.1 Defining Sets

Defining a set by listing its elements is inadequate for all but the simplest situations. How do we define large or infinite sets? A simple device is to use the **ellipsis** '...', which indicates the deliberate omission of certain elements, the identity of which is made clear by the context. For example, the set $\mathbb{N}$ of **natural numbers** is defined as

$$\mathbb{N} := \{1, 2, 3, \ldots\}.$$

Here the ellipsis represents all the integers greater than 3. Some authors regard 0 as a natural number, so the definition

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\}$$

is also found in the literature. Both definitions have merits and drawbacks; mathematicians occasionally argue about it, but this issue will never be resolved. So, when using the symbol $\mathbb{N}$, one may need to clarify which version of this set is employed.[2] The set of **integers**, denoted by $\mathbb{Z}$ (from the German *Zahlen*, meaning numbers), can also be defined using ellipses:

$$\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\} \qquad \text{or} \qquad \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}.$$

To define general sets we need more powerful constructs. A **standard definition** of a set is an expression of the type

$$\{x : x \text{ has } \mathscr{P}\} \tag{2.5}$$

where $\mathscr{P}$ is some unambiguous property that things either have or don't have. This expression identifies the set of all objects x that have property $\mathscr{P}$. The colon ':' separates out the object's symbolic name from its defining properties. The vertical bar '|' or the semicolon ';' may be used for the same purpose.

Thus the empty set may be defined symbolically as

$$\emptyset \stackrel{\text{def}}{=} \{x : x \neq x\}. \tag{2.6}$$

The property $\mathscr{P}$ is 'x is not equal to x', which is not satisfied by any x. Likewise, the cartesian product $A \times B$ of two sets (see Sect. 2.1) may be specified as

$$\{x : x = (a, b) \text{ for some } a \in A \text{ and } b \in B\}.$$

The rule 'x has property $\mathscr{P}$' now reads: 'x is of the form (a, b) with $a \in A$ and $b \in B$'. The same set may be defined more concisely as

[2] Some authors denote the second version by the symbol $\mathbb{N}_0$.

$$\{(a,b)\ :\ a\in A\ \text{and}\ b\in B\}.$$

This is a variant of the standard definition (2.5), where the type of object being considered (ordered pair) is specified at the outset. This form of standard definition can be very effective.

The set $\mathbb{Q}$ of **rational numbers**—ratios of integers with non-zero denominator—is defined as follows:

$$\mathbb{Q} := \Big\{\frac{a}{b}\ :\quad a\in\mathbb{Z},\quad b\in\mathbb{N},\quad \gcd(a,b)=1\Big\}. \tag{2.7}$$

The property $\mathscr{P}$ is phrased in such a way as to avoid repetition of elements. This is the so-called **reduced form** of rational numbers. The rational numbers may also be defined abstractly, as infinite sets of equivalent fractions—see Sect. 4.6.

One might think that in the expression for a set we could choose any property $\mathscr{P}$. Unfortunately this doesn't work for a reason known as the *Russell-Zermelo paradox*[3] (1901). Consider the set definition

$$W := \{x\ :\ x\notin x\} \tag{2.8}$$

in which $\mathscr{P}$ is the property of being a set that is not a member of itself. The quantity

$$x=\{3,\{3,\{3,\{3\}\}\}\}$$

has property $\mathscr{P}$ and hence belongs to W, whereas

$$x=\{3,\{3,\{3,\{3,\ldots\}\}\}\}\qquad\text{or}\qquad x=\{3,x\}$$

does not have property $\mathscr{P}$ and hence does not belong to W. (In the above expression, the nested parentheses must match, so the notation $\{3,\{3,\{3,\{3,\ldots\}$ is incorrect.)

Given that W is a set of sets, we ask: does W belong to W? We see that if $W\in W$, then W has property $\mathscr{P}$, that is, $W\notin W$, and vice-versa. Impossible! Thus the standard definition (2.8), so deceptively similar to (2.6), does not actually define any set.

Fortunately, we can define a set in such a way that the definition guarantees the existence of the set. A **Zermelo definition** identifies a set W by describing it as

The set of members of X that have property $\mathscr{P}$

where the **ambient set** X is given beforehand, and $\mathscr{P}$ is a property that the members of X either have or do not have. In symbols, this is written as

[3] Bertrand Russell (British: 1872–1970); Ernst Zermelo (German: 1871–1953).

$$W := \{x \in X : x \text{ has } \mathscr{P}\}. \tag{2.9}$$

For example, the expression

The set of real numbers strictly between 0 and 1

is a Zermelo definition: the ambient set is the set of real numbers, and we form our set by choosing from it the elements that have the stated property.

Zermelo definitions work because it's a basic principle of mathematics (the so-called *subset axiom*) that for any set X of objects and any property $\mathscr{P}$, there is exactly one set consisting of the objects that are in X and have property $\mathscr{P}$. In Sect. 4.3 we shall see that the definiens of a Zermelo definition—a sentence with a variable x in it—is just a special type of function, called a **predicate**.

Both styles of definitions, standard and Zermelo, are widely used in mathematical writing.

2.1.2 Arithmetic

The notation for arithmetical operations is familiar and established. The **sum** and **difference** of two numbers x and y are always written $x + y$ and $x - y$, respectively. By contrast, their **product** may be written in several equivalent ways:

$$xy \qquad x \cdot y \qquad x \times y, \tag{2.10}$$

and so may their **quotient**:

$$\frac{x}{y} \qquad x/y \qquad x : y.$$

(The notation $x : y$ is used mostly in elementary texts.) Do not confuse the product dot '$\cdot$' with the **decimal point** '.', e.g., $3 \cdot 4 = 12$ and $3.4 = 17/5$.

The **reciprocal** of x, defined for $x \neq 0$, is also written in several ways:

$$\frac{1}{x} \qquad 1/x \qquad x^{-1}$$

while the **opposite** of x is $-x$.

The notation for exponentiation is x^y, where x is the **base** and y the **exponent**. Defining exponentiation for a general exponent is a delicate matter, as it requires the logarithmic and exponential functions. The case of a positive integer exponent is easier, because exponentiation reduces to repeated multiplication:

$$x^n \stackrel{\text{def}}{=} \underbrace{x \cdots x}_{n} \qquad n \geqslant 1.$$

The assignment operator $\stackrel{\text{def}}{=}$ [see (2.2)] indicates that this is a definition, giving meaning to the expression on the left. The use of the under-brace is necessary to specify the number of terms in the product, because all terms are identical. Also note the use of the **raised ellipsis** '$\cdots$' to represent repeated multiplication (or repeated applications of any operator), to be compared with the ordinary ellipsis '$\ldots$', used for sets and sequences (see Sect. 3.1). Thus

$$\underbrace{x \cdots x}_{4} = x \cdot x \cdot x \cdot x \qquad \underbrace{x, \ldots, x}_{4} = x, x, x, x$$

whereas the notation $x \ldots x$ is incorrect.

In integer arithmetic, the symbol '|' is used for **divisibility**.

$$3|x \qquad 3 \textit{ divides } x \qquad x \textit{ is a multiple of } 3.$$

EXAMPLE. Turn symbols into words:

$$\{x \in \mathbb{Z} \,:\, x \geqslant 0,\ 2 \mid x\}.$$

BAD: The set of integers that are greater than or equal to zero, and such that 2 divides them. (*Robotic.*)

GOOD: The set of non-negative even integers.

A positive divisor of n, which is not 1 or n, is called a **proper divisor**, and a **prime** is an integer greater than 1 that has no proper divisors. The acronyms **gcd** and **lcm** are used for **greatest common divisor** and **least common multiple**. (The expression **highest common factor** (hcf)—a variant of gcd which is popular in schools—is seldom used in higher mathematics.) Two integers are **co-prime** (or **relatively prime**) if their greatest common divisor is 1. Some authors use (a, b) for $\gcd(a, b)$; this is to be avoided, since this notation is already overloaded.

The following concise notation represents certain infinite sets of integers (here k and m are any integers):

$$\begin{aligned} k + m\mathbb{Z} &\stackrel{\nabla}{=} \{x \in \mathbb{Z} \,:\, m|(x - k)\} \\ &= \{\ldots, k - 2m, k - m, k, k + m, k + 2m, \ldots\}. \end{aligned} \tag{2.11}$$

This definition gives meaning to the symbolic expression $k + m\mathbb{Z}$ on the left of the assignment operator, which otherwise would be meaningless (you can't form the sum or product of an integer and a set!). The two expressions on the right represent the same object. While any of the two would suffice, their combination adds clarity (we shall expand this idea in Sect. 6.4).

This notation is economical and effective:

$$\begin{array}{ll} x \in 1 + 2\mathbb{Z} & \textit{x is odd} \\ x \in m\mathbb{Z} & \textit{x is a multiple of } m \\ x \in n^2\mathbb{Z} \setminus 2\mathbb{Z} & \textit{x is an odd multiple of } n^2. \end{array}$$

This is a special case of a more general notation for sums and products of sets, to be developed in Sect. 2.3.

2.1.3 Sets of Numbers

The ‘open face’ symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ were introduced in Sect. 2.1.1 to represent the natural numbers, the integers, and the rationals, respectively. Likewise, we denote by $\mathbb{R}$ the set of **real** numbers (its symbolic definition is left as Exercise 2.13), while the set of **complex** numbers is denoted by $\mathbb{C}$. The set $\mathbb{C}$ may be written as

$$\mathbb{C} \stackrel{\text{def}}{=} \{x + \mathrm{i}y \,:\, \mathrm{i}^2 = -1, \quad x, y \in \mathbb{R}\}.$$

The symbol i is called the **imaginary unit**, while x and y are, respectively, the **real part** $\mathrm{Re}(z)$ and the **imaginary part** $\mathrm{Im}(z)$ of the complex number $z = x + \mathrm{i}y$. The sets $\mathbb{R}$ and $\mathbb{C}$ are represented geometrically as the **real line** and the **complex plane** (or **Argand plane**), respectively. A plot of complex numbers in the Argand plane is called an **Argand diagram**. We have the chain of proper inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

We now construct new sets from the sets of numbers introduced above. An **interval** is a subset of $\mathbb{R}$ of the type

$$[a, b] := \{x \in \mathbb{R} \,:\, a \leqslant x \leqslant b\}$$

where a, b are real numbers, with $a < b$. This interval is **closed**, that is, it contains its end points. (A point is sometimes regarded as a degenerate closed interval, by allowing $a = b$ in the definition.) We also have **open** intervals

$$(a, b) := \{x \in \mathbb{R} \,:\, a < x < b\}$$

as well as **half-open** intervals

$$[a, b) \qquad (a, b].$$

Let $X = \{3\}$.
The set whose only element is the integer 3.
Let $X = \{m\}$, *with* $m \in \mathbb{Z}$.
A set whose only element is an integer.
Let $m \in \mathbb{Z}$, *and let* X *be a set such that* $m \in X$.
A set which contains a given integer.
Let X *be a set such that* $X \cap \mathbb{Z} \neq \emptyset$.
A set which contains at least one integer.
Let X *be a set such that* $\#(X \cap \mathbb{Z}) = 1$.
A set which contains precisely one integer.

In the first two examples the combination of 'let' and '=' replaces an assignment operator. An expression such as '*Let* $X \stackrel{\triangledown}{=} \{3\}$' would be overloaded.

The distinction between definite and indefinite articles is essential, the former describing a unique object, the latter an unspecified representative of a class of objects. In the following phrases, a change in one article, highlighted in boldface, has resulted in a logical mistake.

BAD: *A proper infinite subset of* **a** *unit circle.*
BAD: **A** *set whose only element is the integer 3.*
BAD: **The** *set whose only element is an integer.*
BAD: **The** *set which contains precisely one integer.*

As a final exercise, we express some geometric facts using set terminology.

The intersection of a line and a conic section has at most two points.
The set of rational points in any open interval is infinite.
A cylinder is the cartesian product of a segment and a circle.
The complement of the unit circle consists of two disjoint components.

The reader should re-visit familiar mathematics and describe it in the language of sets.

Exercise 2.1 For each of the following topics:

prime numbers, fractions, complex numbers,

(i) write five short sentences; (ii) ask five questions. The sentences should give a definition or state a fact; the questions should have mathematical significance, and preferably possess a certain degree of generality. **[$\notin$]**[4]

Exercise 2.2 Define five interesting finite sets. **[$\notin$]**

[4] The symbol **[$\notin$]** indicates that the exercise must be completed without using any mathematical symbol.

Exercise 2.3 The following expressions define sets. Turn words into symbols, using standard or Zermelo definitions. (Represent geometrical objects, e.g., planar curves, by their cartesian equations.)

1. *The set of negative odd integers.*
2. *The set of natural numbers with three decimal digits.*
3. *The set of rational numbers which are the ratio of consecutive integers.*
4. *The set of rational points in the closed unit cube.*
5. *The complement of the open unit disc in the complex plane.*
6. *The set of vectors of unit length in three-dimensional euclidean space.*
7. *The set of circles in the plane, passing through the origin.*
8. *The set of hyperbolae in the plane, whose asymptotes are the coordinate axes.*
9. *The set of lines tangent to the unit circle.*

Exercise 2.4 The following expressions define sets. Turn symbols into words. [$\not\in$]

1. $\{x \in \mathbb{Q} \,:\, 0 < x < 1\}$
2. $\{1/(2n+1) \,:\, n \in \mathbb{Z}\}$
3. $\{m2^{-k} \,:\, m \in 1 + 2\mathbb{Z},\ \ k \in \mathbb{N}\}$
4. $\{x \in \mathbb{R} \smallsetminus \mathbb{Z} \,:\, x^2 \in \mathbb{Z}\}$
5. $\{z \in \mathbb{C} \smallsetminus \mathbb{R} \,:\, z^2 \in \mathbb{R}\}$
6. $\{z \in \mathbb{C} \,:\, |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leqslant 1\}$
7. $\{(m, n) \in \mathbb{Z}^2 \,:\, m \mid n\}$
8. $\{(x, y, z) \in \mathbb{R}^3 \,:\, x\,y\,z = 0\}$
9. $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, \sum_k x_k = 0\}$
10. $\{x \in \mathbb{R} \,:\, \sin(2\,\pi\,x) = 0\}$
11. $\{(x, y) \in \mathbb{R}^2 \,:\, \sin(\pi\,x)\sin(\pi\,y) = 0\}$.

2.2 Functions

Functions are everywhere. Whenever a process transforms a mathematical object into another object, there is a function in the background. ‘Function’ is arguably the most used word in mathematics.

A **function** consists of two sets together with a rule[5] that assigns to *each* element of the first set a *unique* element of the second set. The first set is called the **domain** of the function and the second set is called the **co-domain**. A function whose domain is a set A may be called a function **over** A or a function **defined on** A. The terms **map** or **mapping** are synonymous with function. The term **operator** is used to describe certain types of functions (see below).

A function is usually denoted by a single letter or symbol, such as f. If x is an element of the domain of a function f, then the **value of** f **at** x, denoted by $f(x)$

[5] Below, we’ll replace the term ‘rule’ with something more rigorous.

is the unique element of the co-domain that the rule defining f assigns to x. The notation

$$f : A \to B \qquad x \mapsto f(x) \tag{2.16}$$

indicates that f is a function with domain A and co-domain B that **maps** $x \in A$ **to** $f(x) \in B$. The symbol x is the **variable** or (the **argument**) of the function. The symbols $\to$ and $\mapsto$ have different meanings, and should not be confused. The function

$$I_A : A \to A \qquad x \mapsto x$$

is called the **identity** (**function**) on A. When explicit reference to the set A is unnecessary, the identity is also denoted by Id or $\mathbb{1}$.

In Definition (2.16) the symbols used for the function's name and variable are inessential; the two expressions

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad x \mapsto \frac{1}{x} \qquad x : \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad f \mapsto \frac{1}{f}$$

define exactly the same function (even though the rightmost expression breaks just about every rule concerning mathematical notation—see Sect. 6.2).

Let us use the word 'function' in short expressions. These are function definitions:

1. *The integer function that squares its argument.*
2. *The function that returns 1 if its argument is rational, and 0 otherwise.*
3. *The function that counts the number of primes smaller than a given real number.*
4. *The function that gives the distance between two points on the unit circle, measured along the circumference.*

We surmise that the function in item 2 is defined over the real numbers. Item 3 is a much-studied function in number theory. The set of values assumed by the function in item 4 is the closed interval $[0, \pi]$.

Functions of several variables are defined over cartesian products of sets. For example, the function

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \qquad (x, y) \mapsto \gcd(x, y)$$

depends on two integer arguments, and hence is defined over the cartesian product of two copies of the integers. This definition requires a value for $\gcd(0, 0)$, which normally is taken to be zero.

Let $f : A \to B$ be a function. The set

$$\{(x, f(x)) \in A \times B \, : \, x \in A\} \tag{2.17}$$

is called the **graph** of f. A function is completely specified by three sets: domain, co-domain and graph. We can now reformulate the definition of a function, replacing the imprecise term 'rule' with the precise term 'graph'. We write a formal definition.

DEFINITION. *A* **function** f *is a triple* (X, Y, G) *of non-empty sets. The sets* X *and* Y *are arbitrary, while* G *is a subset of* $X \times Y$ *with the property that for every* $x \in X$ *there is a unique pair* $(x, y) \in G$*. The quantity* y *is called the* **value of the function at** x*, denoted by* $f(x)$*.*

We see that, besides sets, the definition of a function requires the constructs of ordered pair and triple. It turns out that these quantities can be defined solely in terms of sets (see Exercise 2.14). So, to define functions, all we need are sets after all.

Given a function $f : A \to B$, and a subset $X \subset A$, the set

$$f(X) \stackrel{\text{def}}{=} \{f(x) : x \in X\} \tag{2.18}$$

is called the **image of** X **under** f. The assignment operator gives meaning to the symbolic expression $f(X)$, which otherwise would be meaningless, since we stipulated that the argument of a function is an element of the domain, not a subset of it. Thus $\sin(\mathbb{R})$ is the closed interval $[-1, 1]$.

Clearly, $f(X) \subset B$, and $f(A)$ is the smallest set that can serve as co-domain for f. The set $f(A)$ is often called the **image** or the **range** of the function f. This term is sometimes used to mean co-domain, which should be avoided. A **constant** is a function whose image consists of a single point.

The notation (2.18) is suggestive and widely used. However, in computer algebra, the quantities $f(x)$ and $f(X)$ (with x an element and X a subset of the domain, respectively) are written with a different syntax, e.g., `f(x)` and `map(f,X)` with Maple.

A function is said to be **injective** (or **one-to-one**) if distinct points of the domain map to distinct points of the co-domain. A function is **surjective** (or **onto**) if $f(A) = B$, that is, if the image coincides with the co-domain. A function that is both injective and surjective is said to be **bijective**.

For any non-empty subset X of the domain A, we define the **restriction of** f **to** X as

$$f|_X : X \to B \qquad\qquad x \mapsto f(x).$$

Given two functions $f : A \to B$ and $g : B \to C$, their **composition** is the function

$$g \circ f : A \to C \qquad x \mapsto g(f(x)). \tag{2.19}$$

The notation $g \circ f$ reminds us that f acts before g. The image $g(f(x))$ of x under $g \circ f$ is denoted by $(g \circ f)(x)$, where the parentheses isolate $g \circ f$ as the function's symbolic name. The hybrid notation $g \circ f(x)$ should be avoided.

If $f : A \to B$ is a bijective function, then the **inverse** of f is the function $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = \mathbb{1}_A \qquad f \circ f^{-1} = \mathbb{1}_B$$

where $\mathbb{1}_{A,B}$ are the identities in the respective sets. A function is said to be **invertible** if its inverse exists. If $f : A \to B$ is injective, then we can always define the inverse of f by restricting its domain to $f(A)$ if necessary. In absence of injectivity, it may still be possible to construct an inverse by a suitable restriction of the function. Thus the arcsine may be defined by restricting the sine to the interval $[-\pi/2, \pi/2]$.

Let $f : A \to B$ be a function, and let C be a subset of B. The set of points

$$f^{-1}(C) \stackrel{\text{def}}{=} \{x \in A : f(x) \in C\} \tag{2.20}$$

is called the **inverse image** of the set C.

Since the definition of inverse image does not involve the inverse function, the inverse image exists even if the inverse function does not. These two concepts must be distinguished carefully. When the reciprocal of a function comes into play, things get very confusing, since we now have three unrelated objects represented by closely related notation:

$$f^{-1}(x) \qquad f^{-1}(\{x\}) \qquad f(x)^{-1}.$$

The first expression is well-defined if x belongs to the image of f and f is invertible there. In the second expression there is no condition on f, and x need only be an element of the co-domain. In the third expression the point x must belong to the domain of f, and $f(x)$ must be non-zero. Thus

$$\sin^{-1}(1) = \frac{\pi}{2} \qquad \sin^{-1}(\{1\}) = \frac{\pi}{2} + 2\pi\mathbb{Z} \qquad \sin(1)^{-1} = \csc(1).$$

In the first expression we tacitly assume that $\sin^{-1} = \arcsin : [-1, 1] \to [-\pi/2, \pi/2]$. In the third expression the symbol csc denotes the co-secant ($\csc(x) = 1/\sin(x)$), defined in the domain $\mathbb{R} \setminus \pi\mathbb{Z}$.

With a judicious use of definite and indefinite articles, we can specify a function's type without committing ourselves to a specific object.

1. *The inverse of a trigonometric function.*
2. *The composition of a function with itself.*
3. *An integer-valued bijective function.*
4. *A function which coincides with its own inverse.*

In item 2, we infer that the function maps its domain into itself. Functions of type 3 will be considered in the next section to define cardinality of sets. Functions of type 4 are called **involutions** (e.g., $x \mapsto -x$, over a suitable domain).

Writing about **real functions** is considered in Chap. 5.

with the stipulation that repeated elements are to be ignored. For example, if $X = \{1, 3\}$ and $Y = \{2, 4\}$, then

$$X + Y = \{3, 5, 7\} \qquad XY = \{2, 4, 6, 12\}.$$

The expression 'sum of sets' is always understood as an algebraic sum. In the case of product, it is advisable to use the full expression to avoid confusion with the cartesian product.

If $X = \{x\}$ consists of a single element, then we use the shorthand notation $x + Y$ and xY in place of $\{x\} + Y$ and $\{x\}Y$, respectively (as we did in Sect. 2.1.2 for integers). For example

$$\frac{1}{2} + \mathbb{N} = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\} \qquad \pi\,\mathbb{Z} = \{\dots, -2\,\pi, \pi, 0, \pi, 2\,\pi, \dots\}.$$

This notation leads to concise statements such as

$$m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}$$

which combines algebraic sum and product of sets (see Exercise 7.5).

Elementary—but significant—applications of this construct are found in **modular arithmetic**. Let m be a positive integer. We say that two integers x and y are **congruent modulo** m if m divides $x - y$. This relation is denoted by[8]

$$x \equiv y \pmod{m}.$$

Thus

$$-3 \equiv 7 \pmod{5} \qquad 1 \not\equiv 12 \pmod{7}.$$

The integer m is called the **modulus**. The set of integers congruent to a given integer is called a **congruence** (or **residue**) **class**. One verifies that the congruence class of k modulo m is the infinite set $k + m\mathbb{Z}$ given explicitly in (2.11). The congruence class of k modulo m is also denoted by $[k]_m$, $k \pmod{m}$, or, if the modulus is understood, by $[k]$ or $\overline{k}$.

The set of congruence classes modulo m is denoted by $\mathbb{Z}/m\mathbb{Z}$. If $m = p$ is a prime number, the notation $\mathbb{F}_p$ (meaning 'the field with p elements') may be used in place of $\mathbb{Z}/p\mathbb{Z}$. The set $\mathbb{Z}/m\mathbb{Z}$ contains m elements, which form a **partition** of $\mathbb{Z}$:

$$\mathbb{Z}/m\mathbb{Z} = \{m\mathbb{Z}, 1 + m\mathbb{Z}, 2 + m\mathbb{Z}, \dots, (m-1) + m\mathbb{Z}\}.$$

Variants of this notation are used extensively in algebra, where one defines the sum/product of more general sets, such as groups and rings.

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[8] This notation is due to Carl Friedrich Gauss (German: 1777–1855).

Exercise 2.14 Prove that the definition

$$(a, b) \stackrel{\text{def}}{=} \{\{a\}, \{a, b\}\}$$

satisfies (2.4). (This shows that an ordered pair can be defined in terms of a set, so there's no need to introduce a new object.) Hence define an ordered triple in terms of sets.