MATHEMATICAL AND PHYSICAL FUNDAMENTALS OF CLIMATE CHANGE

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Preface: Interdisciplinary Approaches to Climate Change Research

Climate change is now widely recognized as the major environmental problem facing human societies. Its impacts and costs will be large, serious, and unevenly spread. Owing to the observed increases in temperature, decreases in snow and ice extent, and increases in sea level, global warming is unequivocal.

The main factor causing climate change and global warming is the increase of global carbon dioxide emissions. The Fourth Assessment Report (2007) of the Intergovernmental Panel on Climate Change of the United Nations indicated that most of the observed warming over the last 50 years is likely to have been due to the increasing concentrations of greenhouse gases produced by human activities such as deforestation and burning fossil fuels. This conclusion was made even stronger by the Fifth Assessment Report released in 2013. The concentration of carbon dioxide in the atmosphere increased from a preindustrial value of about 280 to 391 ppm in 2011. Continued increases in carbon dioxide emissions will cause further warming and induce many changes in the global climate system. It is likely that global warming will exceed 2 °C this century unless global carbon dioxide emissions are cut by at least 50% of the 1990 levels by 2050, and by much more thereafter.

In current climate change research, scientists exploit various complicated techniques in order to squeeze useful information out of the available observation data, unravel the causes of climate change, identify significant changes in the climate, interpret the properties of the associated variability, deal with extreme climate events, and make predictions about the future climate.

This book covers the comprehensive range of mathematical and physical techniques used widely in climate change research. The main topics include signal processing, time-frequency analysis, data analysis, statistical diagnosis, power spectra, autoregressive moving average models, data assimilation, atmospheric dynamics, oceanic dynamics, glaciers and sea level rise, and Earth system modeling. This book is self-contained, assuming only a basic knowledge of calculus. Much of the latest research is also included. Various theories and algorithms in this book are used widely not only in climate change research, but also in geoscience and applied science. This book will be of great value to researchers and advanced students in a wide range of disciplines. Researchers in and students of meteorology, climatology, oceanography, and environmental science can grasp advanced mathematical and physical methods used in climate change research and geoscience, and researchers in and students of applied mathematics, statistics, physics, computer science, and electrical engineering can learn how to use advanced mathematical and physical methods in climate change research, geoscience, and applied science.

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Chapter 1

Fourier Analysis

Motivated by the study of heat diffusion, Joseph Fourier claimed that any periodic signals can be represented as a series of harmonically related sinusoids. Fourier's idea has a profound impact in geoscience. It took one and a half centuries to complete the theory of Fourier analysis. The richness of the theory makes it suitable for a wide range of applications such as climatic time series analysis, numerical atmospheric and ocean modeling, and climatic data mining.

1.1 FOURIER SERIES AND FOURIER TRANSFORM

Assume that a system of functions $\{\varphi_n(t)\}_{n\in\mathbb{Z}_+}$ in a closed interval $[a,b]$ satisfies $\int_a^b |\varphi_n(t)|^2\,\mathrm{d}t < \infty$. If

$$\int_a^b \varphi_n(t)\overline{\varphi}_m(t)\,\mathrm{d}t = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m), \end{cases}$$

and there does not exist a nonzero function f such that

$$\int_a^b |f(t)|^2\,\mathrm{d}t < \infty, \qquad \int_a^b f(t)\overline{\varphi}_n(t)\,\mathrm{d}t = 0 \quad (n \in \mathbb{Z}_+),$$

then this system is said to be an *orthonormal basis* in the interval $[a,b]$.

For example, the trigonometric system $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nt), \frac{1}{\sqrt{\pi}}\sin(nt)\}_{n\in\mathbb{Z}_+}$ and the exponential system $\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{int}\}_{n\in\mathbb{Z}}$ are both orthonormal bases in $[-\pi,\pi]$.

Let $f(t)$ be a periodic signal with period 2π and be integrable over $[-\pi,\pi]$, write $f \in L_{2\pi}$. In terms of the above orthogonal basis, let $a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$ and

$$\begin{aligned} a_n(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t \quad (n \in \mathbb{Z}_+), \\ b_n(f) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t \quad (n \in \mathbb{Z}_+). \end{aligned}$$

Then $a_0(f), a_n(f), b_n(f) (n \in \mathbb{Z}_+)$ are said to be *Fourier coefficients* of f. The series

$$\frac{a_0(f)}{2} + \sum_1^{\infty} (a_n(f)\cos(nt) + b_n(f)\sin(nt))$$

is said to be the *Fourier series* of f. The sum

$$S_n(f;t) := \frac{a_0(f)}{2} + \sum_1^n (a_k(f)\cos(kt) + b_k(f)\sin(kt))$$

is said to be the *partial sum* of the Fourier series of f. It can be rewritten in the form

$$S_n(f;t) = \sum_{-n}^{n} c_k(f)\mathrm{e}^{\mathrm{i}kt},$$

where

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t \quad (k \in \mathbb{Z})$$

are also called the Fourier coefficients of f.

It is clear that these Fourier coefficients satisfy

$$a_0(f) = 2c_0(f), \quad a_n(f) = c_{-n}(f) + c_n(f), \quad b_n(f) = i(c_{-n}(f) - c_n(f)).$$

Let $f \in L_{2\pi}$. If f is a real signal, then its Fourier coefficients $a_n(f)$ and $b_n(f)$ must be real. The identity

$$a_n(f)\cos(nt) + b_n(f)\sin(nt) = A_n(f)\sin(nt + \theta_n(f))$$

shows that the general term in the Fourier series of f is a sine wave with circle frequency n, amplitude A_n, and initial phase θ_n. Therefore, the Fourier series of a real periodic signal is composed of sine waves with different frequencies and different phases.

Fourier coefficients have the following well-known properties.

Property. Let $f, g \in L_{2\pi}$ and α, β be complex numbers.

(i) (Linearity). $c_n(\alpha f + \beta g) = \alpha c_n(f) + \beta c_n(g)$.
(ii) (Translation). Let $F(t) = f(t+\alpha)$. Then $c_n(F) = \mathrm{e}^{\mathrm{i}n\alpha} c_n(f)$.
(iii) (Integration). Let $F(t) = \int_0^t f(u)\,\mathrm{d}u$. If $\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = 0$, then $c_n(F) = \frac{c_n(f)}{\mathrm{i}n}\,(n \neq 0)$.
(iv) (Derivative). If $f(t)$ is continuously differentiable, then $c_n(f') = \mathrm{i}nc_n(f)$ $(n \neq 0)$.
(v) (Convolution). Let the convolution $(f * g)(t) = \int_{-\pi}^{\pi} f(t-x)g(x)\,\mathrm{d}x$. Then $c_n(f * g) = 2\pi c_n(f)c_n(g)$.

Proof. Here we prove only (v). It is clear that $f * g \in L_{2\pi}$ and

$$c_n(f * g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f * g)(t)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(t-u)g(u)\,\mathrm{d}u\right)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t.$$

Interchanging the order of integrals, we get

$$c_n(f * g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(t-u)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t\right)g(u)\,\mathrm{d}u.$$

Let $v = t - u$. Since $f(v)\mathrm{e}^{-\mathrm{i}nv}$ is a periodic function with period 2π, the integral in brackets is

$$\int_{-\pi}^{\pi} f(t-u)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}nu}\int_{-\pi-u}^{\pi-u} f(v)\mathrm{e}^{-\mathrm{i}nv}\,\mathrm{d}v$$
$$= \mathrm{e}^{-\mathrm{i}nu}\int_{-\pi}^{\pi} f(v)\mathrm{e}^{-\mathrm{i}nv}\,\mathrm{d}v = 2\pi c_n(f)\mathrm{e}^{-\mathrm{i}nu}.$$

Therefore,

$$c_n(f*g) = c_n(f)\int_{-\pi}^{\pi} g(u)\mathrm{e}^{-\mathrm{i}nu}\,\mathrm{d}u = 2\pi c_n(f)c_n(g).$$

Throughout this book, the notation $f \in L(\mathbb{R})$ means that f is integrable over $\mathbb{R}$ and the notation $f \in L[a,b]$ means that $f(t)$ is integrable over a closed interval $[a,b]$, and the integral $\int_{\mathbb{R}} = \int_{-\infty}^{\infty}$. □

Riemann-Lebesgue Lemma. *If $f \in L(\mathbb{R})$, then $\int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \to 0$ as $|\omega| \to \infty$. Especially,*

(i) *if $f \in L[a,b]$, then $\int_a^b f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \to 0(|\omega| \to \infty)$;*
(ii) *if $f \in L_{2\pi}$, then $c_n(f) \to 0(|n| \to \infty)$ and $a_n(f) \to 0$, $b_n(f) \to 0(n \to \infty)$.*

The Riemann-Lebesgue lemma (ii) states that Fourier coefficients of $f \in L_{2\pi}$ tend to zero as $n \to \infty$.

Proof. If f is a simple step function and

$$f(t) = \begin{cases} c, & a \le t \le b, \\ 0, & \text{otherwise}, \end{cases}$$

where c is a constant, then

$$\left|\int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right| = \left|\int_a^b c\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right| = \left|\frac{c}{\mathrm{i}\omega}(\mathrm{e}^{-\mathrm{i}b\omega} - \mathrm{e}^{-\mathrm{i}a\omega})\right| \le 2\left|\frac{c}{\omega}\right| \quad (\omega \ne 0),$$

and so $\int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \to 0(|\omega| \to \infty)$. Similarly, it is easy to prove that for any step function $s(t)$,

$$\int_{\mathbb{R}} s(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \to 0 \quad (|\omega| \to \infty).$$

If f is integrable over $\mathbb{R}$, then, for $\epsilon > 0$, there exists a step function $s(t)$ such that

$$\int_{\mathbb{R}} |f(t) - s(t)|\,\mathrm{d}t < \epsilon.$$

Since $s(t)$ is a step function, for the above ϵ, there exists an N such that

$$\left|\int_{\mathbb{R}} s(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right| < \epsilon \quad (|\omega| > N).$$

From this and $|e^{-i\omega t}| \le 1$, it follows that

$$\left| \int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt \right| \le \int_{\mathbb{R}} |f(t) - s(t)| \, dt + \left| \int_{\mathbb{R}} s(t) e^{-i\omega t} \, dt \right| < 2\epsilon \quad (|\omega| > N),$$

i.e., $\int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt \to 0 (|\omega| \to \infty)$.

Especially, if $f \in L[a, b]$, take

$$F(t) = \begin{cases} f(t), & a \le t \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F \in L(\mathbb{R})$, and so $\int_{\mathbb{R}} F(t) e^{-i\omega t} \, dt \to 0 (|\omega| \to \infty)$. From

$$\int_{\mathbb{R}} F(t) e^{-i\omega t} \, dt = \int_a^b f(t) e^{-i\omega t} \, dt,$$

it follows that $\int_a^b f(t) e^{-i\omega t} \, dt \to 0 (|\omega| \to \infty)$.

Take $a = -\pi$, $b = \pi$, and $\omega = n$. Then $\int_{-\pi}^{\pi} f(t) e^{-int} \, dt \to 0$ as $|n| \to \infty$, i.e.,

$$c_n(f) \to 0 \quad (|n| \to \infty).$$

Combining this with $a_n(f) = c_{-n}(f) + c_n(f)$ and $b_n(f) = i(c_{-n}(f) - c_n(f))$, we get

$$a_n(f) \to 0, \quad b_n(f) \to 0 \quad (n \to \infty).$$

□

The partial sums of Fourier series can be written in an integral form as follows.

By the definition of Fourier coefficients,

$$\begin{aligned} S_n(f; t) &= \sum_{-n}^{n} c_k(f) e^{ikt} = \sum_{-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iku} \, du \right) e^{ikt} \\ &= \int_{-\pi}^{\pi} f(u) \left(\frac{1}{2\pi} \sum_{-n}^{n} e^{ik(t-u)} \right) du. \end{aligned}$$

Let $v = t - u$. Then

$$S_n(f; t) = \int_{-\pi}^{\pi} f(t - v) D_n(v) \, dv, \qquad (1.1)$$

where $D_n(v) = \frac{1}{2\pi} \sum_{-n}^{n} e^{ikv}$ and is called the *Dirichlet kernel*.

The Dirichlet kernel possesses the following properties:

(i) $D_n(-v) = D_n(v)$, i.e., the Dirichlet kernel is an even function.
(ii) $D_n(v + 2\pi) = D_n(v)$, i.e., the Dirichlet kernel is a periodic function with period 2π.

(iii) $D_n(v) = \frac{\sin\left(n+\frac{1}{2}\right)v}{2\pi \sin\frac{v}{2}}$. This is because

$$D_n(v) = \frac{1}{2\pi}\sum_{-n}^{n} e^{ikv} = \frac{e^{-inv} - e^{i(n+1)v}}{2\pi(1-e^{iv})} = \frac{\sin\left(n+\frac{1}{2}\right)v}{2\pi \sin\frac{v}{2}}.$$

(iv) $\int_{-\pi}^{\pi} D_n(v)\,dv = 1$. This is because

$$\int_{-\pi}^{\pi} D_n(v)\,dv = \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{-n}^{n} e^{ikv}\right)dv = \frac{1}{2\pi}\sum_{-n}^{n}\left(\int_{-\pi}^{\pi} e^{ikv}\,dv\right) = 1.$$

We will give the Jordan criterion for Fourier series. Its proof needs the following proposition.

Proposition 1.1. *For any real numbers a and b, the following inequality holds:*

$$\left|\int_a^b \frac{\sin u}{u}\,du\right| \le 6.$$

Proof. When $1 \le a \le b$, by the second mean-value theorem for integrals, there exists a $\xi (a \le \xi \le b)$ such that

$$\left|\int_a^b \frac{\sin u}{u}\,du\right| = \frac{1}{a}\left|\int_a^{\xi} \sin u\,du\right| \le 2.$$

When $0 \le a \le b \le 1$, with use of the inequality $|\sin u| \le |u|$, it follows that

$$\left|\int_a^b \frac{\sin u}{u}\,du\right| \le \int_a^b \left|\frac{\sin u}{u}\right| du \le 1.$$

When $0 \le a \le 1 \le b$,

$$\left|\int_a^b \frac{\sin u}{u}\,du\right| \le \left|\int_a^1 \frac{\sin u}{u}\,du\right| + \left|\int_1^b \frac{\sin u}{u}\,du\right| \le 3.$$

Noticing that $\frac{\sin u}{u}$ is a even function, it can easily prove that for all cases of real numbers a and b,

$$\left|\int_a^b \frac{\sin u}{u}\,du\right| \le 6.$$

□

If a signal is the difference of two monotone increasing signals in an interval, then this signal is called a signal of *bounded variation* in this interval. Almost all geophysical signals are signals of bounded variation.

Jordan Criterion. *Suppose that a signal $f \in L_{2\pi}$ is of bounded variation in $(t-\eta, t+\eta), \eta > 0$. Then the partial sums of the Fourier series of f*

$$S_n(f;t) \to \frac{1}{2}(f(t+0)+f(t-0)) \quad (n \to \infty) \quad \text{at} t.$$

Proof. The assumption that $f(t)$ is of bounded variation in $(t-\eta, t+\eta)$ shows that $f(t+0)$ and $f(t-0)$ exist. By (1.1) and the properties of Dirichlet kernel, it follows that

$$S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) = \int_{-\pi}^{\pi} \left\{ f(t-v) - \frac{1}{2}(f(t+0)+f(t-0)) \right\}$$

$$D_n(v)\,\mathrm{d}v = \frac{1}{\pi}\int_0^{\pi} \psi_t(v) \frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}}\,\mathrm{d}v,$$

where $\psi_t(v) = f(t+v)+f(t-v)-f(t+0)-f(t-0)$. It is clear that

$$\frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}} = \frac{1}{v}\sin(nv) + \left(\frac{1}{2}\coth\frac{v}{2} - \frac{1}{v}\right)\sin(nv) + \frac{1}{2}\cos(nv).$$

Therefore,

$$\begin{aligned} S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) &= \frac{1}{\pi}\int_0^{\pi} \psi_t(v)\frac{1}{v}\sin(nv)\,\mathrm{d}v \\ &+ \frac{1}{\pi}\int_0^{\pi} \psi_t(v)\left(\frac{1}{2}\coth\frac{v}{2} - \frac{1}{v}\right)\sin(nv)\,\mathrm{d}v \\ &+ \frac{1}{\pi}\int_0^{\pi} \psi_t(v)\frac{1}{2}\cos(nv)\,\mathrm{d}v. \end{aligned} \tag{1.2}$$

Note that $\frac{\psi_t(v)}{v} \in L[\delta,\pi]$. Here δ will be determined, $\psi_t(v)\left(\frac{1}{2}\coth\frac{v}{2} - \frac{1}{v}\right) \in L[0,\pi]$, and $\psi_t(v) \in L[0,\pi]$. By Riemann-Lebesgue Lemma, it follows that

$$\int_{\delta}^{\pi} \frac{\psi_t(v)}{v}\sin(nv)\,\mathrm{d}v \to 0 \quad (n \to \infty),$$

$$\int_0^{\pi} \psi_t(v)\left(\frac{1}{2}\coth\frac{v}{2} - \frac{1}{v}\right)\sin(nv)\,\mathrm{d}v \to 0 \quad (n \to \infty),$$

$$\int_0^{\pi} \psi_t(v)\cos(nv)\,\mathrm{d}v \to 0 \quad (n \to \infty).$$

Combining this with (1.2), we get

$$S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) - \frac{1}{\pi}\int_0^{\delta} \psi_t(v)\frac{1}{v}\sin(nv)\,\mathrm{d}v \to 0 \quad (n \to \infty), \tag{1.3}$$

where $\psi_t(v) = f(t+v)+f(t-v)-f(t+0)-f(t-0)$.

Since $\psi_t(v)$ is of bounded variation in $(-\eta,\eta)$ and $\psi_t(0+0)=0$, there exist two monotone increasing functions $h_1(v)$ and $h_2(v)$ satisfying $h_1(0+0) = h_2(0+0) = 0$ such that

$$\psi_t(v) = h_1(v) - h_2(v).$$

Since $h_1(0+0) = h_2(0+0) = 0$, for any given $\epsilon > 0$, there is a $\delta (0 < \delta < \pi)$ such that

$$0 \le h_1(v) \le \epsilon, \quad 0 \le h_2(v) \le \epsilon \quad (0 < v \le \delta).$$

For the fixed δ, by (1.3), there exists an N such that

$$\left| S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) - \frac{1}{\pi}\int_0^\delta h_1(v)\frac{\sin(nv)}{v}\mathrm{d}v \right.$$
$$\left. + \frac{1}{\pi}\int_0^\delta h_2(v)\frac{\sin(nv)}{v}\,\mathrm{d}v \right| < \epsilon \quad (n \ge N),$$

and so

$$\left| S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) \right| \le \left| \frac{1}{\pi}\int_0^\delta h_1(v)\frac{\sin(nv)}{v}\,\mathrm{d}v \right|$$
$$+ \left| \frac{1}{\pi}\int_0^\delta h_2(v)\frac{\sin(nv)}{v}\,\mathrm{d}v \right| + \epsilon \quad (n \ge N).$$

However, using the second mean-value theorem, there exist $\zeta_i (0 < \zeta_i < \delta)$ such that

$$\frac{1}{\pi}\int_0^\delta h_i(v)\frac{\sin(nv)}{v}\,\mathrm{d}v = \frac{1}{\pi}h_i(\delta)\int_{\zeta_i}^\delta \frac{\sin(nv)}{v}\,\mathrm{d}v \quad (i = 1, 2),$$

and by Proposition 1.1,

$$\left| \frac{1}{\pi}\int_0^\delta h_i(v)\frac{\sin(nv)}{v}\,\mathrm{d}v \right| = \left| \frac{1}{\pi}h_i(\delta)\int_{\zeta_i}^\delta \frac{\sin(nv)}{v}\mathrm{d}v \right|$$
$$\le \frac{\epsilon}{\pi}\left| \int_{n\zeta_i}^{n\delta} \frac{\sin v}{v}\,\mathrm{d}v \right| \le \frac{6\epsilon}{\pi} \quad (i = 1, 2).$$

Therefore,

$$\left| S_n(f;t) - \frac{1}{2}(f(t+0)+f(t-0)) \right| \le \left(\frac{12}{\pi} + 1 \right)\epsilon \quad (n \ge N),$$

i.e., $S_n(f;t) \to \frac{1}{2}(f(t+0)+f(t-0))(n \to \infty)$ at t. □

In general, let $f(t) \in L[-\frac{T}{2}, \frac{T}{2}]$ be a periodic function with period T. Then its Fourier series is

$$\frac{a_0(f)}{2} + \sum_1^\infty \left(a_n(f)\cos\frac{2n\pi t}{T} + b_n(f)\sin\frac{2n\pi t}{T} \right),$$

where the Fourier coefficients are

$$a_0(f) = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t,$$

$$a_n(f) = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,\mathrm{d}t \quad (n \in \mathbb{Z}_+),$$

and

$$b_n(f) = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t \quad (n \in \mathbb{Z}_+).$$

An orthogonal basis and an orthogonal series on $[-1, 1]$ used often are stated as follows.

Denote Legendre polynomials by $X_n(t)(n = 0, 1, \ldots)$:

$$X_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(t^2-1)^n}{\mathrm{d}t^n} \quad (n = 0, 1, \ldots).$$

Especially, $X_0(t) = 1$, $X_1(t) = t$, and $X_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$.

By use of Leibnitz's formula, the Legendre polynomials are

$$\begin{aligned} X_n(t) &= \frac{1}{2^n n!}\left\{(t-1)^n\frac{\mathrm{d}^n(t+1)^n}{\mathrm{d}t^n} + C_n^1 n(t-1)^{n-1}\frac{\mathrm{d}^{n-1}(t+1)^n}{\mathrm{d}t^{n-1}}\right.\\ &\quad \left. + \cdots + C_n^n n!(t+1)^n\right\}, \end{aligned}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$. Let $t = 1$ and $t = -1$. Then

$$X_n(1) = 1, \quad X_n(-1) = (-1)^n \quad (n = 0, 1, 2, \ldots).$$

Legendre polynomials possess the property:

$$\int_{-1}^{1} X_n(t)X_m(t)\mathrm{d}t = \begin{cases} 0, & n \neq m,\\ \frac{2}{2n+1}, & n = m. \end{cases}$$

So Legendre polynomials conform to an orthogonal basis on the interval $[-1, 1]$. In terms of this orthogonal basis, any signal f of finite energy on $[-1, 1]$ can be expanded into a Legendre series $\sum_0^\infty l_n X_n(t)$, where

$$l_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)X_n(t)\,\mathrm{d}t.$$

The coefficients l_n are called *Legendre coefficients*.

Now we turn to introduce the concept of the Fourier transform.

Suppose that $f \in L(\mathbb{R})$. The integral

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t \quad (\omega \in \mathbb{R})$$

is called the *Fourier transform* of f. Suppose that $\widehat{f} \in L(\mathbb{R})$. The integral

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)\mathrm{e}^{\mathrm{i}t\omega}\,\mathrm{d}\omega \quad (t \in \mathbb{R})$$

is called the *inverse Fourier transform*. Suppose that $f \in L(\mathbb{R})$ and $\widehat{f} \in L(\mathbb{R})$. It can be proved easily that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = f(t).$$

Theorem 1.1. *Let $f \in L(\mathbb{R})$. Then*

(i) $\lim_{|\omega|\to\infty} \widehat{f}(\omega) = 0$,
(ii) $|\widehat{f}(\omega)| \le \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t =: \| f \|_1$,
(iii) $\widehat{f}(\omega)$ *is continuous uniformly on* $\mathbb{R}$.

Proof. The first conclusion is just the Riemann-Lebesgue lemma. It follows from the definition that

$$|\widehat{f}(\omega)| = \left|\int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right| \le \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t = \| f \|_1 .$$

Since

$$|\widehat{f}(\omega + h) - \widehat{f}(\omega)| \le \int_{\mathbb{R}} |f(t)||\mathrm{e}^{-\mathrm{i}ht} - 1|\,\mathrm{d}t,$$

with use of the dominated convergence theorem, it follows that for any $\omega \in \mathbb{R}$,

$$\lim_{h\to 0} |\widehat{f}(\omega + h) - \widehat{f}(\omega)| \le \int_{\mathbb{R}} |f(t)| \left(\lim_{h\to 0} |\mathrm{e}^{-\mathrm{i}ht} - 1|\right) \mathrm{d}t = 0,$$

i.e., $\widehat{f}(\omega)$ is continuous uniformly on $\mathbb{R}$. □

Fourier transforms have the following properties.

Property. Let $f, g \in L(\mathbb{R})$. Then

(i) (Linearity). $(\alpha f + \beta g)^\wedge(\omega) = \alpha\widehat{f}(\omega) + \beta\widehat{g}(\omega)$, where α, β be constants.
(ii) (Dilation). $(D_a f)^\wedge(\omega) = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right)$ $(a \ne 0)$, where $D_a f = f(a\,t)$ is the *dilation operator*.
(iii) (Translation). $(T_\alpha f)^\wedge(\omega) = \widehat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\alpha}$, where $T_\alpha f = f(t-\alpha)$ is the *translation operator*.
(iv) (Modulation and conjugate). $\left(f(t)\mathrm{e}^{\mathrm{i}\alpha t}\right)^\wedge(\omega) = \widehat{f}(\omega - \alpha)$, $\widehat{\overline{f}}(\omega) = \overline{\widehat{f}}(-\omega)$.
(v) (Symmetry). If $\widehat{f} \in L(\mathbb{R})$, then $\widehat{\widehat{f}}(t) = 2\pi f(-t)$.
(vi) (Time derivative). If $f^{(j)} \in L(\mathbb{R})\,(j = 1, \ldots, n)$, then $\widehat{f^{(n)}}(\omega) = (\mathrm{i}\omega)^n\widehat{f}(\omega)$.
(vii) (Convolution in time). Let the convolution $(f * g)(t) = \int_{\mathbb{R}} f(t-u)g(u)\,\mathrm{d}u$. Then

$$(f * g)^\wedge(\omega) = \widehat{f}(\omega)\cdot\widehat{g}(\omega),$$

i.e., the Fourier transform of the convolution of two signals equals the product of their Fourier transforms.

Proof. These seven properties are derived easily by the definition. We prove only (ii), (iii), and (vii).

The Fourier transform of $D_a(f)$ is

$$(D_a f)^{\wedge}(\omega) = \int_{\mathbb{R}} f(at)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

If $a > 0$, then $|a| = a$ and

$$\int_{\mathbb{R}} f(at)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}\left(\frac{\omega}{a}\right)u}\frac{\mathrm{d}u}{a} = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right).$$

If $a < 0$, then $|a| = -a$ and

$$\int_{\mathbb{R}} f(at)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = -\int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}\left(\frac{\omega}{a}\right)u}\frac{\mathrm{d}u}{a} = -\frac{1}{a}\widehat{f}\left(\frac{\omega}{a}\right) = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right).$$

We get (ii).

The Fourier transform of $T_\alpha f$ is

$$(T_\alpha f)^{\wedge}(\omega) = \int_{\mathbb{R}} f(t-\alpha)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Let $u = t - \alpha$. Then

$$(T_\alpha f)^{\wedge}(\omega) = \int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}\omega(u+\alpha)}\,\mathrm{d}u = \mathrm{e}^{-\mathrm{i}\omega\alpha}\int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}\omega u}\,\mathrm{d}u = \widehat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\alpha}.$$

We get (iii).

By the definition of the Fourier transform,

$$(f*g)^{\wedge}(\omega) = \int_{\mathbb{R}} (f*g)(t)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(t-u)g(u)\,\mathrm{d}u\right)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t.$$

Interchanging the order of integrals, and then letting $v = t - u$, we get

$$\begin{aligned}(f*g)^{\wedge}(\omega) &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t-u)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t\right)g(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(v)\mathrm{e}^{-\mathrm{i}(v+u)\omega}\,\mathrm{d}v\right)g(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}} f(v)\mathrm{e}^{-\mathrm{i}v\omega}\,\mathrm{d}v\cdot\int_{\mathbb{R}} g(u)\mathrm{e}^{-\mathrm{i}u\omega}\,\mathrm{d}u = \widehat{f}(\omega)\cdot\widehat{g}(\omega).\end{aligned}$$

So we get (vii). □

The notation $f \in L^2(\mathbb{R})$ means that f is a signal of finite energy on $\mathbb{R}$, i.e., $\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t < \infty$. The definition of the Fourier transform of $f \in L^2(\mathbb{R})$ is based on the Schwartz space.

A space consists of the signals f satisfying the following two conditions:

(i) f is infinite-time differentiable on $\mathbb{R}$;
(ii) for any non-negative integers p, q,

$$t^p f^{(q)}(t) \to 0 \quad (|t| \to \infty).$$

This space is called the *Schwartz space*. Denote it by $f \in S$.

From the definition of the Schwartz space, it follows that if $f \in S$, then $f \in L(\mathbb{R})$ and $f \in L^2(\mathbb{R})$. It can be proved easily that if $f \in S$, then $\widehat{f} \in S$.

On the basis of the Schwartz space, the Fourier transform of $f \in L^2(\mathbb{R})$ is defined as follows.

Definition 1.1. Let $f \in L^2(\mathbb{R})$. Take arbitrarily $f_n(t) \in S$ such that $f_n(t) \to f(t)(L^2)$. The limit of $\{\widehat{f}_n(\omega)\}$ in $L^2(\mathbb{R})$ is said to be the Fourier transform of $f(t)$, denoted by $\widehat{f}(\omega)$, i.e., $\widehat{f}_n(\omega) \to \widehat{f}(\omega)(L^2)$.

Remark. $f_n(t) \to f(t)(L^2)$ means that $\int_{\mathbb{R}} (f_n(t) - f(t))^2 \, dt \to 0 (n \to \infty)$.

Similarly, on the basis of Definition 1.1, Fourier transforms for $L^2(\mathbb{R})$ have the following properties.

Property. Let $f, g \in L^2(\mathbb{R})$ and α, β be constants. Then

(i) (Linearity). $(\alpha f + \beta g)^\wedge(\omega) = \alpha \widehat{f}(\omega) + \beta \widehat{g}(\omega)$.
(ii) (Dilation). $(D_a f)^\wedge(\omega) = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right)$, where $D_a f = f(at)$ and $a \neq 0$ is a constant.
(iii) (Translation). $(T_\alpha f)^\wedge(\omega) = \widehat{f}(\omega)e^{-i\omega\alpha}$, where $T_\alpha f = f(t - \alpha)$.
(iv) (Modulation). $(f(t)e^{i\alpha t})^\wedge(\omega) = \widehat{f}(\omega - \alpha)$.
(v) $\widehat{f'}(\omega) = (i\omega)\widehat{f}(\omega)$, $\widehat{\widehat{f}}(t) = 2\pi f(-t)$, and $\widehat{\overline{f}}(\omega) = \overline{\widehat{f}}(-\omega)$.

A linear continuous functional F, which is defined as a linear map from the Schwartz space to the real axis, is called a *generalized distribution* on the Schwartz space. Denote it by $F \in S'$. For any $g \in S$, denote $F(g)$ by $\langle F, g \rangle$. For each $f \in L^2(\mathbb{R})$, we can define a linear continuous functional on the Schwartz space as follows:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(t)g(t)dt \quad \text{for any } g \in S,$$

which implies that $L^2(\mathbb{R}) \subset S'$.

The operation rules for generalized distributions on the Schwartz space are as follows:

(i) (Limit). Let $F_n \in S'(n = 1, 2, \ldots)$ and $F \in S'$. For any $g \in S$, define $F_n \to F(S')(n \to \infty)$ as

$$\langle F_n, g \rangle \to \langle F, g \rangle.$$

(ii) (Multiplier). Let $F \in S'$ and α be a constant. For any $g \in S$, define αF as

$$\langle \alpha F, g \rangle = \langle F, \alpha g \rangle.$$

(iii) (Derivative). Let $F \in S'$. For any $g \in S$, define the derivative $F' \in S'$ as

$$\langle F', g \rangle = -\langle F, g' \rangle.$$

(iv) (Dilation). Let $F \in S'$. For any $g \in S$, define $D_a F = F(at)$ as

$$\langle D_a F, g \rangle = \left\langle F, \frac{1}{|a|} g\left(\frac{t}{a}\right) \right\rangle,$$

where $a \neq 0$ is a constant.

(v) (Translation). Let $F \in S'$. For any $g \in S$, define $T_a F = F(t-a)$ as

$$\langle T_a F, g \rangle = \langle F, g(t+a) \rangle,$$

where a is a constant.

(vi) (Antiderivative). Let $F \in S'$. For any $g \in S$, define the antiderivative F^{-1} as

$$\langle F^{-1}, g \rangle = -\left\langle F, \int_{-\infty}^{t} \Phi_g(u)\, \mathrm{d}u \right\rangle,$$

where $\Phi_g(u) = g(u) - \frac{1}{\sqrt{\pi}} \mathrm{e}^{-u^2} \int_{\mathbb{R}} g(t)\, \mathrm{d}t$.

Definition 1.2. Let $F \in S'$.

(i) The Fourier series of F is defined as $\sum_n C_n \mathrm{e}^{\mathrm{i}nt}$, where the Fourier coefficients are

$$C_n = -\frac{1}{2\pi} \left\{ T_{2\pi} (F\mathrm{e}^{-\mathrm{i}nt})^{-1} - (F\mathrm{e}^{-\mathrm{i}nt})^{-1} \right\},$$

where $T_{2\pi}$ is the translation operator and $(F\mathrm{e}^{-\mathrm{i}nt})^{-1}$ is the antiderivative of $F\mathrm{e}^{-\mathrm{i}nt}$.

(ii) The Fourier transform of F is defined as $\langle \widehat{F}, g \rangle = \langle F, \widehat{g} \rangle$ for any $g \in S$.

Fourier transforms of generalized distributions on the Schwartz space have the following properties.

Property. Let $F \in S'$. Then

(i) (Derivative). $\widehat{F'}(\omega) = \mathrm{i}\omega \widehat{F}(\omega)$.

(ii) (Translation). $(T_a F)^{\wedge}(\omega) = \mathrm{e}^{-\mathrm{i}a\omega} \widehat{F}(\omega)$, where a is a constant and $T_a F = F(t-a)$.

(iii) (Delation). $(D_a F)^{\wedge}(\omega) = \frac{1}{|a|} \widehat{F}(\frac{\omega}{a})$, where $a \neq 0$ and $D_a F = F(at)$.

The Dirac function and the Dirac comb are both important tools in geophysical signal processing. Define the Dirac function δ as a generalized distribution on the Schwartz space which satisfies for any $g \in S$,

$$\langle \delta, g \rangle = g(0).$$

In general, define δ_{t_0} as a generalized distribution on the Schwartz space which satisfies for any $g \in S$,

$$\langle \delta_{t_0}, g\rangle = g(t_0) \quad (t_0 \in \mathbb{R}).$$

Clearly, $\delta_0 = \delta$. Therefore, δ_{t_0} is the generalization of the Dirac function δ.

By operation rule (iv) of generalized distributions on a Schwartz space, it is easy to prove that for any $g \in S$, the first-order generalized derivative of the Dirac function is

$$\langle \delta', g\rangle = -\langle \delta, g'\rangle = -g'(0);$$

and the second-order generalized derivative of the Dirac function is

$$\langle \delta'', g\rangle = -\langle \delta', g'\rangle = \langle \delta, g''\rangle = g''(0).$$

In general, the n-order generalized derivative of the Dirac function is

$$\langle \delta^{(n)}, g\rangle = (-1)^n g^{(n)}(0).$$

Denote the Fourier transform of δ_{t_0} by $\widehat{\delta}_{t_0}$. By Definition 1.2(ii), the Fourier transform of δ_{t_0} satisfies

$$\langle \widehat{\delta}_{t_0}, g\rangle = \langle \delta_{t_0}, \widehat{g}\rangle = \widehat{g}(t_0) \quad \text{for any } g \in S.$$

Since $g \in S \subset L(\mathbb{R})$, by the definition of the Fourier transform, we have

$$\widehat{g}(t_0) = \int_{\mathbb{R}} g(\omega)\mathrm{e}^{-\mathrm{i}t_0\omega}\, \mathrm{d}\omega = \langle \mathrm{e}^{-\mathrm{i}t_0\omega}, g\rangle.$$

Therefore, $\langle \widehat{\delta}_{t_0}, g\rangle = \langle \mathrm{e}^{-\mathrm{i}t_0\omega}, g\rangle$. This means $\widehat{\delta}_{t_0} = \mathrm{e}^{-\mathrm{i}t_0\omega}$. Especially, noticing that $\delta_0 = \delta$, we find that the Fourier transform of the Dirac function is equal to 1.

On the other hand, by Definition 1.2(ii), for any $g \in S$,

$$\left\langle \left(\mathrm{e}^{-\mathrm{i}t_0\omega}\right)^{\wedge}, g\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}t_0\omega}, \widehat{g}\right\rangle = \int_{\mathbb{R}} \widehat{g}(\omega)\mathrm{e}^{-\mathrm{i}t_0\omega}\, \mathrm{d}\omega.$$

Since $g \in L(\mathbb{R})$ and $\widehat{g} \in L(\mathbb{R})$, the identity $\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\omega)\mathrm{e}^{-\mathrm{i}t_0\omega}\, \mathrm{d}\omega = g(-t_0)$ holds. So

$$\left\langle \left(\mathrm{e}^{-\mathrm{i}t_0\omega}\right)^{\wedge}, g\right\rangle = 2\pi g(-t_0).$$

From this and the definition $\langle \delta_{-t_0}, g\rangle = g(-t_0)$, it follows that

$$\left\langle \left(\mathrm{e}^{-\mathrm{i}t_0\omega}\right)^{\wedge}, g\right\rangle = 2\pi \left\langle \delta_{-t_0}, g\right\rangle.$$

This means that $\left(\mathrm{e}^{-\mathrm{i}t_0\omega}\right)^{\wedge} = 2\pi\delta_{-t_0}$. Noticing that $\delta_0 = \delta$, we obtain that the Fourier transform of 1 is equal to $2\pi\delta$.

Summarizing all the results, we have the following.

Formula 1.1.

(i) $\widehat{\delta}_{t_0} = \mathrm{e}^{-\mathrm{i}t_0\omega}$ and $\left(\mathrm{e}^{-\mathrm{i}t_0\omega}\right)^{\wedge} = 2\pi\delta_{-t_0}$,
(ii) $\widehat{\delta} = 1$ and $\widehat{1} = 2\pi\delta$.

Remark. In engineering and geoscience, instead of the rigid definition, one often uses the following alternative definition for the Dirac function δ:

(i) $\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases}$

(ii) $\int_{\mathbb{R}} \delta(t)\,\mathrm{d}t = 1,$

(iii) $\int_{\mathbb{R}} \delta(t)g(t)\,\mathrm{d}t = g(0)$ for any $g(t)$.

The series $\sum_n \delta_{2n\pi}$ is called the *Dirac comb* which is closely related to sampling theory. In order to show that it is well defined, we need to prove that the series $\sum_n \delta_{2n\pi}$ is convergent.

Let S_n be its partial sums and $S_n = \sum_{-n}^{n} \delta_{2k\pi}$. Clearly, S_n are generalized distributions on the Schwartz space, i.e., $S_n \in S'$ and for any $g \in S$,

$$\langle S_n, g\rangle = \left\langle \sum_{-n}^{n} \delta_{2k\pi}, g \right\rangle = \sum_{-n}^{n} \langle \delta_{2k\pi}, g\rangle.$$

Combining this with the definition $\langle \delta_{2k\pi}, g\rangle = g(2k\pi)$, we get

$$\langle S_n, g\rangle = \sum_{-n}^{n} g(2k\pi).$$

Since $g \in S$, the series $\sum_n g(2n\pi)$ converges. So there exists a $\delta^* \in S'$ such that

$$\langle S_n, g\rangle \to \langle \delta^*, g\rangle \quad \text{or} \quad S_n \to \delta^*(S') \quad (n \to \infty),$$

i.e., the series $\sum_n \delta_{2n\pi}$ converges to δ^*, and $\langle \delta^*, g\rangle = \sum_n g(2n\pi)$ for any $g \in S$.

Secondly, we prove that δ^* is a 2π-periodic generalized distribution.

By operation rule (v) of generalized distributions on a Schwartz space, for any $g \in S$,

$$\langle T_{2\pi}\delta^*, g\rangle = \langle \delta^*, g(t + 2\pi)\rangle = \sum_n g(2(n+1)\pi) = \sum_n g(2n\pi) = \langle \delta^*, g\rangle.$$

This means that δ^* is a periodic generalized distribution with period 2π.

Third, by Definition 1.2(i), we will find the Fourier series of δ^*. We only need to find its Fourier coefficients.

Denote the Fourier coefficients of δ^* by C_n. Since $\delta^* \in S'$, by Definition 1.2(i), for any $g \in S$,

$$\langle C_n, g\rangle = -\frac{1}{2\pi}\langle T_{2\pi}(\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1} - (\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1}, g\rangle.$$

Using operation rule (v) of generalized distributions on a Schwartz space, we get

$$\langle T_{2\pi}(\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1} - (\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1}, g\rangle = \langle (\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1}, \widetilde{g}(t)\rangle,$$

where $\widetilde{g}(t) = g(t + 2\pi) - g(t)$. Therefore

$$\langle C_n, g\rangle = -\frac{1}{2\pi}\langle (\delta^* \mathrm{e}^{-\mathrm{i}nt})^{-1}, \widetilde{g}(t)\rangle.$$

Using operation rule (vi) of generalized distributions on a Schwartz space, we get

$$\langle C_n, g\rangle = \frac{1}{2\pi}\left\langle \delta^* \mathrm{e}^{-\mathrm{i}nt}, \int_{-\infty}^{t} \Phi_{\widetilde{g}}(u)\,\mathrm{d}u\right\rangle,$$

where

$$\Phi_{\widetilde{g}}(u) = \widetilde{g}(u) - \frac{1}{\sqrt{\pi}}\mathrm{e}^{-u^2}\int_{\mathbb{R}} \widetilde{g}(t)\,\mathrm{d}t.$$

Since $\int_{\mathbb{R}} \widetilde{g}(t)\mathrm{d}t = \int_{\mathbb{R}} g(t+2\pi)\,\mathrm{d}t - \int_{\mathbb{R}} g(t)\,\mathrm{d}t = 0$, we get

$$\int_{-\infty}^{t} \Phi_{\widetilde{g}}(u)\,\mathrm{d}u = \int_{-\infty}^{t} \widetilde{g}(u)\,\mathrm{d}u = \int_{-\infty}^{t} (g(u+2\pi) - g(u))\,\mathrm{d}u = \int_{t}^{t+2\pi} g(u)\,\mathrm{d}u,$$

and so

$$\langle C_n, g\rangle = \frac{1}{2\pi}\left\langle \delta^* \mathrm{e}^{-\mathrm{i}nt}, \int_{t}^{t+2\pi} g(u)\,\mathrm{d}u\right\rangle.$$

Using operation rule (ii) of generalized distributions on a Schwartz space, we get

$$\left\langle \delta^* \mathrm{e}^{-\mathrm{i}nt}, \int_{t}^{t+2\pi} g(u)\,\mathrm{d}u\right\rangle = \left\langle \delta^*, \quad \mathrm{e}^{-\mathrm{i}nt}\int_{t}^{t+2\pi} g(u)\,\mathrm{d}u\right\rangle,$$

and so

$$\langle C_n, g\rangle = \frac{1}{2\pi}\left\langle \delta^*, \quad \mathrm{e}^{-\mathrm{i}nt}\int_{t}^{t+2\pi} g(u)\,\mathrm{d}u\right\rangle.$$

We have proved $\langle \delta^*, g\rangle = \sum_k g(2k\pi)$ for any $g \in S$. Noticing that $\mathrm{e}^{-\mathrm{i}n2k\pi} = 1$, we find the right-hand side is

$$\begin{aligned}\frac{1}{2\pi}\left\langle \delta^*, \mathrm{e}^{-\mathrm{i}nt}\int_{t}^{t+2\pi} g(u)\,\mathrm{d}u\right\rangle &= \frac{1}{2\pi}\sum_k \mathrm{e}^{-\mathrm{i}n2k\pi}\int_{2k\pi}^{2k\pi+2\pi} g(u)\mathrm{d}u \\ &= \frac{1}{2\pi}\sum_k \int_{2k\pi}^{2(k+1)\pi} g(u)\,\mathrm{d}u,\end{aligned}$$

and so

$$\langle C_n, g\rangle = \frac{1}{2\pi}\sum_k \int_{2k\pi}^{2(k+1)\pi} g(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} g(u)\,\mathrm{d}u = \left\langle \frac{1}{2\pi}, g\right\rangle,$$

i.e., $C_n = \frac{1}{2\pi}\,(n \in \mathbb{Z})$. By Definition 1.2(i), the Fourier series of δ^* is $\frac{1}{2\pi}\sum_n \mathrm{e}^{\mathrm{i}nt}$.

Finally, we prove the Fourier series $\frac{1}{2\pi}\sum_n \mathrm{e}^{\mathrm{i}nt}$ converges to δ^*, i.e., $\frac{1}{2\pi}\sum_n \mathrm{e}^{\mathrm{i}nt} = \delta^*(t)\,(S')$.

Its partial sum is $S_n(t) = \frac{1}{2\pi}\sum_{-n}^{n} \mathrm{e}^{\mathrm{i}kt}$. This is the Dirichlet kernel $D_n(t)$. Using property (ii) of the Dirichlet kernel, we get

$$\begin{aligned}\langle S_n, g\rangle = \langle D_n, g\rangle &= \int_{\mathbb{R}} D_n(t)g(t)\,\mathrm{d}t = \sum_k \int_{(2k-1)\pi}^{(2k+1)\pi} D_n(t)g(t)\,\mathrm{d}t \\ &= \sum_k \int_{-\pi}^{\pi} D_n(t)g(t+2k\pi)\,\mathrm{d}t = \int_{-\pi}^{\pi} D_n(t)\sum_k g(t+2k\pi)\,\mathrm{d}t.\end{aligned}$$

By the Jordan criterion for Fourier series, we have

$$\int_{-\pi}^{\pi} D_n(t)\sum_k g(t+2k\pi)\,\mathrm{d}t \to \sum_k g(2k\pi) \quad (n\to\infty),$$

and so $\langle S_n, g\rangle \to \sum_k g(2k\pi)(n\to\infty)$. From this and $\langle \delta^*, g\rangle = \sum_k g(2k\pi)$, it follows that

$$\langle S_n, g\rangle \to \langle \delta^*, g\rangle \quad (n\to\infty).$$

This means that $S_n \to \delta^*(S')(n\to\infty)$. From this and $\delta^* = \sum_n \delta_{2n\pi}$, we get

$$\sum_n \delta_{2n\pi} = \frac{1}{2\pi}\sum_n \mathrm{e}^{int} \quad (S').$$

Taking the Fourier transform on both sides and using Formula 1.1, we get

$$\left(\sum_n \delta_{2n\pi}\right)^{\wedge} = \frac{1}{2\pi}\sum_n \left(\mathrm{e}^{int}\right)^{\wedge} = \frac{1}{2\pi}\sum_n \delta_n.$$

Formula 1.2. The Fourier transform of a Dirac comb is still a Dirac comb, i.e.,

$$\left(\sum_n \delta_{2n\pi}\right)^{\wedge} = \frac{1}{2\pi}\sum_n \delta_n.$$

The Laplace transform is a generalization of the Fourier transform. Since it can convert differential or integral equations into algebraic equations, the Laplace transform can be used to solve differential/integral equations with initial conditions.

Let $f \in L[0,\infty]$. The *Laplace transform* of a signal $f(t)$ is defined as

$$L[f(t)] := \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \quad (\mathrm{Re}\,s \geq 0).$$

It is sometimes called the *one-sided Laplace transform*.

Laplace transforms possess the following properties:

(i) Let $f, g \in L[0,\infty]$ and c, d be constants. Then $L[cf(t) + dg(t)] = cL[f(t)] + dL[g(t)]$.

(ii) Let $f^{(j)} \in L[0,\infty]\,(j = 1,\dots,N)$. Then

$$\begin{aligned} L[f^{(N)}(t)] &= -f^{(N-1)}(0) - \cdots - s^{N-3}f''(0) - s^{N-2}f'(0) \\ &\quad - s^{N-1}f(0) + s^N L[f(t)]. \end{aligned}$$

(iii) Let $f \in L[0,\infty]$. Then $L\left[\int_0^t f(u)\,\mathrm{d}u\right] = \frac{1}{s}L[f(t)]$.

By the definition and properties of Laplace transforms, it follows further that

$$L[1] = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{s},$$

$$L[\mathrm{e}^{-at}] = \int_0^\infty \mathrm{e}^{-(a+s)t}\,\mathrm{d}t = \frac{1}{s+a},$$

$$\begin{aligned} L\left[\frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{a-b}\right] &= \frac{1}{a-b}\{L\{\mathrm{e}^{-at}\} - L\{\mathrm{e}^{-bt}\}\} \\ &= \frac{1}{a-b}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -\frac{1}{(s+a)(s+b)}, \end{aligned}$$

$$\begin{aligned} L\left[\frac{a\mathrm{e}^{-at} - b\mathrm{e}^{-bt}}{a-b}\right] &= \frac{1}{a-b}\{aL\{\mathrm{e}^{-at}\} - bL\{\mathrm{e}^{-bt}\}\} \\ &= \frac{1}{a-b}\left\{\frac{a}{s+a} - \frac{b}{s+b}\right\} = \frac{s}{(s+a)(s+b)}, \\ L[t^N] &= \int_0^\infty t^N \mathrm{e}^{-st}\,\mathrm{d}t = \frac{N!}{s^{N+1}}. \end{aligned}$$

Finally, we consider the two-dimensional case. If $f(t_1, t_2) \in L(\mathbb{R}^2)$, the *two-dimensional Fourier transform* is defined as

$$\widehat{f}(\omega_1, \omega_2) := \int\int_{\mathbb{R}^2} f(t_1\ t_2)\mathrm{e}^{-\mathrm{i}(\omega_1 t_1 + \omega_2 t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

The *two-dimensional inverse Fourier transform* is defined as

$$\frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \widehat{f}(\omega_1, \omega_2)\mathrm{e}^{\mathrm{i}(\omega_1 t_1 + \omega_2 t_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2.$$

It can be proved that if $f \in L(\mathbb{R}^2)$ and $\widehat{f} \in L(\mathbb{R}^2)$, then

$$f(t_1, t_2) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \widehat{f}(\omega_1, \omega_2)\mathrm{e}^{\mathrm{i}(\omega_1 t_1 + \omega_2 t_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2.$$

Two-dimensional Fourier transforms have the following similar properties:

(i) (Translation). Let $f \in L(\mathbb{R}^2)$ and $a = (a_1, a_2) \in \mathbb{R}^2$. Then

$$(f(t_1 + a_1, t_2 + a_2))^{\wedge}(\omega_1, \omega_2) = \mathrm{e}^{\mathrm{i}(\omega_1 a_1 + \omega_2 a_2)} \widehat{f}(\omega_1, \omega_2).$$

(ii) (Delation). Let $f \in L(\mathbb{R}^2)$ and λ be a real constant. Then

$$(f(\lambda t_1, \lambda t_2))^{\wedge}(\omega_1, \omega_2) = \frac{1}{|\lambda|^2} \widehat{f}\left(\frac{\omega_1}{\lambda}, \frac{\omega_2}{\lambda}\right).$$

(iii) (Convolution). Let $f, g \in L(\mathbb{R}^2)$ and the convolution

$$(f * g)(t_1, t_2) = \int\int_{\mathbb{R}^2} f(t_1 - u_1, t_2 - u_2) g(u_1, u_2)\, \mathrm{d}u_1 \mathrm{d}u_2.$$

Then

$$(f * g)^{\wedge}(\omega_1, \omega_2) = \widehat{f}(\omega_1, \omega_2) \widehat{g}(\omega_1, \omega_2).$$

1.2 BESSEL'S INEQUALITY AND PARSEVAL'S IDENTITY

Bessel's inequality and Parseval's identity are fundamental results of Fourier series and Fourier transform. Bessel's inequality is a stepping stone to the more powerful Parseval's identity.

Bessel's Inequality for Fourier Series. *Let $f \in L_{2\pi}$ and a_n, b_n, c_n be its Fourier coefficients. Then*

$$\left(\frac{a_0}{2} + \sum_{1}^{n} (a_k^2 + b_k^2)\right) \le \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(t)\, \mathrm{d}t$$

or

$$\sum_{-n}^{n} |c_k|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(t)\, \mathrm{d}t.$$

Proof. Denote partial sums of the Fourier series of f by $S_n(f;t)$. Since

$$(S_n(f;t) - f(t))^2 = S_n^2(f;t) - 2f(t)S_n(f;t) + f^2(t),$$

integrating over the interval $[-\pi, \pi]$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} (S_n(f;t) - f(t))^2\, \mathrm{d}t &= \int_{-\pi}^{\pi} S_n^2(f;t)\, \mathrm{d}t - 2\int_{-\pi}^{\pi} f(t)S_n(f;t)\, \mathrm{d}t + \int_{-\pi}^{\pi} f^2(t)\, \mathrm{d}t \\ &= I_1 - I_2 + \int_{-\pi}^{\pi} f^2(t)\, \mathrm{d}t.\end{aligned}$$

We compute I_1. The partial sums of the Fourier series of f are

$$S_n(f;t) = \frac{a_0}{2} + \sum_{1}^{n} (a_k \cos(kt) + b_k \sin(kt)).$$

So

$$
\begin{aligned}
I_1 &= \int_{-\pi}^{\pi} S_n^2(f;\ t)\,\mathrm{d}t = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_1^n (a_k \cos(kt) + b_k \sin(kt)) \right)^2 \mathrm{d}t \\
&= \int_{-\pi}^{\pi} \frac{a_0^2}{4}\,\mathrm{d}t + \int_{-\pi}^{\pi} a_0 \left(\sum_1^n (a_k\ \cos(kt) + b_k \sin(kt)) \right) \mathrm{d}t \\
&\quad + \int_{-\pi}^{\pi} \left(\sum_1^n (a_k \cos(kt) + b_k \sin(kt)) \right)^2 \mathrm{d}t.
\end{aligned}
$$

By the orthogonality of trigonometric system $\{1, \cos(nt),\ \sin(nt)\}_{n\in\mathbb{Z}_+}$, we obtain that

$$
I_1 = \pi \left(\frac{a_0^2}{2} + \sum_1^n (a_k^2 + b_k^2) \right).
$$

We compute I_2. Since

$$
\begin{aligned}
I_2 &= 2\int_{-\pi}^{\pi} f(t) S_n(f;t)\,\mathrm{d}t = 2\int_{-\pi}^{\pi} f(t) \left(\frac{a_0}{2} + \sum_1^n (a_k \cos(kt) + b_k \sin(kt)) \right) \mathrm{d}t \\
&= a_0 \int_{-\pi}^{\pi} f(t)\,\mathrm{d}t + 2\sum_1^n \left(a_k \int_{-\pi}^{\pi} f(t)\cos(kt)\mathrm{d}t + b_k \int_{-\pi}^{\pi} f(t)\sin(kt)\,\mathrm{d}t \right),
\end{aligned}
$$

by the definition of the Fourier coefficients, we get

$$
I_2 = 2\pi \left(\frac{a_0^2}{2} + \sum_1^n (a_k^2 + b_k^2) \right).
$$

Therefore,

$$
\int_{-\pi}^{\pi} (S_n(f;t) - f(t))^2\,\mathrm{d}t = -\pi \left(\frac{a_0^2}{2} + \sum_1^n (a_k^2 + b_k^2) \right) + \int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t. \quad (1.4)
$$

Noticing that $a_0 = 2c_0$, $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$, and

$$
\begin{aligned}
a_k^2 + b_k^2 &= |c_{-k} + c_k|^2 + |i\,(c_{-k} - c_k)|^2 \\
&= (c_{-k} + c_k)\,(\overline{c}_{-k} + \overline{c}_k) + (c_{-k} - c_k)\,(\overline{c}_{-k} - \overline{c}_k) \\
&= 2\,(c_{-k}\overline{c}_{-k} + c_k\overline{c}_k) = 2\left(|c_{-k}|^2 + |c_k|^2\right),
\end{aligned}
$$

the first term on the right-hand side of (1.4):

$$
\begin{aligned}
-\pi \left(\frac{a_0^2}{2} + \sum_1^n (a_k^2 + b_k^2) \right) &= -\pi \left(2|c_0|^2 + \sum_1^n 2(|c_{-k}|^2 + |c_k|^2) \right) \\
&= -2\pi \sum_{-n}^n |c_k|^2.
\end{aligned}
$$

From this and (1.4), it follows that

$$\int_{-\pi}^{\pi} (S_n(f;t) - f(t))^2 \,\mathrm{d}t = -2\pi \sum_{-n}^{n} |c_k|^2 + \int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t. \tag{1.5}$$

Noticing that $\int_{-\pi}^{\pi} (S_n(f;t) - f(t))^2 \,\mathrm{d}t \geq 0$, we find from (1.4) and (1.5) that

$$\left(\frac{a_0}{2} + \sum_{1}^{n} (a_k^2 + b_k^2)\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t$$

and

$$\sum_{-n}^{n} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t.$$

□

Parseval's Identity for Fourier Series. *Let $f \in L_{2\pi}$ and a_n, b_n, c_n be its Fourier coefficients. If the partial sums of its Fourier series $S_n(f;t)$ tend to $f(t)$ as $n \to \infty$, then*

$$\int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t = \pi \left(\frac{a_0^2}{2} + \sum_{1}^{\infty} (a_n^2 + b_n^2)\right)$$

and

$$\int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t = 2\pi \sum_{n} |c_n|^2.$$

Parseval's identity is sometimes called the law of conservation of energy.

Proof. In the proof of Bessel's inequality, we have obtained (1.4) and (1.5).

Letting $n \to \infty$ in (1.4) and (1.5), and using the assumption $S_n(f;t) \to f(t)$ $(n \to \infty)$, we obtain immediately the desired results:

$$\int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t = \pi \left(\frac{a_0^2}{2} + \sum_{1}^{\infty} (a_k^2 + b_k^2)\right)$$

and

$$\int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t = 2\pi \sum_{k} |c_k|^2.$$

□

For a Schwartz space, the original signals and their Fourier transforms have the following relation.

Theorem 1.2. *If $f, g \in S$, then*

$$\int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{g}}(\omega)\,\mathrm{d}\omega.$$

Proof. It follows from $g \in S$ that $g \in L(\mathbb{R})$ and $\widehat{g} \in L(\mathbb{R})$. Thus,

$$g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\omega) e^{i\omega t} \, d\omega.$$

Taking the conjugate on both sides, we get

$$\overline{g}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\widehat{g}}(\omega) e^{-i\omega t} \, d\omega,$$

and so

$$\int_{\mathbb{R}} f(t)\overline{g}(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} \overline{\widehat{g}}(\omega) e^{-i\omega t} \, d\omega \right) dt.$$

Interchanging the order of integrals and using the definition of the Fourier transform, the right-hand side is

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(t) \left(\int_{\mathbb{R}} \overline{\widehat{g}}(\omega) e^{-i\omega t} \, d\omega \right) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt \right) \overline{\widehat{g}}(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}}(\omega) \, d\omega.$$

Therefore,

$$\int_{\mathbb{R}} f(t)\overline{g}(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}}(\omega) \, d\omega.$$

□

Let $f(t) = g(t)$ in Theorem 1.2. Then the following identity holds.

Parseval's Identity for a Schwartz Space. *If $f \in S$, then*

$$\int_{\mathbb{R}} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 \, d\omega.$$

Theorem 1.2 can be extended from S to $L^2(\mathbb{R})$ as follows.

Theorem 1.3. *If $f, g \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f(t)\overline{g}(t) \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}}(\omega) \, d\omega.$$

Proof. Take arbitrarily $f_n \in S$, $g_n \in S$ such that $f_n \to f(L^2)$, $g_n \to g(L^2)$ as $n \to \infty$. By Definition 1.1,

$$\widehat{f_n}(\omega) \to \widehat{f}(\omega)(L^2),$$

$$\widehat{g}_n(\omega) \to \widehat{g}(\omega)(L^2),$$

and so

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f_n}(\omega) \overline{\widehat{g}}_n(\omega) \, d\omega \to \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \overline{\widehat{g}}(\omega) \, d\omega.$$

On the other hand, since $f_n \in S$ and $g_n \in S$, Theorem 1.2 shows that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}_n(\omega)\overline{\widehat{g}}_n(\omega)\,\mathrm{d}\omega = \int_{\mathbb{R}} f_n(t)\overline{g}_n(t)\,\mathrm{d}t.$$

Since $f_n \to f$ and $g_n \to g$, the integral on the right-hand side has a limit, i.e., as $n \to \infty$

$$\int_{\mathbb{R}} f_n(t)\overline{g}_n(t)\,\mathrm{d}t \to \int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t,$$

and so

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}_n(\omega)\overline{\widehat{g}}_n(\omega)\,\mathrm{d}\omega \to \int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t.$$

Since the limit is unique, we get

$$\int_{\mathbb{R}} f(t)\overline{g}(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{g}}(\omega)\,\mathrm{d}\omega.$$

□

Let $g(t) = f(t)$ in Theorem 1.3. Then the following identity holds.

Parseval's Identity of the Fourier Transform. *If $f \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

In a similar way, for the two-dimensional signal, the following theorem can be derived.

Theorem 1.4. *If $f, g \in L^2(\mathbb{R}^2)$, then*

$$\int\int_{\mathbb{R}^2} f(t_1,t_2)\overline{g}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} \widehat{f}(\omega_1,\omega_2)\overline{\widehat{g}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Let $f = g$ in Theorem 1.4. Then the following identity holds.

Parseval's Identity. *Let $f(t_1,t_2) \in L^2(\mathbb{R}^2)$. Then*

$$\int\int_{\mathbb{R}^2} |f(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} |\widehat{f}(\omega_1,\omega_2)|^2\,\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

1.3 GIBBS PHENOMENON

If a function $f(t)$ is defined in a neighborhood of t_0 and $f(t_0+0), f(t_0-0)$ exist but $f(t_0+0) \neq f(t_0-0)$, then t_0 is called the *first kind of discontinuity* of $f(t)$.

Suppose that functions $\{f_n(t)\}_{n\in\mathbb{Z}_+}$ and $f(t)$ are defined in a neighborhood of t_0 and $f_n(t) \to f(t)$ as $n \to \infty$ in the neighborhood, and t_0 is the first kind of discontinuity of $f(t)$. Without loss of generality, we may assume $f(t_0-0) < f(t_0+0)$. If $\{f_n(t)\}$ has a double sublimit lying outside the closed interval $[f(t_0-0), f(t_0+0)]$ as $t \to t_0, n \to \infty$, then we say that for the sequence of functions $\{f_n(t)\}$ the *Gibbs phenomenon* occurs at t_0.

Example 1.1. Consider a function

$$\varphi(t)=\begin{cases}\frac{\pi-t}{2}, & 0<t<2\pi,\\ 0, & t=0,\end{cases}\qquad \text{and}\quad \varphi(t+2\pi)=\varphi(t),\quad \text{and}\quad t_0=0.$$

Clearly, $\varphi(t)$ is continuous in $0<|t|<\pi$ and $\varphi(0+0)=\frac{\pi}{2}$, $\varphi(0-0)=-\frac{\pi}{2}$, and the point $t_0=0$ is the first kind of discontinuity of $\varphi(t)$. It is well known that the Fourier series of $\varphi(t)$ is

$$\sum_1^\infty \frac{\sin(kt)}{k}\quad (t\in\mathbb{R}).$$

Consider the sequence of partial sums of the Fourier series of $\varphi(t)$:

$$S_n(\varphi;t)=\sum_1^n \frac{\sin(kt)}{k}\quad (t\in\mathbb{R}).$$

Since $\varphi(t)\in L_{2\pi}$ and is of bounded variation in $0<|t|<\pi$, the Jordan criterion shows that the sequence of partial sums of its Fourier series converges at $t_0=0$ and

$$S_n(\varphi;0)\to\frac{1}{2}(\varphi(0+0)+\varphi(0-0))\quad (n\to\infty).$$

Since $\varphi(0+0)=\frac{\pi}{2}$ and $\varphi(0-0)=-\frac{\pi}{2}$, we get $S_n(\varphi;0)\to 0(n\to\infty)$.

Now we prove $S_n(\varphi;t)$ has a double sublimit lying outside the closed interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ as $n\to\infty, t\to 0$.

Note that

$$\sum_1^n \cos(kv)=\sum_1^n \frac{e^{-ikv}+e^{ikv}}{2}=\frac{1}{2}\left(\sum_{-n}^n e^{ikv}-1\right)=\pi D_n(v)-\frac{1}{2},$$

where $D_n(v)$ is the Dirichlet kernel. Using property (iii) of the Dirichlet kernel, the partial sums of the Fourier series of $\varphi(t)$ can be rewritten as follows:

$$\begin{aligned}S_n(\varphi;t) &= \sum_1^n \frac{\sin(kt)}{k}=\sum_1^n\int_0^t \cos(kv)\,dv=\int_0^t\sum_1^n \cos(kv)\,dv\\ &=\int_0^t \frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}}\,dv-\frac{t}{2}\\ &=\int_0^t \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\,dv+\int_0^t\left(\frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}}-\frac{\sin\left(n+\frac{1}{2}\right)v}{v}\right)\\ &\quad dv-\frac{t}{2}.\end{aligned}\tag{1.6}$$

Let $u = (n + \frac{1}{2})v$. Then the first integral on the right-hand side of (1.6) is

$$\int_0^t \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\,\mathrm{d}v = \int_0^{\left(n+\frac{1}{2}\right)t} \frac{\sin u}{u}\,\mathrm{d}u.$$

Take $t = t_n = \frac{a}{n}$, where a is any real number. Then, as $n \to \infty$ and $t \to 0$,

$$\int_0^{t_n} \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\,\mathrm{d}v = \int_0^{\left(n+\frac{1}{2}\right)\frac{a}{n}} \frac{\sin u}{u}\,\mathrm{d}u \to \int_0^a \frac{\sin u}{u}\,\mathrm{d}u.$$

By inequalities $|\sin\left(n+\frac{1}{2}\right)v| \le 1$ and $|v - 2\sin\frac{v}{2}| \le \left|\frac{v^3}{24}\right|$, and $\sin v \ge \frac{2}{\pi}v\left(0 < v \le \frac{\pi}{2}\right)$, it follows that

$$\left|\sin\left(n+\frac{1}{2}\right)v\left(\frac{1}{2\sin\frac{v}{2}} - \frac{1}{v}\right)\right| = \left|\sin(n+\tfrac{1}{2})v\,\frac{v-2\sin\frac{v}{2}}{2v\sin\frac{v}{2}}\right|$$
$$\le \left|\frac{\frac{v^3}{24}}{\frac{2}{\pi}v^2}\right| = \frac{\pi}{12}|v|,$$

and so the second integral on the right-hand side of (1.6) is

$$\left|\int_0^t \left(\frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\right)\mathrm{d}v\right| \le \frac{\pi}{24}t^2.$$

Take $t = t_n = \frac{a}{n}$. Then

$$\left|\int_0^{t_n} \left(\frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\right)\mathrm{d}v\right| \le \frac{\pi a^2}{24n^2}.$$

As $n \to \infty$ and $t \to 0$,

$$\int_0^{t_n} \left(\frac{\sin\left(n+\frac{1}{2}\right)v}{2\sin\frac{v}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)v}{v}\right)\mathrm{d}v \to 0$$

It is clear that the last term on the right-hand side of (1.6) $\frac{t_n}{2} \to 0$ as $n \to \infty$ and $t \to 0$.

Therefore, take $t = t_n = \frac{a}{n}$, where a is any real number. By (1.6), we have

$$S_n(\varphi; t_n) \to \int_0^a \frac{\sin u}{u}\,\mathrm{d}u =: I(a) \quad (n \to \infty, \quad t \to 0),$$

i.e., $S_n(\varphi; t)$ has double sublimits $I(a)$ as $n \to \infty, t \to 0$. Since a is any real number, all values of $I(a)$ consist of a closed interval $[I(-\pi), I(\pi)]$, and

$$I(\pi) = \int_0^{\pi} \frac{\sin u}{u}\,\mathrm{d}u > \frac{\pi}{2}, \quad I(-\pi) = \int_0^{-\pi} \frac{\sin u}{u}\,\mathrm{d}u < -\frac{\pi}{2},$$

and so $[I(-\pi), I(\pi)] \supset [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Therefore, for the sequence of partial sums $\{S_n(\varphi;t)\}$ the Gibbs phenomenon occurs at $t_0 = 0$.

Theorem 1.5. *Suppose that $f(t)$ is a 2π-periodic function of bounded variation and continuous in a neighborhood of t_0, and t_0 is the first kind of discontinuity of $f(t)$. Then for the sequence of partial sums of the Fourier series of $f(t)$ the Gibbs phenomenon occurs at t_0.*

Proof. Without loss of generality, assume that $f(t)$ is continuous in $0 < |t - t_0| < \delta$ and $f(t_0+0) > f(t_0-0)$. Let $\varphi(t)$ be stated as in Example 1.1, and let

$$g(t) = f(t) - \frac{d}{\pi}\varphi(t-t_0), \tag{1.7}$$

where $d = f(t_0+0) - f(t_0-0) > 0$. By the assumption, we see that $g(t)$ is a 2π-periodic function of bounded variation and continuous in $0 < |t-t_0| < \delta$. According to the Jordan criterion, the partial sums of the Fourier series of $g(t)$ converge and

$$S_n(g;t) \to \frac{1}{2}(g(t_0+0) + g(t_0-0)) \quad (n \to \infty, 0 < |t-t_0| < \delta).$$

Since $\varphi(0+0) = \frac{\pi}{2}$ and $\varphi(0-0) = -\frac{\pi}{2}$ (see Example 1.1), it follows from (1.7) that

$$\begin{aligned} g(t_0+0) &= f(t_0+0) - \frac{d}{2}, \\ g(t_0-0) &= f(t_0-0) + \frac{d}{2}, \end{aligned}$$

and so

$$S_n(g;t) \to \frac{1}{2}(f(t_0+0) + f(t_0-0)), \quad 0 < |t-t_0| < \delta \quad (n \to \infty). \tag{1.8}$$

Now we prove that $S_n(f;t)$ has a double sublimit lying outside the closed interval $[f(t_0-0), f(t_0+0)]$ as $n \to \infty, t \to t_0$.

Denote the partial sums of the Fourier series of $\varphi(t)$ by $S_n(\varphi;t)$. By (1.7), it follows that

$$S_n(f;t) = S_n(g;t) + \frac{d}{\pi}S_n(\varphi;t-t_0).$$

Take $t - t_0 = t_n = \frac{a}{n}$, where a is any real number. Then

$$S_n(f;t_0+t_n) = S_n(g;t_0+t_n) + \frac{d}{\pi}S_n(\varphi;t_n).$$

By Example 1.1,

$$S_n(\varphi;t_n) \to I(a) \quad (n \to \infty, t \to t_0),$$

where $I(a) = \int_0^a \frac{\sin u}{u}\,\mathrm{d}u$. Denote $f(t_0) = \frac{1}{2}(f(t_0+0) + f(t_0-0))$. By (1.8),

$$S_n(g;t_0+t_n) \to f(t_0) \quad (n \to \infty, t \to t_0).$$

Therefore,

$$S_n(f;t_0+t_n) \to f(t_0) + \frac{d}{\pi}I(a) \quad (n \to \infty, t \to t_0),$$

i.e., $S_n(f;t)$ has double sublimits $f(t_0)+\frac{d}{\pi}I(a)$ as $n\to\infty, t\to t_0$. Since a can be any real number, all values of $f(t_0)+\frac{a}{\pi}I(a)$ consist of the closed interval $\left[f(t_0)+\frac{d}{\pi}I(-\pi), f(t_0)+\frac{d}{\pi}I(\pi)\right]$. Noticing that $I(\pi)>\frac{\pi}{2}$ and $I(-\pi)<-\frac{\pi}{2}$, we have

$$\left[f(t_0)+\frac{d}{\pi}I(-\pi), f(t_0)+\frac{d}{\pi}I(\pi)\right]\supset\left[f(t_0)-\frac{d}{2}, f(t_0)+\frac{d}{2}\right].$$

From $f(t_0)=\frac{1}{2}(f(t_0+0)+f(t_0-0))$ and $d=f(t_0+0)-f(t_0-0)$, it follows that

$$\left[f(t_0)+\frac{d}{\pi}I(-\pi), f(t_0)+\frac{d}{\pi}I(\pi)\right]\supset[f(t_0-0), f(t_0+0)].$$

Therefore, for the sequence of partial sums of the Fourier series of $f(t)$ the Gibbs phenomenon occurs at t_0. □

1.4 POISSON SUMMATION FORMULAS AND SHANNON SAMPLING THEOREM

We will introduce three important theorems: the Poisson summation formula in $L(\mathbb{R})$, the Poisson summation formula in $L^2(\mathbb{R})$, and the Shannon sampling theorem. In signal processing, the Poisson summation formula leads to the Shannon sampling theorem and the discrete-time Fourier transform.

To prove the Poisson summation formula in $L(\mathbb{R})$, we first give a relation between Fourier transforms in $L(\mathbb{R})$ and Fourier coefficients in $L_{2\pi}$.

Lemma 1.1. *Let $f\in L(\mathbb{R})$. Then*

- **(i)** *the series $\sum_n f(t+2n\pi)$ is absolutely convergent almost everywhere. Denote its sum by $F(t)$;*
- **(ii)** *$F(t)\in L_{2\pi}$;*
- **(iii)** *for any integer n,*

$$c_n(F)=\frac{1}{2\pi}\widehat{f}(n),$$

where $c_n(F)$ is the Fourier coefficient of $F(t)$ and $\widehat{f}(\omega)$ is the Fourier transform of $f(t)$.

Proof. Consider the series $\sum_n f(t+2n\pi)$. By the assumption that $f\in L(\mathbb{R})$, we have

$$\begin{aligned}\left|\int_0^{2\pi}\sum_n f(t+2n\pi)\,\mathrm{d}t\right| &\le \sum_n\int_0^{2\pi}|f(t+2n\pi)|\,\mathrm{d}x\\ &=\sum_n\int_{2n\pi}^{2(n+1)\pi}|f(y)|\,\mathrm{d}y\\ &=\int_{\mathbb{R}}|f(y)|\,\mathrm{d}y<\infty.\end{aligned}$$

So the series is integrable over $[0, 2\pi]$. Since

$$\sum_n f((t+2\pi)+2n\pi) = \sum_n f(t+2(n+1)\pi) = \sum_n f(t+2n\pi),$$

the series is a 2π-periodic function. Therefore, the series is absolutely convergent almost everywhere. Denote its sum by $F(t)$, i.e.,

$$F(t) = \sum_n f(t+2n\pi) \quad \text{almost everywhere,}$$

and so $F(t)$ is integrable over $[0, 2\pi]$ and is a 2π-periodic function, i.e., $F \in L_{2\pi}$

By the definition of the Fourier coefficients and $\mathrm{e}^{\mathrm{i}n(2k\pi)} = 1$, we have

$$\begin{aligned} c_n(F) &= \frac{1}{2\pi}\int_0^{2\pi} F(t)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_k f(t+2k\pi)\right)\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\sum_k \int_{2k\pi}^{2(k+1)\pi} f(u)\mathrm{e}^{-\mathrm{i}n(u-2k\pi)}\,\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}nu}\,\mathrm{d}u. \end{aligned}$$

However, since $f \in L(\mathbb{R})$, by the definition of the Fourier transform, we have

$$\int_{\mathbb{R}} f(u)\mathrm{e}^{-\mathrm{i}nu}\,\mathrm{d}u = \widehat{f}(n).$$

Therefore, $c_n(F) = \frac{1}{2\pi}\widehat{f}(n)$. □

Poisson Summation Formula I. If $f \in L(\mathbb{R})$ and f satisfies one of the following two conditions:

(i) $f(t)$ is of bounded variation on $\mathbb{R}$ and $f(t) := \frac{1}{2}(f(t+0) + f(t-0))$;

(ii) $|f(t)| \le K_1(1+|t|)^{-\alpha}$ and $|\widehat{f}(\omega)| \le K_2(1+|\omega|)^{-\alpha}$, where $\alpha > 1$ and K_1, K_2 are constants,

then

$$\sum_n f(t+2n\pi) = \frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{int} \quad (t \in \mathbb{R}).$$

Specially,

$$\sum_n f(2n\pi) = \frac{1}{2\pi}\sum_n \widehat{f}(n).$$

Proof. Suppose that $f(t)$ satisfies the first condition. Lemma 1.1 has shown that the series $\sum_n f(t+2n\pi)$ is absolutely convergent almost everywhere. Now we prove that the series $\sum_n f(t+2n\pi)$ is absolutely, uniformly convergent everywhere on $[0, 2\pi]$.

Take $t_0 \in [0, 2\pi]$ such that $\sum_n f(t_0+2n\pi)$ converges. When $0 \le t \le 2\pi$,

$$\begin{aligned}\left|\sum_{|n|>N} f(t+2n\pi)\right| &= \left|\sum_{|n|>N} f(t_0+2n\pi)+f(t+2n\pi)-f(t_0+2n\pi)\right| \\ &\le \left|\sum_{|n|>N} f(t_0+2n\pi)\right| + \left|\sum_{|n|>N} (f(t+2n\pi)-f(t_0+2n\pi))\right| \\ &= I_N(t_0)+\widetilde{I}_N(t).\end{aligned}$$

Since the series $\sum_n f(t_0+2n\pi)$ is convergent and is independent of t,

$$I_N(t_0) \to 0 \quad (N \to \infty)$$

uniformly on $[0, 2\pi]$. Note that $f(t)$ is a function of bounded variation on $\mathbb{R}$. Denote its variation by

$$V_n = \bigvee_{2n\pi}^{2(n+1)\pi} (f).$$

So the total variation is

$$\sum_n V_n = \sum_n \left(\bigvee_{2n\pi}^{2(n+1)\pi} (f) \right) = \bigvee_{-\infty}^{\infty} (f) < \infty,$$

and so for $0 \le t \le 2\pi$,

$$\widetilde{I}_N(t) \le \sum_{|n|>N} |f(t+2n\pi)-f(t_0+2n\pi)| \le \sum_{|n|>N} V_n \to 0 \quad (N \to \infty),$$

i.e., $\widetilde{I}_N(t) \to 0 (N \to \infty)$ uniformly on $[0, 2\pi]$. Therefore,

$$\left|\sum_{|n|>N} f(t+2n\pi)\right| \to 0 \quad (N \to \infty)$$

uniformly on $[0, 2\pi]$, i.e., the series $\sum_n f(t+2n\pi)$ is absolutely, uniformly convergent everywhere on $[0, 2\pi]$. Denote

$$F(t) = \sum_n f(t+2n\pi) \quad (t \in [0, 2\pi]),$$

where $F(t) := \frac{1}{2}(F(t+0)+F(t-0))$ since $f(t) := \frac{1}{2}(f(t+0)+f(t-0))$. Then $F(t)$ is an integrable periodic function of bounded variation with period 2π and its total variation on $[0, 2\pi]$ is

$$\bigvee_0^{2\pi}(F) = \bigvee_0^{2\pi}\left(\sum_n f(t+2n\pi)\right) \le \sum_n\left(\bigvee_0^{2\pi} f(t+2n\pi)\right)$$
$$= \sum_n\left(\bigvee_{2n\pi}^{2(n+1)\pi} f(t)\right) = \bigvee_{-\infty}^{\infty}(f) < \infty,$$

According to the Jordan criterion, the Fourier series of $F(t)$ converges to $F(t)$, i.e.,

$$F(t) = \sum_n c_n(F)\mathrm{e}^{\mathrm{i}nt} \quad (t \in \mathbb{R}),$$

where $c_n(F)$ are the Fourier coefficients of F. By Lemma 1.1, we get $c_n(F) = \frac{1}{2\pi}\widehat{f}(n)$, and so

$$F(t) = \frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{\mathrm{i}nt} \quad (t \in \mathbb{R}).$$

Noticing that $F(t) = \sum_n f(t+2n\pi)$, we have

$$\sum_n f(t+2n\pi) = \frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{\mathrm{i}nt} \quad (t \in \mathbb{R}).$$

Let $t = 0$. Then

$$\sum_n f(2n\pi) = \frac{1}{2\pi}\sum_n \widehat{f}(n),$$

i.e., under condition (i), Poisson summation formula I holds.

Suppose that the function $f(t)$ satisfies condition (ii). Clearly, $f \in L(\mathbb{R})$ and $\widehat{f} \in L(\mathbb{R})$.

Consider the series $\sum_n f(t+2n\pi)$. Since $\widehat{f} \in L(\mathbb{R})$ and $2\pi f(-t) = \widehat{\widehat{f}}(t)$ (Property (v) of the Fourier transform), it follows from Theorem 1.1(iii) that $f(t)$ is uniformly continuous on $\mathbb{R}$. Since $|f(t)| \le K_1(1+|t|)^{-\alpha}(\alpha > 1)$, the series $\sum_n f(t+2n\pi)$ converges uniformly on $\mathbb{R}$. Denote its sum by $F(t)$, i.e., $F(t) = \sum_n f(t+2n\pi)$ on $\mathbb{R}$ uniformly and $F(t)$ is a continuous 2π-periodic function.

Denote the Fourier coefficients of $F(t)$ by $c_n(F)$. Then the Fourier series of $F(t)$ is $\sum_n c_n(F)\mathrm{e}^{\mathrm{i}nt}$. Since $f \in L(\mathbb{R})$, by Lemma 1.1(iii), $c_n(F) = \frac{1}{2\pi}\widehat{f}(n)$. So the Fourier series of $F(t)$ is $\frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{\mathrm{i}nt}$.

By the condition (ii), $|\widehat{f}(n)| \le K_2(1+|n|)^{-\alpha}(\alpha > 1)$. So $\widehat{f}(n) \to 0$ monotonously as $n \to \infty$. By use of the Dirichlet criterion in calculus, it follows that $\frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{\mathrm{i}nt} = F(t)(t \in \mathbb{R})$, i.e.,

$$\sum_n f(t+2n\pi) = \frac{1}{2\pi}\sum_n \widehat{f}(n)\mathrm{e}^{\mathrm{i}nt} \quad (t \in \mathbb{R}).$$

Let $t = 0$. Then

$$\sum_n f(2n\pi) = \frac{1}{2\pi} \sum_n \widehat{f}(n),$$

i.e., under condition (ii), Poisson summation formula I holds. □

The derivation of the Poisson summation formula in $L^2(\mathbb{R})$ needs the following lemma.

Lemma 1.2 (Convolution in Frequency)**.** *Suppose that* $f, g \in L^2(\mathbb{R})$*. Then*

$$2\pi (fg)^\wedge(\omega) = (\widehat{f} * \widehat{g})(\omega).$$

i.e., the convolution of Fourier transforms of two functions is equal to 2π *times the Fourier transform of the product of these two functions.*

Proof. By $f, g \in L^2(\mathbb{R})$, it follows that $fg \in L(\mathbb{R})$. So the Fourier transform of fg is

$$(fg)^\wedge(\omega) = \int_{\mathbb{R}} f(t) g(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t.$$

Let $h(t) = \overline{g}(t)\mathrm{e}^{\mathrm{i}\omega t}$, and then using Theorem 1.3, we get

$$(fg)^\wedge(\omega) = \int_{\mathbb{R}} f(t)\overline{h}(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u)\overline{\widehat{h}}(u) \, \mathrm{d}u.$$

However, by the definition of the Fourier transform, the factor of the integrand on the right-side hand

$$\overline{\widehat{h}}(u) = \overline{\int_{\mathbb{R}} h(t)\mathrm{e}^{-\mathrm{i}ut} \, \mathrm{d}t} = \overline{\int_{\mathbb{R}} \overline{g}(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}ut} \, \mathrm{d}t} = \int_{\mathbb{R}} g(t)\mathrm{e}^{-\mathrm{i}(\omega-u)t} \, \mathrm{d}t = \widehat{g}(\omega - u).$$

Therefore,

$$(fg)^\wedge(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u)\widehat{g}(\omega - u) \, \mathrm{d}u = \frac{1}{2\pi}(\widehat{f} * \widehat{g})(\omega).$$

We get the desired result. □

On the basis of Lemma 1.2 and Poisson summation formula I, we have

Poisson Summation Formula II. If $f \in L^2(\mathbb{R})$ and f satisfies one of the following two conditions:

(i) $\widehat{f}(\omega)$ is a function of bounded variation on $\mathbb{R}$;
(ii) $|f(t)| \le K_1|t|^{-\beta}\,(\beta > 1)$ and $|\widehat{f}(\omega)| \le K_2|\omega|^{-\alpha}\,(\alpha > \frac{1}{2})$, where K_1 and K_2 are constants, then

$$\sum_n |\widehat{f}(\omega + 2n\pi)|^2 = \sum_n \left(\int_{\mathbb{R}} f(t)\overline{f}(n+t) \, \mathrm{d}t \right) \mathrm{e}^{\mathrm{i}n\omega} \quad (\omega \in \mathbb{R}).$$

Proof. Let

$$\varphi(\omega) = |\widehat{f}(\omega)|^2 = \widehat{f}(\omega)\overline{\widehat{f}}(\omega).$$

By the assumption $f \in L^2(\mathbb{R})$ and Definition 1.1, $\widehat{f} \in L^2(\mathbb{R})$, and so $\varphi \in L(\mathbb{R})$.

Suppose that $f(t)$ satisfies the first condition. Then φ is a function of bounded variation on $\mathbb{R}$. Define $\varphi(\omega) = \frac{1}{2}(\varphi(\omega+0)+\varphi(\omega-0))$. So $\varphi(\omega)$ satisfies the first condition of Poisson summation formula I.

Suppose that $f(t)$ satisfies the second condition. By the assumption $|\widehat{f}(\omega)| \le K_2|\omega|^{-\alpha}(\alpha > \frac{1}{2})$, we get $|\varphi(\omega)| \le K_2^2|\omega|^{-2\alpha}(2\alpha > 1)$. By using Lemma 1.2, we get

$$\widehat{\varphi}(u) = \left(\widehat{f}\overline{\widehat{f}}\right)^{\wedge}(u) = \frac{1}{2\pi}\left(\widehat{\widehat{f}} * \widehat{\overline{\widehat{f}}}\right)(u).$$

By Properties (iv) and (v) of the Fourier transform, $\widehat{\widehat{f}}(u) = 2\pi f(-u)$ and $\widehat{\overline{\widehat{f}}}(u) = \overline{\widehat{\widehat{f}}}(-u) = 2\pi\overline{f}(u)$, and so

$$\widehat{\varphi}(u) = 2\pi f(-u) * \overline{f}(u) = 2\pi \int_{\mathbb{R}} f(t)\overline{f}(u+t)\,\mathrm{d}t, \tag{1.9}$$

which can be rewritten in the form

$$\widehat{\varphi}(u) = 2\pi\left(\int_{|t|\le\frac{|u|}{2}} + \int_{|t|>\frac{|u|}{2}}\right) f(t)\overline{f}(u+t)\,\mathrm{d}t = I_1(u) + I_2(u).$$

When $|t| \le \frac{|u|}{2}$, we have $|u+t| \ge |u| - |t| \ge \frac{|u|}{2}$. From this and the assumption $|f(t)| \le K_1|t|^{-\beta}(\beta > 1)$, we get

$$\begin{aligned}
|I_1(u)| &\le 2\pi \int_{|t|\le\frac{|u|}{2}} |f(t)\overline{f}(u+t)|\,\mathrm{d}t \\
&\le 2\pi K_1^2 \int_{|t|\le\frac{|u|}{2}} \frac{1}{|t(u+t)|^\beta}\,\mathrm{d}t \\
&\le 2\pi \frac{2^\beta K_1^2}{|u|^\beta}\int_{\mathbb{R}} \frac{1}{|t|^\beta}\,\mathrm{d}t \le K_3|u|^{-\beta} \quad (\beta > 1),
\end{aligned}$$

where K_3 is a constant.

When $|t| > \frac{|u|}{2}$, by the assumption $|f(t)| \le K_1|t|^{-\beta}(\beta > 1)$, we get

$$\begin{aligned}
|I_2(u)| &\le 2\pi \int_{|t|>\frac{|u|}{2}} |f(t)\overline{f}(u+t)|\,\mathrm{d}t \\
&\le 2\pi K_1^2 \int_{|t|>\frac{u}{2}} \frac{1}{|t(u+t)|^\beta}\,\mathrm{d}t \\
&\le 2\pi \frac{2^\beta K_1^2}{|u|^\beta}\int_{\mathbb{R}} \frac{1}{|u+t|^\beta}\,\mathrm{d}t \le K_4|u|^{-\beta}, \quad \beta > 1,
\end{aligned}$$

where K_4 is a constant.

Therefore, $\widehat{\varphi}(u) \leq K|u|^{-\beta}(\beta > 1)$, where K is a constant. Therefore, φ satisfies the second condition of Poisson summation formula I.

Using Poisson summation formula I, we get

$$\sum_n \varphi(\omega + 2n\pi) = \frac{1}{2\pi} \sum_n \widehat{\varphi}(n)\, \mathrm{e}^{\mathrm{i}n\omega}.$$

By (1.9), $\widehat{\varphi}(n) = 2\pi \int_{\mathbb{R}} f(t)\overline{f}(n+t)\,\mathrm{d}t$, noticing that $\varphi(\omega) = |\widehat{f}(\omega)|^2$, we can rewrite this equality in the form

$$\sum_n |\widehat{f}(\omega + 2n\pi)|^2 = \sum_n \left(\int_{\mathbb{R}} f(t)\overline{f}(n+t)\,\mathrm{d}t \right) \mathrm{e}^{\mathrm{i}n\omega}.$$

So Poisson summation formula II holds. □

The following lemma is used to prove the Shannon sampling theorem.

Lemma 1.3. *Let $X(\omega)$ be the characteristic function of $[-\pi, \pi]$, i.e.,*

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \pi, \\ 0, & |\omega| > \pi. \end{cases}$$

Then the inverse Fourier transform of $X(\omega)\mathrm{e}^{-\mathrm{i}n\omega}(n \in \mathbb{Z})$ is equal to $\frac{\sin \pi(t-n)}{\pi(t-n)}$, i.e.,

$$(X(\omega)\mathrm{e}^{-\mathrm{i}n\omega})^\vee(t) = \frac{\sin \pi(t-n)}{\pi(t-n)} \quad (n \in \mathbb{Z}).$$

Proof. It is clear that $X(\omega)\mathrm{e}^{-\mathrm{i}n\omega} \in L(\mathbb{R})$, and its inverse Fourier transform is

$$\begin{aligned} (X(\omega)\mathrm{e}^{-\mathrm{i}n\omega})^\vee(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} (X(\omega)\mathrm{e}^{-\mathrm{i}n\omega})\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} X(\omega)\mathrm{e}^{\mathrm{i}(t-n)\omega}\,\mathrm{d}\omega. \end{aligned}$$

Since $X(\omega) = 1(|\omega| \leq \pi)$ and $X(\omega) = 0(|\omega| > \pi)$, we get

$$\begin{aligned} (X(\omega)\mathrm{e}^{-\mathrm{i}n\omega})^\vee(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(t-n)\omega}\,\mathrm{d}\omega = \frac{1}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\pi(t-n)} - \mathrm{e}^{-\mathrm{i}\pi(t-n)}}{\mathrm{i}(t-n)} \\ &= \frac{\sin \pi(t-n)}{\pi(t-n)}. \end{aligned}$$

□

Shannon Sampling Theorem. *Let $f \in L^2(\mathbb{R})$ and its Fourier transform $\widehat{f}(\omega) = 0(|\omega| \geq \pi)$. Then the interpolation formula*

$$f(t) = \sum_n f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \quad (L^2)$$

holds, and the series $\sum_n f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$ converges uniformly to a continuous function $g(t)$ in every closed interval on $\mathbb{R}$ and $g(t) = f(t)$ almost everywhere.

Proof. From $\widehat{f}(\omega) = 0(|\omega| \geq \pi)$, it follows that $\widehat{f} \in L^2(\mathbb{R})$ and $\widehat{f} \in L(\mathbb{R})$. Take a 2π-periodic function $f_p(\omega)$ such that $f_p(\omega) = \widehat{f}(\omega)(|\omega| \leq \pi)$. Then $f_p(\omega) \in L_{2\pi}$ and $\widehat{f}(\omega) = f_p(\omega)X(\omega)$, where $X(\omega)$ is the characteristic function of $[-\pi, \pi]$.

We expand $f_p(\omega)$ into the Fourier series

$$f_p(\omega) = \sum_n c_n(f_p)e^{in\omega}(L^2), \tag{1.10}$$

where $c_n(f_p)$ are Fourier coefficients and

$$c_n(f_p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_p(\omega)e^{-in\omega}\,d\omega \quad (n \in \mathbb{Z}).$$

By $\widehat{f}(\omega) = f_p(\omega)(|\omega| \leq \pi)$ and the assumption $\widehat{f}(\omega) = 0(|\omega| \geq \pi)$, and $\widehat{\widehat{f}}(t) = 2\pi f(-t)$ (property of the Fourier transform), it follows that

$$\begin{aligned} c_n(f_p) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{-in\omega}\,d\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)e^{-in\omega}d\omega \\ &= \frac{1}{2\pi}\widehat{\widehat{f}}(n) = f(-n) \quad (n \in \mathbb{Z}). \end{aligned} \tag{1.11}$$

Combining this with (1.9), we get

$$f_p(\omega) = \sum_n f(-n)e^{in\omega} \quad (L^2).$$

Noticing that $\widehat{f}(\omega) = f_p(\omega)X(\omega)$, we get

$$\widehat{f}(\omega) = \sum_n f(-n)X(\omega)e^{in\omega} = \sum_n f(n)X(\omega)e^{-in\omega}.$$

Taking the inverse Fourier transform on both sides, we get

$$f(t) = \sum_n f(n)(X(\omega)e^{-in\omega})^{\vee}(t) \quad (L^2).$$

By Lemma 1.3, we get an interpolation formula:

$$f(t) = \sum_n f(n)\frac{\sin(\pi(t-n))}{\pi(t-n)} \quad (L^2). \tag{1.12}$$

From this, the Riesz theorem shows that the series $\sum_n f(n)\frac{\sin(\pi(t-n))}{\pi(t-n)}$ converges to $f(t)$ almost everywhere.

On the other hand, for Fourier series (1.10), by using Bessel's inequality, we get

$$\sum_n |c_n(f_p)|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_p(\omega)|^2\,d\omega.$$

By (1.11), $c_n(f_p) = f(-n)(n \in \mathbb{Z})$, the left-hand side is

$$\sum_n |c_n(f_p)|^2 = \sum_n |f(-n)|^2 = \sum_n |f(n)|^2.$$

By $f_p(\omega) = f(\omega)(|\omega| \leq \pi)$ and $\widehat{f}(\omega) = 0(|\omega| \geq \pi)$, the right-hand side is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_p(\omega)|^2\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2\,\mathrm{d}\omega = \int_{\mathbb{R}} |\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

Therefore,

$$\sum_n |f(n)|^2 \leq \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

From $\widehat{f} \in L^2(\mathbb{R})$, it follows that $\sum_n |f(n)|^2 < \infty$. So the series $\sum_n |f(n)|^2$ converges. Since $\left|\frac{\sin(\pi(t-n))}{\pi(t-n)}\right| \leq \frac{1}{|t-n|}$, the series $\sum_n |\frac{\sin \pi(t-n)}{\pi(t-n)}|^2$ converges uniformly in every closed interval on $\mathbb{R}$.

According to Cauchy's principle of convergence in calculus, for $\epsilon > 0$, there is an $N > 0$ such that when $M \geq m > N$,

$$\sum_{m \leq |k| \leq M} |f(k)|^2 < \epsilon, \qquad \sum_{m \leq |k| \leq M} \left|\frac{\sin \pi(t-k)}{\pi(t-k)}\right|^2 < \epsilon$$

hold simultaneously in every closed interval on $\mathbb{R}$. By using Cauchy's inequality, we have

$$\left|\sum_{m \leq |k| \leq M} f(k)\frac{\sin \pi(t-k)}{\pi(t-k)}\right|^2 \leq \left(\sum_{m \leq |k| \leq M} |f(k)|^2\right)\left(\sum_{m \leq |k| \leq M} \left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2\right).$$

Therefore, for the above $\epsilon > 0$ and $N > 0$, when $M \geq m > N$,

$$\left|\sum_{m \leq |k| \leq M} f(k)\frac{\sin \pi(t-k)}{\pi(t-k)}\right| < \epsilon$$

in every closed interval on $\mathbb{R}$. According to Cauchy's principle of convergence, the series

$$\sum_n f(n)\frac{\sin \pi(t-n)}{\pi(t-n)}$$

converges uniformly in every closed interval on $\mathbb{R}$ to a continuous function, denoted by $g(t)$. By (1.12), we get $g(t) = f(t)$ almost everywhere. □

1.5 DISCRETE FOURIER TRANSFORM

Discrete Fourier transforms are used in discrete signal or discrete time series. The discrete Fourier transform is defined as follows.

Given an N-point time series $x = (x_0, x_1, \ldots, x_{N-1})$, the *discrete Fourier transform* of x is defined as

$$X_k = \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} \quad (k = 0, 1, \ldots, N-1).$$

In this definition, x_n is called the *sample*, N is called the *number of samples*, $\Delta\omega = \frac{2\pi}{N}$ is called the *sampling frequency interval*, $\omega_k = \frac{2\pi k}{N}$ is called the *discrete frequency*, X_k is called the *frequency coefficient*, and $\{|X_k|^2\}_{k=0,\ldots,N-1}$ is called the *Fourier power spectrum* of x. In detail, the discrete Fourier transform gives a system of equations as follows:

$$\begin{aligned}
X_0 &= \frac{1}{N} \sum_{0}^{N-1} x_n = \frac{1}{N}(x_0 + x_1 + x_2 + \cdots + x_{N-1}), \\
X_1 &= \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi}{N}} = \frac{1}{N}\left(x_0 + x_1 \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}} + x_2 \mathrm{e}^{-\mathrm{i}\frac{4\pi}{N}} + \cdots + x_{N-1} \mathrm{e}^{-\mathrm{i}\frac{2(N-1)\pi}{N}}\right), \\
&\vdots \\
X_{N-1} &= \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi(N-1)}{N}} \\
&= \frac{1}{N}\left(x_0 + x_1 \mathrm{e}^{-\mathrm{i}\frac{2(N-1)\pi}{N}} + x_2 \mathrm{e}^{-\mathrm{i}\frac{4(N-1)\pi}{N}} + \cdots + x_{N-1} \mathrm{e}^{-\mathrm{i}\frac{2(N-1)^2\pi}{N}}\right).
\end{aligned}$$

It can also be rewritten in the matrix form $\mathbf{X} = \frac{1}{N}\mathbf{F}\mathbf{x}$, where

$$\mathbf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

and

$$\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}} & \mathrm{e}^{-\mathrm{i}\frac{4\pi}{N}} & \cdots & \mathrm{e}^{-\mathrm{i}\frac{2(N-1)\pi}{N}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{e}^{-\mathrm{i}\frac{2(N-1)\pi}{N}} & \mathrm{e}^{-\mathrm{i}\frac{4(N-1)\pi}{N}} & \cdots & \mathrm{e}^{-\mathrm{i}\frac{2(N-1)^2\pi}{N}} \end{pmatrix} = \left(\mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}}\right)_{k,n=0,1,\ldots,N-1}.$$

If all x_n are real, the discrete Fourier transform possesses the property of symmetry:

$$X_{N-k} = \overline{X}_k \quad (k = 0, 1, \ldots, N-1).$$

In fact, by the definition of the discrete Fourier transform,

$$X_{N-k} = \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n \frac{2\pi(N-k)}{N}}.$$

Since $\mathrm{e}^{-\mathrm{i}n \frac{2\pi(N-k)}{N}} = \mathrm{e}^{-\mathrm{i}(2\pi n - \frac{2n\pi k}{N})} = \mathrm{e}^{\mathrm{i}n \frac{2\pi k}{N}}$, we get

$$X_{N-k} = \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{\mathrm{i}n \frac{2\pi k}{N}}.$$

On the other hand, since x_n are real, it follows by the definition that

$$\overline{X}_k = \frac{1}{N} \sum_{0}^{N-1} \overline{x}_n \overline{\mathrm{e}^{-\mathrm{i}n \frac{2\pi k}{N}}} = \frac{1}{N} \sum_{0}^{N-1} x_n \mathrm{e}^{\mathrm{i}n \frac{2\pi k}{N}}.$$

Therefore, $X_{N-k} = \overline{X}_k (k = 0, 1, \ldots, N-1)$.

Given N frequency coefficients $\{X_k\}_{k=0,\ldots,N-1}$, the *inverse discrete Fourier transform* is defined as

$$x_n = \sum_{0}^{N-1} X_k \mathrm{e}^{\mathrm{i}k \frac{2\pi n}{N}} \quad (n = 0, \ldots, N-1).$$

In detail, the inverse discrete Fourier transform gives the system of equations as follows:

$$\begin{aligned}
x_0 &= \sum_{0}^{N-1} X_k = X_0 + X_1 + X_2 + \cdots + X_{N-1}, \\
x_1 &= \sum_{0}^{N-1} X_k \mathrm{e}^{\mathrm{i}k \frac{2\pi}{N}} = X_0 + X_1 \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}} + X_2 \mathrm{e}^{\mathrm{i}\frac{4\pi}{N}} + \cdots + X_{N-1} \mathrm{e}^{\mathrm{i}\frac{2(N-1)\pi}{N}}, \\
&\vdots \\
x_{N-1} &= \sum_{0}^{N-1} X_k \mathrm{e}^{\mathrm{i}k \frac{2(N-1)\pi}{N}} = X_0 + X_1 \mathrm{e}^{\mathrm{i}\frac{2(N-1)\pi}{N}} + X_2 \mathrm{e}^{\mathrm{i}\frac{4(N-1)\pi}{N}} + \cdots + X_{N-1} \mathrm{e}^{\mathrm{i}\frac{2(N-1)^2\pi}{N}}.
\end{aligned}$$

It can also be written in the matrix form $\mathbf{x} = \overline{\mathbf{F}}\mathbf{X}$, where

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{pmatrix},$$

and

$$\overline{\mathbf{F}} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}} & \mathrm{e}^{\mathrm{i}\frac{4\pi}{N}} & \dots & \mathrm{e}^{\mathrm{i}\frac{2\pi(N-1)}{N}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{e}^{\mathrm{i}\frac{2(N-1)\pi}{N}} & \mathrm{e}^{\mathrm{i}\frac{4(N-1)\pi}{N}} & \dots & \mathrm{e}^{\mathrm{i}\frac{2(N-1)^2\pi}{N}} \end{pmatrix} = \left(\mathrm{e}^{\mathrm{i}n\frac{2\pi k}{N}}\right)_{k,n=0,1,\dots,N-1}.$$

If all X_n are real, the inverse discrete Fourier transform possesses the property of symmetry:

$$x_{N-k} = \overline{x}_k \quad (k = 0, 1, \dots, N-1).$$

In fact, the definition of the inverse discrete Fourier transform shows that

$$x_{N-k} = \sum_{0}^{N-1} X_n \mathrm{e}^{\mathrm{i}n\frac{2\pi(N-k)}{N}}.$$

Since $\mathrm{e}^{\mathrm{i}n\frac{2\pi(N-k)}{N}} = \mathrm{e}^{\mathrm{i}2\pi n}\mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} = \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}}$, we have

$$x_{N-k} = \sum_{0}^{N-1} X_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}}.$$

On the other hand, since X_n are real, it follows by the definition that

$$\overline{x}_k = \sum_{0}^{N-1} \overline{X}_n \overline{\mathrm{e}^{\mathrm{i}n\frac{2\pi k}{N}}} = \sum_{0}^{N-1} X_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}}.$$

Therefore, $x_{N-k} = \overline{x}_k (k = 0, 1, \dots, N-1)$.

Similarly, we consider the two-dimensional case. Given a two-dimensional discrete $M \times N$-point time series,

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,N-1} \\ x_{1,0} & x_{1,1} & \cdots & x_{1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{M-1,0} & x_{M-1,1} & \cdots & x_{M-1,N-1} \end{pmatrix},$$

the *two-dimensional discrete Fourier transform* of x is defined as

$$X_{k,l} = \frac{1}{MN} \sum_{m=0}^{M-1} \left(\sum_{n=0}^{N-1} x_{m,n} \mathrm{e}^{-\mathrm{i}n\frac{2\pi l}{N}} \right) \mathrm{e}^{-\mathrm{i}m\frac{2\pi k}{M}}$$

$$(k = 0, \dots, M-1; l = 0, \dots, N-1).$$

In this definition, $x_{m,n}$ is called the *sample* and $X_{k,l}$ are called the *frequency coefficient*. The two-dimensional discrete Fourier transform is a transform from an $M \times N$-point time series to $M \times N$ frequency coefficients.

Conversely, given $M \times N$ frequency coefficients:

$$\begin{array}{cccc} X_{0,0} & X_{0,1} & \cdots & X_{0,N-1}, \\ X_{1,0} & X_{1,1} & \cdots & X_{1,N-1}, \\ \vdots & \vdots & \vdots & \vdots \\ X_{M-1,0} & X_{M-1,1} & \cdots & X_{M-1,N-1}. \end{array}$$

The *two-dimensional inverse discrete Fourier transform* of $\{X_{k,l}\}$ is defined as

$$x_{m,n} = \sum_{k=0}^{M-1} \left(\sum_{l=0}^{N-1} X_{k,l} e^{il\frac{2\pi n}{N}} \right) e^{ik\frac{2\pi m}{M}}$$
$$(m = 0, 1, \ldots, M-1; n = 0, 1, \ldots, N-1).$$

The two-dimensional inverse discrete Fourier transform is a transform from $M \times N$ frequency coefficients to an $M \times N$-point time series.

1.6 FAST FOURIER TRANSFORM

The fast Fourier transform is not a new transform, but is a fast algorithm for computing discrete Fourier transform. It is based on the halving trick. This trick halves a given $2N$-point time series into two N-point time subseries, and then discrete Fourier transforms of these two N-point time subseries are used to compute the discrete Fourier transform of the given $2N$-point time series.

Given a $2N$-point time series $z = (z_0, z_1, \ldots, z_{2N-1})$, its discrete Fourier transform is

$$Z_k = \frac{1}{2N} \sum_{0}^{2N-1} z_n e^{-in\frac{2\pi k}{2N}} \quad (k = 0, \ldots, 2N-1).$$

We use the halving trick to halve the $2N$ frequency coefficients $\{Z_k\}$.

First, we compute the first half: $Z_0, Z_1, \ldots, Z_{N-1}$. We decompose the $2N$-point time series z into two N-point time series x and y as follows:

$$x = (z_0, z_2, z_4, \ldots, z_{2N-2}) =: (x_0, x_1, x_2 \ldots, x_{N-1}),$$
$$y = (z_1, z_3, z_5 \ldots, z_{2N-1}) =: (y_0, y_1, y_2, \ldots, y_{N-1}),$$

i.e., x consists of even samples of $\{z_n\}$ and y consists of odd samples of $\{z_n\}$. So

$$Z_k = \frac{1}{2N} \sum_{0}^{N-1} z_{2n} e^{i2n\frac{2\pi k}{2N}} + \frac{1}{2N} \sum_{0}^{N-1} z_{2n+1} e^{i(2n+1)\frac{2\pi k}{2N}}$$
$$= \frac{1}{2N} \sum_{0}^{N-1} x_n e^{i2n\frac{2\pi k}{2N}} + \frac{1}{2N} \sum_{0}^{N-1} y_n e^{i(2n+1)\frac{2\pi k}{2N}} \quad (k = 0, \ldots, N-1),$$

where $x_n = z_{2n}$ and $y_n = z_{2n+1}$. Since $e^{-i(2n+1)\frac{2\pi k}{2N}} = e^{-in\frac{2\pi k}{N}} e^{-i\frac{\pi k}{N}}$, we get

$$\begin{aligned} Z_k &= \frac{1}{2N}\sum_0^{N-1} x_n e^{-in\frac{2\pi k}{N}} + e^{-i\frac{\pi k}{N}} \frac{1}{2N}\sum_0^{N-1} y_n e^{-in\frac{2\pi k}{N}} \\ &= \frac{1}{2}(X_k + e^{-i\frac{\pi k}{N}} Y_k) \quad (k = 0, \dots, N-1), \end{aligned}$$

where

$$X_k = \frac{1}{N}\sum_0^{N-1} x_n e^{-in\frac{2\pi k}{N}} \quad (k = 0, \dots, N-1),$$

$$Y_k = \frac{1}{N}\sum_0^{N-1} y_n e^{-in\frac{2\pi k}{N}} \quad (k = 0, \dots, N-1).$$

Clearly, they are just discrete Fourier transforms of $x = (x_0, x_1, \dots, x_{N-1})$ and $y = (y_0, y_1, \dots, y_{N-1})$, respectively.

Second, we compute the second half: $Z_N, Z_{N+1}, \dots, Z_{2N-1}$. Note that

$$Z_k = \frac{1}{2N}\sum_0^{2N-1} z_n e^{-in\frac{2\pi k}{2N}} \quad (k = N, \dots, 2N-1).$$

Taking the substitution $k = j + N$, we get

$$Z_{j+N} = \frac{1}{2N}\sum_0^{2N-1} z_n e^{-in\frac{2\pi(j+N)}{2N}} \quad (j = 0, \dots, N-1).$$

We decompose Z_{j+N} into two sums according to even samples and odd samples as follows:

$$\begin{aligned} Z_{j+N} &= \frac{1}{2N}\sum_0^{N-1} z_{2n} e^{-i2n\frac{2\pi(j+N)}{2N}} + \frac{1}{2N}\sum_0^{N-1} z_{2n+1} e^{-i(2n+1)\frac{2\pi(j+N)}{2N}} \\ & (j = 0, \dots, N-1). \end{aligned}$$

Noticing that

$$\begin{aligned} e^{-in\frac{2\pi(j+N)}{N}} &= e^{-in\frac{2\pi j}{N}} e^{-i2\pi n} = e^{-in\frac{2\pi j}{N}}, \\ e^{-i(2n+1)\frac{2\pi(j+N)}{2N}} &= e^{-i(2n+1)\frac{\pi j}{N}} e^{-i\pi(2n+1)} = -e^{-in\frac{2\pi j}{N}} e^{-i\frac{\pi j}{N}}, \end{aligned}$$

we get

$$Z_{j+N} = \frac{1}{2N}\sum_0^{N-1} z_{2n} e^{-in\frac{2\pi j}{N}} - e^{-i\frac{\pi j}{N}} \frac{1}{2N}\sum_0^{N-1} z_{2n+1} e^{-in\frac{2\pi j}{N}} \quad (j = 0, \dots, N-1).$$

Noticing that $x_n = z_{2n}$ and $y_n = z_{2n+1}$, and replacing j by k, we get

$$\begin{aligned} Z_{k+N} &= \frac{1}{2N}\sum_0^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} - \mathrm{e}^{-\mathrm{i}\frac{\pi k}{N}} \frac{1}{2N}\sum_0^{N-1} y_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} \\ &= \frac{1}{2}(X_k - \mathrm{e}^{-\mathrm{i}\frac{\pi k}{N}} Y_k) \quad (k = 0, \ldots, N-1), \end{aligned}$$

where X_k and Y_k are stated as above.

Summarizing the above procedure, we see that for the given $2N$-point time series $z = (z_0, z_1, \ldots, z_{2N-1})$, its discrete Fourier transform is

$$Z_k = \frac{1}{2N}\sum_0^{2N-1} z_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{2N}} \quad (k = 0, \ldots, 2N-1).$$

Halving these frequency coefficients $Z_k (k = 0, \ldots, 2N-1)$, we obtain a pair of frequency coefficients:

$$\begin{aligned} Z_k &= \frac{1}{2}(X_k + \mathrm{e}^{-\mathrm{i}\frac{\pi k}{N}} Y_k) \quad (k = 0, \ldots, N-1), \\ Z_{k+N} &= \frac{1}{2}(X_k - \mathrm{e}^{-\mathrm{i}\frac{\pi k}{N}} Y_k) \quad (k = 0, \ldots, N-1), \end{aligned}$$

where

$$\begin{aligned} X_k &= \frac{1}{N}\sum_0^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} \quad (k = 0, \ldots, N-1), \\ Y_k &= \frac{1}{N}\sum_0^{N-1} y_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} \quad (k = 0, \ldots, N-1), \end{aligned}$$

are discrete Fourier transforms of two N-point time subseries x and y which consist of even samples and odd samples of z, respectively.

Now we explain the procedure of the fast Fourier transform.

Given a 2^N-point time series $z = (z_0, z_1, \ldots, z_{2^N-1})$, its discrete Fourier transform is

$$Z_k = \frac{1}{2^N}\sum_0^{2^N-1} z_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{2^N}} \quad (k = 0, 1, \ldots, 2^N-1).$$

Halving these frequency coefficients $Z_k (k = 0, 1, \ldots, 2^N-1)$, we obtain a pair of frequency coefficients:

$$\begin{aligned} Z_k &= \frac{1}{2}(X_k + \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^N}} Y_k) \quad (k = 0, \ldots, 2^{N-1}-1), \\ Z_{k+2^{N-1}} &= \frac{1}{2}(X_k - \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^N}} Y_k) \quad (k = 0, \ldots, 2^{N-1}-1), \end{aligned}$$

where X_k $(k = 0, \ldots, 2^{N-1} - 1)$ and Y_k $(k = 0, \ldots, 2^{N-1} - 1)$ are the discrete Fourier transforms of two 2^{N-1}-point time subseries x and y which consist of even samples and odd samples of z, respectively.

Again, halving $X_k, Y_k (k = 0, \ldots, 2^{N-1} - 1)$, we obtain two pairs of frequency coefficients:

$$\begin{aligned} X_k &= \frac{1}{2}(X'_k + \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^{N-1}}} X''_k) \quad (k = 0, \ldots, 2^{N-2} - 1), \\ X_{k+2^{N-2}} &= \frac{1}{2}(X'_k - \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^{N-1}}} X''_k) \quad (k = 0, \ldots, 2^{N-2} - 1), \end{aligned}$$

and

$$\begin{aligned} Y_k &= \frac{1}{2}(Y'_k + \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^{N-1}}} Y''_k) \quad (k = 0, \ldots, 2^{N-2} - 1), \\ Y_{k+2^{N-2}} &= \frac{1}{2}(Y'_k - \mathrm{e}^{-\mathrm{i}\frac{2\pi k}{2^{N-1}}} Y''_k) \quad (k = 0, \ldots, 2^{N-2} - 1), \end{aligned}$$

where X'_k, X''_k and $Y'_k, Y''_k (k = 0, \ldots, 2^{N-2} - 1)$ are the discrete Fourier transforms of two 2^{N-2}-point time subseries which consist of even samples and odd samples of x and y, respectively.

If the above procedure is continued again and again, the fast Fourier transform algorithm terminates at the one-point time subseries.

For this fast algorithm, the total number of multiplication operations is equal to $N2^{N-1}$. Let $2^N = M$. Then the total number of multiplication operations is equal to $\frac{1}{2}(\log_2 M)M$. For the original discrete Fourier transform algorithm, the total number of multiplication operations is equal to $2^{2N} = M^2$. So the fast Fourier transform has better computationally efficiency than the discrete Fourier transform.

Zero padding is another trick. It can be used to decrease the frequency interval. The zero padding trick is as follows.

Given an N-point time series $x = (x_0, x_1, \ldots, x_{N-1})$, the discrete Fourier transform of x is

$$X_k = \frac{1}{N}\sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{N}} \quad (k = 0, 1, \ldots, N-1).$$

The sampling frequency interval $\Delta\omega = \frac{2\pi}{N}$, where N is the number of samples. From this, we see that the sampling frequency interval is controlled by the number of samples of the time series.

Let $M > N$. We define a new M-point time series as follows:

$$x^{\text{new}} = (x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots, 0).$$

The discrete Fourier transform of the new M-point time series is

$$X_k^{\text{new}} = \frac{1}{M}\sum_{0}^{M-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{M}} \quad (k = 0, 1, \ldots, M-1).$$

Note that $x_n = 0 (n = N, \ldots, M-1)$, and the discrete Fourier transform of the new M-point time series is

$$X_k^{\text{new}} = \frac{1}{M}\sum_{0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}n\frac{2\pi k}{M}} \quad (k = 0, \ldots, M-1).$$

The new sampling frequency interval $\Delta\omega^{\text{new}} = \frac{2\pi}{M}$. By $M > N$, we see that

$$\Delta\omega^{\text{new}} = \frac{2\pi}{M} < \frac{2\pi}{N} = \Delta\omega.$$

This means that when the zero padding trick is used, the sampling frequency interval decreases.

1.7 HEISENBERG UNCERTAINTY PRINCIPLE

The Heisenberg uncertainty principle is the fundament of time-frequency analysis in Chapter 2. This principle is related to the temporal variance and the frequency variance of signals of finite energy.

Heisenberg Uncertainty Principle. *If $f \in L^2(\mathbb{R})$, then*

$$\left(\int_{\mathbb{R}} t^2 |f(t)|^2 \,\mathrm{d}t\right)\left(\int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega\right) \geq \frac{\pi}{2} \parallel f \parallel_2^4 .$$

In particular, the necessary and sufficient condition that the sign of equality holds is $f(t) = C\mathrm{e}^{-t^2/4a}$, where C is a constant and $a > 0$.

Proof. By the assumption $f \in L^2(\mathbb{R})$ and Definition 1.1, it is clear that $\widehat{f} \in L^2(\mathbb{R})$. When

$$\int_{\mathbb{R}} t^2 |f(t)|^2 \,\mathrm{d}t = \infty \quad \text{or} \quad \int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega = \infty,$$

the conclusion holds clearly. Therefore, we may assume that

$$\int_{\mathbb{R}} t^2 |f(t)|^2 \,\mathrm{d}t < \infty, \quad \int_{\mathbb{R}} \omega^2 |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega < \infty.$$

Based on this assumption and noticing that

$$\left(\int_{\mathbb{R}} |f(t)| \,\mathrm{d}t\right)^2 \leq \int_{\mathbb{R}} \frac{1}{1+|t|^2}\mathrm{d}t \int_{\mathbb{R}} (1+|t|^2)|f(t)|^2 \,\mathrm{d}t < \infty,$$

it follows that $f \in L(\mathbb{R})$. Similarly, $\widehat{f} \in L(\mathbb{R})$.

Note that

$$\mathrm{Re}(tf(t)\overline{f'}(t)) \le |tf(t)\overline{f'}(t)| = |tf(t)f'(t)|,$$

and using Cauchy's inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathrm{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t\right|^2 &\le \left(\int_{\mathbb{R}} |tf(t)f'(t)|\mathrm{d}t\right)^2 \\ &\le \left(\int_{\mathbb{R}} t^2|f(t)|^2\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} |f'(t)|^2\,\mathrm{d}t\right).\end{aligned}$$

Using Parseval's equality and $\widehat{f'}(\omega) = (\mathrm{i}\omega)\widehat{f}(\omega)$ (Property (vi) of the Fourier transform), we find the integral in the second set of brackets on the right-hand side is

$$\int_{\mathbb{R}} |f'(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{f'}(\omega)\right|^2\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \omega^2|\widehat{f}(\omega)|^2\,\mathrm{d}\omega.$$

Therefore,

$$\left|\int_{\mathbb{R}} \mathrm{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t\right|^2 \le \frac{1}{2\pi}\left(\int_{\mathbb{R}} t^2|f(t)|^2\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} \omega^2|\widehat{f}(\omega)|^2\,\mathrm{d}\omega\right). \quad (1.13)$$

Let $f(t) = u(t) + \mathrm{i}v(t)$, where u and v are real functions. Then $u(t) = \mathrm{Re}f(t)$ and $v(t) = \mathrm{Im}f(t)$, and

$$\begin{aligned}f'(t) &= u'(t) + \mathrm{i}v'(t), \\ \frac{\mathrm{d}|f(t)|^2}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(u^2(t) + v^2(t)) = 2(u(t)u'(t) + v(t)v'(t)),\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Re}(tf(t)\overline{f'}(t)) &= \mathrm{Re}\{t(u(t) + \mathrm{i}v(t))(u'(t) - \mathrm{i}v'(t))\} \\ &= t(u(t)u'(t) + v(t)v'(t)) = \frac{t}{2}\frac{\mathrm{d}|f(t)|^2}{\mathrm{d}t}.\end{aligned}$$

Integrating both sides over $\mathbb{R}$, we get

$$\int_{\mathbb{R}} \mathrm{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t = \int_{\mathbb{R}} \frac{t}{2}\frac{\mathrm{d}|f(t)|^2}{\mathrm{d}t}\,\mathrm{d}t.$$

Using integration by parts and noticing that $\lim_{r\to\infty}\left(r|f(r)|^2\right) \to 0$ and $\lim_{r\to\infty}\left(r|f(-r)|^2\right) \to 0$, we obtain for the right-hand side

$$\begin{aligned}\int_{\mathbb{R}} \frac{t}{2}\frac{\mathrm{d}|f(t)|^2}{\mathrm{d}t}\,\mathrm{d}t &= \lim_{r\to\infty}\left(\frac{r}{2}|f(r)|^2 + \frac{r}{2}|f(-r)|^2\right) \\ &\quad -\frac{1}{2}\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t = -\frac{1}{2}\parallel f \parallel_2^2 .\end{aligned}$$

So

$$\int_{\mathbb{R}} \operatorname{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t = -\frac{1}{2}\parallel f \parallel_2^2,$$

and so

$$\left|\int_{\mathbb{R}} \operatorname{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t\right|^2 = \frac{1}{4}\parallel f \parallel_2^4 .$$

Combining this with (1.13), we get the desired result:

$$\left(\int_{\mathbb{R}} t^2|f|^2\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} \omega^2|\widehat{f}(\omega)|^2\,\mathrm{d}\omega\right) \geq \frac{\pi}{2}\parallel f \parallel_2^4 . \tag{1.14}$$

Now we give the necessary and sufficient condition that the sign of equality of (1.14) holds. According to the line of the proof of the inequality (1.14), we need to prove only that the necessary and sufficient condition that the equalities

$$\begin{aligned}\left|\int_{\mathbb{R}} \operatorname{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t\right|^2 &= \left(\int_{\mathbb{R}} |tf(t)f'(t)|\,\mathrm{d}t\right)^2 \\ &= \left(\int_{\mathbb{R}} t^2|f(t)|^2\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} |f'(t)|^2\,\mathrm{d}t\right) \end{aligned} \tag{1.15}$$

hold is $f(t) = C\mathrm{e}^{-t^2/4a}$, where C is a constant and $a > 0$.

If the first sign of equality holds, then

$$\int_{\mathbb{R}} \operatorname{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t = \int_{\mathbb{R}} |tf(t)f'(t)|\mathrm{d}t \quad \text{or}$$

$$-\int_{\mathbb{R}} \operatorname{Re}(tf(t)\overline{f'}(t))\,\mathrm{d}t = \int_{\mathbb{R}} |tf(t)f'(t)|\,\mathrm{d}t.$$

From $\pm\operatorname{Re}(tf(t)f'(t)) \leq |tf(t)f'(t)|$, it follows that

$$\operatorname{Re}(tf(t)\overline{f'}(t)) = |tf(t)\overline{f'}(t)| \quad \text{or} \quad \operatorname{Re}(tf(t)\overline{f'}(t)) = -|tf(t)\overline{f'}(t)|,$$

and so

$$tf(t)\overline{f'}(t) \geq 0 \quad \text{or} \quad tf(t)\overline{f'}(t) \leq 0.$$

If the second sign of equality holds, then $|tf(t)| = 2a|f'(t)|(a > 0)$, and so $tf(t) = 2af'(t)\mathrm{e}^{\mathrm{i}\theta(t)}$. Multiplying both sides by $\overline{f'}(t)$, we get

$$tf(t)\overline{f'}(t) = 2a|f'(t)|^2\mathrm{e}^{\mathrm{i}\theta(t)}.$$

If these two signs of equality hold simultaneously, i.e., (1.15) holds, then the results

$$tf(t)\overline{f'}(t) \geq 0 \quad \text{or} \quad tf(t)\overline{f'}(t) \leq 0,$$

$$tf(t)\overline{f'}(t) = 2a|f'(t)|^2\mathrm{e}^{\mathrm{i}\theta(t)}$$

hold simultaneously. So $e^{i\theta(t)} = \pm 1$, and so

$$tf(t) = 2af'(t) \quad \text{or} \quad tf(t) = -2af'(t).$$

Solving these two equations, we obtain that $f(t) = Ce^{t^2/4a}$ or $f(t) = Ce^{-t^2/4a}$. Noticing that $e^{t^2/4a} \notin L^2(\mathbb{R})$, we obtain finally that the necessary and sufficient condition that the sign of equality of (1.14) holds is $f(t) = Ce^{-t^2/4a}$, where C is a constant and $a > 0$. □

1.8 CASE STUDY: ARCTIC OSCILLATION INDICES

The Arctic Oscillation (AO) is a key aspect of climate variability in the Northern Hemisphere (see Figure 1.1). The AO indices are defined as the leading empirical orthogonal function of Northern Hemisphere sea level pressure anomalies poleward of 20°N and are characterized by an exchange of atmospheric mass between the Arctic and middle latitudes (Thompson and Wallace, 1998). We research the Fourier power spectrum of AO indices (December to February 1851-1997) with the help of the discrete Fourier transform (see Figure 1.2). The highest peak in the Fourier power spectrum occurs with a period of about 2.2 years.

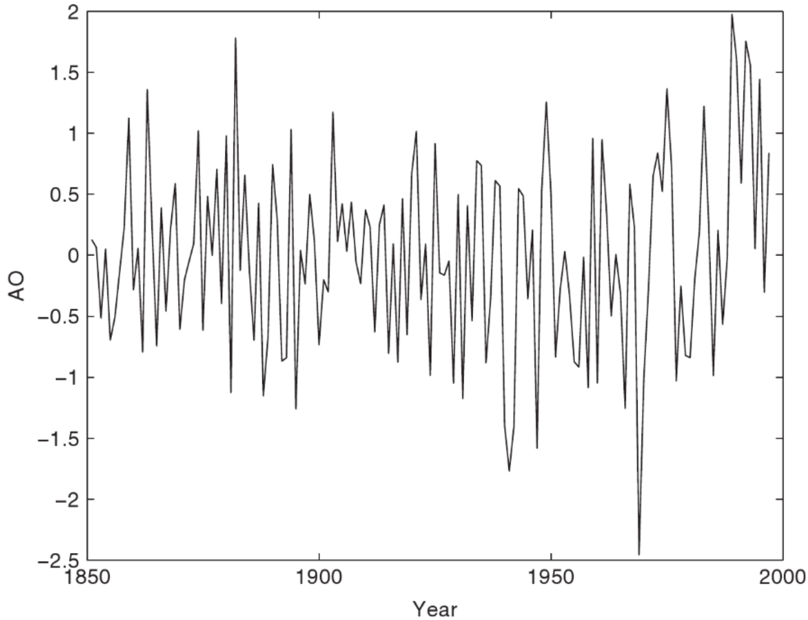

FIGURE 1.1 AO indices.

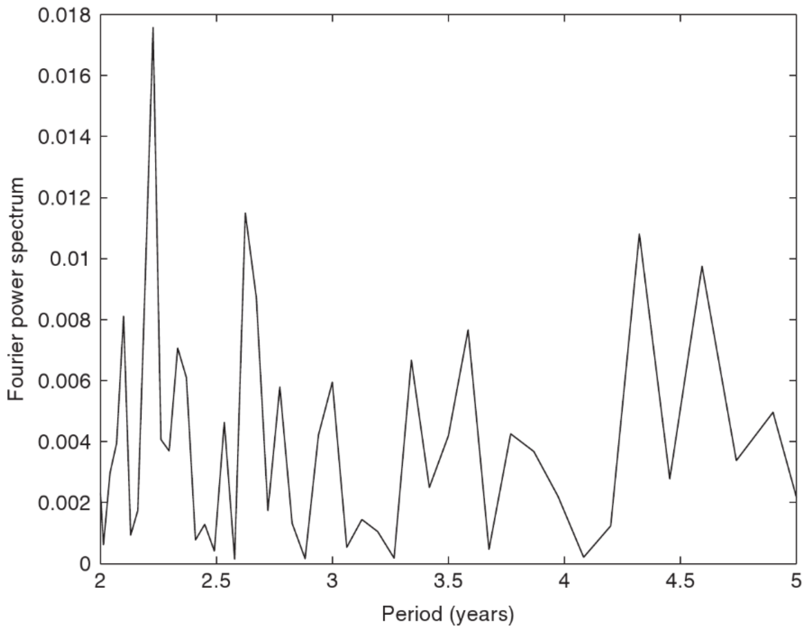

FIGURE 1.2 Fourier power spectrum of AO indices.

In Chapter 7, using the statistical significant test, we will do further research on it.

PROBLEMS

1.1 Let f be a 2π-periodic signal and $f(t) = |t| (t \in [-\pi, \pi])$. Find its Fourier series and Parseval's equality.

1.2 Show that the Legendre polynomials $X_n(t) (n = 0, 1, \ldots)$ satisfy $\int_{-1}^{1} X_n^2(t)\, dt = \frac{2}{2n+1}$.

1.3 Find the Fourier transform of the Gaussian function $f(t) = e^{-t^2/2}$.

1.4 Given a four-point time series $x = (i, 1, -i, 1+i)$, find its discrete Fourier transform.

1.5 Compute the one-sided Laplace transform of te^{-2t}.

1.6 Let $t = (t_1, t_2)$. Find the two-dimensional Fourier transform of $e^{-|t|^2/2}$.

1.7 The North Atlantic Oscillation (NAO) index is based on the surface sea level pressure difference between the Subtropical (Azores) High and the Subpolar Low. Download the monthly mean NAO index from http://www.cpc.ncep.noaa.gov/products/precip/CWlink/pna/new.nao.shtml and then research the Fourier power spectrum of the NAO index.

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Chapter 2

Time-Frequency Analysis

The Fourier transform of a signal can provide only global frequency information. While a time-frequency distribution of a signal can provide information about how the frequency content of the signal evolves with time. This is performed by mapping a one-dimensional time domain signal into a two-dimensional time-frequency representation of the signal. A lot of techniques have been developed to extract local time-frequency information. In this chapter, we introduce basic concepts and theory in time-frequency analysis, including windowed Fourier transform, wavelet transform, multiresolution analysis, wavelet basis, Hilbert transform, instantaneous frequency, Wigner-Ville distribution, and empirical mode decomposition.

2.1 WINDOWED FOURIER TRANSFORM

In order to compute the Fourier transform of a signal, we must have full knowledge of this signal in the whole time domain. However, in practice, since one does not know the information of the signal in the past or in the future, the Fourier transform alone is quite inadequate.

The windowed Fourier transform of $f \in L^2(\mathbb{R})$ is defined as

$$\left(G_b^\alpha f\right)(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\omega} f(t) g_\alpha(t-b)\,\mathrm{d}t,$$

where $g_\alpha(t)$ is the Gaussian function $g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}}\mathrm{e}^{-(t^2/4\alpha)}\,(\alpha > 0)$. Since

$$\int_{\mathbb{R}} (G_b^\alpha f)(\omega)\,\mathrm{d}b = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\omega} f(t)\,\mathrm{d}t \int_{\mathbb{R}} g_\alpha(t-b)\,\mathrm{d}b = \int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}t\omega}\,\mathrm{d}t = \widehat{f}(\omega),$$

the windowed Fourier transform is a nice tool to extract local-frequency information from a signal.

In general, the windowed Fourier transform is defined as

$$(S_b f)(\omega) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\omega} f(t)\overline{W}(t-b)\,\mathrm{d}t =: (f, W_{b,\omega}), \tag{2.1}$$

where $W(t)$ is a window function and $W_{b,\omega}(t) = \mathrm{e}^{\mathrm{i}t\omega}W(t-b)$.

The main window functions are as follows:

1. Rectangular window $\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)$;
2. Hamming window $(0.54 + 0.46\cos(2\pi t))\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)$;
3. Gaussian window e^{-18t^2};
4. Hanning window $\cos^2(\pi t)\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)$;
5. Blackman window $(0.42 + 0.5\cos(2\pi t) + 0.08\cos(4\pi t))\chi_{[-\frac{1}{2},\frac{1}{2}]}$,

   where $\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)$ is the characteristic function on $[-\frac{1}{2},\frac{1}{2}]$.

From (2.1), we see that the windowed Fourier transform $(S_b f)(\omega)$ is the Fourier transform of $f(t)\overline{W(t-b)}$, i.e.

$$(S_b f)(\omega) = (f(t)\overline{W(t-b)})^{\wedge}(\omega).$$

Let $\| W \|_2 = \left(\int_{\mathbb{R}} |W(t)|^2 \,\mathrm{d}t\right)^{1/2}$. Define the center t^* and the radius Δ_W of a window function W as follows:

$$t^* = \frac{1}{\| W \|_2} \int_{\mathbb{R}} t|W(t)|^2 \,\mathrm{d}t,$$

$$\Delta_W = \frac{1}{\| W \|_2} \left(\int_{\mathbb{R}} (t-t^*)^2 |W(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

So the windowed Fourier transform gives local-time information of f in the time window:

$$[t^* + b - \Delta_W, t^* + b + \Delta_W].$$

On the other hand, by (2.1) and Theorem 1.3, it follows that

$$(S_b f)(\omega) = \frac{1}{2\pi}(\widehat{f}, \widehat{W}_{b,\omega}).$$

So the windowed Fourier transform also gives local-frequency information of f in the frequency window:

$$[\omega^* + \omega - \Delta_{\widehat{W}}, \omega^* + \omega + \Delta_{\widehat{W}}],$$

where ω^* and $\Delta_{\widehat{W}}$ are the center and the radius of $\widehat{W}$, respectively. Furthermore, the windowed Fourier transform possesses a time-frequency window:

$$[t^* + b - \Delta_W, t^* + b + \Delta_W] \times [\omega^* + \omega - \Delta_{\widehat{W}}, \omega^* + \omega + \Delta_{\widehat{W}}]$$

with window area $4\Delta_W\Delta_{\widehat{W}}$. If W is the Gaussian function g_α, then $\Delta_W = \sqrt{\alpha}$ and $\Delta_{\widehat{W}} = \frac{1}{2\sqrt{\alpha}}$. So the window area $4\Delta_W\Delta_{\widehat{W}} = 2$. The Heisenberg uncertainty principle in Section 1.7 shows that it is not possible to construct a window function W such that the window area is less than 2.

Therefore, the windowed Fourier transform with a Gabor function has the smallest time-frequency window.

Theorem 2.1. *Let the window function W satisfy $\| W \|_2 = 1$. Then, for any $f, h \in L^2(\mathbb{R})$,*

$$\int\int_{\mathbb{R}^2} (S_b f)(\omega)\overline{(S_b h)}(\omega)\, \mathrm{d}\omega\, \mathrm{d}b = 2\pi (f, h).$$

Proof. For any $f, h \in L^2(\mathbb{R})$, by Theorem 1.3, it follows that

$$\int_{\mathbb{R}} (S_b f)(\omega)\overline{(S_b h)}(\omega)\, \mathrm{d}\omega = 2\pi \int_{\mathbb{R}} (S_b f)^{\vee}(t)\overline{(S_b h)^{\vee}}(t)\, \mathrm{d}t,$$

where $\xi^{\vee}$ is the inverse Fourier transform of ξ. Since

$$(S_b f)^{\vee}(t)\overline{(S_b h)^{\vee}}(t) = f(t)\overline{h}(t)|W(t-b)|^2,$$

it follows that

$$\int_{\mathbb{R}} (S_b f)(\omega)\overline{(S_b h)}(\omega)\, \mathrm{d}\omega = 2\pi \int_{\mathbb{R}} f(t)\overline{h}(t)|W(t-b)|^2\, \mathrm{d}t.$$

Integrating on both sides over $\mathbb{R}$ with respect to b, we get

$$\int\int_{\mathbb{R}^2} (S_b f)(\omega)\overline{(S_b h)}(\omega)\, \mathrm{d}\omega\, \mathrm{d}b = 2\pi \int_{\mathbb{R}} f(t)\overline{h}(t) \left(\int_{\mathbb{R}} |W(t-b)|^2\, \mathrm{d}b \right) \mathrm{d}t.$$

By $\int_{\mathbb{R}} |W(t-b)|^2\, \mathrm{d}b = \| W \|_2^2 = 1$, we get the desired result. □

Taking $h = g_\alpha(\cdot - t)$ in Theorem 2.1, where $g_\alpha(t)$ is the Gaussian function $g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} \mathrm{e}^{-(t^2/4\alpha)} (\alpha > 0)$, and then letting $\alpha \to 0+$, we derived the following theorem immediately.

Theorem 2.2. *Under the conditions of Theorem 2.1, we have*

$$f(t) = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}t\omega} (S_b f)(\omega) W(x-b)\, \mathrm{d}\omega \mathrm{d}b.$$

The formula in Theorem 2.2 is called the reconstruction formula of the windowed Fourier transform.

2.2 WAVELET TRANSFORM

The wavelet transform possesses the ability to construct a time-frequency representation of a signal that offers very good time and frequency localization, so wavelet transforms can analyze localized intermittent periodicity of geophysical time series very well.

A wavelet is a function $\psi \in L^2(\mathbb{R})$ with zero-average $\int_{\mathbb{R}} \psi(t)\, \mathrm{d}t = 0$. The wavelet transform of $f \in L^2(\mathbb{R})$ is defined as

$$(W_\psi f)(b, a) = \frac{1}{\sqrt{|a|}} \int_{\mathbb{R}} f(t)\overline{\psi}\left(\frac{t-b}{a}\right) \mathrm{d}t = (f, \psi_{b,a}) \quad (a \neq 0, b \in \mathbb{R}), \tag{2.2}$$

where a is called the *dilation parameter*, b is called the *translation parameter*, and $\psi_{b,a}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$.

From Theorem 1.3, it follows that $(f, \psi_{b,a}) = \frac{1}{2\pi}(\widehat{f}, \widehat{\psi}_{b,a})$, and so

$$(W_\psi f)(b,a) = \frac{1}{2\pi}(\widehat{f}, \widehat{\psi}_{b,a}) \quad (a > 0, b \in \mathbb{R}),$$

where $\widehat{\psi}_{b,a}$ is the Fourier transform of $\psi_{b,a}$ and

$$\widehat{\psi}_{b,a}(\omega) = \frac{1}{\sqrt{|a|}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\omega}\psi\left(\frac{t-b}{a}\right)\mathrm{d}t = \sqrt{|a|}\mathrm{e}^{-\mathrm{i}b\omega}\widehat{\psi}(a\omega).$$

The wavelet transform $(W_\psi f)(b,a)$ possesses the time-frequency window

$$[b + at^* - |a|\Delta_\psi, b + at^* + |a|\Delta_\psi] \times \left[\frac{\omega^*}{a} - \frac{\Delta_{\widehat{\psi}}}{|a|}, \frac{\omega^*}{a} + \frac{\Delta_{\widehat{\psi}}}{|a|}\right],$$

where t^* and ω^* are the centers of ψ and $\widehat{\psi}$, respectively, and $\Delta_{\widehat{\psi}}$ and $\Delta_{\widehat{\psi}^*}$ are the radii of ψ and $\widehat{\psi}$, respectively. This time-frequency window automatically narrows when detecting high-frequency information (i.e., small $|a|$) and widens when detecting low-frequency information (i.e., large $|a|$). Similarly to the Fourier power spectrum, the wavelet power spectrum of a signal f is defined as the square of the modulus of the wavelet transform of the signal, i.e., $|W_\psi f(b,a)|^2$.

To reconstruct the signals from their wavelet transform, we need to assume only that wavelet ψ satisfies the *admissibility condition*:

$$C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|}\mathrm{d}\omega < \infty. \tag{2.3}$$

A wavelet ψ with an admissibility condition is called a *basic wavelet*.

If $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 0$ and for some constant K and $\epsilon > 0$,

$$|\psi(t)| \le K\frac{1}{(1+|t|)^{1+\epsilon}} \quad (t \in \mathbb{R}),$$

then ψ is a basic wavelet.

Theorem 2.3. *Let ψ be a basic wavelet. Then any signal $f \in L^2(\mathbb{R})$ satisfies*

$$f(t) = \frac{1}{C_\psi}\int\int_{\mathbb{R}^2}(W_\psi f)(b,a)\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b.$$

The formula in Theorem 2.3 is called the reconstruction formula of the wavelet transform.

Proof. Denote the integral on the right-hand side by $\lambda(t)$. Let $\psi_a(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t}{a}\right)$. Then

$$\lambda(t) = \frac{1}{C_\psi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(W_\psi f)(b,a)\psi_a(t-b)\,\mathrm{d}b\right)\frac{\mathrm{d}a}{a^2}.$$

The integral in brackets can be represented by a convolution:

$$\int_{\mathbb{R}} (W_\psi f)(b,a)\psi_a(t-b)\,\mathrm{d}b = ((W_\psi f)(\cdot,a) * \psi_a)(t),$$

and so

$$\lambda(t) = \frac{1}{C_\psi}\int_{\mathbb{R}} ((W_\psi f)(\cdot,a) * \psi_a)(t)\frac{\mathrm{d}a}{a^2}.$$

However, by (2.2),

$$(W_\psi f)(b,a) = \frac{1}{\sqrt{a}}\int_{\mathbb{R}} f(t)\overline{\psi}\left(\frac{t-b}{a}\right)\mathrm{d}t = (f * \widetilde{\psi}_a)(b),$$

where $\widetilde{\psi}_a(t) = \frac{1}{\sqrt{|a|}}\overline{\psi}\left(\frac{-t}{a}\right)$. Therefore,

$$\lambda(t) = \frac{1}{C_\psi}\int_{\mathbb{R}} ((f * \widetilde{\psi}_a) * \psi_a)(t)\frac{\mathrm{d}a}{a^2}.$$

Taking the Fourier transform on both sides, using the convolution property in frequency, we get

$$\begin{aligned}\widehat{\lambda}(\omega) &= \frac{1}{C_\psi}\int_{\mathbb{R}} \widehat{f}(\omega)\sqrt{|a|}\overline{\widehat{\psi}}(a\omega)\sqrt{|a|}\widehat{\psi}(a\omega)\frac{\mathrm{d}a}{a^2}\\ &= \frac{\widehat{f}(\omega)}{C_\psi}\int_{\mathbb{R}}\frac{|\widehat{\psi}(a\omega)|^2}{a}\mathrm{d}a = \frac{\widehat{f}(\omega)}{C_\psi}\int_{\mathbb{R}}\frac{|\widehat{\psi}(u)|^2}{|u|}\,\mathrm{d}u.\end{aligned}$$

Note that $C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(u)|^2}{|u|}\,\mathrm{d}u$. Then

$$\widehat{\lambda}(\omega) = \widehat{f}(\omega).$$

Taking the inverse Fourier transform on both sides, we get the desired result: $\lambda(t) = f(t)$. □

Let

$$K(b_0, b, a_0, a) = (\psi_{b,a}, \psi_{b_0,a_0}),$$

where $\psi_{b,a}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$. A wavelet transform is a redundant representation whose redundancy is characterized by the *reproducing equation*:

$$(W_\psi f)(b_0,a_0) = \frac{1}{C_\psi}\int\int_{\mathbb{R}^2} K(b,b_0,a,a_0)(W_\psi f)(b,a)\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b,$$

where $K(b_0,b,a_0,a)$ is called the *reproducing kernel*. It measures the correlation of two wavelets $\psi_{b,a}$ and ψ_{b_0,a_0}. The reproducing equation can be derived directly by Theorem 2.3 and the definition of the wavelet transform.

Example 2.1. In geoscience, the Morlet wavelet and the Mexican hat wavelet are often used. Morlet wavelets consist of a plane wave modulated by a Gaussian function:

$$\psi^{\mathrm{M}}(t) = \pi^{-(1/4)} \mathrm{e}^{\mathrm{i}t\theta} \mathrm{e}^{-(t^2/2)}.$$

When $\theta \geq 6$, the value of its Fourier transform at the origin approximates to 0, i.e., the Morlet wavelet has zero mean and is localized in both time and frequency space. The Mexican hat wavelet is

$$\psi^{\mathrm{H}}(t) = -\frac{1}{\sqrt{\Gamma(2.5)}}(1-t^2)\mathrm{e}^{-(t^2/2)},$$

where $\Gamma(t)$ is the Gamma function.

To measure the degree of uncertainty of a random signal, the continuous wavelet entropy is defined as

$$S(t) = -\int_0^\infty P(a,b)\log P(a,b)\,\mathrm{d}a,$$

where $P(a,b) = \frac{|W_\psi f(b,a)|^4}{\int_{\mathbb{R}} |W_\psi f(\tau,a)|^4 \,\mathrm{d}\tau}$. The wavelet entropy of a white noise is maximal.

Theorem 2.3 shows a signal is reconstructed by all the values of wavelet transform $W_\psi f(b,a)(a \neq 0, t \in \mathbb{R})$. Since the wavelet transform provides redundant information, a signal may be reconstructed by discretizing the wavelet transform. If a wavelet ψ satisfies the stability condition

$$A \leq \sum_m |\widehat{\psi}(2^{-m}\omega)|^2 \leq B \quad (\omega \in \mathbb{R}),$$

where $\widehat{\psi}$ is the Fourier transform of ψ, then the half-discrete values $W_\psi f(b, 2^{-m})(b \in \mathbb{R}, m \in \mathbb{Z})$ can reconstruct the signal f. Such a wavelet ψ is called a dyadic wavelet.

Taking $a = 2^{-m}$ and $b = 2^{-m}n$ in $\psi_{b,a}(t) = \frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})$, we get

$$\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n),$$

where m is the *dilation parameter* and n is the *translation parameter*.

For any signal $f \in L^2(\mathbb{R})$, the discrete values $W_\psi f(2^{-m}n, 2^{-m})(m,n \in \mathbb{Z})$ can reconstruct the signal if and only if the wavelet ψ satisfies the *frame condition*:

$$A \parallel f \parallel^2 \leq |(f, \psi_{m,n})|^2 \leq B \parallel f \parallel^2 .$$

The family $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is called a *wavelet frame* with *upper bound* A and *lower bound* B. If $A = B = 1$, then it is called the *Parseval wavelet frame*.

If $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis, then $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is called a *wavelet basis* and ψ is called an *orthonormal wavelet*.

2.3 MULTIRESOLUTION ANALYSES AND WAVELET BASES

All orthonormal wavelets can be characterized by their Fourier transforms as follows.

A wavelet $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if ψ satisfies the following equations:

$$\| \psi \|^2 = 1, \quad \sum_m |\widehat{\psi}(2^m \omega)|^2 = 1 \quad (\omega \in \mathbb{R})$$

and for each odd integer k,

$$\sum_{m=0}^{\infty} \widehat{\psi}(2^m \omega)\overline{\widehat{\psi}}(2^m(\omega + 2k\pi)) = 0 \quad (\omega \in \mathbb{R}).$$

However, orthonormal wavelets cannot be constructed easily by this characterization.

2.3.1 Multiresolution Analyses

To construct orthonormal wavelets, multiresolution analysis is the most important method.

A sequence of closed subspaces $\{V_m\}_{m\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is a multiresolution analysis if

- **(i)** $V_m \subset V_{m+1} \quad (m \in \mathbb{Z})$;
- **(ii)** $f \in V_m$ if and only if $f(2\cdot) \in V_{m+1} \quad (m \in \mathbb{Z})$;
- **(iii)** $\overline{\bigcup_m V_m} = L^2(\mathbb{R})$;
- **(iv)** $\bigcap_m V_m = \{0\}$;
- **(v)** there exists a function $\varphi \in V_0$ such that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0.

Here the function φ is called a *scaling function* and V_0 is called the center space.

Proposition 2.1. *Let $\varphi \in L^2(\mathbb{R})$. Then $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if*

$$\sum_n |\widehat{\varphi}(\omega + 2n\pi)|^2 = 1 \quad (\omega \in \mathbb{R}).$$

Proof. We know that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\int_{\mathbb{R}} \varphi(t)\overline{\varphi}(t-n)\,\mathrm{d}t = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

However, by Theorem 1.3, it follows that

$$\begin{aligned}\int_{\mathbb{R}} \varphi(t)\overline{\varphi}(t-n)\,\mathrm{d}t &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\varphi}(\omega)|^2 \mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega = \frac{1}{2\pi}\sum_k \int_{2k\pi}^{2(k+1)\pi} |\widehat{\varphi}(\omega)|^2 \mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\sum_k \int_0^{2\pi} |\widehat{\varphi}(\omega+2k\pi)|^2 \mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_k |\widehat{\varphi}(\omega+2k\pi)|^2 \mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega.\end{aligned}$$

Denote $g(\omega) = \sum_k |\widehat{\varphi}(\omega+2k\pi)|^2$. Then

$$\int_{\mathbb{R}} \varphi(t)\overline{\varphi}(t-n)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} g(\omega)\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega.$$

Therefore, $\{\varphi(\cdot-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} g(\omega)\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega = \begin{cases} 1, & n=0, \\ 0, & n\neq 0, \end{cases}$$

that is, the Fourier coefficients of $g(\omega)$ vanish at $n \neq 0$ and equal 1 at $n = 0$. So $g(\omega) = 1$, i.e.

$$\sum_k |\widehat{\varphi}(\omega+2k\pi)|^2 = 1.$$

□

By Proposition 2.1 and (v), it follows that $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ must satisfy $\sum_n |\widehat{\varphi}(\omega+2n\pi)|^2 = 1$. Since $\varphi \in V_0$ and $\frac{1}{2}\varphi(\frac{t}{2}) \in V_{-1} \subset V_0$, we expand $\frac{1}{2}\varphi(\frac{t}{2})$ in terms of the orthonormal basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ as follows:

$$\frac{1}{2}\varphi\left(\frac{t}{2}\right) = \sum_n c_n\varphi(t-n).$$

This equation is called the *bi-scale equation* and $\{c_n\}_{n\in\mathbb{Z}}$ are called *bi-scale coefficients*. Taking the Fourier transform on both sides of the bi-scale equation, we get

$$\widehat{\varphi}(2\omega) = \widehat{\varphi}(\omega)\sum_n c_n\mathrm{e}^{-\mathrm{i}n\omega} = \widehat{\varphi}(\omega)H(\omega),$$

where $H(\omega) = \sum_n c_n\mathrm{e}^{-\mathrm{i}n\omega}$ is called the *transfer function* associated with the scaling function φ. It is clear that $H(\omega)$ is a 2π-periodic function.

Theorem 2.4. *Let $H(\omega)$ be the transfer function associated with the scaling function φ. Then*

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \quad (\omega \in [0, 2\pi]).$$

Proof. Since $\widehat{\varphi}(2\omega) = \widehat{\varphi}(\omega)H(\omega)$, it is clear that

$$\widehat{\varphi}(2\omega + 2n\pi) = \widehat{\varphi}(\omega + n\pi)H(\omega + n\pi).$$

Since φ is a scaling function, by Proposition 2.1, $\sum_n |\widehat{\varphi}(\omega + 2n\pi)|^2 = 1$, and so

$$\sum_n |\widehat{\varphi}(2\omega + 2n\pi)|^2 = 1, \quad \sum_n |\widehat{\varphi}(\omega + \pi + 2n\pi)|^2 = 1.$$

Since $H(\omega)$ is a 2π-periodic function, $H(\omega + 2l\pi) = H(\omega)(l \in \mathbb{Z})$. Therefore,

$$\begin{aligned}
1 &= \sum_n |\widehat{\varphi}(2\omega + 2n\pi)|^2 = \sum_n |\widehat{\varphi}(\omega + n\pi)|^2 |H(\omega + n\pi)|^2 \\
&= \sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 |H(\omega + 2k\pi)|^2 \\
&\quad + \sum_k |\widehat{\varphi}(\omega + (2k+1)\pi)|^2 |H(\omega + (2k+1)\pi)|^2 \\
&= |H(\omega)|^2 \sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 + |H(\omega + \pi)|^2 \sum_k |\widehat{\varphi}(\omega + \pi + 2k\pi)|^2 \\
&= |H(\omega)|^2 + |H(\omega + \pi)|^2.
\end{aligned}$$

We get Theorem 2.4. □

Since φ is the scaling function, by (v), $\{\varphi(t - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0. Let

$$\varphi_{m,n}(t) = 2^{m/2}\varphi(2^m t - n) \quad (m, n \in \mathbb{Z}).$$

Then $\{\varphi_{m,n}(t)\}_{n\in\mathbb{Z}}$ is the orthonormal basis of V_m.

To construct an orthonormal wavelet by using a multiresolution analysis $\{V_m\}_{m\in\mathbb{Z}}$, we consider the orthogonal complement space W_0 of the center space V_0 in V_1, i.e.

$$V_1 = V_0 \bigoplus W_0,$$

where $\bigoplus$ represents the orthogonal sum. The following theorem gives a construction method for the orthonormal wavelet.

Theorem 2.5. *Suppose that for a multiresolution analysis, φ is the scaling function, H is the transfer function, and $\{c_n\}_{n\in\mathbb{Z}}$ are bi-scale coefficients. Let ψ satisfy $\widehat{\psi}(\omega) = \widetilde{H}\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right)$, where $\widetilde{H}(\omega) = \mathrm{e}^{-\mathrm{i}\omega}\overline{H}(\omega + \pi)$, i.e.,*

$$\psi(t) = -2\sum_n (-1)^n \overline{c}_{1-n}\varphi(2t - n),$$

Then $\{\psi(t - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of W_0 and $\{2^{m/2}\psi(2^m t - n)\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, i.e., ψ is an orthonormal wavelet.

Theorem 2.5 is called the *existence theorem of orthonormal wavelets*.

As an example, let

$$N_1(t) = \begin{cases} 1, & t \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$N_k(t) = (N_{k-1} * N_1)(t) = \int_0^1 N_{k-1}(t-x)\,\mathrm{d}x \quad (k \geq 2)$$

and call $N_k(t)$ the *k-order cardinal B-spline*. Its Fourier transform is

$$\widehat{N}_k(\omega) = \left(\frac{1-\mathrm{e}^{-\mathrm{i}\omega}}{\mathrm{i}\omega}\right)^k = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^k \mathrm{e}^{-\mathrm{i}(k\omega/2)}.$$

A direct computation shows that

$$\sum_l |\widehat{N}_k(2\omega + 2l\pi)|^2 = -\frac{\sin^{2k}\omega}{(2k-1)!}\frac{\mathrm{d}^{2k-1}(\cot\omega)}{\mathrm{d}\omega^{2k-1}} =: F_k(2\omega).$$

Especially, $F_1(\omega) = 1$ and $F_2(\omega) = \frac{1}{3}\sin^2\frac{\omega}{2} + \cos^2\frac{\omega}{2}$. Let φ_k satisfy the condition

$$\widehat{\varphi}_k(\omega) = \frac{\widehat{N}_k(\omega)}{\left(\sum_l |\widehat{N}_k(\omega + 2l\pi)|^2\right)^{1/2}} = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^k \mathrm{e}^{-\mathrm{i}(k\omega/2)} F_k^{-(1/2)}(\omega).$$

Then φ_k is a scaling function. By Theorem 2.5, the corresponding orthonormal wavelet $\psi_k(t)$ satisfies

$$\widehat{\psi}_k(\omega) = \left(\frac{4}{i\omega}\right)^k \mathrm{e}^{-\mathrm{i}(\omega/2)} \sin^{2k}\frac{\omega}{4}\left(\frac{F_k((\omega/2)+\pi)}{F_k(\omega/2)F_k(\omega)}\right)^{1/2}.$$

The wavelet ψ_k is called the Battle-Lemarié wavelet of order k.

A function f is called a *compactly supported function* if there exists a $c > 0$ such that $f(t) = 0(|t| > c)$. Daubechies constructed a lot of compactly supported orthonormal wavelets and applied them widely in signal processing.

For any $N \in \mathbb{Z}_+$, Daubechies constructed a rational function $P(z) = \sum_{-N+1}^{N-1} c_n z^n$ with real-valued coefficients $c_n \in \mathbb{R}$ such that

$$P(1) = 1, \quad |P(\mathrm{e}^{-\mathrm{i}\omega})|^2 = \sum_0^{N-1} C_{N+k-1}^k \left(\sin\frac{\omega}{2}\right)^{2k},$$

where $C_m^n = \frac{m!}{n!(m-n)!}$. Denote the filter

$$H_N^{\mathrm{D}}(\omega) := \left(\frac{1+\mathrm{e}^{-\mathrm{i}\omega}}{2}\right)^N P(\mathrm{e}^{-\mathrm{i}\omega}) = \sum_{n=0}^{2N-1} h_{n,N}\mathrm{e}^{-\mathrm{i}n\omega}.$$

On the basis of $\{h_{n,N}\}_{n=0,\ldots,2N-1}$, the scaling function φ_N^{D} can be obtained numerically by the bi-scale equation:

$$\frac{1}{2}\varphi_N^{\mathrm{D}}(t) = \sum_{n=0}^{2N-1} h_{n,N}\varphi_N^{\mathrm{D}}(2t-n),$$

and φ_N^{D} is compactly supported. By Theorem 2.5, the corresponding orthonormal wavelet is

$$\psi_N^{\mathrm{D}}(t) = -2\sum_{n=2-2N}^{1}(-1)^n\overline{h}_{1-n,N}\varphi_N^{\mathrm{D}}(2t-n).$$

It is compactly supported. The wavelet $\psi_N^{\mathrm{D}}(t)$ is called the Daubechies wavelet.

Let ψ be an orthonormal wavelet and $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)(m,n \in \mathbb{Z})$. Then any $f \in L^2(\mathbb{R})$ can be expanded into a wavelet series:

$$f = \sum_{m,n} d_{m,n}\psi_{m,n} \quad (L^2(\mathbb{R})),$$

where the coefficients are

$$d_{m,n} = (f, \psi_{m,n}) = \int_{\mathbb{R}} f(t)\,\overline{\psi}_{m,n}(t)\,\mathrm{d}t \quad (m,n \in \mathbb{Z})$$

and $d_{m,n}$ are called *wavelet coefficients*, and Parseval's identity $\sum_{m,n}|d_{m,n}|^2 = \| f \|_2^2$ holds. Notice that the coefficient formula can be written as

$$d_{m,n} = 2^{m/2}\int_{\mathbb{R}} f(t)\,\overline{\psi}\left(\frac{t-2^{-m}n}{2^{-m}}\right)\mathrm{d}t.$$

Therefore, when we regard ψ as a basic wavelet, the wavelet coefficients are just the values of wavelet transform at $a = 2^{-m}$ and $b = 2^{-m}n$. If φ is the scaling function corresponding to ψ, then any $f \in L^2(\mathbb{R})$ can also be expanded into another wavelet series:

$$f(t) = \sum_n c_n\varphi(t-n) + \sum_{m=0}^{\infty}\sum_n d_{m,n}\,\psi_{m,n}(t) \quad (L^2(\mathbb{R})),$$

where

$$c_n = (f, \varphi) = \int_{\mathbb{R}} f(t)\overline{\varphi}(t-n)\,\mathrm{d}t \quad (n \in \mathbb{Z}),$$

$$d_{m,n} = (f, \psi_{m,n}) = \int_{\mathbb{R}} f(t)\overline{\psi}_{m,n}(t)\,\mathrm{d}t \quad (m,\ n \in \mathbb{Z}).$$

For any $f \in L^2(\mathbb{R})$, since $L^2(\mathbb{R}) = \overline{\bigcup_{m\in\mathbb{Z}} V_m}$, the projection of f on space V_m

$$\mathrm{Proj}_{V_m} f \to f \quad (m \to \infty),$$

that is, $f \approx \mathrm{Proj}_{V_m} f$ when m is sufficiently large. Denote the orthogonal complement space of V_m in V_{m+1} by W_m, i.e., $V_{m+1} = V_m \bigoplus W_m$. So

$$\mathrm{Proj}_{V_{m+1}} f = \mathrm{Proj}_{V_m} f + \mathrm{Proj}_{W_m} f,$$

where $\mathrm{Proj}_{V_m} f$ and $\mathrm{Proj}_{W_m} f$ are the low-frequency part and the high-frequency part of the projection $\mathrm{Proj}_{V_{m+1}} f$, respectively. Note that

$$\begin{aligned} \varphi_{m,n}(t) &= 2^{m/2}\varphi(2^m t - n) \quad (m, n \in \mathbb{Z}), \\ \psi_{m,n}(t) &= 2^{m/2}\psi(2^m t - n) \quad (m, n \in \mathbb{Z}). \end{aligned}$$

Since $\{\varphi_{m,n}\}_{n\in\mathbb{Z}}$ and $\{\psi_{m,n}\}_{n\in\mathbb{Z}}$ are orthonormal bases of V_m and W_m, respectively,

$$\sum_n c_{m+1,n}\, \varphi_{m+1,n} = \sum_n c_{m,n}\, \varphi_{m,n} + \sum_n d_{m,n}\, \psi_{m,n} \quad (m \in \mathbb{Z}), \tag{2.4}$$

where $c_{m,n} = (f, \varphi_{m,n})$ and $d_{m,n} = (f, \psi_{m,n})$. This formula is called the *decomposition formula*.

Replacing m by $m-1$ in (2.4), we get

$$\sum_n c_{m,n}\, \varphi_{m,n} = \sum_n c_{m-1,n}\, \varphi_{m-1,n} + \sum_n d_{m-1,n}\, \psi_{m-1,n},$$

and then substituting it into the first term on the right-hand side of (2.4), we get

$$\sum_n c_{m+1,n}\, \varphi_{m+1,n} = \left(\sum_n c_{m-1,n}\, \varphi_{m-1,n} + \sum_n d_{m-1,n}\, \psi_{m-1,n} \right) + \sum_n d_{m,n}\, \psi_{m,n}.$$

Continuing this procedure l times, when m is sufficiently large, we have

$$f \approx \sum_n c_{m+1,n}\, \varphi_{m+1,n} = \sum_n c_{m-l,n}\, \varphi_{m-l,n} + \sum_{j=m-l}^{m} \sum_n d_{j,n}\, \psi_{j,n}.$$

In application, one often uses such a decomposition.

2.3.2 Discrete Wavelet Transform

To avoid computing each coefficient $c_{m,n}$, $d_{m,n} (n \in \mathbb{Z})$ in (2.4), by using integrals,

$$\begin{aligned} c_{m,n} &= \int_{\mathbb{R}} f(t)\overline{\varphi}_{m,n}(t)\, \mathrm{d}t \quad (n \in \mathbb{Z}), \\ d_{m,n} &= \int_{\mathbb{R}} f(t)\overline{\psi}_{m,n}(t)\, \mathrm{d}t \quad (n \in \mathbb{Z}), \end{aligned}$$

the discrete wavelet transform provides a fast algorithm that can compute coefficients $\{c_{m,n}\}$ and $\{d_{m,n}\}$ with the help of $\{c_{m+1,n}\}_{n\in\mathbb{Z}}$.

Now we introduce the discrete wavelet transform.

Let φ be a scaling function and ψ be the corresponding orthonormal wavelet, and

$$c_{m,n} = (f, \varphi_{m,n}), \quad d_{m,n} = (f, \psi_{m,n}),$$

where

$$\begin{aligned} \varphi_{m,n}(t) &= 2^{m/2}\varphi(2^m t - n) \quad (m, n \in \mathbb{Z}), \\ \psi_{m,n}(t) &= 2^{m/2}\psi(2^m t - n) \quad (m, n \in \mathbb{Z}). \end{aligned}$$

From the bi-scale equation and Theorem 2.5, it follows that

$$\varphi(t) = \sum_k p_k \varphi(2t - k),$$

$$\psi(t) = \sum_k q_k \varphi(2t - k), \tag{2.5}$$

where $p_k = 2c_k$ and $q_k = (-1)^{k+1}2\overline{c}_{1-k}$, and c_k is the bi-scale coefficient. Since $\{\varphi_{m+1,l}\}_{l\in\mathbb{Z}}$ and $\{\psi_{m+1,l}\}_{l\in\mathbb{Z}}$ are an orthonormal basis of V_{m+1} and W_{m+1}, respectively, and $\varphi_{m,n} \in V_m \subset V_{m+1}$, $\psi_{m,n} \in W_m \subset V_{m+1}$,

$$\begin{aligned} \varphi_{m,n} &= \sum_l (\varphi_{m,n}, \varphi_{m+1,l})\varphi_{m+1,l}, \\ \psi_{m,n} &= \sum_l (\psi_{m,n}, \varphi_{m+1,l})\varphi_{m+1,l}. \end{aligned}$$

By (2.5), it follows that

$$\begin{aligned} (\varphi_{m,n}, \varphi_{m+1,l}) &= \sqrt{2}\int_{\mathbb{R}} \varphi(u - n)\overline{\varphi}(2u - l)\,\mathrm{d}u = \frac{1}{\sqrt{2}}p_{l-2n}, \\ (\psi_{m,n}, \varphi_{m+1,l}) &= \sqrt{2}\int_{\mathbb{R}} \psi(u - n)\overline{\varphi}(2u - l)\,\mathrm{d}u = \frac{1}{\sqrt{2}}q_{l-2n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi_{m,n} &= \frac{1}{\sqrt{2}}\sum_l p_{l-2n}\varphi_{m+1,l}, \\ \psi_{m,n} &= \frac{1}{\sqrt{2}}\sum_l q_{l-2n}\varphi_{m+1,l}. \end{aligned}$$

Noticing that $c_{m,n} = (f, \varphi_{m,n})$ and $d_{m,n} = (f, \psi_{m,n})$, we find

$$c_{m,n} = \frac{1}{\sqrt{2}} \sum_l p_{l-2n} c_{m+1,l},$$

$$d_{m,n} = \frac{1}{\sqrt{2}} \sum_l q_{l-2n}\, c_{m+1,l}.$$

These formulas are called the *discrete wavelet transform*.

Since the union of $\{\varphi_{m,n}\}_{n\in\mathbb{Z}}$ and $\{\psi_{m,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_{m+1} and $\varphi_{m+1,n} \in V_{m+1}$,

$$\varphi_{m+1,n} = \sum_l (\varphi_{m+1,n}, \varphi_{m,l})\, \varphi_{m,l} + \sum_l (\varphi_{m+1,n}, \psi_{m,l})\, \psi_{m,l},$$

and so

$$(f, \varphi_{m+1,n}) = \sum_l (\varphi_{m+1,n}, \varphi_{m,l})(f, \varphi_{m,l}) + \sum_l (\varphi_{m+1,n}, \psi_{m,l})(f, \psi_{m,l}),$$

that is, the *inverse discrete wavelet transform* is

$$c_{m+1,n} = \frac{1}{\sqrt{2}} \left(\sum_l \overline{p}_{n-2l}\, c_{m,l} + \sum_l \overline{q}_{n-2l}\, d_{m,l} \right).$$

2.3.3 Biorthogonal Wavelets, Bivariate Wavelets, and Wavelet Packet

Biorthogonal wavelets are a kind of wavelet that are used often. Their constructions depend on the concept of the Riesz basis. Let $\{g_n\}$ be a basis for $L^2(\mathbb{R})$, and for any sequence $c_n(\sum_n |c_n|^2 < \infty)$ there exists $B \geq A > 0$ such that

$$A \sum_n |c_n|^2 \leq \| \sum_n c_n g_n \|_2^2 \leq B \sum_n |c_n|^2,$$

then $\{g_n\}$ is called a *Riesz basis* for $L^2(\mathbb{R})$.

Let $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$. If their integral translations and dyadic dilations satisfy $(\psi_{m,n}, \widetilde{\psi}_{m',n'}) = \delta_{m,m'}\delta_{n,n'}$ and both $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{\widetilde{\psi}_{m,n}\}_{m,n\in\mathbb{Z}}$ are Riesz bases of $L^2(\mathbb{R})$, then $\{\psi, \widetilde{\psi}\}$ is called a *pair of biorthogonal wavelets*, where $\delta_{k,l} = 0(k \neq l)$, $\delta_{k,l} = 1(k = l)$.

If $\{\psi, \widetilde{\psi}\}$ is a pair of biorthogonal wavelets, then, for $f \in L^2(\mathbb{R})$, the *reconstruction formula* holds:

$$f = \sum_{m,n} (f, \psi_{m,n})\widetilde{\psi}_{m,n} = \sum_{m,n} \left(f, \widetilde{\psi}_{m,n}\right) \psi_{m,n}.$$

Symmetric or antisymmetric compactly supported spline biorthogonal wavelets are applied widely. The construction method is as follows. First, a pair of trigonometric polynomials $H(\omega)$ and $\widetilde{H}(\omega)$ are defined as

$$\begin{aligned} H(\omega) &= \mathrm{e}^{-\mathrm{i}(\epsilon\,\omega/2)}\left(\cos\frac{\omega}{2}\right)^{p} L(\omega), \\ \widetilde{H}(\omega) &= \mathrm{e}^{-\mathrm{i}(\epsilon\,\omega/2)}\left(\cos\frac{\omega}{2}\right)^{\widetilde{p}} \widetilde{L}(\omega), \end{aligned}$$

where $\epsilon = 0$ for even numbers p and $\widetilde{p}$, and $\epsilon = 1$ for odd numbers p and $\widetilde{p}$, and

$$L(\cos\omega)\widetilde{L}(\cos\omega) = \sum_{0}^{q-1} C_{q-1+k}^{k}\,\sin^{2k}\frac{\omega}{2}, \quad \left(q = \frac{1}{2}(p+\widetilde{p})\right).$$

Next, the bi-scale coefficients $\{h_n\}$ and $\{\widetilde{h}_n\}$ are computed using

$$\begin{aligned} H(\omega) &= \sum_{-p}^{p} h_n \mathrm{e}^{-\mathrm{i}n\omega}, \\ \widetilde{H}(\omega) &= \sum_{-\widetilde{p}}^{\widetilde{p}} \widetilde{h}_n \mathrm{e}^{-\mathrm{i}n\omega}. \end{aligned}$$

For example, let $p = 2, \widetilde{p} = 4$, and $L(\omega) = 1$. Then

$$h_2 = h_{-2} = 0, \quad h_1 = h_{-1} = 0.35355, \quad h_0 = 1$$

and

$$\widetilde{h}_0 = 0.9944, \quad \widetilde{h}_{-1} = \widetilde{h}_1 = 0.4198, \quad \widetilde{h}_{-2} = \widetilde{h}_2 = -0.1767,$$

$$\widetilde{h}_{-3} = \widetilde{h}_3 = -0.0662, \quad \widetilde{h}_{-4} = \widetilde{h}_4 = 0.0331.$$

From this, with use of bi-scale equations,

$$\begin{aligned} \varphi(t) &= \sum_{-p}^{p} 2h_n\,\varphi(2t-n), \\ \widetilde{\varphi}(t) &= \sum_{-\widetilde{p}}^{\widetilde{p}} 2\widetilde{h}_n\,\widetilde{\varphi}(2t-n), \end{aligned}$$

the scaling functions $\varphi(t)$ and $\widetilde{\varphi}(t)$ can be solved numerically. Finally, the corresponding biorthogonal wavelets $\psi(t)$ and $\widetilde{\psi}(t)$ are obtained.

If φ is a scaling function and ψ is the corresponding wavelet, define

$$\begin{aligned} \psi^{(1)}(t) &= \varphi(t_1)\psi(t_2), \\ \psi^{(2)}(t) &= \psi(t_1)\varphi(t_2), \\ \psi^{(3)}(t) &= \psi(t_1)\psi(t_2). \end{aligned}$$

Denote $\psi^{(k)}_{m,n}(t) = \frac{1}{2^m}\,\psi^{(k)}(2^m t - n)$, where $m \in \mathbb{Z}, n \in \mathbb{Z}^2$ and $k = 1, 2, 3$. Then

$$\{\psi^{(1)}_{m,n}, \psi^{(2)}_{m,n}, \psi^{(3)}_{m,n}\}_{(m,n)\in\mathbb{Z}^3}$$

forms an orthonormal basis of $L^2(\mathbb{R}^2)$. Such a basis is called a *bivariate wavelet basis*.

A multiresolution analysis can generate not only an orthogonal basis but also a library of functions, called a *wavelet packet*, from which infinitely many wavelet packet bases can be constructed. The Heisenberg uncertainty principle considers only the minimal area of time-frequency windows and does not mention their shapes. For a wavelet basis, the shape of the time-frequency window has been predetermined by the choice of the wavelet function. However, in a wavelet packet, the time-frequency windows are rectangular with arbitrary aspect ratios.

For a multiresolution analysis, let $\varphi(t)$ be the scaling function, $H(\omega)$ be the transfer function, and ψ be the corresponding wavelet. Define $\mu_0 = \varphi, \mu_1 = \psi$, and

$$\widehat{\mu}_{2l}(\omega) = H\left(\frac{\omega}{2}\right)\widehat{\mu}_l\left(\frac{\omega}{2}\right),$$

$$\widehat{\mu}_{2l+1}(\omega) = \mathrm{e}^{-\mathrm{i}(\omega/2)}\,\overline{H}\left(\frac{\omega}{2}+\pi\right)\widehat{\mu}_l\left(\frac{\omega}{2}\right) \quad (l = 0, 1, \ldots).$$

The sequence $\{\mu_l\}_{l=0,1,\ldots}$ is called the *wavelet packet* determined by the scale function φ, where l is called the *modulation parameter*. The integral translations and dyadic dilations of all wavelet packet functions,

$$\mu_{l,m,n} = 2^{\frac{m}{2}}\mu_l(2^m t - n) \quad (l = 0, 1, \ldots; m, n \in \mathbb{Z}),$$

are called the *dictionary*. The choice of the modulation parameter l and the dilation parameter m, and the translation parameter n can give a lot of orthonomal bases. These orthonomal bases are called *wavelet packet bases*. A signal f can be expanded into an orthogonal series with respect to a wavelet packet basis of order $k(0 \le k \le j_0)$ as follows:

$$f(t) = P_{j_0}f + \sum_{j=j_0}^{\infty}\sum_{m=0}^{2^k-1}\sum_{n} c_{j,k,m,n}\,\mu_{2^k+m}(2^{j-k}t - n),$$

where $P_{j_0}f$ is the projection of f on the space V_{j_0}, and $c_{j,k,m,n} = (f, \mu_{2^k+m}(2^{j-k}t - n))$.

Recently, great advances in wavelet analysis have resulted from the study of Parseval wavelet frames (see Section 2.2).

The Parseval wavelet frame has now become an alternative to the wavelet basis and it is anticipated that Parseval wavelet frames will soon be applied in the analysis of geophysical processes. For any signal f of finite energy, if $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is a Parseval wavelet frame, then

$$f = \sum_{m,n} d_{m,n}\psi_{m,n}, \quad \text{where} \quad d_{m,n} = (f, \psi_{m,n}).$$

This is similar to the orthogonal expansion of a signal with respect to a wavelet basis. However, Parseval wavelet frames $\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ may not be orthogonal or linear independent. Their construction is easier than that of wavelet bases. It is well known that a univariate wavelet basis is generated by one function, and a bivariate wavelet basis is generated by three functions. However, the number of functions generating a Parseval wavelet frame may be arbitrary. Their construction method is based on the following unitary extension principle.

Let a function φ satisfy $\widehat{\varphi}(\omega) = P\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right)$, where P is a trigonometric polynomial. One constructs r trigonometric polynomials $\{Q_j\}_{j=1,\dots,r}$ such that

$$P(\omega)\overline{P}(\omega + l) + \sum_{1}^{r} Q_j(\omega)\,\overline{Q}_j(\omega + l) = \begin{cases} 1, & l = 0, \\ 0, & l = 1, \end{cases}$$

and then defines $\{\psi_j\}_{j=1,\dots,r}$ as

$$\widehat{\psi}_j(\omega) = Q_j\left(\frac{\omega}{2}\right)\widehat{\varphi}\left(\frac{\omega}{2}\right).$$

The integral translations and dyadic dilations of these functions form a Parseval wavelet frame.

2.4 HILBERT TRANSFORM, ANALYTICAL SIGNAL, AND INSTANTANEOUS FREQUENCY

For a function $f(t)(t \in \mathbb{R})$, if the Cauchy principal value

$$\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\tau)}{t-\tau}\,d\tau = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|t-\tau|>\epsilon} \frac{f(\tau)}{t-\tau}\,d\tau$$

exists, then it is called the *Hilbert transform* of $f(t)$, denoted by $\widetilde{f}(t)$, i.e.

$$\widetilde{f}(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\tau)}{t-\tau}\,d\tau.$$

Hilbert transforms have the following properties:

(i) (Linearity). Let $F = \alpha f_1 + \beta f_2$, where α, β are constants. Then $\widetilde{F} = \alpha\widetilde{f}_1 + \beta\widetilde{f}_2$.

(ii) (Translation). Let $F(t) = f(t-\alpha)$. Then the Hilbert transform $\widetilde{F} = \widetilde{f}(t-\alpha)$.

In fact,

$$\widetilde{F}(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\tau-\alpha)}{t-\tau}\,d\tau = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(u)}{t-\alpha-u}\,du = \widetilde{f}(t-\alpha).$$

(iii) (Dilation). Let $F(t) = f(\lambda t)$, where λ is a real number. Then the Hilbert transform

$$\widetilde{F}(t) = \widetilde{f}(\lambda t)\,\mathrm{sgn}\lambda,$$

where $\mathrm{sgn}\lambda = 1(\lambda > 0)$ and $\mathrm{sgn}\lambda = -1(\lambda < 0)$, and $\mathrm{sgn}\,0 = 0$.

For $\lambda > 0$,

$$\widetilde{F}(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(\lambda\tau)}{t-\tau}\,\mathrm{d}\tau = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(u)}{\lambda t-u}\,\mathrm{d}u = \widetilde{f}(\lambda t) = \widetilde{f}(\lambda t)\,\mathrm{sgn}\lambda,$$

and for $\lambda < 0$,

$$\begin{aligned}\widetilde{F}(t) &= \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(\lambda\tau)}{t-\tau}\,\mathrm{d}\tau \\ &= -\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(u)}{\lambda t-u}\,\mathrm{d}u = -\widetilde{f}(\lambda t) = \widetilde{f}(\lambda t)\,\mathrm{sgn}\lambda.\end{aligned}$$

The following theorem shows that the Hilbert transform of a harmonic wave is also a harmonic wave.

Theorem 2.6. *Let $f(t)$ be a periodic signal with period 2π. Then its Hilbert transform is*

$$\widetilde{f}(t) = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\frac{f(t+\tau)-f(t-\tau)}{\tan(\tau/2)}\,\mathrm{d}\tau. \tag{2.6}$$

Especially, if $f(t) = \cos t$, then $\widetilde{f}(t) = \sin t$; if $f(t) = \sin t$, then $\widetilde{f}(t) = -\cos t$.

Proof. The Hilbert transform of f is

$$\begin{aligned}\widetilde{f}(t) &= \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau = \lim_{N\to\infty}\text{p.v.}\frac{1}{\pi}\sum_{-N}^{N}\int_{(2k-1)\pi}^{(2k+1)\pi}\frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau \\ &= \lim_{N\to\infty}\text{p.v.}\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{-N}^{N}\frac{f(u+2k\pi)}{t-(u+2k\pi)}\,\mathrm{d}u.\end{aligned}$$

From $f(u+2k\pi) = f(u)$, it follows that

$$\begin{aligned}\widetilde{f}(t) &= \lim_{N\to\infty}\text{p.v.}\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{-N}^{N}\frac{f(u)}{(t-u)-2k\pi}\,\mathrm{d}u \\ &= \text{p.v.}\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\left(\lim_{N\to\infty}\sum_{-N}^{N}\frac{1}{(t-u)-2k\pi}\right)\mathrm{d}u.\end{aligned}$$

By using the known formula $\frac{1}{2\tan(t/2)} = \lim_{N\to\infty}\sum_{-N}^{N}\frac{1}{t-2k\pi}$, we find the right-hand side is equal to

$$\text{p.v.}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{\tan((t-u)/2)}\,\mathrm{d}u = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right)\frac{f(t-\tau)}{\tan(\tau/2)}\mathrm{d}\tau$$
$$= -\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\frac{f(t+\tau)-f(t-\tau)}{\tan(\tau/2)}\,\mathrm{d}\tau.$$

So we get (2.6).

If $f(t)=\cos t$, from (2.6), it follows that

$$\widetilde{f}(t) = -\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\frac{\cos(t+\tau)-\cos(t-\tau)}{\tan(\tau/2)}\mathrm{d}\tau$$
$$= \frac{2}{\pi}\sin t\int_{0}^{\pi}\cos^2\frac{\tau}{2}\,\mathrm{d}\tau = \sin t.$$

If $f(t)=\sin t$, then $f(t)=-\cos(t+\frac{\pi}{2})$, and so $\widetilde{f}(t)=-\sin(t+\frac{\pi}{2})=-\cos t$. □

By Theorem 2.6 and the properties of the Hilbert transform, if a signal $f(t)$ is a trigonometric polynomial and

$$f(t)=\sum_{0}^{N}(c_n\cos(nt)+d_n\sin(nt)),$$

then its Hilbert transform is also a trigonometric polynomial and

$$\widetilde{f}(t)=\sum_{0}^{N}(c_n\sin(nt)-d_n\cos(nt)).$$

If a signal $f\in L_{2\pi}$ can be expanded into a Fourier series,

$$f(t)=\sum_{n}c_n(f)\mathrm{e}^{\mathrm{i}nt}=\frac{a_0}{2}(f)+\sum_{1}^{\infty}(a_n(f)\cos(nt)+b_n(f)\sin(nt)),$$

then its Hilbert transform satisfies

$$\widetilde{f}(t)=\sum_{n}-\mathrm{i}c_n(f)\,\mathrm{sgn}\,n\mathrm{e}^{\mathrm{i}nt}=\sum_{1}^{\infty}(a_n(f)\sin(nt)-b_n(f)\cos(nt)),$$

where the series on the right-hand side is called the *conjugate Fourier series*. So

$$c_n(\widetilde{f})=-\mathrm{i}\,c_n(f)\,\mathrm{sgn}\,n\quad(n\in\mathbb{Z}),$$

$$f(t)+\mathrm{i}\widetilde{f}(t)=c_0(f)+\sum_{1}^{\infty}2c_n(f)\,z^n\quad(z=\mathrm{e}^{\mathrm{i}t}).$$

From this, we get the following theorem.

Theorem 2.7. *Let $f\in L_{2\pi}$ and $\widetilde{f}$ be its Hilbert transform, and $c_n(f)$, $c_n(\widetilde{f})$ be their Fourier coefficients. Then $c_n(\widetilde{f})=-\mathrm{i}\,c_n(f)\,\mathrm{sgn}\,n(n\in\mathbb{Z})$ and $f(t)+\mathrm{i}\widetilde{f}(t)=c_0(f)+\sum_1^\infty 2c_n(f)\,z^n(z=\mathrm{e}^{\mathrm{i}t})$.*

From Theorem 2.7, we see that for a real-valued periodic signal f, adding the Hilbert transform $\widetilde{f}$ as the imaginary part, we obtain an analytic function in the unit disk $f_\alpha(z) = c_0(f) + \sum_1^\infty 2c_n(f)\, z^n (|z| < 1)$.

For a nonperiodic signal of finite energy, replacing Fourier coefficients by Fourier transforms, we obtain a result similar to Theorem 2.7, as follows.

Theorem 2.8. *Let $f \in L^2(\mathbb{R})$ and $\widetilde{f}$ be its Hilbert transform. Then their Fourier transforms satisfy*

$$\widehat{\widetilde{f}}(\omega) = -\mathrm{i}\widehat{f}(\omega)\,\mathrm{sgn}\,\omega.$$

Proof. Denote

$$K_{\delta,\eta}(t) = \begin{cases} \frac{1}{t}, & 0 < \delta \le |t| \le \eta < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{f}_{\delta,\eta}(t) = \frac{1}{\pi} \int_{\delta \le |u| \le \eta} \frac{f(t-u)}{u}\, \mathrm{d}u.$$

From these two representation, it follows that

$$\widetilde{f}_{\delta,\eta}(t) = \frac{1}{\pi} \int_{\mathbb{R}} f(t-u)\, K_{\delta,\eta}(u)\, \mathrm{d}u = \frac{1}{\pi}(f * K_{\delta,\eta})(t).$$

By the convolution property of the Fourier transform, we get

$$\widehat{\widetilde{f}}_{\delta,\eta}(\omega) = \frac{1}{\pi}(f * K_{\delta,\eta})^{\wedge}(\omega) = \frac{1}{\pi}\widehat{f}(\omega)\widehat{K}_{\delta,\eta}(\omega). \tag{2.7}$$

With use of the Euler formula, $\mathrm{e}^{-\mathrm{i}v} - \mathrm{e}^{\mathrm{i}v} = -2\mathrm{i}\sin v$, the Fourier transform of $K_{\delta,\eta}$ is

$$\begin{aligned} \widehat{K}_{\delta,\eta}(\omega) &= \int_{\delta<|t|\le\eta} \frac{1}{t}\mathrm{e}^{-\mathrm{i}t\omega}\, \mathrm{d}t = \left(\int_{-\eta}^{-\delta} + \int_{\delta}^{\eta} \right) \frac{\mathrm{e}^{-\mathrm{i}t\omega}}{t} \mathrm{d}t \\ &= \int_{\delta}^{\eta} \frac{\mathrm{e}^{-\mathrm{i}t\omega} - \mathrm{e}^{\mathrm{i}t\omega}}{t}\, \mathrm{d}t = -2\mathrm{i} \int_{\delta\omega}^{\eta\omega} \frac{\sin u}{u}\, \mathrm{d}u. \end{aligned}$$

By the formula $\int_0^\infty \frac{\sin u}{u}\,\mathrm{d}u = \frac{\pi}{2}$, we deduce that for $\omega > 0$,

$$\lim_{\substack{\delta\to 0\\ \eta\to\infty}} \widehat{K}_{\delta,\eta}(\omega) = \lim_{\substack{\delta\to 0\\ \eta\to\infty}} \left(-2\mathrm{i} \int_{\delta\omega}^{\eta\omega} \frac{\sin u}{u}\, \mathrm{d}u \right) = -2\mathrm{i} \int_0^\infty \frac{\sin u}{u}\, \mathrm{d}u = -\pi\mathrm{i}.$$

Similarly, for $\omega < 0$, we can deduce that $\lim_{\substack{\delta\to 0\\ \eta\to\infty}} \widehat{K}_{\delta,\eta}(\omega) = \pi\mathrm{i}$. Therefore,

$$\lim_{\substack{\delta\to 0\\ \eta\to\infty}} \widehat{K}_{\delta,\eta}(\omega) = -\pi\mathrm{i}\,\mathrm{sgn}\,\omega.$$

From this and

$$\widetilde{f}(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau = \lim_{\substack{\delta\to 0\\ \eta\to\infty}} \widetilde{f}_{\delta,\eta}(t),$$

by (2.7), we get

$$\widehat{\widetilde{f}}(\omega) = \lim_{\substack{\delta\to 0\\ \eta\to\infty}} \widehat{\widetilde{f}}_{\delta,\eta}(\omega) = \lim_{\substack{\delta\to 0\\ \eta\to\infty}} \left(\frac{1}{\pi}\widehat{f}(\omega)\widehat{K}_{\delta,\eta}(\omega)\right) = -\mathrm{i}\widehat{f}(\omega)\,\mathrm{sgn}\,\omega.$$

□

From Theorem 2.8, it follows that

$$\widehat{f}(\omega) + \mathrm{i}\widehat{\widetilde{f}}(\omega) = \widehat{f}(\omega) + \widehat{f}(\omega)\,\mathrm{sgn}\,\omega = \begin{cases} 2\widehat{f}(\omega), & \omega > 0,\\ 0, & \omega < 0 \end{cases} =: F(\omega).$$

So

$$f(t) + \mathrm{i}\widetilde{f}(t) = F^{\vee}(t),$$

where $F^{\vee}(t)$ is the inverse Fourier transform of $F(\omega)$. This implies that $\widetilde{f}(t) = \mathrm{Im}\, F^{\vee}(t)$.

Corollary 2.1. *Let $f \in L^2(\mathbb{R})$ and $\widetilde{f}$ be the Hilbert transform of f. Then $\widetilde{\widetilde{f}}(t) = -f(t)$.*

Proof. Let $\varphi(t) = \widetilde{f}(t)$. Then, by Theorem 2.8, we have

$$\widehat{\widetilde{\varphi}}(\omega) = -\mathrm{i}\widehat{\varphi}(\omega)\,\mathrm{sgn}\,\omega = -\mathrm{i}\widehat{\widetilde{f}}(\omega)\,\mathrm{sgn}\,\omega = (-\mathrm{i}\,\mathrm{sgn}\,\omega)^2\widehat{f}(\omega) = -\widehat{f}(\omega),$$

and so $\widetilde{\widetilde{f}}(t) = \widetilde{\varphi}(t) = -f(t)$. □

Bedrosian studied the Hilbert transform of products of two signals as follows.

Bedrosian Identity. *Let $f, g \in L^2(\mathbb{R})$ and the Fourier transforms of f, g satisfy $\widehat{f}(\omega) = 0 (\omega \in \mathbb{R}\setminus(-a,a))$ and $\widehat{g}(\omega) = 0 (\omega \in [-a,a])$ for some $a > 0$. Then $\widetilde{fg} = f\widetilde{g}$.*

Proof. By the assumption and the convolution property in frequency, it follows that

$$\widehat{fg}(\omega) = \frac{1}{2\pi}(\widehat{f} * \widehat{g})(\omega) = \frac{1}{2\pi}\int_{-a}^{a} \widehat{f}(u)\widehat{g}(\omega-u)\,\mathrm{d}u.$$

By Theorem 2.8, $\widehat{\widetilde{fg}} = -\mathrm{i}\,\mathrm{sgn}\,\omega\widehat{fg}(\omega)$, and so

$$\begin{aligned}
\widetilde{fg}(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} (-\mathrm{i}\,\mathrm{sgn}\,\omega)\widehat{fg}(\omega)\mathrm{e}^{\mathrm{i}t\omega}\,\mathrm{d}\omega\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} (-\mathrm{i}\,\mathrm{sgn}\,\omega)\mathrm{e}^{\mathrm{i}t\omega}\int_{-a}^{a} \widehat{f}(u)\widehat{g}(\omega-u)\,\mathrm{d}u\,\mathrm{d}\omega\\
&= \frac{1}{(2\pi)^2}\int_{-a}^{a} \widehat{f}(u)\left(\int_{\mathbb{R}} \widehat{g}(v)(-\mathrm{i}\,\mathrm{sgn}(u+v))\mathrm{e}^{\mathrm{i}t(u+v)}\,\mathrm{d}v\right)\mathrm{d}u.
\end{aligned}$$

Consider $\operatorname{sgn}(u+v)$. Note that $-a \le u \le a$. If $v \ge a$, then $u+v \ge 0, v \ge 0$, and so $\operatorname{sgn}(u+v) = \operatorname{sgn} v$. If $v \le -a$, then $u+v \le 0, v \le 0$, and so $\operatorname{sgn}(u+v) = \operatorname{sgn} v$. Note that $\widehat{g}(\omega) = 0 (\omega \in [-a, a])$. The integral in brackets is equal to

$$\begin{aligned}
&\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) \widehat{g}(v)(-\mathrm{i}\operatorname{sgn}(u+v))\mathrm{e}^{\mathrm{i}t(u+v)}\,\mathrm{d}v \\
&= -\mathrm{i}\int_{\mathbb{R}\setminus[-a,a]} \widehat{g}(v)\operatorname{sgn} v\mathrm{e}^{\mathrm{i}t(u+v)}\,\mathrm{d}v = -\mathrm{i}\int_{\mathbb{R}} \widehat{g}(v)\operatorname{sgn} v\mathrm{e}^{\mathrm{i}t(u+v)}\,\mathrm{d}v.
\end{aligned}$$

Therefore,

$$\widetilde{fg}(t) = \frac{1}{(2\pi)^2}\int_{-a}^{a}\widehat{f}(u)\mathrm{e}^{\mathrm{i}tu}\,\mathrm{d}u\int_{\mathbb{R}}\widehat{g}(v)(-\mathrm{i}\operatorname{sgn} v)\mathrm{e}^{\mathrm{i}vt}\,\mathrm{d}v = f(t)\widetilde{g}(t).$$

□

A signal of finite energy is called an *analytic signal* if its Fourier transform is zero for negative frequency.

Proposition 2.2. *Let $f \in L^2(\mathbb{R})$ and $\widetilde{f}$ be the Hilbert transform of f. Then $f_\alpha(t) = f(t) + \mathrm{i}\widetilde{f}(t)$ is an analytic signal.*

Proof. By Theorem 2.8: $\widehat{\widetilde{f}} = -\mathrm{i}\widehat{f}(\omega)\operatorname{sgn}\omega$, it follows that

$$\widehat{f_\alpha}(\omega) = \widehat{f}(\omega) + \mathrm{i}\widehat{\widetilde{f}}(\omega) = \widehat{f}(\omega) + \widehat{f}(\omega)\operatorname{sgn}\omega = \begin{cases} 2\widehat{f}(\omega), & \omega \ge 0, \\ 0, & \omega < 0. \end{cases}$$

that is, $f_\alpha(t)$ is an analytic signal. □

Complex analysis shows that $f_\alpha(t) = f(t) + \mathrm{i}\widetilde{f}(t)$ can be extended to an analytic function $f_\alpha(z)$ on the upper-half plane. Denote $f_\alpha(t) = A(t)\mathrm{e}^{\mathrm{i}\theta(t)}$. Then

$$\begin{aligned}
A(t) &= ((f(t))^2 + (\widetilde{f}(t))^2)^{1/2}, \\
\theta(t) &= \tan^{-1}\left(\frac{\widetilde{f}(t)}{f(t)}\right),
\end{aligned}$$

where $A(t)$ and $\theta(t)$ are called the *modulus* and *argument* of $f_\alpha(t)$, respectively.

Definition 2.1. Let $f \in L^2(\mathbb{R})$ be a real signal and

$$f_\alpha(t) = f(t) + \mathrm{i}\widetilde{f}(t) = A(t)\mathrm{e}^{\mathrm{i}\theta(t)} \quad (A(t) \ge 0).$$

Then $\theta'(t)$ is called the *instantaneous frequency* of $f(t)$.

Example 2.2. Let $f(t) = a(t)\cos(\omega_0 t + \varphi)$, where $a(t) \in L^2(\mathbb{R})$ and $a(t) > 0$, $\omega_0 > 0$, and $\widehat{a}(\omega) = 0 \ (|\omega| > \omega_0)$. Then the instantaneous frequency of f is ω_0.

Let $g(t) = \cos(\omega_0 t + \varphi)$. By Theorem 2.6 and properties of the Hilbert transform, it follows that

$$\widetilde{g}(t) = \sin(\omega_0 t + \varphi)\operatorname{sgn}\omega_0 = \sin(\omega_0 t + \varphi) \quad (\omega_0 > 0).$$

Noticing that $\widehat{a}(\omega) = 0(|\omega| > \omega_0)$ and $\text{supp}\,\widehat{g}(\omega) = \{\omega_0, -\omega_0\}$, by the Bedrosian identity, we find that

$$\widetilde{f}(t) = a(t)\widetilde{g}(t) = a(t)\sin(\omega_0 t + \varphi).$$

Therefore,

$$f_\alpha(t) = f(t) + \mathrm{i}\widetilde{f}(t) = a(t)(\cos(\omega_0 t + \varphi) + \mathrm{i}\sin(\omega_0 t + \varphi)) = a(t)\mathrm{e}^{\mathrm{i}(\omega_0 t + \varphi)}.$$

By Definition 2.1, the instantaneous frequency is ω_0.

Let a signal f be the sum of two cosine waves with the same amplitude:

$$f(t) = a\cos(\omega_1 t) + a\cos(\omega_2 t) \quad (\omega_1 > \omega_2 > 0).$$

Then its Hilbert transform is

$$\widetilde{f}(t) = a\sin(\omega_1 t)\text{sgn}\,\omega_1 + a\sin(\omega_2 t)\text{sgn}\,\omega_2 = a\sin(\omega_1 t) + a\sin(\omega_2 t).$$

Then the corresponding analytic signal is

$$f_\alpha(t) = f(t) + \mathrm{i}\widetilde{f}(t) = a(\mathrm{e}^{\mathrm{i}\omega_1 t} + \mathrm{e}^{\mathrm{i}\omega_2 t}) = 2a\cos\frac{(\omega_1 - \omega_2)t}{2}\mathrm{e}^{\mathrm{i}((\omega_1+\omega_2)t/2)}.$$

By Definition 2.1, the instantaneous frequency is $\frac{1}{2}(\omega_1 + \omega_2)$. This does not reveal that the signal includes two cosine waves with frequency ω_1 and ω_2, respectively, so the Hilbert transform can deal only with narrow-band signals

2.5 WIGNER-VILLE DISTRIBUTION AND COHEN'S CLASS

The windowed Fourier transform and the wavelet transform analyze the time-frequency structure by using a window function, while the Wigner-Ville distribution analyzes the time-frequency structure by translations. The Wigner-Ville distribution is defined as

$$W_{\mathrm{V}}f(u,\omega) = \int_{\mathbb{R}} f\left(u + \frac{\tau}{2}\right)\overline{f}\left(u - \frac{\tau}{2}\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\,\mathrm{d}\tau \quad (f \in L^2(\mathbb{R})).$$

If $f(t) = \mathrm{e}^{ibt}$, then $W_{\mathrm{V}}f(u,\omega) = \frac{1}{2\pi}\delta(\omega - b)$, where δ is the Dirac function.

The Wigner-Ville distribution possesses the following properties:

- **(i)** (Phase translation). If $f(t) = \mathrm{e}^{\mathrm{i}\varphi}\,g(t)$, then $W_{\mathrm{V}}f(u,\omega) = W_{\mathrm{V}}g(u,\omega)$.
- **(ii)** (Time translation). If $f(t) = g(t - u_0)$, then $W_{\mathrm{V}}f(u,\omega) = W_{\mathrm{V}}g(u - u_0,\omega)$.
- **(iii)** (Frequency translation). If $f(t) = \mathrm{e}^{\mathrm{i}t\omega_0}\,g(t)$, then $W_{\mathrm{V}}f(u,\omega) = W_{\mathrm{V}}\,g(u,\omega - \omega_0)$.
- **(iv)** (Scale dilation). If $f(t) = \frac{1}{\sqrt{s}}\,g\left(\frac{t}{s}\right)$, then $W_{\mathrm{V}}f(u,\omega) = W_{\mathrm{V}}f(\frac{u}{s}, s\omega)$.

The Wigner-Ville distribution can localize the time-frequency structure of the signal f. In the Wigner-Ville distribution, time and frequency have a symmetrical role, i.e., the following proposition holds.

Proposition 2.3. $W_{\mathrm{V}}f(u,\omega) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\omega + \frac{r}{2})\overline{\widehat{f}}(\omega - \frac{r}{2})\mathrm{e}^{\mathrm{i}\tau u}\,\mathrm{d}r.$

Proof. Denote

$$\varphi(\tau) = f\left(u + \frac{\tau}{2}\right) \mathrm{e}^{-\mathrm{i}\tau\xi}, \quad g(\tau) = f\left(u - \frac{\tau}{2}\right).$$

By Theorem 1.3, the Wigner-Ville distribution is

$$W_{\mathrm{V}} f(u,\xi) = \int_{\mathbb{R}} \varphi(\tau)\,\overline{g}(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\omega)\overline{\widehat{g}}(\omega)\,\mathrm{d}\omega.$$

Let $u + \frac{\tau}{2} = t$. Then

$$\begin{aligned}\widehat{\varphi}(\omega) &= \int_{\mathbb{R}} f\left(u + \frac{\tau}{2}\right) \mathrm{e}^{-\mathrm{i}\tau\xi}\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau \\ &= 2\int_{\mathbb{R}} f(t)\mathrm{e}^{-2\mathrm{i}(t-u)(\xi+\omega)}\,\mathrm{d}t = 2\widehat{f}(2\omega+2\xi)\mathrm{e}^{2\mathrm{i}u(\xi+\omega)}\end{aligned}$$

and

$$\overline{\widehat{g}}(\omega) = \overline{\int_{\mathbb{R}} f\left(u - \frac{\tau}{2}\right) \mathrm{e}^{-\mathrm{i}\tau\omega}\,\mathrm{d}\tau} = 2\overline{\widehat{f}}(-2\omega)\mathrm{e}^{2\mathrm{i}u\omega}.$$

Using the substitution $\xi + 2\omega = \frac{r}{2}$, we have

$$\begin{aligned}W_V f(u,\xi) &= \frac{2}{\pi}\int_{\mathbb{R}} \widehat{f}(2(\omega+\xi))\overline{\widehat{f}}(-2\omega)\mathrm{e}^{2\mathrm{i}u(\xi+2\omega)}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}\left(\xi+\frac{r}{2}\right)\overline{\widehat{f}}\left(\xi-\frac{r}{2}\right)\mathrm{e}^{\mathrm{i}ru}\,\mathrm{d}r.\end{aligned}$$

□

From Proposition 2.3 and the definition of the Wigner-Ville distribution, we get the following proposition.

Proposition 2.4. **(i)** *If* $\operatorname{supp} f(u) = [u_0 - \mu, u_0 + \mu]$, *then* $\operatorname{supp} W_{\mathrm{V}} f(\cdot,\omega) \subset [u_0 - \mu,\, u_0 + \mu]\ (\omega \in \mathbb{R})$.

(ii) *If* $\operatorname{supp} \widehat{f}(\omega) = [\omega_0 - \eta,\, \omega_0 + \eta]$, *then* $\operatorname{supp} W_{\mathrm{V}} f(u,\cdot) \subset [\omega_0 - \eta,\, \omega_0 + \eta]$ $(u \in \mathbb{R})$.

Proof. Let $g(t) = f(-t)$. The Wigner-Ville distribution is written in the form

$$W_{\mathrm{V}} f(u,\omega) = \int_{\mathbb{R}} f\left(\frac{\tau+2u}{2}\right)\overline{g}\left(\frac{\tau-2u}{2}\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\,\mathrm{d}\tau.$$

Since $\operatorname{supp} f = [u_0 - \mu, u_0 + \mu]$ and $\operatorname{supp} g = [-u_0 - \mu, -u_0 + \mu]$, it follows that

$$\begin{aligned}\operatorname{supp} f\left(\frac{\tau+2u}{2}\right) &= [2(u_0-u)-2\mu,\, 2(u_0-u)+2\mu], \\ \operatorname{supp} g\left(\frac{\tau-2u}{2}\right) &= [-2(u_0-u)-2\mu,\, -2(u_0-u)+2\mu].\end{aligned}$$

Therefore, $W_V f(\cdot, \omega) \neq 0$ only if these two intervals overlap. This is equivalent to $|u_0 - u| \leq \mu$. So we get (i). Similarly, by Proposition 2.3, we can get (ii). □

Since the Fourier transform of a Gaussian function is still a Gaussian function, a direct computation shows that the Wigner-Ville distribution of Gaussian function $f(t) = (\sigma^2\pi)^{-(1/4)} e^{-(t^2/2\sigma^2)}$ is a bivariate Gaussian function,

$$W_V f(u, \omega) = \frac{1}{\pi} e^{-(u^2/\sigma^2) - \sigma^2\omega^2} \quad \text{i.e.,} \quad W_V f(u, \omega) = |f(u)|^2 |\widehat{f}(\omega)|^2.$$

For a signal $f(t)$, we know that $f_\alpha(t) = f(t) + i\widetilde{f}(t) = A(t)e^{i\theta(t)}\,(A(t) \geq 0)$ is an analytic signal and $\theta'(t)$ is the instantaneous frequency of $f(t)$. The formula

$$\theta'(u) = \frac{\int_{\mathbb{R}} \omega W_V f_\alpha(u, \omega)\, d\omega}{\int_{\mathbb{R}} W_V f_\alpha(u, \omega)\, d\omega}$$

gives an *equivalent definition of the instantaneous frequency* computed by the Wigner-Ville distribution. This shows that the instantaneous frequency is the average frequency. Moreover, the Wigner-Ville distribution is a unitary transform which can imply the energy conservation property.

Theorem 2.9. *For $f, g \in L^2(\mathbb{R})$,*

$$\left| \int_{\mathbb{R}} f(t)\overline{g}(t)\, dt \right|^2 = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} W_V f(u, \omega) W_V g(u, \omega)\, du\, d\omega.$$

Proof. Note that

$$\begin{aligned} A &= \int\int_{\mathbb{R}^2} W_V f(u, \omega) W_V g(u, \omega)\, du\, d\omega \\ &= \int\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f\left(u + \frac{\tau}{2}\right) \overline{f}\left(u - \frac{\tau}{2}\right) e^{-i\omega\tau}\, d\tau \right) \left(\int_{\mathbb{R}} g\left(u + \frac{\tau'}{2}\right) \right. \\ &\quad \left. \overline{g}\left(u - \frac{\tau'}{2}\right) e^{-i\omega\tau'}\, d\tau' \right) du\, d\omega. \end{aligned}$$

By Formula 1.1, the Fourier transform of 1 is $2\pi\delta$, we get

$$\int_{\mathbb{R}} e^{-i\omega(\tau+\tau')}\, d\omega = 2\pi\delta(\tau + \tau').$$

Moreover, we have

$$\begin{aligned} A &= 2\pi \int\int_{\mathbb{R}^2} f\left(u + \frac{\tau}{2}\right) \overline{f}\left(u - \frac{\tau}{2}\right) \left(\int_{\mathbb{R}} \delta(\tau + \tau')\, g\left(u - \frac{\tau'}{2}\right) \overline{g}\left(u + \frac{\tau'}{2}\right) d\tau' \right) d\tau\, du \\ &= 2\pi \int\int_{\mathbb{R}^2} f\left(u + \frac{\tau}{2}\right) \overline{f}\left(u - \frac{\tau}{2}\right) g\left(u - \frac{\tau}{2}\right) \overline{g}\left(u + \frac{\tau}{2}\right) d\tau\, du \end{aligned}$$

Let $t = u + (\tau/2)$ and $s = u - (\tau/2)$. Then

$$A = 2\pi \int\int_{\mathbb{R}^2} f(t)\overline{f}(s)g(s)\overline{g}(t)\, dt\, ds = 2\pi \left| \int_{\mathbb{R}} f(t)\overline{g}(t)\, dt \right|^2.$$

□

Proposition 2.5. *The Wigner-Ville distribution satisfies*

$$\int_{\mathbb{R}} W_V f(u,\omega)\,\mathrm{d}u = |\widehat{f}(\omega)|^2 \quad \text{and} \quad \frac{1}{2\pi}\int_{\mathbb{R}} W_V f(u,\omega)\,\mathrm{d}\omega = |f(u)|^2.$$

Proof. Let $g_\omega(u) = (W_V f)(u,\omega)$. Note that $\widehat{g}_\omega(0) = \int_{\mathbb{R}} g_\omega(u)\mathrm{e}^{-\mathrm{i}0u}\,\mathrm{d}u = \int_{\mathbb{R}} g_\omega(u)\,\mathrm{d}u$. Then

$$\int_{\mathbb{R}} (W_V f)(u,\omega)\,\mathrm{d}u = \int_{\mathbb{R}} g_\omega(u)\,\mathrm{d}u = \widehat{g}_\omega(0).$$

By Proposition 2.3, the Fourier transform of g_ω is $\widehat{g}_\omega(r) = \widehat{f}(\omega + \frac{r}{2})\overline{\widehat{f}}(\omega - \frac{r}{2})$. Therefore,

$$\int_{\mathbb{R}} (W_V f)(u,\omega)\,\mathrm{d}u = |\widehat{f}(\omega)|^2.$$

Similarly, let $h_u(\omega) = W_V f(u,\omega)$. Then

$$\int_{\mathbb{R}} W_V f(u,\omega)\,\mathrm{d}\omega = \int_{\mathbb{R}} h_u(\omega)\,\mathrm{d}\omega = \widehat{h}_u(0).$$

By the definition of the Wigner-Ville distribution, the Fourier transform $\widehat{h}_u(\tau) = 2\pi f(u + \frac{\tau}{2})\overline{f}(u - \frac{\tau}{2})$. Therefore,

$$\int_{\mathbb{R}} W_V f(u,\omega)\,\mathrm{d}\omega = 2\pi\,|f(u)|^2.$$

□

However, the Wigner-Ville distribution may take negative values. For example, let $f = \chi_{[-T,T]}$. Since f is a real even function,

$$\begin{aligned}(W_V f)(u,\omega) &= \int_{\mathbb{R}} f\left(\frac{\tau}{2} + u\right) f\left(\frac{\tau}{2} - u\right)\mathrm{e}^{-\mathrm{i}\tau\omega}\mathrm{d}\tau \\ &= 2\int_{\mathbb{R}} f(\tau + u) f(\tau - u)\mathrm{e}^{-2\mathrm{i}\tau\omega}\,\mathrm{d}\tau\end{aligned}$$

and $f(\tau + u)f(\tau - u) = \chi_{[-T+|u|,T-|u|]}(\tau)$, and its Fourier transform

$$\begin{aligned}(f(\tau + u)f(\tau - u))^\wedge(\omega) &= \int_{-T+|u|}^{T-|u|} \mathrm{e}^{-\mathrm{i}t\omega}\mathrm{d}t \\ &= \frac{\mathrm{e}^{-\mathrm{i}\omega(T-|u|)} - \mathrm{e}^{\mathrm{i}\omega(T-|u|)}}{-i\omega} = \frac{2\sin((T-|u|)\omega)}{\omega}.\end{aligned}$$

Note that $(W_V f)(u,\omega) = 0(|u| > \frac{T}{2})$. Then

$$\frac{1}{2}W_V f(u,\omega) = (f(\tau + u)f(\tau - u))^\wedge(2\omega) = \frac{2\sin(2(T-|u|)\omega)}{\omega}\,\chi_{\left[-\frac{T}{2},\frac{T}{2}\right]}(u).$$

Clearly, $W_V f(u,\omega)$ takes negative values. A Gaussian function is the only function whose Wigner-Ville distribution remains positive.

To obtain a positive energy distribution, one needs to average the Wigner-Ville distribution and introduce the *Cohen's class distributions* as follows

$$Kf(u,\omega) := \int\int_{\mathbb{R}^2} W_V f(u',\omega')\, k(u-u',\omega-\omega')\ \mathrm{d}u'\,\mathrm{d}\omega',$$

where $k(u,v)$ is a smooth kernel function. The windowed Fourier transform belongs to Cohen's class distributions, and the corresponding smooth kernel is

$$k(u,\omega) = \frac{1}{2\pi} W_V g(u,\omega),$$

where $g(t)$ is a window function.

2.6 EMPIRICAL MODE DECOMPOSITIONS

Spline functions play a key role in the empirical mode decomposition (EMD) algorithm. If f is a polynomial of degree $k-1$ on each interval $[x_n, x_{n+1}](n \in \mathbb{Z})$ and f is a $k-2$-order continuously differentiable function on $\mathbb{R}$, then f is called a *spline function* of degree $k(k \geq 2)$ with knots $\{x_n\}_{n\in\mathbb{Z}}$.

Let a function f on $\mathbb{R}$ have local maximal values on $\{\alpha_n\}$:

$$\cdots < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 < \cdots.$$

Define the *upper envelope* $M(f)$ of f as follows:

(i) $M(f)(\alpha_n) = f(\alpha_n)(n \in \mathbb{Z})$;
(ii) $M(f)$ is a 3-order spline function with knots $\{\alpha_n\}$.

Let a function f on $\mathbb{R}$ have local minimal values on $\{\beta_n\}$:

$$\cdots < \beta_{-1} < \beta_0 < \beta_1 < \beta_2 < \cdots.$$

Define the *lower envelope* $m(f)$ of f as follows:

(i) $m(f)(\beta_n) = f(\beta_n)(n \in \mathbb{Z})$;
(ii) $m(f)$ is 3-order spline function with knots $\{\beta_n\}$.

The *local mean* of a function f on $\mathbb{R}$ is defined as

$$V(f)(t) = \frac{1}{2}(M(f)(t) + m(f)(t)).$$

For example, $f(t) = 3\sin(2t + \frac{\pi}{4})$ attains the maximal values on $\alpha_n = \frac{1}{2}((2n+\frac{1}{2})\pi - \frac{\pi}{4})(n \in \mathbb{Z})$ and attains the minimal values on $\beta_n = \frac{1}{2}((2n-\frac{1}{2})\pi - \frac{\pi}{4})(n \in \mathbb{Z})$, and attains the crossing zeros on $\gamma_n = \frac{1}{2}(n\pi - \frac{\pi}{4})(n \in \mathbb{Z})$. Clearly,

$$\cdots < \gamma_{2n-1} < \beta_n < \gamma_{2n} < \alpha_n < \gamma_{2n+1} < \beta_{n+1} < \cdots.$$

So $f(t)$ has the upper envelope $M(f)(t) = 3$ and the lower envelope $m(f)(t) = -3$, and its local mean $V(f)(t) = 0$.

A function f is called an *intrinsic mode function* (IMF) if it satisfies the following conditions:

(i) The number of extrema and the number of crossing zeros are equal or differ at most by one.
(ii) Its local mean is zero.

Empirical mode decomposition is used to decompose a signal f into several IMFs. If a discrete signal $f(t)$ has more than one oscillatory mode, then it can be decomposed into a sum of several IMFs and a monotonic signal as follows:

(i) Take the upper envelope $M(f)$ and lower envelope $m(f)$ of $f(t)$.
(ii) Compute the mean $V(f)(t) = \frac{1}{2}(M(f)(t) + m(f)(t))$ and the residual $r(t) = f(t) - V(f)(t)$.
(iii) Let $r(t)$ be the new signal. Follow this procedure until the local mean of $r(t)$ is equal to zero.
(iv) Once we have the zero-mean $r(t)$, it is designated as the first IMF, $c_1(t)$.
(v) Denote $f_1(t) = f(t) - c_1(t)$. We start from $f_1(t)$. Repeating the procedure from (i) to (iv), we get the second IMF, $c_2(t)$.
(vi) Continuing this procedure, we get $c_1(t), c_2(t), \ldots, c_n(t)$.

This process is stopped when the residual $r_n(t)$ is a monotonic function.

The procedure from (i) to (vi) gives an empirical mode decomposition of the signal $f(t)$ as follows:

$$f(t) = \sum_{1}^{n} c_k(t) + r_n(t),$$

where each $c_k(t)$ is an IMF and $r_n(t)$ is monotonic.

Let $\widetilde{c}_k(t)$ be the Hilbert transform of $c_k(t)$:

$$\widetilde{c}_k(t) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{c_k(u)}{t-u}\,\mathrm{d}u.$$

Then $Z_k(t) := c_k(t) + \mathrm{i}\widetilde{c}_k(t) = A_k(t)\mathrm{e}^{\mathrm{i}\theta_k(t)}$ is an analytic signal, where

$$A_k(t) = (c_k^2(t) + \widetilde{c}_k^2(t))^{1/2}, \quad \theta_k(t) = \arctan\left(\frac{\widetilde{c}_k(t)}{c_k(t)}\right).$$

Denote by $\omega_k(t)$ the instantaneous frequency of $c_k(t)$. Then the instantaneous frequency $\omega_k(t) = \theta_k'(t)$. This process is also called the *Hilbert-Huang transform.*

PROBLEMS

2.1 Let $f(t) = \mathrm{e}^{-t^2}$. Compute its Gabor transform $\left(G_0^{\alpha} f\right)(\omega)$.
2.2 Compare the time-frequency window of the windowed Fourier transform with that of the wavelet transform.

2.3 Download the monthly mean North Atlantic Oscillation index from http://www.cpc.ncep.noaa.gov/products/precip/CWlink/pna/new.nao.shtml and then research the wavelet power spectrum of the North Atlantic Oscillation index at different scales.

2.4 Let $\chi(x)$ be a Haar wavelet, i.e.

$$\chi(t) = \begin{cases} -1, & 0 \le t < \frac{1}{2}, \\ 1, & \frac{1}{2} < t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove $\{2^{m/2}\chi(2^m - n)\}_{m,n\in\mathbb{Z}}$ is a wavelet basis of $L^2(\mathbb{R})$.

2.5 Given a multiresolution analysis $\{V_m\}$,

$$V_m = \{f \in L^2(\mathbb{R}), \quad \widehat{f}(\omega) = 0, \quad |\omega| \ge 2^m\pi\},$$

try to find the scaling function and the corresponding orthonormal wavelet.

2.6 Let $H(\omega)$ be the filter of a scaling function and

$$H(\omega) = \sum_n a_n \mathrm{e}^{-\mathrm{i}n\omega}.$$

Prove that

(i) $\sum_n a_{2n} = \sum_n a_{2n+1} = \frac{1}{2}$;

(ii) $\sum_n a_n \overline{a}_{n-2k} = \begin{cases} 0, & k \ne 0, k \in \mathbb{Z}, \\ \frac{1}{2}, & k = 0. \end{cases}$

2.7 Perform empirical mode decomposition of local temperature data and analyze when significant warming occurs.

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Chapter 3

Filter Design

The purpose of filtering is to extract the information of geophysical signals for a given frequency band or restore the original signal details as much as possible by removing the unwanted noise produced by measurement imperfections. A lot of filters have been proposed, each of which has its own advantages and limitations. Implementation of these filters is easy, fast, and cost-effective by using a linear time-invariant system. In this chapter, we first focus on continuous linear time-invariant systems and the corresponding analog filters, including Butterworth filters, Chebeshev filters, and elliptic filters. Then we turn to discrete linear time-invariant systems, finite impulse response (FIR) filters, infinite impulse response (IIR) filters, and conjugate mirror filters.

3.1 CONTINUOUS LINEAR TIME-INVARIANT SYSTEMS

Linear time-invariant systems play a key role in the construction of filters. To explain this concept, we use the notation $y(t) = T[x(t)]$ to represent a system, where $x(t)$ is the input to the system and $y(t)$ is the output from the system.

If, for arbitrary constants a and b,

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)],$$

then the system $y(t) = T[x(t)]$ is called a *linear system*. If, for $x_n(t) \to x(t)$ (L^2),

$$T[x_n(t)] \to T[x(t)] \quad (t \in \mathbb{R}),$$

then the system $y(t) = T[x(t)]$ is *continuous*.

Let $y(t) = T[x(t)]$ be a linear system and τ be a constant. If $y(t-\tau) = T[x(t-\tau)]$ for any τ, then this linear system is called a *linear time-invariant system*.

In order to study linear time-invariant systems, we first define convolution.

Let $g(t) \in L^2(\mathbb{R})$ and $x(t) \in L^2(\mathbb{R})$ be two continuous signals. The *convolution* of $g(t)$ and $x(t)$ is

$$(g * x)(t) = \int_{\mathbb{R}} g(t-u)\, x(u)\, \mathrm{d}u \quad (t \in \mathbb{R}).$$

It has the following properties:

$$(g * (c\,x + d\,y))(t) = c\,(g * x)(t) + d\,(g * y)(t) \quad (t \in \mathbb{R}),$$

$$(g * x)(t) = (x * g)(t), \quad (g * (x * y))(t) = ((g * x) * y)(t) \quad (t \in \mathbb{R}),$$

where $g(t)$, $x(t)$, and $y(t)$ are continuous signals and c and d are constants.

Proposition 3.1. *Let $g \in L^2(\mathbb{R})$. A system $y(t) = T[x(t)]$ determined by the convolution*

$$y(t) = (g * x)(t) = \int_{\mathbb{R}} g(t-u)\, x(u)\, \mathrm{d}u \quad (x \in L^2(\mathbb{R}))$$

is a linear time-invariant system and is continuous. Here g is often called a filter.

Proof. Take $y_1(t) = T[x_1(t)]$ and $y_2(t) = T[x_2(t)]$. For any two constants α and β, it is clear that

$$\alpha y_1(t) + \beta y_2(t) = \alpha T[x_1(t)] + \beta T[x_2(t)].$$

On the other hand, by the assumption

$$y_1(t) = (g * x_1)(t) = \int_{\mathbb{R}} g(t-u)\, x_1(u)\, \mathrm{d}u,$$

$$y_2(t) = (g * x_2)(t) = \int_{\mathbb{R}} g(t-u)\, x_2(u)\, \mathrm{d}u,$$

it follows that

$$\begin{aligned} \alpha y_1(t) + \beta y_2(t) &= \alpha \int_{\mathbb{R}} g(t-u)\, x_1(u)\, \mathrm{d}u + \beta \int_{\mathbb{R}} g(t-u)\, x_2(u)\, \mathrm{d}u \\ &= \int_{\mathbb{R}} g(t-u)\, (\alpha x_1(u) + \beta x_2(u))\, \mathrm{d}u \\ &= T[\alpha x_1(t) + \beta x_2(t)]. \end{aligned}$$

Therefore,

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)],$$

i.e., the system T is a linear system.

Let $x_n(t) \to x(t)$ (L^2). By the Schwarz inequality,

$$\begin{aligned} |T[x_n(t)] - T[x(t)]| &= \left| \int_{\mathbb{R}} g(t-u)(x_n(u) - x(u))\, \mathrm{d}u \right| \\ &\le \left(\int_{\mathbb{R}} |g(t-u)|^2\, \mathrm{d}u \int_{\mathbb{R}} |x_n(u) - x(u)|^2\, \mathrm{d}u \right)^{1/2} \\ &= \left(\int_{\mathbb{R}} |g(u)|^2\, \mathrm{d}u \right)^{1/2} \left(\int_{\mathbb{R}} |x_n(u) - x(u)|^2\, \mathrm{d}u \right)^{1/2} \\ &\to 0 \quad (n \to \infty). \end{aligned}$$

So $T[x_n(t)] \to T[x(t)] (t \in \mathbb{R})$, i.e., the system is continuous.

By the assumption and $g * x = x * g$, it follows that

$$y(t) = (g * x)(t) = (x * g)(t) = \int_{\mathbb{R}} x(t-u)g(u)\,\mathrm{d}u,$$

and so, for any τ,

$$y(t-\tau) = \int_{\mathbb{R}} x(t-\tau-u)g(u)\,\mathrm{d}u = T[x(t-\tau)],$$

i.e., the system $y(t) = T[x(t)]$ determined by the convolution is a time-invariant system. □

For a continuous linear time-invariant system, the inverse proposition of Proposition 3.1 holds.

Proposition 3.2. *If a linear time-invariant system T is continuous, then there exists a filter $g(t)$ such that the input $x(t)$ and the output $y(t)$ of the system satisfy*

$$y(t) = \int_{\mathbb{R}} g(t-u)x(u)\,\mathrm{d}u = (g * x)(t). \tag{3.1}$$

Proof. Since the system is a linearly continuous system, from

$$x(t) =< \delta(t-u), x(u) >= \int_{\mathbb{R}} \delta(t-u)x(u)\,\mathrm{d}u,$$

where δ is the Dirac function, it follows that

$$T[x(t)] = \int_{\mathbb{R}} T[\delta(t-u)]x(u)\,\mathrm{d}u.$$

Let $g(t) = T[\delta(t)]$. Since T is a time-invariant system, $T[\delta(t-u)] = g(t-u)$, and so

$$y(t) = \int_{\mathbb{R}} g(t-u)x(u)\,\mathrm{d}u.$$

□

Propositions 3.1 and 3.2 state that a continuous system T is a linear time-invariant system if and only if T can be represented by a convolution form, i.e., $y(t) = T[x(t)] = (g * x)(t)$, where the filter g is the response of the Dirac impulse, i.e., $g(t) = T[\delta(t)]$.

A linear time-invariant system T is *causal* if the output $y(t)$ depends only on the input $x(u)$ $(u \le t)$. Proposition 3.2 shows that T is causal if and only if the filter $g(u) = 0$ $(u < 0)$. A linear time-invariant system T is *stable* if any bounded input produces a bounded output. By (3.1), we have

$$|y(t)| \le \sup_{u\in\mathbb{R}} |x(u)| \int_{\mathbb{R}} |g(u)|\,\mathrm{d}u.$$

So T is stable if and only if $\int_{\mathbb{R}} |g(u)|\,\mathrm{d}u < \infty$. Suppose that T is a continuous linear time-invariant system with the filter g. For complex exponent $\mathrm{e}^{\mathrm{i}\Omega t}$, the output of the system T is

$$T[\mathrm{e}^{\mathrm{i}\Omega t}] = \int_{\mathbb{R}} g(u)\,\mathrm{e}^{\mathrm{i}\Omega(t-u)}\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}\Omega t}\int_{\mathbb{R}} g(u)\,\mathrm{e}^{-\mathrm{i}\Omega u}\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}\Omega t}\hat{g}(\Omega), \qquad (3.2)$$

so $\hat{g}(\Omega)$ is called the frequency response of the system T. If T is regarded as a linear continuous operator, then each $\mathrm{e}^{\mathrm{i}\Omega t}$ is the eigenfunction of T corresponding to the eigenvalue $\hat{g}(\Omega)$.

Now we introduce an ideal low-pass filter $g_{\mathrm{d}}(t)$ which passes low-frequency signals and completely eliminates all high-frequency information.

Let

$$G_{\mathrm{d}}(\omega) = \chi_{[-\Omega_{\mathrm{c}},\Omega_{\mathrm{c}}]}(\omega) =: \begin{cases} 1, & |\omega| \le \Omega_{\mathrm{c}}, \\ 0, & |\omega| > \Omega_{\mathrm{c}} \end{cases}$$

and the filter $g_{\mathrm{d}}(t)$ be the inverse Fourier transform of $G_{\mathrm{d}}(\omega)$, i.e.,

$$g_{\mathrm{d}}(t) = \frac{1}{2\pi}\int_{-\Omega_{\mathrm{c}}}^{\Omega_{\mathrm{c}}} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{\mathrm{e}^{\mathrm{i}t\Omega_{\mathrm{c}}} - \mathrm{e}^{-\mathrm{i}t\Omega_{\mathrm{c}}}}{2\pi\mathrm{i}t} = \frac{\sin(t\Omega_{\mathrm{c}})}{\pi t}.$$

Define a linear time-invariant system T by $y(t) = T[x(t)] = (g_{\mathrm{d}} * x)(t)$. Taking Fourier transforms on both sides, by the convolution property of the Fourier transform, we get

$$Y(\omega) = G_{\mathrm{d}}(\omega)X(\omega) = \begin{cases} X(\omega), & |\omega| \le \Omega_{\mathrm{c}}, \\ 0, & |\omega| > \Omega_{\mathrm{c}}, \end{cases}$$

where $X(\omega)$ and $Y(\omega)$ are Fourier transforms of the input $x(t)$ and the output $y(t)$, respectively. This equality states that the frequency spectrum of low-frequency waves remains invariant, while that of high-frequency waves vanishes. Therefore, $g_{\mathrm{d}}(t)$ is called an *ideal low-pass filter* and $\mathrm{e}^{\mathrm{i}\omega t}\chi_{[-\Omega_{\mathrm{c}},\Omega_{\mathrm{c}}]}(\omega)$ is the frequency response. However, the continuous linear time-invariant system with an ideal low-pass filter is not stable, and this implies that bounded input does not imply bounded output, moreover, it is also not causal, so it cannot be used in practice.

3.2 ANALOG FILTERS

Three classical analog filters are follows:

(i) *Butterworth filter*. A Butterworth filter $g_{\mathrm{b}}(t)$ is a filter whose Laplace transform $G_{\mathrm{b}}(s)$ satisfies

$$|G_{\mathrm{b}}(\mathrm{i}\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_{\mathrm{c}})^{2N}} = \frac{\Omega_{\mathrm{c}}^{2N}}{\Omega^{2N}+\Omega_{\mathrm{c}}^{2N}}, \qquad (3.3)$$

where Ω_{c} is the *width* of the passband, N is an integer, and N is the *order* of the filter.

When $\Omega = \Omega_{\mathrm{c}}$, $|G_{\mathrm{b}}(\mathrm{i}\Omega)|^2 = \frac{1}{2}$. When N is increasing, $|G_{\mathrm{b}}(\mathrm{i}\Omega)|$ approximates to an ideal low-pass filter. It does not have a zero. Its poles s_{p} are determined as follows.

From $\Omega^{2N}+\Omega_c^{2N}=0$, it follows that $\Omega^{2N}=-\Omega_c^{2N}$, and so $\Omega=(-1)^{\frac{1}{2N}}\Omega_c$. Since $(-1)^{\frac{1}{2N}}$ has $2N$ values, $\{e^{i\frac{(2k+1)\pi}{2N}}\}_{k=0,\ldots,2N-1}$. Therefore,

$$|G_b(i\Omega)|^2=\frac{\Omega_c^{2N}}{\prod_0^{2N-1}(\Omega-s_k)}\quad(\Omega\in\mathbb{R}),$$

where $s_k=e^{i\frac{(2k+1)\pi}{2N}}\Omega_c$. Let $s=i\Omega$. Then

$$\frac{\Omega_c^{2N}}{\prod_0^{2N-1}(\Omega-s_k)}=\frac{(-1)^N\Omega_c^{2N}}{\prod_0^{2N-1}(s-is_k)}.$$

These poles $\{is_k\}_{k=0,\ldots,2N-1}$ are symmetric about the origin. Let $p_1,\ldots,p_N$ lie in the left-half plane. Then the other poles are $-p_1,\ldots,-p_N$. So

$$\frac{\Omega_c^{2N}}{\prod_0^{2N-1}(\Omega-s_k)}=\frac{\Omega_c^N}{\prod_1^N(s-p_k)}\,\frac{(-1)^N\Omega_c^N}{\prod_1^N(s+p_k)}.$$

Noticing that $|G_b(i\Omega)|^2=G_b(i\Omega)G_b(-i\Omega)$ for a real filter $g_b(t)$, we know that

$$G_b(s)=\frac{\Omega_c^N}{\prod_1^N(s-p_k)}.$$

Taking the inverse Laplace transform, we have $g_b(t)=L^{-1}(G_b(s))$.

(ii) *Chebeshev filter*. The Chebyshev polynomial of order N is defined as

$$T_N(x)=\begin{cases}\cos(N\theta), & \theta=\cos^{-1}x \quad (|x|\le 1),\\ \cosh(N\tau), & \tau=\cosh^{-1}x \quad (|x|>1).\end{cases}$$

Especially,

$$T_0(x)=1,\quad T_1(x)=x.$$

From this and the recurrence formula $T_{N+1}(x)=2x\,T_n(x)-T_{N-1}(x)$, it follows that

$$T_2(x)=2x^2-1, T_3(x)=4x^3-3x,\ldots.$$

A type I Chebyshev filter $g_c^1(t)$ is a filter whose Laplace transform satisfies

$$|G_c^1(i\Omega)|^2=\frac{1}{1+\epsilon^2T_N^2(\Omega/\Omega_c)},$$

where N is the order of the filter and $0<\epsilon<1$. The larger ϵ is, the larger the ripple is. Since all zeros of Chebyshev polynomials $T_N(x)$ lie in $[-1,1]$, when $0\le\Omega\le\Omega_c$,

$$\frac{1}{1+\epsilon^2}\le|G_c^1(i\Omega)|^2\le 1.$$

When $\Omega \geq \Omega_c$, $|G_c^1(i\Omega)|^2$ increases monotonically as Ω increases.

We compute the bandwidth Ω_A, which is defined as $|G_c^1(i\Omega_A)|^2 = \frac{1}{2}$. It is clear that $|G_c^1(i\Omega_A)|^2 = \frac{1}{2}$ is equivalent to $\epsilon^2 T_N^2(\Omega_A/\Omega_c) = 1$. If $\Omega_A \leq \Omega_c$, then $T_N^2(\Omega_A/\Omega_c) \leq 1$, and so $\epsilon \geq 1$. This is contrary to $0 < \epsilon < 1$. Therefore, $\Omega_A > \Omega_c$, i.e., $\frac{\Omega_A}{\Omega_c} > 1$. From this and $T_N(\Omega_A/\Omega_c) = \frac{1}{\epsilon}$, we get

$$\cosh\left[N\cosh^{-1}\left(\frac{\Omega_A}{\Omega_c}\right)\right] = \frac{1}{\epsilon}.$$

Therefore,

$$\Omega_A = \Omega_c \cosh\left(\frac{1}{N}\cosh^{-1}\frac{1}{\epsilon}\right).$$

The poles of $|G_c^1(i\Omega)|^2$ are

$$p_k = \sigma_k + i\tau_k,$$

where $\sigma_k = -\Omega_c \sinh\zeta \sin\frac{(2k-1)\pi}{2N}$ and $\tau_k = \Omega_c \cosh\zeta \cos\frac{(2k-1)\pi}{2N}$, and $\zeta = \frac{1}{N}\sinh^{-1}\frac{1}{\epsilon}$. This implies that each pole p_k of the type I Chebyshev filter lies on the ellipse $\frac{\sigma^2}{a^2} + \frac{\tau^2}{b^2} = 1$, where $a = \Omega_c \sinh\zeta$ and $b = \Omega_c \cosh\zeta$. Similarly to the Butterworth filter, with the help of these poles, we can obtain the type I Chebyshev filter $g_c^1(t)$ and its Laplace transform $G_c^1(s)$.

The type II Chebeshev filter $g_c^2(t)$ is a filter whose Laplace transform $G_c^2(s)$ satisfies

$$|G_c^2(i\Omega)|^2 = \frac{1}{1 + \epsilon^2\left(\frac{T_N(\Omega_r/\Omega_c)}{T_N(\Omega_r/\Omega)}\right)^2} \quad (0 < \epsilon < 1).$$

It is clear that it decreases monotonically in the passband and $|G(0)| = 1$, and $G(i\Omega_c) = \frac{1}{1+\epsilon^2}$, and it has equiripple in $\Omega \geq \Omega_r$.

$|G_c^2(i\Omega)|^2$ has $2N$ zeros $z_k = \frac{i\,\Omega_r}{\cos((2k-1)\pi/2N)}\,(k = 1, \ldots, 2N)$ and $2N$ poles $p_k = \sigma_k + i\tau_k$ $(k = 1, \ldots, 2N)$, where

$$\sigma_k = \frac{\Omega_r \alpha_k}{\alpha_k^2 + \beta_k^2}, \quad \tau_k = -\frac{\Omega_r \beta_k}{\alpha_k^2 + \beta_k^2},$$

and

$$\begin{aligned}\alpha_k &= -\sinh\xi \sin\frac{(2k-1)\pi}{2N},\\ \beta_k &= \cosh\xi \cos\frac{(2k-1)\pi}{2N},\\ \xi &= \frac{1}{N}\sinh^{-1}\left(\epsilon T_N\left(\frac{\Omega_r}{\Omega_c}\right)\right).\end{aligned}$$

Similarly to the Butterworth filter, with the help of these poles and zeros, we can obtain the type II Chebyshev filter $g_c^2(t)$ and its Laplace transform $G_c^1(s)$.

(iii) *Elliptic filter*. An elliptic filter $g_e(t)$ is a filter whose Laplace transform $G_e(s)$ satisfies

$$|G_e(i\Omega)|^2 = \frac{1}{1+\epsilon^2 J_N^2(\Omega)},$$

where $J_N(\Omega)$ is the Jacobian ellipse function of order N. The elliptic filter allows equiripple for both the passband and the stopband.

3.3 DISCRETE LINEAR TIME-INVARIANT SYSTEMS

A discrete signal comes from sampling or discretization of a continuous signal. A discrete signal is also called a digital signal. If the input and output signals of a system are both discrete signals, then the system is called a *discrete system*.

3.3.1 Discrete Signals

A one-dimensional *discrete signal* is a sequence $\{x(n)\}_{n\in\mathbb{Z}}$. For example, the unit step signal is

$$u(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

The rectangular signal is

$$r_N(n) = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise}. \end{cases}$$

An exponential signal is expressed by $e(n) = a^n u(n)$, where a is a real constant.

The two-dimensional unit step signal is

$$u(n_1, n_2) = \begin{cases} 1, & n_1 \geq 0,\, n_2 \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

The two-dimensional exponential signal is $\{a^{n_1} b^{n_2}\}_{n_1,n_2\in\mathbb{Z}}$, where a and b are real constants. The two-dimensional sinusoidal sequence is $\{A\cos(n_1\omega_1 + \theta_1)\cos(n_2\omega_2 + \theta_2)\}_{n_1,n_2\in\mathbb{Z}}$.

Now we discuss frequency domain representations of discrete signals.

The Fourier transform of a one-dimensional discrete signal $\{x(n)\}_{n\in\mathbb{Z}}$ is defined as

$$F[x(n)] := X(e^{i\omega}) = \sum_k x(k)\, e^{-ik\omega}.$$

If the series on the right-hand side is absolutely convergent, i.e., $\sum_k |x(k)| < \infty$, then the Fourier transform $X(e^{i\omega})$ is a periodic continuous function with period 2π. The inverse Fourier transform is defined as

$$F^{-1}(X(\mathrm{e}^{\mathrm{i}\omega})) := \{x(n)\},$$

where $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega$.

For example, the Fourier transform of the rectangular sequence $\{r_N(n)\}$ is

$$F[r_N(n)] = \sum_0^{N-1} \mathrm{e}^{-\mathrm{i}n\omega} = \frac{1-\mathrm{e}^{-\mathrm{i}N\omega}}{1-\mathrm{e}^{-\mathrm{i}\omega}} = \mathrm{e}^{-\mathrm{i}\frac{(N-1)\omega}{2}}\frac{\sin(N\omega/2)}{\sin(\omega/2)},$$

and the inverse Fourier transform $F^{-1}\left(\mathrm{e}^{-\mathrm{i}\frac{(N-1)\omega}{2}}\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right) = \{r_N(n)\}$.

Fourier transforms of discrete signals have the following properties.

Property. Let $\{x(n)\}_{n\in\mathbb{Z}}$ and $\{y(n)\}_{n\in\mathbb{Z}}$ be two discrete signals and c and d be two constants. Then

(i) $F[cx(n) + dy(n)] = cF[x(n)] + dF[y(n)]$;
(ii) $F[(x * y)(n)] = F[x(n)]F[y(n)]$.

The Fourier transform of a two-dimensional discrete signal $\{x(n_1, n_2)\}_{n_1,n_2\in\mathbb{Z}}$ is defined as

$$F[x(n_1, n_2)] := X(\mathrm{e}^{\mathrm{i}\omega_1}, \mathrm{e}^{\mathrm{i}\omega_2}) = \sum_{n_1}\sum_{n_2} x(n_1, n_2)\,\mathrm{e}^{-\mathrm{i}n_1\omega_1}\,\mathrm{e}^{-\mathrm{i}n_2\omega_2}.$$

If $\sum_{n_1}\sum_{n_2} |x(n_1, n_2)| < \infty$, then the Fourier transform $X(\mathrm{e}^{\mathrm{i}\omega_1}, \mathrm{e}^{\mathrm{i}\omega_2})$ is a continuous function of ω_1, ω_2 and

$$X(\mathrm{e}^{\mathrm{i}\omega_1}, \mathrm{e}^{\mathrm{i}\omega_2}) = X(\mathrm{e}^{\mathrm{i}(\omega_1+2k\pi)}, \mathrm{e}^{\mathrm{i}(\omega_2+2l\pi)}) \quad (k,\ l \in \mathbb{Z}).$$

The inverse Fourier transform is defined as

$$F^{-1}[X(\mathrm{e}^{\mathrm{i}\omega_1}, \mathrm{e}^{\mathrm{i}\omega_2})] := \{x(n_1, n_2)\},$$

where $x(n_1, n_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{\mathrm{i}\omega_1}, \mathrm{e}^{\mathrm{i}\omega_2})\,\mathrm{e}^{\mathrm{i}n_1\omega_1}\,\mathrm{e}^{\mathrm{i}n_2\omega_2}\,\mathrm{d}\omega_1\mathrm{d}\omega_2$.

3.3.2 Discrete Convolution

Let $h = \{h(n)\}_{n\in\mathbb{Z}}$ and $x = \{x(n)\}_{n\in\mathbb{Z}}$ be two infinite discrete signals. The *discrete convolution* of $h(n)$ and $x(n)$ is defined as

$$(h * x)(k) = \sum_n h(k-n)x(n) \quad (k \in \mathbb{Z}).$$

If $h = \{h(n)\}_{n=0,\ldots,N_h-1}$ and $x = \{x(n)\}_{n=0,\ldots,N_x-1}$ are two finite signals with lengths N_h and N_x, respectively, then the length of discrete convolution $h * x$ is $N_x + N_h - 1$.

The discrete convolution has the following properties:

$$(h * (cx + dy))(k) = c(h * x)(k) + d(h * y)(k) \quad (k \in \mathbb{Z}),$$

$$(h * x)(k) = (x * h)(k), \quad (h * (x * y))(k) = ((h * x) * y)(k) \quad (k \in \mathbb{Z}),$$

where $h = \{h(n)\}_{n\in\mathbb{Z}}$, $x = \{x(n)\}_{n\in\mathbb{Z}}$, and $y = \{y(n)\}_{n\in\mathbb{Z}}$ are discrete signals and c and d are constants.

Let $h = \{h(n_1, n_2)\}_{n_1,n_2\in\mathbb{Z}}$ and $x = \{x(n_1, n_2)\}_{n_1,n_2\in\mathbb{Z}}$ be the two-dimensional discrete signals. Then the convolution of $h(n_1, n_2)$ and $x(n_1, n_2)$ is defined as

$$(h * x)(n_1, n_2) = \sum_{m_1}\sum_{m_2} h(n_1 - m_1, n_2 - m_2)x(m_1, m_2) \quad (n_1,\ n_2 \in \mathbb{Z}).$$

The following convolution properties also hold in the two-dimensional case:

$$(h * (cx + dy))(n_1, n_2) = c(h * x)(n_1, n_2) + d(h * y)(n_1, n_2) \quad (n_1,\ n_2 \in \mathbb{Z}),$$

$$(h * x)(n_1, n_2) = (x * h)(n_1, n_2), (h * (x * y))(n_1, n_2) \ = \ ((h * x) * y)(n_1, n_2)$$
$$(n_1,\ n_2 \in \mathbb{Z}).$$

3.3.3 Discrete System

To define discrete time-invariant systems, we use the notation $y(n) = T[x(n)]$ $(n \in \mathbb{Z})$ to represent a discrete system, where $\{x(n)\}_{n\in\mathbb{Z}}$ is the input sequence and $\{y(n)\}_{n\in\mathbb{Z}}$ is the output sequence.

If, for arbitrary constants α and β,

$$T[\alpha x_1(n) + \beta x_2(n)] = \alpha T[x_1(n)] + \beta T[x_2(n)] \quad (n \in \mathbb{Z}),$$

then the system $y(n) = T[x(n)]$ $(n \in \mathbb{Z})$ is called a *discrete linear system.*

Let $y(n) = T[x(n)]$ $(n \in \mathbb{Z})$ be a discrete linear system. If

$$y(n - k) = T[x(n - k)]\ (k \in \mathbb{Z}),$$

then the system $y(n) = T[x(n)]$ $(n \in \mathbb{Z})$ is called a *discrete time-invariant system.*

The sequence $\{\delta(n)\}_{n\in\mathbb{Z}}$, where $\delta(n) = 0$ $(n \neq 0)$ and $\delta(0) = 1$, is called the *unit impulse*. The unit impulse response $h(n) = T[\delta(n)]$ $(n \in \mathbb{Z})$ is called the *filter* of the system T.

Proposition 3.3. *Any discrete linear time-invariant system* $y(n) = T[x(n)]$ $(n \in \mathbb{Z})$ *can be represented by the discrete convolution of the input and the unit impulse response, i.e.,* $y(n) = (h * x)(n)(n \in \mathbb{Z})$.

Proof. Note that $\delta(n) = 0$ $(n \neq 0)$ and $\delta(0) = 1$. Any input $\{x(n)\}_{n\in\mathbb{Z}}$ can be represented by

$$x(n) = \sum_k x(k)\delta(n - k) \quad (n \in \mathbb{Z}).$$

The system T is a linear system, so

$$y(n) = T[x(n)] = \sum_k x(k)T[\delta(n - k)].$$

Note that $h(n) = T[\delta(n)]$. Since the system T is time invariant, we get $h(n-k) = T[\delta(n-k)]$, and so

$$y(n) = \sum_k x(k)h(n-k) = (h * x)(n).$$

□

For a linear time-invariant system $y(n) = (h * x)(n)$ $(n \in \mathbb{Z})$, if the output $y(n)$ depends only on the input $x(k)$ $(k \leq n)$, then this system is called *causal.*

Proposition 3.4. *A system $y(n) = (h * x)(n)$ $(n \in \mathbb{Z})$ is causal if and only if $h(n) = 0$ $(n < 0)$.*

Proof. Assume that $h(n) = 0$ $(n < 0)$. Then $h(n-k) = 0$ $(k > n)$, and so

$$y(n) = \sum_{k \leq n} x(k)h(n-k).$$

Therefore, the output $y(n)$ depends only on the input $x(k)$ $(k \leq n)$, i.e., the system is causal.

Assume that the system is causal. If $h(-l) \neq 0$ for some $l \in \mathbb{Z}_+$, we take $x(k) = 0$ $(k \neq n + l)$ and $x(n+l) \neq 0$, then

$$y(n) = \sum_k x(k)h(n-k) = x(n+l)h(-l) \neq 0,$$

so the output $y(n)$ cannot be determined by the input $x(k)$ $(k \leq n)$. This is contrary to the assumption. Hence, $h(n) = 0$ $(n < 0)$. □

If a linear time-invariant system is such that any bounded input products a bounded output, then this system is called a *stable system.*

Proposition 3.5. *A linear time-invariant system*

$$y(n) = (h * x)(n) \ (n \in \mathbb{Z})$$

is stable if and only if $\sum_n |h(n)| < \infty$.

Proof. Assume that $\sum_n |h(n)| = M < \infty$. If $|x(n)| < A$ $(n \in \mathbb{Z})$, then

$$|y(n)| = \left| \sum_k x(k)h(n-k) \right| \leq A \sum_k |h(n-k)| = A \sum_n |h(n)| \leq AM,$$

and so the system is stable.

Assume that $\sum_n |h(n)| = \infty$. Take

$$x(n) = \begin{cases} \dfrac{\overline{h}(-n)}{|h(-n)|}, & h(-n) \neq 0, \\ 0, & h(-n) = 0. \end{cases}$$

Then

$$|y(0)| = \left| \sum_k x(k)h(-k) \right| = \sum_{h(-k) \neq 0} |h(-k)| = \sum_k |h(k)| = \infty,$$

and so the output is unbounded. This is contrary to the assumption. Hence $\sum_n |h(n)| < \infty$. □

For a linear time-invariant system with the unit impulse response h, we consider the frequency response. If the input is a complex exponent sequence with frequency ω: $x(n) = \mathrm{e}^{\mathrm{i}n\omega}$ $(n \in \mathbb{Z})$, then its output response is

$$y(n) = (h * x)(n) = \sum_k h(k)\mathrm{e}^{\mathrm{i}(n-k)\omega} = \mathrm{e}^{\mathrm{i}n\omega}H(\mathrm{e}^{\mathrm{i}\omega}), \tag{3.4}$$

where $H(\mathrm{e}^{\mathrm{i}\omega}) = \sum_k h(k)\mathrm{e}^{-\mathrm{i}k\omega}$. The function $H(\mathrm{e}^{\mathrm{i}\omega})$ is called the *frequency response* of the system. It is clear that $H(\mathrm{e}^{\mathrm{i}\omega})$ is a 2π-periodic function. If the system is stable, then $\sum_n |h(n)| < \infty$. So $H(\mathrm{e}^{\mathrm{i}\omega})$ exists and is continuous, and is the Fourier transform of the filter h. The inverse Fourier transform is

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega.$$

Now consider the two-dimensional case. For a two-dimensional linear system $y(n_1,n_2) = T[x(n_1,n_2)]$. Noticing that

$$\delta(n_1,n_2) = \begin{cases} 1, & n_1 = n_2 = 0, \\ 0, & n_1 \text{ or } n_2 \neq 0, \end{cases}$$

we can represent the input x by

$$x(n_1,n_2) = \sum_k \sum_l x(k,l)\delta(n_1-k,n_2-l),$$

so

$$y(n_1,n_2) = \sum_k \sum_l x(k,l)\,T[\delta(n_1-k,n_2-l)].$$

Let h be the response of the two-dimensional unit impulse δ, i.e.,

$$h(n_1,n_2) = T[\delta(n_1,n_2)].$$

The unit impulse response h is also called the *filter* of the system. If T is a time-invariant system, then

$$T[\delta(n_1-k,n_2-l)] = h(n_1-k,n_2-l),$$

and so

$$y(n_1,n_2) = \sum_k \sum_l x(k,l)h(n_1-k,n_2-l) = (h*x)(n_1,n_2), \tag{3.5}$$

i.e., any two-dimensional linear time-invariant system $y(n_1,n_2) = T[x(n_1,n_2)]$ can be represented by the two-dimensional discrete convolution of the input and the filter. If the filter h of a two-dimensional discrete time-invariant system satisfies $\sum_{n_1}\sum_{n_2} |h(n_1,n_2)| < \infty$, then the system is called *a stable system*.

If the output $y(n_1, n_2)$ depends only on the input $x(k, l)$ ($k \leq n_1,\ l \leq n_2$), then the system is *causal*. Similarly, a two-dimensional time-invariant system is causal if and only if its filter $h(n_1, n_2) = 0$ ($n_1 < 0, n_2 < 0$).

3.3.4 Ideal Digital Filters

For a discrete linear time-invariant system, let h be the filter and $H(\mathrm{e}^{\mathrm{i}\omega}) = \sum_n h(n)\,\mathrm{e}^{-\mathrm{i}n\omega}$ be its frequency response.

Case 1. If $H(\mathrm{e}^{\mathrm{i}\omega}) = 0$ ($|\omega_\mathrm{c}| < |\omega| \leq \pi$), then the filter h is called a *low-pass filter*. Let

$$\begin{aligned} Y(\mathrm{e}^{\mathrm{i}\omega}) &= \sum_n y(n)\,\mathrm{e}^{-\mathrm{i}n\omega}, \\ X(\mathrm{e}^{\mathrm{i}\omega}) &= \sum_n x(n)\,\mathrm{e}^{-\mathrm{i}n\omega}. \end{aligned}$$

Then, by the property of the Fourier transform of discrete signals,

$$Y(\mathrm{e}^{\mathrm{i}\omega}) = H(\mathrm{e}^{\mathrm{i}\omega})\,X(\mathrm{e}^{\mathrm{i}\omega}).$$

So a low-pass filter only passes low-frequency signals. The inverse Fourier transform gives

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_\mathrm{c}}^{\omega_\mathrm{c}} \mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega = \begin{cases} \frac{\sin(n\omega_\mathrm{c})}{\pi n}, & n \neq 0, \\ \frac{\omega_\mathrm{c}}{\pi}, & n = 0. \end{cases}$$

Case 2. If $H(\mathrm{e}^{\mathrm{i}\omega}) = 0$ ($|\omega| \leq \omega_\mathrm{c} < \pi$), then the filter h is called a *high-pass filter*. We can see that a high-pass filter passes only high-frequency signals. The inverse Fourier transform gives

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{e}^{\mathrm{i}n\omega}\,\mathrm{d}\omega = \begin{cases} -\frac{\sin(n\omega_\mathrm{c})}{\pi n}, & n \neq 0, \\ 1 - \frac{\omega_\mathrm{c}}{\pi}, & n = 0. \end{cases}$$

Case 3. If $H(e^{\mathrm{i}\omega}) = 0$ ($0 < |\omega_\mathrm{c}| \leq |\omega| \leq |\omega_\mathrm{d}| < \pi$), then the filter h is called a *band-pass filter*.

3.3.5 *Z*-Transforms

For a discrete signal $x = \{x(n)\}_{n\in\mathbb{Z}}$, its *Z-transform* is defined as

$$X(z) = \sum_n x(n)\,z^{-n}.$$

It is sometimes called the two-sided Z-transform. Denote it by $Z\{x(n)\}$, i.e., $Z\{x(n)\} = X(z)$.

If the limits

$$r_1 = \lim_{n\to\infty} \sqrt[n]{|x(n)|},$$

$$r_2 = \lim_{n\to-\infty} \sqrt[n]{|x(n)|}$$

exist and $\frac{1}{r_1} < r_2$, then the convergence domain of its Z-transform is the annular region $\frac{1}{r_1} < |z| < r_2$. Let $z = r\mathrm{e}^{\mathrm{i}\theta}$ $(\frac{1}{r_1} < r < r_2)$. Then the Z-transform of $\{x(n)\}_{n\in\mathbb{Z}}$ can be rewritten as

$$X(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_n x(n) r^{-n} \mathrm{e}^{-\mathrm{i}n\theta}.$$

By the orthogonality of exponential sequence $\{\mathrm{e}^{-\mathrm{i}n\theta}\}_{n\in\mathbb{Z}}$, it follows that

$$\int_{-\pi}^{\pi} X(r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}n\theta}\, \mathrm{d}\theta = \sum_k x(k) r^{-k} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n-k)\theta}\, \mathrm{d}\theta = 2\pi x(n) r^{-n}.$$

So the inverse Z-transform of $X(z)$ is

$$x(n) = Z^{-1}[X(z)] = \frac{r^n}{2\pi} \int_{-\pi}^{\pi} X(r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}n\theta}\, \mathrm{d}\theta.$$

If the Z-transform $X(z)$ is a rational function which has only simple poles p_{k} $(k = 1, \ldots, N)$, then $X(z)$ can be decomposed into a sum of partial fractions and a polynomial $p(z)$, i.e.,

$$X(z) = \sum_1^N \frac{A_{\mathrm{k}}}{z - p_{\mathrm{k}}} + p(z),$$

where $A_{\mathrm{k}} = \lim_{z\to p_{\mathrm{k}}} X(z)(z - p_{\mathrm{k}})$. Expanding each $1/(z - p_{\mathrm{k}})$ into the positive power series or the negative power series, we can also obtain the inverse Z-transform of $X(z)$.

The Z-transforms have the following properties.

Property. Let $\{x(n)\}_{n\in\mathbb{Z}}$ and $\{y(n)\}_{n\in\mathbb{Z}}$ be two discrete signals. Denote their Z-transforms by $X(z)$ and $Y(z)$, respectively. Then

(i) $Z\{ax(n) + by(n)\}$ is $aX(z) + bY(z)$;
(ii) $Z\{x(n - n_0)\}$ is $z^{-n_0}X(z)$;
(iii) $Z\{a^n x(n)\}$ is $X\left(\frac{z}{a}\right)$;
(iv) $Z\{\overline{x}(n)\}$ is $\overline{X}(\overline{z})$; and
(v) $Z\{nx(n)\}$ is $-zX'(z)$.

Proposition 3.6. *For a discrete linear time-invariant system, let $X(z)$, $H(z)$, and $Y(z)$ be the Z-transforms of the input x, the output y, and the filter h, respectively. Then $Y(z) = H(z)X(z)$.*

The Z-transform of the filter h is called the *transfer function* of the system.

Proof. By Proposition 3.4, we have $y(k) = (h * x)(k) = \sum_n h(k-n)x(n)$. It follows that

$$Y(z) = \sum_k y(k)\, z^{-k} = \sum_n x(n) z^{-n} \left(\sum_k h(k-n) z^{-(k-n)} \right).$$

Since $\sum_k h(k-n)z^{-(k-n)} = \sum_k h(k)z^{-k}$, we get

$$Y(z) = \left(\sum_k h(k) z^{-k} \right) \left(\sum_n x(n) z^{-n} \right) = H(z)X(z).$$

□

The concept of the one-dimensional Z-transform may be generalized to the two-dimensional case.

Let $\{x(m,n)\}_{m,n\in\mathbb{Z}}$ be a two-dimensional discrete signal. Then its Z-transform is defined as

$$X(z_1, z_2) = \sum_{m,n} x(m,n) z_1^{-m} z_2^{-n}.$$

Let $z_1 = r_1 \mathrm{e}^{-\mathrm{i}\omega_1}$ and $z_2 = r_2 \mathrm{e}^{\mathrm{i}\omega_2}$. Then

$$X(r_1\mathrm{e}^{\mathrm{i}\omega_1}, r_2\mathrm{e}^{\mathrm{i}\omega_2}) = \sum_{m,n} x(m,n)\, r_1^{-m} r_2^{-n}\, \mathrm{e}^{-\mathrm{i}m\omega_1}\, \mathrm{e}^{-\mathrm{i}n\omega_2}.$$

If $\sum_{m,n} |x(m,n)| r_1^{-m} r_2^{-n} < \infty$, then the series on the right-hand side converges.

The inverse Z-transform of $X(z_1, z_2)$ is defined as

$$x(m,n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(r_1\, \mathrm{e}^{\mathrm{i}\omega_1}, r_2\, \mathrm{e}^{\mathrm{i}\omega_2})\, r_1^m r_2^n\, \mathrm{e}^{\mathrm{i}m\omega_1}\, \mathrm{e}^{\mathrm{i}n\omega_2}\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2 (m,\ n \in \mathbb{Z}).$$

3.3.6 Linear Difference Equations

We will discuss the discrete linear time-invariant system which can be represented by a linear difference equation:

$$\sum_0^N b(k)\, y(u-k) = \sum_0^N a(k)\, x(u-k), \tag{3.6}$$

where $x(u)$ is the input signal, $y(u)$ is the output signal, and $a(k)$ and $b(k)$ are constants. Taking the Z-transform on both sides of (3.6), we get

$$\sum_0^N b(k)\, z^{-k}\, Y(z) = \sum_0^N a(k)\, z^{-k}\, X(z),$$

where $X(z) = \sum_0^N x(n)\, z^{-n}$ and $Y(z) = \sum_0^N y(n)\, z^{-n}$, and so

$$Y(z) = H(z)\, X(z), \tag{3.7}$$

where the transfer function

$$H(z) = \frac{\sum_0^N a(k)\, z^{-k}}{\sum_0^N b(k)\, z^{-k}}.$$

Expand $H(z)$ into the two-sided power series $H(z) = \sum_n h(n) z^n$. By the convolution property of the Z-transform, it follows from (3.7) that

$$y(n) = (h * x)(n).$$

If there exist infinitely many nonzero terms in $\{h(n)\}_{n\in\mathbb{Z}}$, then it is called an *infinite impulse response* (IIR) *filter*. Otherwise, if there exist only finitely many nonzero terms in $\{h(n)\}_{n\in\mathbb{Z}}$, then it is called a *finite impulse response* (FIR) *filter*.

3.4 LINEAR-PHASE FILTERS

Let T be a discrete linear time-variant system with the FIR filter h. The input $x = \{x(n)\}_{n\in\mathbb{Z}}$ and the output $y = \{y(n)\}_{n\in\mathbb{Z}}$ of the system satisfy

$$y(n) = (h * x)(n) \quad (n \in \mathbb{Z}).$$

Without loss of generality, we assume that $h(n) = 0$ $(n \neq 0, \ldots, N-1)$. The transfer function

$$H(z) = \sum_0^{N-1} h(n)\, z^{-n}$$

is an $N-1$ degree polynomial of z^{-1}. This is a causal stable discrete system.

Let $z = \mathrm{e}^{\mathrm{i}\,\omega}$. The *frequency response* is

$$H(\mathrm{e}^{\mathrm{i}\,\omega}) = \sum_0^{N-1} h(n)\, \mathrm{e}^{-\mathrm{i}n\omega}. \tag{3.8}$$

Clearly, this is a 2π-periodic function. It can be expressed as

$$H(\mathrm{e}^{\mathrm{i}\omega}) = |H(\mathrm{e}^{\mathrm{i}\omega})|\, \mathrm{e}^{\mathrm{i}\theta(\omega)}, \quad \text{where } \tan\theta(\omega) = \frac{\mathrm{Im}(H(\mathrm{e}^{\mathrm{i}\omega}))}{\mathrm{Re}(H(\mathrm{e}^{\mathrm{i}\omega}))}. \tag{3.9}$$

Here $|H(\mathrm{e}^{\mathrm{i}\omega})|$ is called the *frequency spectrum* and $\theta(\omega)$ is called the *phase*.

When $\theta(\omega) = -\tau\,\omega$, where τ is a constant, we say the filter has a rigorous linear phase. When $\theta(\omega) = b - \tau\omega$, where τ and b are constants, we say the filter has a generalized linear phase. Now we study the FIR filter with a linear phase. It is very important in geophysical signal processing.

From (3.8), it follows that

$$\mathrm{Im}(H(\mathrm{e}^{\mathrm{i}\omega})) = -\sum_0^{N-1} h(n) \sin(n\omega),$$

$$\mathrm{Re}(H(\mathrm{e}^{\mathrm{i}\omega})) = \sum_0^{N-1} h(n)\cos(n\omega).$$

Combining this with (3.9), we have

$$\tan\theta(\omega) = -\frac{\sum_0^{N-1} h(n)\sin(n\,\omega)}{\sum_0^{N-1} h(n)\cos(n\,\omega)}.$$

This implies that for any ω, $\theta(\omega) = -\tau\omega$ if and only if

$$\sum_0^{N-1} h(n)(\cos(n\omega)\sin(\tau\omega) - \sin(n\omega)\cos(\tau\omega)) = 0,$$

i.e., for any ω, $\theta(\omega) = -\tau\omega$ if and only if $\sum_0^{N-1} h(n)\ \sin(\tau - n)\omega = 0$.

Similarly, we can deduce that for any ω, $\theta(\omega) = \frac{\pi}{2} - \tau\omega$ if and only if $\sum_0^{N-1} h(n)\cos(\tau - n)\omega = 0$.

Proposition 3.7. *Let the filter $h = \{h(n)\}_{n=0,\dots,N-1}$ be an FIR digital filter.*

(i) *If*

$$h(n) = h(N-1-n) \quad (n = 0,\dots,N-1),$$

then the filter h is a rigorous linear-phase filter and $\arg H(\mathrm{e}^{\mathrm{i}\omega}) = -\frac{N-1}{2}\omega$.

(ii) *If*

$$h(n) = -h(N-1-n) \quad (n = 0,\dots,N-1),$$

then the filter h is a generalized linear-phase filter and $\arg H(\mathrm{e}^{\mathrm{i}\omega}) = \frac{\pi}{2} - \frac{N-1}{2}\omega$.

Proof. Let $\tau = \frac{N-1}{2}$.

(i) By the assumption

$$h(n) = h(N-1-n) \quad (n = 0,\dots,N-1),$$

it follows that $\{h(n)\}_{n=0,\dots,N-1}$ is an even symmetric sequence with center τ. So $\{\sin\omega(\tau - n)\}_{n=0,\dots,N-1}$ is an odd sequence with center τ, and so $h(n)\sin\omega(\tau - n)$ is an odd sequence with center τ. This implies that

$$\sum_0^{N-1} h(n)\ \sin(\tau - n)\omega = 0,$$

which is equivalent to $\theta(\omega) = -\tau\omega$, so the filter h is a rigorous linear phase filter and $\arg H(\mathrm{e}^{\mathrm{i}\theta}) = -\tau\omega$.

(ii) By the assumption

$$h(n) = -h(N-1-n) \quad (n = 0,\dots,N-1),$$

it follows that $\{h(n)\}_{n=0,\dots,N-1}$ is an odd symmetric sequence with center τ. Since $\{\cos(\tau - n)\omega\}_{n=0,\dots,N-1}$ is an even sequence with center τ,

$$\sum_0^{N-1} h(n)\cos(\tau - n)\omega = 0,$$

which is equivalent to $\theta(\omega) = \frac{\pi}{2} - \tau\omega$. So the filter h is a generalized linear phase filter and $\arg H(e^{i\omega}) = \frac{\pi}{2} - \tau\omega$.

□

3.4.1 Four Types of Linear-Phase Filters

Assume that $\{h(n)\}_{n=0,\dots,N-1}$ is an FIR filter and its frequency response $H(e^{i\omega}) = \sum_0^{N-1} h(n)\, e^{-in\omega}$.

(i) $\{h(n)\}_{n=0,\dots,N-1}$ has even symmetry and N is odd.

Its frequency response is

$$H(e^{i\omega}) = \sum_0^{N-1} h(n)\, e^{-in\omega} = \sum_0^{\frac{N-3}{2}} h(n)\, e^{-in\omega} + h\left(\frac{N-1}{2}\right) e^{-i\frac{N-1}{2}\omega} + \sum_{\frac{N+1}{2}}^{N-1} h(n)\, e^{-in\omega}.$$

By $h(n) = h(N-1-n)$, the third term on the right-hand side becomes

$$\sum_0^{\frac{N-3}{2}} h(n)\, e^{in\omega}\, e^{-i(N-1)\omega}.$$

So

$$\begin{aligned} H(e^{i\omega}) &= e^{-i\frac{N-1}{2}\omega}\left\{\sum_0^{\frac{N-3}{2}} 2h(n)\cos\left(\frac{N-1}{2} - n\right)\omega + h\left(\frac{N-1}{2}\right)\right\} \\ &= e^{-i\frac{N-1}{2}\omega}\left\{\sum_1^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - m\right)\cos(m\omega) + h\left(\frac{N-1}{2}\right)\right\}. \end{aligned}$$

Let $a(0) = h\left(\frac{N-1}{2}\right)$ and $a(m) = 2h\left(\frac{N-1}{2} - m\right)$ $\left(m = 1, \dots, \frac{N-1}{2}\right)$. Then

$$H(e^{i\omega}) = e^{-i\frac{N-1}{2}\omega}\left\{\sum_1^{\frac{N-1}{2}} a(m)\cos(m\omega) + a(0)\right\} = e^{-i\frac{N-1}{2}\omega}\sum_0^{\frac{N-1}{2}} a(n)\cos(n\omega).$$

(ii) $\{h(n)\}_{n=0,\ldots,N-1}$ has even symmetry and N is even.

Similarly to the argument in (i), its frequency response is

$$H(e^{i\omega}) = e^{-i\frac{N-1}{2}\omega} \sum_{1}^{\frac{N}{2}} b(n) \cos\left(n - \frac{1}{2}\right)\omega,$$

where $b(n) = 2h(\frac{N}{2} - n)(n = 1, 2, \ldots, \frac{N}{2})$.

(iii) $\{h(n)\}_{n=0,\ldots,N-1}$ has odd symmetry and N is odd.

$\{h(n)\}_{n=0,\ldots,N-1}$ has odd symmetry with center $\frac{N-1}{2}$, so $h(\frac{N-1}{2}) = 0$. Similarly to the argument in (i), its frequency response is

$$H(e^{i\omega}) = e^{i(\frac{\pi}{2} - \frac{N-1}{2}\omega)} \sum_{1}^{\frac{N-1}{2}} c(n)\ \sin(n\omega),$$

where $c(n) = 2h(\frac{N-1}{2} - n)(n = 1, 2, \ldots, \frac{N-1}{2})$.

(iv) $\{h(n)\}_{n=0,\ldots,N-1}$ has odd symmetry and N is even.

Its frequency response is

$$H(e^{i\omega}) = e^{i(\frac{\pi}{2} - \frac{N-1}{2}\omega)} \sum_{1}^{\frac{N}{2}} d(n)\ \sin \frac{2n-1}{2}\omega,$$

where $d(n) = 2h(\frac{N}{2} - n)(n = 1, 2, \ldots, \frac{N}{2})$.

3.4.2 Structure of Linear-Phase Filters

For an FIR digital filter with a rigorous linear phase, $\{h(n)\}_{n=0,\ldots,N-1}$, its transfer function

$$H(z) = \sum_{0}^{N-1} h(n)\, z^{-n},$$

where $h(n) = h(N-1-n)$ and $\arg H(e^{i\,\omega}) = -\tau\,\omega$, and $\tau = \frac{N-1}{2}$. Therefore,

$$H(z) = \sum_{0}^{N-1} h(N-1-n)\, z^{-n} = z^{-N+1} \sum_{0}^{N-1} h(n)\, z^{n} = z^{-(N-1)}\, H(z^{-1}).$$

From this, we see that if z_k is a zero of $H(z)$, then $H(z_k^{-1}) = z_k^{N-1}\, H(z_k) = 0$. Since each $h(n)$ is real,

$$\overline{H}(z) = \sum_{0}^{N-1} h(n)\, \overline{z}^{n} = H(\overline{z}).$$

From this, we see that if z_k is a zero of $H(z)$, then $H(\overline{z}_k) = 0$. Therefore, we obtain the following conclusion.

Suppose that z_k is a zero of $H(z)$:

(i) If $|z_k| < 1$ and z_k is not real, then z_k, z_k^{-1}, $\bar{z}_k$, and $\bar{z}_k^{-1}$ are four different zeros of $H(z)$. This constitutes a system of order 4. Denote it by $H_k(z)$, i.e.,

$$H_k(z) = (1 - z^{-1} z_k)(1 - z^{-1} \bar{z}_k)(1 - z^{-1} z_k^{-1})(1 - z^{-1} \bar{z}_k^{-1}).$$

Denote $z_k = r_k\, e^{i\theta_k}$. This equality can be expanded into

$$\begin{aligned} H_k(z) \;&=\; 1 - 2z^{-1}\left(r_k + \frac{1}{r_k}\right)\cos\theta_k + z^{-2}\left(r_k^2 + \frac{1}{r_k^2} + 4\cos^2\theta_k\right) \\ &\quad -2z^{-3}\left(r_k + \frac{1}{r_k}\right)\cos\theta_k + z^{-4}. \end{aligned}$$

(ii) If $|z_k| < 1$ and $z_k = r_k$ is real, then r_k and r_k^{-1} are two different zeros of $H(z)$. This constitutes a system of order 2. Denote it by $H_m(z)$, i.e.,

$$H_m(z) = (1 - z^{-1} r_k)\left(1 - \frac{z^{-1}}{r_k}\right) = 1 - z^{-1}\left(r_k + \frac{1}{r_k}\right) + z^{-2}.$$

(iii) If $|z_k| = 1$ and z_k is not real, then $z_k = z_k^{-1}$ and $\bar{z}_k = \bar{z}_k^{-1}$. So z_k and $\bar{z}_k$ are two different zeros of $H(z)$. This also constitutes a system of order 2. Denote it by $H_l(z)$, i.e.,

$$\begin{aligned} H_l(z) \;&=\; (1 - z_k z^{-1})(1 - \bar{z}_k z^{-1}) = 1 + z^{-1}(z_k + \bar{z}_k) + z_k \bar{z}_k\, z^{-2} \\ &=\; 1 + 2\, z^{-1}\,\mathrm{Re}(z_k) + z^{-2}. \end{aligned}$$

(iv) If $|z_k| = 1$ and z_k is real, then $z_k = z_k^{-1} = \bar{z}_k = \bar{z}_k^{-1}$. So only z_k is a zero of $H(z)$. This constitutes the simplest system of order 1. Denote it by $H_s(z)$, i.e.,

$$H_s(z) = 1 - z_k z^{-1}.$$

In this way, for the FIR digital filter with a linear phase, its transfer function $H(z)$ can be expressed as

$$H(z) = \left(\prod_k H_k(z)\right)\left(\prod_m H_m(z)\right)\left(\prod_l H_l(z)\right)\left(\prod_s H_s(z)\right),$$

where H_k, H_m, H_l, and H_s are subsystems with a rigorous linear phase.

3.5 DESIGNS OF FIR FILTERS

Now we give three methods for designing FIR digital filters.

3.5.1 Fourier Expansions

From Section 3.3.4, for some $0 < \omega_c < \pi$,

$$h_d(n) = \begin{cases} \frac{\sin(n\omega_c)}{\pi n}, & n \neq 0, \\ \frac{\omega_c}{\pi}, & n = 0 \end{cases}$$

is an ideal low-pass filter, and the corresponding frequency response is

$$H_d(e^{i\omega}) = \sum_n h_d(n)\, e^{-in\omega} = \begin{cases} 1, & |\omega| \leq \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases}$$

The ideal low-pass filter is noncausal and infinite in duration. Clearly, it cannot be implemented in practice. In order to obtain an FIR filter, we may approximate to $H_d(e^{i\omega})$ by $H_{d,N}(e^{i\omega})$:

$$H_{d,N}(e^{i\omega}) = \sum_{-\tau}^{\tau} h_d(n)\, e^{-in\omega} \quad \left(\tau = \frac{N-1}{2}\right). \tag{3.10}$$

By Parseval's identity of Fourier series, the approximation error is

$$r_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{i\omega}) - H_{d,N}(e^{i\omega})|^2\, d\omega = \sum_{|n|>\tau} |h_d(n)|^2 \quad \left(\tau = \frac{N-1}{2}\right).$$

Take an odd number N large enough such that the error $\sum_{|n|>\tau} |h_d(n)|^2 < \epsilon$.

To avoid the noncausal problem, we multiply both sides of (3.10) by $e^{-i\tau\omega}$ to get a new filter:

$$H(e^{i\omega}) = e^{-i\tau\omega} H_{d,N}(e^{i\omega}) = e^{-i\tau\omega} \sum_{-\tau}^{\tau} h_d(n)\, e^{-in\omega} = \sum_{0}^{2\tau} h_d(n-\tau)\, e^{-in\omega}.$$

Let $h(n) = h_d(n-\tau)$. Then the frequency response $H(e^{i\omega}) = \sum_0^{N-1} h(n)\, e^{-in\omega}$, where

$$h(n) = \frac{\sin(n-\tau)\omega_c}{\pi(n-\tau)} \quad (n = 0, \ldots, N-1; n \neq \tau), \quad h(\tau) = \frac{\omega_c}{\pi}, \tag{3.11}$$

and $H(e^{i\omega})$ has rigorous linear phase $\arg H(e^{i\omega}) = -\tau\omega$, where $\tau = \frac{N-1}{2}$. By $\sum_{|n|>\tau} |h_d(n)|^2 < \epsilon$, it follows that the filter $\{h(n)\}_{n=0,\ldots,N-1}$ is a linear-phase filter and approximates to an ideal low-pass filter.

Since $h(n+\tau) = h(\tau - n)$ and $e^{in\omega} + e^{-in\omega} = 2\cos(n\omega)$, by (3.11), the frequency response $H(e^{i\omega})$ can be rewritten in the real form

$$\begin{aligned} H(e^{i\omega}) &= e^{-i\tau\omega}\left(h(\tau) + 2\sum_1^{\tau} h(n+\tau)\cos(n\omega)\right) \\ &= e^{-i\tau\omega}\left(\frac{\omega_c}{\pi} + 2\sum_1^{\tau} \frac{\sin(n\omega_c)}{\pi n}\cos(n\omega)\right). \end{aligned}$$

Since the frequency response of the ideal digital filter has points of discontinuity, the convergence rate of partial sums of its Fourier series is low and truncating the ideal digital filter introduces undesirable ripples and overshoots in the frequency response. Therefore, the filter constructed as above cannot approximate well to the ideal filter and makes the Gibbs phenomenon occur. To solve this problem, window functions are introduced.

3.5.2 Window Design Method

Suppose that $\{h_{\rm d}(n)\}$ is an ideal digital filter and $H_{\rm d}(e^{i\omega})$ is its frequency response. To reduce the Gibbs phenomenon, we need to choose a window sequence ω_n with finite length and then multiply $h_{\rm d}(n)$ by ω_n, i.e.,

$$h(n) = h_{\rm d}(n)\,\omega_n.$$

Denote the frequency responses corresponding to $h(n)$, $h_{\rm d}(n)$, and ω_n by $H(e^{i\omega})$, $H_{\rm d}(e^{i\omega})$, and $W(e^{i\omega})$, respectively. By using the convolution theorem, we get

$$H(e^{i\omega}) = H_{\rm d}(e^{i\omega}) * W(e^{i\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\rm d}(e^{i\theta})\, W(e^{i(\omega-\theta)})\, d\theta.$$

This shows that the frequency response equals the convolution of the frequency response of the ideal digital filter and the frequency response of the window sequence. We choose window sequence $\{\omega_n\}$ such that $H(e^{i\omega})$ is smooth and approximates well to $H_{\rm d}(e^{i\omega})$. Several window sequences are often used, as follows:

(i) Rectangular window

$$\omega_{{\rm R},n} = \begin{cases} 1, & n = 0, \dots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Its frequency response is

$$W_{\rm R}(e^{i\omega}) = \sum_{0}^{N-1} e^{-in\omega} = \frac{1 - e^{-iN\omega}}{1 - e^{-i\omega}} = e^{-i\frac{N-1}{2}\omega}\frac{\sin(N\omega/2)}{\sin(\omega/2)}.$$

From this equality, we see that its phase is linear.

(ii) Bartlett window

$$\omega_{{\rm B},n} = \begin{cases} \frac{2n}{N-1}, & n = 0, \dots, \frac{N-1}{2}, \\ 2 - \frac{2n}{N-1}, & n = \frac{N-1}{2}, \dots, N-1. \end{cases}$$

(iii) Hanning window

$$\omega_{{\rm H},n} = \begin{cases} 0.5 - 0.5\cos\frac{2\pi n}{N-1}, & n = 0, \dots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$