Mathematics
The Loss of Certainty

MORRIS KLINE

"A thinker who understands numbers better than anyone since Euclid delivers a ringing indictment of modern mathematics" *Omni*

MATHEMATICS

The Loss of Certainty

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Mathematics

The gods have not revealed all things from the beginning,
But men seek and so find out better in time.

• • •

Let us suppose these things are like the truth.

• • •

But surely no man knows or ever will know
The truth about the gods and all I speak of.
For even if he happens to tell the perfect truth,
He does not know it, but appearance is fashioned over everything.

XENOPHANES

Introduction: The Thesis

To foresee the future of mathematics, the true method is to study its history and its present state. HENRI POINCARÉ

There are tragedies caused by war, famine, and pestilence. But there are also intellectual tragedies caused by limitations of the human mind. This book relates the calamities that have befallen man's most effective and unparalleled accomplishment, his most persistent and profound effort to utilize human reason—mathematics.

Put in other terms, this book treats on a non-technical level the rise and decline of the majesty of mathematics. In view of its present immense scope, the increasing, even flourishing, mathematical activity, the thousands of research papers published each year, the rapidly growing interest in computers, and the expanded search for quantitative relationships especially in the social and biological sciences, how can we talk about the decline of mathematics? Wherein lies the tragedy? To answer these questions we must consider first what values won for mathematics its immense prestige, respect, and glory.

From the very birth of mathematics as an independent body of knowledge, fathered by the classical Greeks, and for a period of over two thousand years, mathematicians pursued truth. Their accomplishments were magnificent. The vast body of theorems about number and geometric figures offered in itself what appeared to be an almost endless vista of certainty.

Beyond the realm of mathematics proper, mathematical concepts and derivations supplied the essence of remarkable scientific theories. Though the knowledge obtained through the collaboration of mathematics and science employed physical principles, these seemed to be as secure as the principles of mathematics proper because the predictions in the mathematical theories of astronomy, mechanics, optics, and hydrodynamics were in remarkably accurate accord with observation and experiment. Mathematics, then, provided a firm grip on the workings of nature, an understanding which dissolved mystery and replaced it by law and order. Man could pridefully survey the world about him and boast that he had grasped many of the secrets of the universe, which in essence were a series of mathematical laws. The conviction that mathematicians were securing truths is epitomized in Laplace's remark that Newton was a most fortunate man because there is just one universe and Newton had discovered its laws.

To achieve its marvelous and powerful results, mathematics relied upon a special method, namely, deductive proof from self-evident principles called axioms, the methodology we still learn, usually in high school geometry. Deductive reasoning, by its very nature, guarantees the truth of what is deduced if the axioms are truths. By utilizing this seemingly clear, infallible, and impeccable logic, mathematicians produced apparently indubitable and irrefutable conclusions. This feature of mathematics is still cited today. Whenever someone wants an example of certitude and exactness of reasoning, he appeals to mathematics.

The successes mathematics achieved with its methodology attracted the greatest intellectuals. Mathematics had demonstrated the capacities, resources, and strengths of human reason. Why should not this methodology be employed, they asked, to secure truths in fields dominated by authority, custom, and habit, fields such as philosophy, theology, ethics, aesthetics, and the social sciences? Man's reason, so evidently effective in mathematics and mathematical physics, could surely be the arbiter of thought and action in these other fields and obtain for them the beauty of truths and the truths of beauty. And so, during the period called the Enlightenment or the Age of Reason, mathematical methodology and even some mathematical concepts and theorems were applied to human affairs.

The most fertile source of insight is hindsight. Creations of the early 19th century, strange geometries and strange algebras, forced mathematicians, reluctantly and grudgingly, to realize that mathematics proper and the mathematical laws of science were not truths. They found, for example, that several differing geometries fit spatial experience equally well. All could not be truths. Apparently mathematical design was not inherent in nature, or if it was, man's mathematics was not necessarily the account of that design. The key to reality had been lost. This realization was the first of the calamities to befall mathematics.

The creation of these new geometries and algebras caused mathematicians to experience a shock of another nature. The conviction that they were obtaining truths had entranced them so much that they had rushed impetuously to secure these seeming truths at the cost of sound reasoning. The realization that mathematics was not a body of truths shook their confidence in what they had created, and they undertook to reexamine their creations. They were dismayed to find that the logic of mathematics was in sad shape.

In fact mathematics had developed illogically. Its illogical development contained not only false proofs, slips in reasoning, and inadvertent mistakes which with more care could have been avoided. Such blunders there were aplenty. The illogical development also involved inadequate understanding of concepts, a failure to recognize all the principles of logic required, and an inadequate rigor of proof; that is, intuition, physical arguments, and appeal to geometrical diagrams had taken the place of logical arguments.

However, mathematics was still an effective description of nature. And mathematics itself was certainly an attractive body of knowledge and in the minds of many, the Platonists especially, a part of reality to be prized in and for itself. Hence mathematicians decided to supply the missing logical structure and to rebuild the defective portions. During the latter half of the 19th century the movement often described as the rigorization of mathematics became the outstanding activity.

By 1900 the mathematicians believed they had achieved their goal. Though they had to be content with mathematics as an approximate description of nature and many even abandoned the belief in the mathematical design of nature, they did gloat over their reconstruction of the logical structure of mathematics. But before they had finished toasting their presumed success, contradictions were discovered in the reconstructed mathematics. Commonly these contradictions were referred to as paradoxes, a euphemism that avoids facing the fact that contradictions vitiate the logic of mathematics.

The resolution of the contradictions was undertaken almost immediately by the leading mathematicians and philosophers of the times. In effect four different approaches to mathematics were conceived, formulated, and advanced, each of which gathered many adherents. These foundational schools all attempted not only to resolve the known contradictions but to ensure that no new ones could ever arise, that is, to establish the consistency of mathematics. Other issues arose in the foundational efforts. The acceptability of some axioms and some principles of deductive logic also became bones of contention on which the several schools took differing positions.

As late as 1930 a mathematician might perhaps have been content with accepting one or another of the several foundations of mathematics and declared that his mathematical proofs were at least in accord with the tenets of that school. But disaster struck again in the form of a famous paper by Kurt Gödel in which he proved, among other significant and disturbing results, that the logical principles accepted by the several schools could not prove the consistency of mathematics. This, Gödel showed, cannot be done without involving logical principles so dubious as to question what is accomplished. Gödel's theorems produced a debacle. Subsequent developments brought further complications. For example, even the axiomatic-deductive method so highly regarded in the past as *the* approach to exact knowledge was seen to be flawed. The net effect of these newer developments was to add to the variety of possible approaches to mathematics and to divide mathematicians into an even greater number of differing factions.

The current predicament of mathematics is that there is not one but many mathematics and that for numerous reasons each fails to satisfy the members of the opposing schools. It is now apparent that the concept of a universally accepted, infallible body of reasoning—the majestic mathematics of 1800 and the pride of man—is a grand illusion. Uncertainty and doubt concerning the future of mathematics have replaced the certainties and complacency of the past. The disagreements about the foundations of the "most certain" science are both surprising and, to put it mildly, disconcerting. The present state of mathematics is a mockery of the hitherto deep-rooted and widely reputed truth and logical perfection of mathematics.

There are mathematicians who believe that the differing views on what can be accepted as sound mathematics will some day be reconciled. Prominent among these is a group of leading French mathematicians who write under the pseudonym of Nicholas Bourbaki:

> Since the earliest times, all critical revisions of the principles of mathematics as a whole, or of any branch of it, have almost invariably followed periods of uncertainty, where contradictions did appear and had to be resolved. . . . There are now twenty-five centuries during which the mathematicians have had the practice of correcting their errors and thereby seeing their science enriched, not impoverished; this gives them the right to view the future with serenity.

However, many more mathematicians are pessimistic. Hermann Weyl, one of the greatest mathematicians of this century, said in 1944:

> The question of the foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.

In the words of Goethe, "The history of a science is the science itself."

The disagreements concerning what correct mathematics is and the variety of differing foundations affect seriously not only mathematics proper but most vitally physical science. As we shall see, the most well-developed physical theories are entirely mathematical. (To be sure, the conclusions of such theories are interpreted in sensuous or truly physical objects, and we hear voices over our radios even though we have not the slightest physical understanding of what a radio wave is.) Hence scientists, who do not personally work on foundational problems, must nevertheless be concerned about what mathematics can be confidently employed if they are not to waste years on unsound mathematics.

The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians to abandon science. With a "plague on all your houses" they have retreated to specialties in areas of mathematics where the methods of proof seem to be safe. They also find problems concocted by humans more appealing and manageable than those posed by nature.

The crises and conflicts over what sound mathematics is have also discouraged the application of mathematical methodology to many areas of our culture such as philosophy, political science, ethics, and aesthetics. The hope of finding objective, infallible laws and standards has faded. The Age of Reason is gone.

Despite the unsatisfactory state of mathematics, the variety of approaches, the disagreements on acceptable axioms, and the danger that new contradictions, if discovered, would invalidate a great deal of mathematics, some mathematicians are still applying mathematics to physical phenomena and indeed extending the applied fields to economics, biology, and sociology. The continuing effectiveness of mathematics suggests two themes. The first is that effectiveness can be used as the criterion of correctness. Of course such a criterion is provisional. What is considered correct today may prove wrong in the next application.

The second theme deals with a mystery. In view of the disagreements about what sound mathematics is, why is it effective at all? Are we performing miracles with imperfect tools? If man has been deceived, can nature also be deceived into yielding to man's mathematical dictates? Clearly not. Yet, do not our successful voyages to the moon and our explorations of Mars and Jupiter, made possible by technology which itself depends heavily on mathematics, confirm mathematical theories of the cosmos? How can we, then, speak of the artificiality and varieties of mathematics? Can the body live on when the mind and spirit are bewildered? Certainly this is true of human beings and it is true of mathematics. It behooves us therefore to learn why, despite its uncertain foundations and despite the conflicting theories of mathematicians, mathematics has proved to be so incredibly effective.

I

The Genesis of Mathematical Truths

Thrice happy souls! to whom 'twas given to rise
To truths like these, and scale the spangled skies!
Far distant stars to clearest view they brought,
And girdled ether with their chains of thought.
So heaven is reached—not as of old they tried
By mountains piled on mountains in their pride.
OVID

Any civilization worthy of the appelation has sought truths. Thoughtful people cannot but try to understand the variety of natural phenomena, to solve the mystery of how human beings came to dwell on this earth, to discern what purpose life should serve, and to discover human destiny. In all early civilizations but one, the answers to these questions were given by religious leaders, answers that were generally accepted. The ancient Greek civilization is the exception. What the Greeks discovered—the greatest discovery made by man—is the power of reason. It was the Greeks of the classical period, which was at its height during the years from 600 to 300 B.C., who recognized that man has an intellect, a mind which, aided occasionally by observation or experimentation, can discover truths.

What led the Greeks to this discovery is a question not readily answered. The initiators of the plan to apply reason to human affairs and concerns lived in Ionia, a Greek settlement in Asia Minor, and many historians have sought to account for the happenings there on the basis of political and social conditions. For example, the Ionians were rather freer to disregard the religious beliefs that dominated the European Greek culture. However, our knowledge of Greek history before about 600 B.C. is so fragmentary that no definitive explanation is available.

In the course of time the Greeks applied reason to political systems, ethics, justice, education, and numerous other concerns of man. Their chief contribution, and the one which decisively influenced all later cultures, was to undertake the most imposing challenge facing reason, learning the laws of nature. Before the Greeks made this contribution, they and the other civilizations of ancient times regarded nature as chaotic, capricious, and even terrifying. Acts of nature were either unexplained or attributed to the arbitrary will of gods who could be propitiated only by prayers, sacrifices, and other rituals. The Babylonians and Egyptians, who had notable civilizations as far back as 3000 B.C., did note some periodicities in the motions of the sun and moon and indeed based their calendars on these periodicities but saw no deeper significance in them. These few exceptional observations did not influence their attitude toward nature.

The Greeks dared to look nature in the face. Their intellectual leaders, if not the people at large, rejected traditional doctrines, supernatural forces, superstitions, dogma, and other trammels on thought. They were the first people to examine the multifarious, mysterious, and complex operations of nature and to attempt to understand them. They pitted their minds against the welter of seemingly haphazard occurrences in the universe and undertook to throw the light of reason upon them.

Possessed of insatiable curiosity and courage, they asked and answered the questions that occur to many, are tackled by few, and are resolved only by individuals of the highest intellectual caliber. Is there any plan underlying the workings of the entire universe? Are plants, animals, men, planets, light, and sound mere physical accidents or are they part of a grand design? Because they were dreamers enough to arrive at new points of view, the Greeks fashioned a conception of the universe which has dominated all subsequent Western thought.

The Greek intellectuals adopted a totally new attitude toward nature. This attitude was rational, critical, and secular. Mythology was discarded as was the belief that the gods manipulate man and the physical world according to their whims. The intellectuals eventually arrived at the doctrine that nature is orderly and functions invariably according to a grand design. All phenomena apparent to the senses, from the motions of the planets to the stirrings of the leaves on a tree, can be fitted into a precise, coherent, intelligible pattern. In short, nature is rationally designed and that design, though unaffected by human actions, can be apprehended by man's mind.

The Greeks were not only the first people with the audacity to conceive of law and order in the welter of phenomena but also the first with the genius to uncover some of the underlying patterns to which nature apparently conforms. Thus they dared to ask for, and found, design underlying the greatest spectacle man beholds, the motion of the brilliant sun, the changing phases of the many hued moon, the brightness of the planets, the broad panorama of lights from the canopy of stars, and the seemingly miraculous eclipses of the sun and moon.

It was the Ionian philosophers of the 6th century B.C. who also made the first attempts to secure a rational explanation of the nature and functioning of the universe. The famous philosophers of this period, Thales, Anaximander, Anaximenes, Heraclitus, and Anaxagoras, each fixed on a single substance to explain the constitution of the universe. Thales, for example, argued that everything is made up of water in either gaseous, liquid, or solid state, and he attempted explanations of many phenomena in terms of water—a not unreasonable choice because clouds, fog, dew, rain, and hail are forms of water and water is necessary to life, nourishes the crops, and supports much animal life. Even the human body, we now know, is 90 percent water.

The natural philosophy of the Ionians was a series of bold speculations, shrewd guesses, and brilliant intuitions rather than the outcome of extensive and careful scientific investigations. These men were perhaps a little over-eager to see the whole picture and so jumped to broad conclusions. But they did discard the older, largely mythical accounts and substituted materialistic and objective explanations of the design and operation of the universe. They offered a reasoned approach in place of fanciful and uncritical accounts and they defended their contentions by reason. These men dared to tackle the universe with their minds and refused to rely on gods, spirits, ghosts, devils, angels, and other mythical agents who might maintain or disrupt nature's happenings. The spirit of these rational explanations can be expressed in the words of Anaxagoras: "Reason rules the world."

The decisive step in dispelling the mystery, mysticism, and seeming chaos in the workings of nature and in replacing them by an understandable pattern was the application of mathematics. Here the Greeks displayed an insight almost as pregnant and as original as the discovery of the power of reason. The universe is mathematically designed, and through mathematics man can penetrate to that design. The first major group to offer a mathematical plan of nature was the Pythagoreans, a school led by Pythagoras (*c.* 585–*c.* 500 B.C.) and rooted in southern Italy. While they did draw inspiration and doctrines from the prevailing Greek religion centering on purification of the soul and its redemption from the taint and prison of the body, Pythagorean natural philosophy was decidedly rational. The Pythagoreans were struck by the fact that phenomena most diverse from a qualitative point of view exhibit identical mathematical properties. Hence mathematical properties must be the essence of these phenomena. More specifically, the Pythagoreans found this essence in number and in numerical relationships. Number was the first principle in their explanation of nature. All objects were made up of elementary particles of matter or "units of existence" in combinations corresponding to the various geometrical figures. The total number of units represented, in fact, the material object. Number was the matter and form of the universe. Hence the Pythagorean doctrine, "All things are numbers." Since number is the "essence" of all objects, the explanation of natural phenomena could be achieved only through number.

This early Pythagorean doctrine is puzzling because to us numbers are abstract ideas, and things are physical objects or substance. But we have made an abstraction of number which the early Pythagoreans did not make. To them, numbers were points or particles. When they spoke of triangular numbers, square numbers, pentagonal numbers, and others, they were thinking of collections of points, pebbles, or point-like objects arranged in those shapes (Figs. 1.1–1.4).

Though historical fragments do not afford precise chronological data, there is no doubt that as the Pythagoreans developed and refined their own doctrines they began to understand numbers as abstract concepts, whereas objects were merely concrete realizations of numbers. With this later distinction we can make sense of the statement of Philolaus, a famous 5th-century Pythagorean: "Were it not for number and its nature, nothing that exists would be clear to anybody either in itself or in its relation to other things. . . . You can observe the power of number exercising itself . . . in all the acts and the thoughts of men, in all handicrafts and music."

The reduction of music, for example, to simple relationships among numbers became possible for the Pythagoreans when they discovered two facts: first that the sound caused by a plucked string depends upon the length of the string; and second that harmonious sounds are given off by equally taut strings whose lengths are to each other as the ratios of whole numbers. For example, a harmonious sound is produced by plucking two equally taut strings, one twice as long as the other. In our language the interval between the two notes is an octave. Another harmonious combination is formed by two strings whose lengths are in the ratio 3 to 2; in this case the shorter one gives off a note, called the fifth, above that given off by the first string. In fact, the relative lengths in every harmonious combination of plucked strings can be expressed as a ratio of whole numbers. The Pythagoreans also developed a famous musical scale. Though we shall not devote space to music of the Greek period, we would like to note that many Greek mathematicians, including Euclid and Ptolemy, wrote on the subject, especially on harmonious combinations of sounds and the construction of scales.

The Pythagoreans reduced the motions of the planets to number

Figure 1.1
Triangular numbers

Figure 1.2
Square numbers

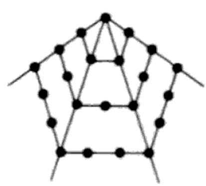


Figure 1.3
Pentagonal numbers

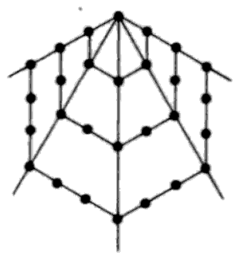


Figure 1.4
Hexagonal numbers

relations. They believed that bodies moving in space produce sounds. Perhaps this was suggested by the swishing of an object whirled on the end of a string. They believed, further, that a body which moves rapidly gives forth a higher note than one which moves slowly. Now according to their astronomy, the greater the distance of a planet from the earth the more rapidly it moved. Hence the sounds produced by the planets varied with their distances from the earth and these sounds all harmonized. But this "music of the spheres," like all harmony, reduced to no more than number relationships and hence so did the motions of the planets. We do not hear this music because we are accustomed to it from birth.

Other features of nature were "reduced" to number. The numbers 1, 2, 3, and 4, the *tetractys,* were especially valued. In fact, the Pythagorean oath is reported to be: "I swear in the name of the Tetractys which has been bestowed on our soul. The source and roots of the everflowing nature are contained in it." Nature was composed of fournesses such as the four geometric elements, point, line, surface, and solid; and the four material elements Plato later emphasized, earth, air, fire, and water.

The four numbers of the *tetractys* added up to ten and so ten was the ideal number and represented the universe. Because ten was ideal there must be ten bodies in the heavens. To fill out the required number the Pythagoreans introduced a central fire around which the earth, sun, moon, and the five planets then known revolved and a counter-earth on the opposite side of the central fire. We do not see this central fire and the counter-earth because the area of the earth on which we live faces away from them. The details are not worth pursuing; the main point is that the Pythagoreans tried to build an astronomical theory based on numerical relationships.

Because the Pythagoreans "reduced" astronomy and music to number, music and astronomy came to be linked with arithmetic and geometry, and all four subjects were regarded as mathematical. The four became part of the school curriculum and remained so even into medieval times, where they were labelled the quadrivium.

Aristotle in his *Metaphysics* sums up the Pythagorean identification of number and the real world:

> In numbers they seemed to see resemblances to things that exist and come into being—more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity—and similarly almost all other things being numerically expressible); since, again, that the modifications and the ratios of the musical scales were expressible in numbers;—since, then, all other things seemed in their whole nature

> to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.

The natural philosophy of the Pythagoreans is hardly substantial. Aesthetic considerations commingled with an obsession to find number relationships certainly led to assertions transcending observational evidence. Nor did the Pythagoreans develop any one branch of physical science very far. One could justifiably call their theories superficial. But, whether by luck or by intuitive genius, the Pythagoreans did hit upon two doctrines which proved later to be all-important: the first is that nature is built according to mathematical principles; the second that number relationships underlie, unify, and reveal the order in nature. Actually modern science adheres to the Pythagorean emphasis on number, though, as we shall see, the modern doctrines are a much more sophisticated form of Pythagoreanism.

The philosophers who chronologically succeeded the Pythagoreans were as much concerned with the nature of reality and the underlying mathematical design. Leuccipus (*c.* 440 B.C.) and Democritus (*c.* 460–*c.* 370 B.C.) are notable because they were most explicit in affirming the doctrine of atomism. Their common philosophy was that the world is composed of an infinite number of simple, eternal atoms. These differ in shape, size, hardness, order, and position. Every object is some combination of these atoms. Though geometrical magnitudes such as a line segment are infinitely divisible, the atoms are ultimate, indivisible particles. Properties such as shape, size, and the others just mentioned were properties of the atoms. All other properties such as taste, heat, and color were not in the atoms but in the effect of the atoms on the perceiver. This sensuous knowledge was unreliable because it varied with the perceiver. Like the Pythagoreans, the atomists asserted that the reality underlying the constantly changing diversity of the physical world was expressible in terms of mathematics. Moreover, the happenings in this world were strictly determined by mathematical laws.

After the Pythagoreans the most influential group to expound and propagate the doctrine of the mathematical design of nature was the Platonists, led, of course, by Plato. Though Plato (427–347 B.C.) took over some Pythagorean doctrines, he was a master who dominated Greek thought in the momentous 4th century B.C. He was the founder of the Academy in Athens, a center which attracted leading thinkers of his day and endured for nine hundred years.

Plato's belief in the rationality of the universe is perhaps best expressed in his dialogue the *Philebus:*

Protarchus: What question?

Socrates: Whether all this which they call the universe is left to the guidance of unreason and chance medley, or, on the contrary, as our fathers have declared, ordered and governed by a marvellous intelligence and wisdom.

Protarchus: Wide asunder are the two assertions, illustrious Socrates, for that which you were just now saying to me appears to be blasphemy, but the other assertion, that mind orders all things, is worthy of the aspect of the world, and of the sun, and of the moon, and of the stars and of the whole circle of the heavens; and never will I say or think otherwise.

The later Pythagoreans and the Platonists distinguished sharply between the world of things and the world of ideas. Objects and relationships in the material world were subject to imperfections, change, and decay and hence did not represent the ultimate truth, but there was an ideal world in which there were absolute and unchanging truths. These truths were the proper concern of the philosopher. About the physical world we can only have opinions. The visible and sensuous world is just a vague, dim, and imperfect realization of the ideal world. "Things are the shadows of ideas thrown on the screen of experience." Reality then was to be found in the ideas of sensuous, physical objects. Thus Plato would say that there is nothing real in a horse, a house, or a beautiful woman. The reality is in the universal type or idea of a horse, a house, or a woman. Infallible knowledge can be obtained only about pure ideal forms. These ideas are in fact constant and invariable, and knowledge concerning them is firm and indestructible.

Plato insisted that the reality and intelligibility of the physical world could be comprehended only through the mathematics of the ideal world. There was no question that this world was mathematically structured. Plutarch reports Plato's famous, "God eternally geometrizes." In the *Republic,* Plato said "the knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient." Mathematical laws were not only the essence of reality but eternal and unchanging. Number relations, too, were part of reality, and collections of things were mere imitations of numbers. Whereas with the earlier Pythagoreans numbers were immanent in things, with Plato they transcended things.

Plato went further than the Pythagoreans in that he wished not merely to understand nature through mathematics but to substitute mathematics for nature herself. He believed that a few penetrating glances at the physical world would suggest basic truths with which reason could then carry on unaided. From that point on there would be just mathematics. Mathematics would substitute for physical investigation.

Plutarch relates in his "Life of Marcellus" that Eudoxus and Archytas, famous contemporaries of Plato, used physical arguments to "prove" mathematical results. But Plato indignantly denounced such proofs as a corruption of geometry; they utilized sensuous facts in place of pure reasoning.

Plato's attitude toward astronomy illustrates his position on the knowledge to be sought. This science, he said, is not concerned with the movements of the visible heavenly bodies. The arrangement of the stars in the heavens and their apparent movements are indeed wonderful and beautiful to behold, but mere observations and explanation of the motions fall far short of true astronomy. Before we can attain to this true science we "must leave the heavens alone," for true astronomy deals with the laws of motion of true stars in a mathematical heaven of which the visible heaven is but an imperfect expression. He encouraged devotion to a theoretical astronomy whose problems please the mind and not the eye and whose objects are apprehended by the mind and not by vision. The varied figures the sky presents to the eye are to be used only as diagrams to assist the search for higher truths. We must treat astronomy, like geometry, as a series of problems merely suggested by visible things. The uses of astronomy in navigation, calendar-reckoning, and the measurement of time were of no interest to Plato.

Aristotle, though a student of Plato from whom he derived many ideas, had a quite different concept of the study of the real world and of the relation of mathematics to reality. He criticized Plato's otherworldliness and his reduction of science to mathematics. Aristotle was a physicist in the literal sense of the word. He believed in material things as the primary substance and source of reality. Physics, and science generally, must study the physical world and obtain truths from it. Genuine knowledge is obtained from sense experience by intuition and abstraction. These abstractions have no existence independent of human minds.

Aristotle did emphasize universals, general qualities that are abstracted from real things. To obtain these he said we "start with things which are knowable and observable to us and proceed toward those things which are clearer and more knowable by nature." He took the obvious sensuous qualities of objects, hypostatized them, and elevated them to independent, mental concepts.

Where was mathematics in Aristotle's scheme of things? The physical sciences were fundamental. Mathematics helped in the study of nature by describing formal properties such as shape and quantity. Also mathematics provided the reasons for facts observed in material phenomena. Thus geometry could provide the explanation of facts provided by optics and astronomy, and arithmetical ratios could give the basis for harmony. But mathematical concepts and principles are definitely abstractions from the real world. Because they are abstracted from the world, they are applicable to it. There is a faculty of the mind which enables us to arrive at these idealized properties of physical objects from sensations and these abstractions are necessarily true.

This brief survey of the philosophers who forged and molded the Greek intellectual world may serve to show that all of them stressed the study of nature for comprehension, understanding, and appreciation of the underlying reality. Moreover, from the time of the Pythagoreans practically all philosophers asserted that nature was designed mathematically. By the end of the classical period the doctrine of the mathematical design of nature was established and the search for mathematical laws had been instituted. Though this belief did not motivate all later mathematics, once accepted it was acted on by most of the great mathematicians, even those who had no contact with the belief. Of all the triumphs of the speculative thought of the Greeks, the most truly novel was their conception of the cosmos operating in accordance with mathematical laws discoverable by human thought.

The Greeks, then, were determined to seek truths and in particular truths about the mathematical design of nature. How does one go about seeking truths and guaranteeing that they are truths? Here, too, the Greeks provided the plan. Though this evolved gradually during the period from 600 to 300 B.C., and though there is some question as to when and by whom it was first conceived of, by 300 B.C. it was perfected.

Mathematics in a loose sense of the term, in the sense of utilizing numbers and geometrical figures, antedates the work of the classical Greeks by several thousand years. In this loose sense the term mathematics includes the contributions of many bygone civilizations among which the Egyptian and Babylonian are most prominent. In all of these, except the Greek civilization, mathematics was hardly a distinct discipline—it had no methodology nor was it pursued for other than immediate, practical ends. It was a tool, a series of disconnected, simple rules which enabled people to answer questions of daily life: calendar-reckoning, agriculture, and commerce. These rules were arrived at by trial and error, experience, and simple observation, and many were only approximately correct. About the best one can say for the mathematics of these civilizations is that it showed some vigor if not rigor of thought and more perseverance than brilliance. This mathematics is characterized by the word empirical. The empirical mathematics of the

Babylonians and the Egyptians also served as a prelude to the work of the Greeks.

Though the Greek culture was not entirely free of outside influences—Greek thinkers did travel and study in Egypt and Babylonia—and though mathematics in the modern sense of the word had to undergo a period of gestation even in the congenial intellectual atmosphere of Greece, what the Greeks created differs as much from what they took over as gold from tin.

Having decided to search for mathematical truths, the Greeks could not build upon the crude, empirical, limited, disconnected, and, in many instances, approximate results that their predecessors, notably the Egyptians and Babylonians, had compiled. Mathematics itself, the basic facts about number and geometrical figures, must be a body of truths, and mathematical reasoning, aimed at arriving at truths about physical phenomena, the motions of the heavens for example, must produce indubitable conclusions. How were these objectives to be attained?

The first principle was that mathematics was to deal with abstractions. For the philosophers who molded Greek mathematics, truth by its very meaning could pertain only to permanent, unchanging entities and relationships. Fortunately, the intelligence of man excited to reflection by the impressions of sensuous objects can rise to higher conceptions; these are the ideas, the eternal realities and the true object of thought. There was another reason for the preference for abstractions. If mathematics was to be powerful it must embrace in one abstract concept the essential feature of all the physical occurrences of that concept. Thus the mathematical straight line must embrace stretched strings, ruler's edges, boundaries of fields, and the paths of light rays. Accordingly, the mathematical line was to have no thickness, color, molecular structure, or tension. The Greeks were explicit in asserting that their mathematics dealt with abstractions. Speaking of geometricians, Plato said in *The Republic:*

> Do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter . . . they are really seeking to behold the things themselves, which can be seen only with the eye of the mind?

Hence mathematics would deal first of all with abstract concepts such as point, line, and whole number. Other concepts such as triangle, square, and circle could then be defined in terms of the basic ones, which as Aristotle pointed out must be undefined or else there would be no starting point. The acuity of the Greeks is evident in the requirement that defined concepts must be shown to have counterparts in reality, either by demonstration or construction. Thus one could not define an angle trisector and prove theorems about it. It might not exist. And in fact, since the Greeks did not succeed in constructing an angle trisector under the limitations they imposed on constructions, they did not introduce this concept.

To reason about the concepts of mathematics the Greeks started with axioms, truths so self-evident that no one could doubt them. Surely such truths were available. Plato justified acceptance of the axioms by his theory of recollection or *anamnesis.* There was for him, as we noted earlier, an objective world of truths. Humans had experience as souls in another world before coming to earth and the soul had but to be stimulated to recall its prior experience in order to know that the axioms of geometry were truths. No experience on earth was necessary. Aristotle put it otherwise. The axioms are intelligible principles which appeal to the mind beyond possibility of doubt. The axioms, Aristotle said in *Posterior Analytics,* are known to be true by our infallible intuition. Moreover, we must have these truths on which to base our reasoning. If, instead, reasoning were to use some facts not known to be truths, further reasoning would be needed to establish these facts and this process would have to be repeated endlessly. There would then be an infinite regress. Among the axioms, he distinguished common notions and postulates. Common notions are true in all fields of thought and include statements such as "Equals added to equals give equals." Postulates apply to a specific subject such as geometry. Thus, "Two points determine a unique line." Aristotle did say that postulates need not be self-evident but when not must be supported by the consequences which follow from them. However, self-evidency was required by the mathematicians.

From the axioms, conclusions were to be derived by reasoning. There are many types of reasoning, for example, induction, reasoning by analogy, and deduction. Of the many types, only one guarantees the correctness of the conclusion. The conclusion that all apples are red because one thousand apples are found to be red is inductive and therefore not absolutely reliable. Likewise the argument that John should be able to graduate from college because his brother who inherited the same faculties did so, is reasoning by analogy and certainly not reliable. Deductive reasoning, on the other hand, though it can take many forms does guarantee the conclusion. Thus, if one grants that all men are mortal and Socrates is a man, one must accept that Socrates is mortal. The principle of logic involved here is one form of what Aristotle called syllogistic reasoning. Among other laws of deductive reasoning, Aristotle included the law of contradiction (a proposition cannot be both true and false) and the law of excluded middle (a proposition must be either true or false).

He and the world at large accepted unquestioningly that these deductive principles when applied to any premise yielded conclusions as reliable as the premise. Hence if the premises were truths, so would the conclusions be. It is worthy of note, especially in the light of what we shall be discussing later, that Aristotle abstracted the principles of deductive logic from the reasoning already practiced by mathematicians. Deductive logic is, in effect, the child of mathematics.

Though deductive reasoning was advocated by almost all the Greek philosophers as the only reliable method of obtaining truths, Plato's view was somewhat different. Though he would not object to deductive proof, he did regard it as superfluous, for the axioms and theorems of mathematics exist in some objective world independent of man, and in accordance with Plato's doctrine of anamnesis, man has but to recall them to recognize their indubitable truth. The theorems, to use Plato's own analogy in his *Theaetetus,* are like birds in an aviary. They exist and one has only to reach in to grasp them. Learning is but a process of recollection. In Plato's dialogue *Meno,* Socrates by skillful questioning elicits from a young slave the assertion that the square erected on the diagonal of an isosceles right triangle has twice the area of a square erected on a side. Socrates then triumphantly concludes that the slave, since he was not educated in geometry, recalled it under the proper suggestions.

It is important to appreciate *how radical the insistence on deductive proof was.* Suppose a scientist should measure the sum of the angles of a hundred different triangles in different locations and of different size and shape and find that sum to be 180° to within the limits of experimental accuracy. Surely he would conclude that the sum of the angles of any triangle is 180°. But his proof would be inductive, not deductive, and would therefore not be mathematically acceptable. Likewise, one can test as many even numbers as he pleases and find that each is a sum of two prime numbers. But this test is not a deductive proof and so the result is not a theorem of mathematics. Deductive proof is, then, a very stringent requirement. Nevertheless, the Greek mathematicians, who were in the main philosophers, insisted on the exclusive use of deductive reasoning because this yields truths, eternal verities.

There is another reason that philosophers favor deductive reasoning. Philosophers are concerned with broad knowledge about man and the physical world. To establish universal truths such as that man is basically good, that the world is designed, or that man's life has purpose, deductive reasoning from acceptable first principles is far more feasible than induction or analogy.

Still another reason for the classical Greeks' preference for deduction may be found in the organization of their society. Philosophical, mathematical, and artistic activities were carried on by the wealthier class. These people did no manual work. Slaves, metics (non-citizens), and free citizen-artisans were employed in business and in the household, and they even practiced the most important professions. Educated freemen did not use their hands and rarely engaged in commercial pursuits. Plato declared that the trade of a shopkeeper was a degradation to a freeman and wished that his engagement in such a trade be punished as a crime. Aristotle said that in the perfect state no citizen (as opposed to slaves) would practice any mechanical art. Among the Boeotians, one of the Greek tribes, those who defiled themselves with commerce were excluded from all state offices for ten years. To thinkers in such a society, experimentation and observation would be alien. Hence no results scientific or mathematical would be derived from such sources.

Though there are many reasons for the Greeks' insistence on deductive proof there is some question as to which philosopher or group of philosophers first laid down this requirement. Unfortunately our knowledge of the teachings and writings of the pre-Socratic philosophers is fragmentary and though various answers have been given there is no universally accepted one. By Aristotle's time the requirement was certainly in effect, for he is explicit about standards of rigor such as the need for undefined terms and the laws of reasoning.

How successful were the Greeks in executing their plan of obtaining mathematical laws of the universe? The cream of the mathematics created by such men as Euclid, Apollonius, Archimedes, and Claudius Ptolemy has fortunately come down to us. Chronologically these men belonged to the second great period of Greek culture, the Hellenistic or Alexandrian (300 B.C.–A.D. 600). During the 4th century B.C. King Philip of Macedonia undertook to conquer the Persians, who controlled the Near East and had been traditional enemies of the European Greeks. Philip was assassinated and was succeeded by his son Alexander. Alexander did defeat the Persians and moved the cultural center of the enlarged Greek empire to a new city which he modestly named after himself. Alexander died in 323 B.C. but his plan to develop the new center was continued by his successors in Egypt who adopted the royal title of Ptolemy.

It is quite certain that Euclid lived in Alexandria about 300 B.C. and trained students there, though his own education was probably acquired in Plato's Academy. This information, incidentally, is about all we have on Euclid's personal life. Euclid's work has the form of a systematic, deductive, and vast account of the separate discoveries of many classical Greeks. His chief work, the *Elements*, offers the laws of space and figures in space.

Euclid's *Elements* was by no means all of his contribution to the geometry of space. Euclid took up the theme of conic sections in a book no longer extant, and Apollonius (262–190 B.C.), a native of Pergamum in Asia Minor who learned mathematics in Alexandria, carried on this study of the parabola, ellipse, and hyperbola and wrote the classic work on the subject, the *Conic Sections*.

To this purely geometrical knowledge Archimedes (287–212 B.C.), who was educated in Alexandria but lived in Sicily, added several works, *On the Sphere and Cylinder*, *On Conoids and Spheroids*, and *The Quadrature of the Parabola*, all of which deal with the calculation of complex areas and volumes by a method introduced by Eudoxus (390–337 B.C.) and later known as the method of exhaustion. Nowadays these problems are solved by the methods of the calculus.

The Greeks made one more major addition to the study of space and figures in space—trigonometry. The originator of this work was Hipparchus, who lived in Rhodes and in Alexandria and died about 125 B.C. It was extended by Menelaus (*c.* A.D. 98) and given a complete and authoritative version by the Egyptian Claudius Ptolemy (d. A.D. 168), who worked in Alexandria. His major work was *Mathematical Composition*, known more popularly by the Arabic title, *Almagest*. Trigonometry concerns the quantitative relationships among the sides and angles of a triangle. The Greeks were concerned mainly with triangles on the surface of a sphere, the sides of which are formed by arcs of great circles (circles with centers at the center of the sphere) because the major application was to the motion of planets and stars, which in Greek astronomy moved along great circles. However, the same theory, when translated, readily applies to triangles in a plane, the form in which trigonometry is approached in our schools today. The introduction of trigonometry required of its users rather advanced arithmetic and some algebra. Just how the Greeks operated in these areas will be a later concern (Chapter V).

With these several creations mathematics emerged from obscure, empirical, disconnected fragments to brilliant, huge, systematic, and deep intellectual creations. However, the classics of Euclid, Apollonius, and Archimedes—Ptolemy's *Almagest* is an exception—that deal with the properties of space and of figures in space seem to be limited in scope and give little indication of the broader significance of their material. These works seem to have little relation to revealing truths about the workings of nature. In fact, these classics give only the formal, polished deductive mathematics. In this respect Greek mathematical texts are no different from modern mathematical textbooks and treatises. Such books seek only to organize and present the mathematical results that have been attained and so omit the motivations for the mathematics, the clues and suggestions for the theorems, and the uses to which the mathematical knowledge is put. Hence many writers on classical Greek mathematics assert that the mathematicians of the period were concerned only with mathematics for its own sake and they arrive at and defend this assertion by pointing to Euclid's *Elements* and Apollonius's *Conic Sections,* the two greatest compilations of work in that period. However, these writers have narrowed their focus. To look only at the *Elements* and the *Conic Sections* is like looking at Newton's paper on the binomial theorem and concluding that Newton was a pure mathematician.

The real goal was the study of nature. Insofar as the study of the physical world was concerned, even the truths of geometry were highly significant. It was clear to the Greeks that geometric principles were embodied in the entire structure of the universe, of which space was the primary component. Hence the study of space and figures in space was an essential contribution to the investigation of nature. Geometry was in fact part of the larger study of cosmology. For example, the study of the geometry of the sphere was undertaken when astronomy became mathematical, which happened in Plato's time. In fact, the Greek word for sphere meant astronomy for the Pythagoreans. And Euclids's *Phaenomena,* which was on the geometry of the sphere, was specifically intended for use in astronomy. With such evidence and with the fuller knowledge of how developments in mathematics took place in more recent times, we may be certain that the scientific investigations must have suggested mathematical problems and that the mathematics was part and parcel of the investigation of nature. But we need not speculate. We have only to examine what the Greeks accomplished in the study of nature and who were the men involved.

The greatest success in the field of physical science proper was achieved in astronomy. Plato, though fully aware of the impressive number of astronomical observations made by the Babylonians and Egyptians, emphasized that they had no underlying or unifying theory and no explanation of the seemingly irregular motions of the planets. Eudoxus, who was a student at the Academy and whose purely geometrical work is incorporated in Books V and XII of Euclid's *Elements,* took up the problem of "saving the appearances." His answer is the first reasonably complete astronomical theory known to history.

We shall not describe Eudoxus's theory except to state that it was thoroughly mathematical and involved the motions of interacting

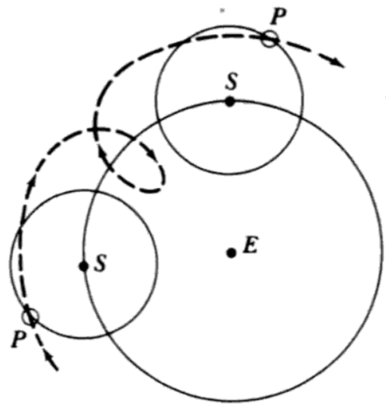


Figure 1.5

spheres. These spheres were, except for the "sphere" of fixed stars, not material bodies but mathematical constructions. Nor did he try to account for forces which would make the spheres rotate as he said they did. His theory is thoroughly modern in spirit, for today mathematical description and not physical explanation is the goal in science. This theory was superseded by the theory credited to the three greatest theoretical astronomers after Eudoxus, namely, Apollonius, Hipparchus and Ptolemy and incorporated in Ptolemy's *Almagest.*

Apollonius left no extant work in astronomy. However, his contributions are cited by Greek writers including Ptolemy in his *Almagest* (Book XII). He was so famous as an astronomer that he was nicknamed ε (epsilon) because he had done much work on the motion of the moon and ε was the symbol for the moon. Only one minor work of Hipparchus is known but he, too, is cited and credited in the *Almagest.*

The basic scheme of what is now referred to as Ptolemaic astronomy had entered Greek astronomy between the times of Eudoxus and Apollonius. In this scheme a planet P moves at a constant speed on a circle (Fig. 1.5) with center S while S itself moves with constant speed on a circle with center at the earth E. The circle on which S moves is called the deferent while the circle on which P moves is called an epicycle. The point S in the cases of some planets was the sun but in other cases it was just a mathematical point. The direction of the motion of P could agree with or be opposite to the direction of motion of S. The latter was the case for the sun and moon. Ptolemy also used a variation on this scheme to describe the motion of some of the planets. By properly selecting the radii of the epicycle and deferent, the speed of a body on its epicycle, and the speed of the center of the epicycle on the deferent, Hipparchus and Ptolemy were able to get descriptions of the motions which were quite in accord with the observations of their times. From the time of Hipparchus an eclipse of the moon could be predicted to within an hour or two, though eclipses of the sun were predicted somewhat less accurately. These predictions were possible because Ptolemy used trigonometry, which he said he created for astronomy.

From the standpoint of the search for truths, it is noteworthy that Ptolemy, like Eudoxus, fully realized that his theory was just a convenient mathematical description which fit the observations and was not necessarily the true design of nature. For some planets he had a choice of alternative schemes and he chose the mathematically simpler one. Ptolemy says in Book XIII of his *Almagest* that in astronomy one ought to seek as simple a mathematical model as possible. But Ptolemy's mathematical model was received as the truth by the Christian world.

Ptolemaic theory offered the first reasonably complete evidence of the uniformity and invariability of nature and is the final Greek answer to Plato's problem of rationalizing the apparent motions of the heavenly bodies. No other product of the entire Greek era rivals the *Almagest* for its profound influence on conceptions of the universe and none, except Euclid's *Elements,* achieved such unquestioned authority.

This brief account of Greek astronomy does not of course cover many other contributions to the subject nor does it reveal the depth and extent of the work even of the men treated. Greek astronomy was masterful and comprehensive and it employed a vast amount of mathematics. Moreover, almost every Greek mathematician devoted himself to the subject, including the masters Euclid and Archimedes.

The attainment of physical truths did not end with the mathematics of space and astronomy. The Greeks founded the science of mechanics. Mechanics deals with the motion of objects that may be considered as particles, the motion of extended bodies, and the forces that cause these motions. In his *Physics* Aristotle put together a theory of motion which is the high point of Greek mechanics. Like all of his physics, his mechanics is based on rational, seemingly self-evident principles, entirely in accord with observation. Though this theory held sway for almost two thousand years, we shall not review it because it was superseded by Newtonian mechanics. Notable additions to Aristotle's theory of motion were Archimedes' works on centers of gravity of bodies and his theory of the lever. What is relevant in all of this work is that mathematics played a leading role and thereby added to the conviction that mathematics was fundamental in penetrating the design of nature.

Next to astronomy and mechanics optics has been the subject most constantly pursued. This mathematical science, too, was founded by the Greeks. Almost all of the Greek philosophers, beginning with the Pythagoreans, speculated on the nature of light, vision, and color. Our concern, however, is with mathematical accomplishments in these areas. The first was the assertion on a priori grounds by Empedocles of Agrigentum (*c.* 490 B.C.)—Agrigentum was in Sicily—that light travels with finite velocity. The first systematic treatments of light that we have are Euclid's *Optics* and *Catoptrica.** The *Optics* is concerned with the problem of vision and with the use of vision to determine sizes of objects. The *Catoptrica* (theory of mirrors) shows how light rays behave when reflected from plane, concave, and convex mirrors and the effect of this behavior on what we see. Like the *Optics* it starts with definitions which are really postulates. Theorem 1 (an axiom in modern texts) is fundamental in geometrical optics and is known as the law of reflection. It says that the angle *A* that a ray incident from point *P* makes with a mirror (Fig. 1.6) equals the angle *B* which the reflected ray makes with the mirror. Euclid also proves the law for a ray striking a convex or a concave mirror (Fig. 1.7). At the point of contact he substitutes the tangent *R* for the mirror. Both books are thoroughly mathematical not only in content but in organization. Definitions, axioms and theorems dominate as in Euclid's *Elements.*

From the law of reflection, the mathematician and engineer Heron (1st century A.D.) drew an important consequence. If *P* and *Q* in Figure 1.6 are any two points on one side of the line *ST*, then of all the paths one could follow in going from point *P* to the line and then to point *Q*, the shortest path is by way of the point *R* such that the two line segments *PR and QR* make equal angles with the line. And this is exactly the path a light ray takes. Hence, the light ray takes the shortest path in going from *P* to the mirror to *Q*. Apparently nature is well acquainted with geometry and employs it to full advantage. This proposition appears in Heron's *Catoptrica* which also treats concave and convex mirrors and combinations of mirrors.

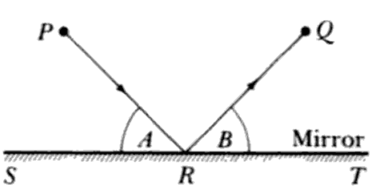


Figure 1.6

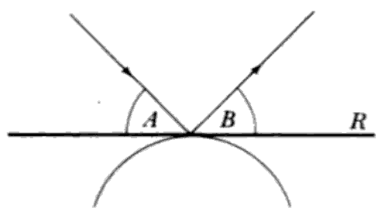


Figure 1.7

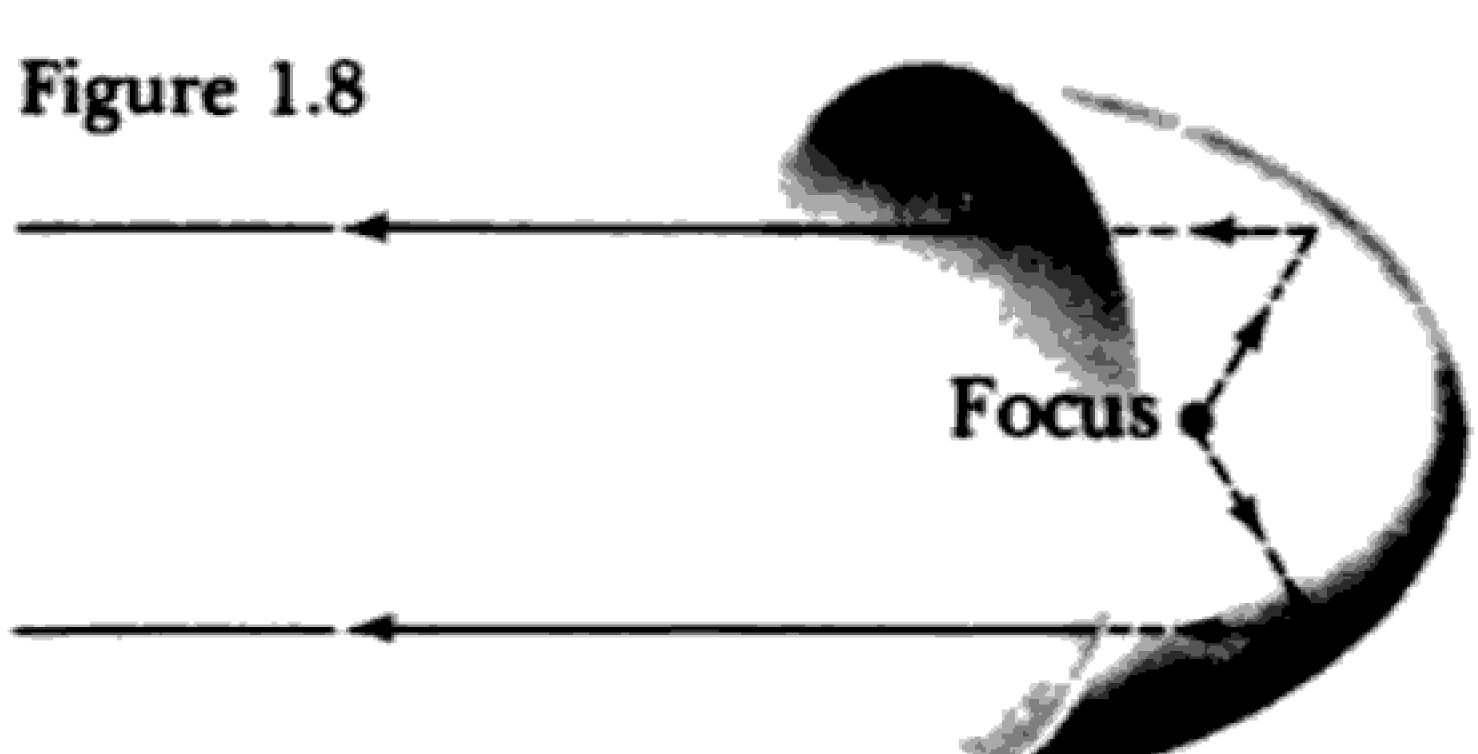
*The version we have today is probably a compilation of several works including Euclid's.

Any number of works were written on the reflection of light by mirrors of various shapes. Among these are the now lost works, Archimedes' *Catoptrica* and Apollonius's *On the Burning Mirror* (*c.* 190 B.C.), and the extant work of Diocles, *On Burning-Mirrors* (*c.* 190 B.C.). Burning mirrors were concave mirrors in the form of portions of a sphere, paraboloids of revolution (formed by revolving a parabola about its axis), and ellipsoids of revolution. Apollonius knew and Diocles' book contains the proof that a paraboloidal mirror will reflect light emanating from the focus into a beam parallel to the axis of the mirror (Fig. 1.8). Conversely, rays coming in parallel to the axis will after reflection be concentrated at the focus. The sun's rays thus concentrated produce great heat at the focus and hence the term burning mirror. This is the property of the paraboloidal mirror which Archimedes is reported to have used to concentrate the sun's rays on the Roman ships besieging his home city Syracuse and to set them afire. Apollonius also knew the reflection properties of the other conic sections, such as that all rays emanating from one focus of an ellipsoidal mirror will be reflected to the other focus. He gives the relevant geometrical properties of the ellipse and hyperbola in Book III of his *Conic Sections*.

The Greeks founded many other sciences, notably geography and hydrostatics. Eratosthenes of Cyrene (*c.* 284–*c.* 192 B.C.), one of the most learned men of antiquity and director of the library at Alexandria, made numerous calculations of distances between significant places on the portion of our earth known to the Greeks. He also made a now famous and quite accurate calculation of the circumference of the earth and wrote his *Geography*, in which beyond describing his mathematical methods he also gave his explanation of causes for the changes which had taken place on the earth's surface.

The most extensive work on geography was Ptolemy's *Geography*, in eight books. Ptolemy not only extended Eratosthenes' work but located eight thousand places on the earth in terms of the very same latitude and longitude we now use. Ptolemy also gave methods of mapmaking,

Figure 1.8



some of which are still used, particularly the method of stereographic projection. In all of this work in geography the geometry of figures on a sphere, applied from the 4th century B.C. onward, was basic.

As for hydrostatics, the subject which deals with the pressure on bodies which are placed in water, Archimedes' book *On Floating Bodies* is the foundational work. Like all of the works we have been describing it is thoroughly mathematical in approach and derivation of results. In particular it contains what is now known as Archimedes' principle, that a body immersed in water is buoyed up by a force equal to the weight of the water displaced. Thus we owe to Archimedes the explanation of how man can remain afloat in a world of forces that tend to submerge him.

Though the deductive approach to mathematics and the mathematical representation of the laws of nature dominated the Alexandrian Greek period, we should note that the Alexandrians, unlike the classical Greeks, also resorted to experimentation and observation. The Alexandrians took over and utilized the remarkably accurate astronomical observations which the Babylonians had made over a period of two thousand years. Hipparchus made a catalogue of the stars observable in his time. Inventions (notably by Archimedes and the mathematician and engineer Heron) included sun-dials, astrolabes, and uses for steam and water power.

Particularly famous was the Alexandrian Museum, which was started by Ptolemy Soter, the immediate successor of Alexander in Egypt. The Museum was a home for scholars and included a famous library of about 400,000 volumes. Since it could not house all the manuscripts an additional 300,000 were housed in the Temple of Serapis. The scholars also gave instruction to students.

With their mathematical work and many scientific investigations, the Greeks gave substantial evidence that the universe is mathematically designed. Mathematics is immanent in nature; it is the truth about nature's structure, or, as Plato would have it, the reality about the physical world. There is law and order in the universe and mathematics is the key to this order. Moreover, human reason can penetrate the plan and reveal the mathematical structure.

The impetus for the conception of a logical, mathematical approach to nature must be credited primarily to Euclid's *Elements*. Though this work was intended to be a study of physical space, its organization, ingenuity, and clarity inspired the axiomatic-deductive approach not only to other areas of mathematics such as the theory of numbers but to all of the sciences. Through this work the notion of a logical organization of all physical knowledge based on mathematics entered the intellectual world.

Thus the Greeks founded the alliance between mathematics and the study of nature's design which has since become the very basis of modern science. Until the latter part of the 19th century, the search for mathematical design was the search for truth. The belief that mathematical laws were the truth about nature attracted the deepest and noblest thinkers to mathematics.

II

The Flowering of Mathematical Truths

> The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics.
>
> JOHANNES KEPLER

The majestic Greek civilization was destroyed by several forces. The first was the gradual conquest by the Romans of Greece, Egypt, and the Near East. The Roman objective in extending its political power was not to spread its materialistic culture. The subjugated areas became colonies from which great wealth was extracted by expropriation and by taxation.

The rise of Christianity was another blow to pagan Greek culture. Though Christian leaders adopted many Greek and Oriental myths and customs with the intent of making Christianity more acceptable to converts, they opposed pagan learning and even ridiculed mathematics, astronomy, and physical science. Despite cruel persecution by the Romans, Christianity spread and became so powerful that the Roman emperor Constantine the Great in his Edict of Milan of A.D. 313 recognized Christianity as the official religion of the Empire. Later, Theodosius (ruled A.D. 379–396) proscribed the pagan religions and in 392 ordered that their temples be destroyed.

Thousands of Greek books were burned by the Romans and the Christians. In 47 B.C., the Romans set fire to the Egyptian ships in the harbor of Alexandria; the fire spread and burned the library—the most extensive of ancient libraries. In the year that Theodosius banned the pagan religions, the Christians destroyed the temple of Serapis in Alexandria, which housed the only remaining sizable collection of Greek works. Many other works written on parchment were expunged by the Christians so that they could use the parchment for their own writings.

The late history of the Roman Empire is also relevant. The Emperor

Theodosius divided the extensive empire between his two sons, Honorius, who was to rule Italy and western Europe, and Arcadius, who was to rule Greece, Egypt, and the Near East. The western part was conquered by the Goths in the 5th century A.D. and its subsequent history belongs to the history of medieval Europe. The eastern part preserved its independence. Since the Eastern Roman Empire, known also as the Byzantine Empire, included Greece proper and Egypt, Greek culture and Greek works were to some extent preserved.

The final blow to the Greek civilization was the conquest of Egypt by the upsurging Moslems in A.D. 640. The remaining books were destroyed on the ground that, as Omar, the Arab conqueror, put it, "Either the books contain what is in the Koran, in which case we don't have to read them, or they contain the opposite of what is in the Koran, in which case we must not read them." And so for six months the baths of Alexandria were heated by burning rolls of parchment.

After the capture of Egypt by the Mohammedans the majority of scholars migrated to Constantinople, which had become the capital of the Eastern Roman Empire. Though no activity along the lines of Greek thought could flourish in the unfriendly Christian atmosphere of Byzantium, this inflow of scholars and their works to comparative safety increased the treasury of knowledge that was to reach Europe 800 years later.

India and Arabia contributed to the continuity of mathematical activity and introduced some ideas that were to play a larger role later.* During the years from A.D. 200 to about 1200 the Hindus, influenced somewhat by the Greek works, made some original contributions to arithmetic and algebra. The Arabs, whose empire at its height extended over all the lands bordering the Mediterranean and into the Near East and embraced many races united by Mohammedanism, absorbed the Greek and Hindu contributions and also made some advances of their own. These, in the spirit of the Alexandrian Greeks, commingled deductive reasoning and experimentation. The Arabs contributed to algebra, geography, astronomy, and optics. They also built colleges and schools for the transmission of knowledge. It is to the credit of the Arabs that though they were firm adherents of their own religion, they did not allow religious doctrines to restrict their mathematical and scientific investigations.

Despite the fact that both the Hindus and the Arabs were able to profit from the magnificent foundations erected by the Greeks and though they furthered Greek mathematics and science, they were not possessed as were the Greeks to understand the structure of the uni-

* We shall say more about the work of the Hindus and Arabs in Chapter V.

verse. The Arabs translated and commented extensively and even critically on Greek works but nothing of great moment or magnitude was added to the truths already known. By A.D. 1500 their empire was destroyed by the Christians in the West and by internal strife in the East.

While the Arabs were building and expanding their civilization, another civilization was being founded in Western Europe. A high level of culture in this region was attained in the medieval period, which extended from about A.D. 500 to 1500. This culture was dominated by the Catholic Church, and its teachings, however deep and meritorious, did not favor the study of the physical world. The Christian God ruled the universe and man's role was to serve and please Him and by so doing win salvation, whereupon the soul would live in an after-life of joy and splendor. The conditions of life on this earth were immaterial and hardship and suffering were not only to be tolerated but were in fact to be undergone as a test of man's faith in God. Understandably, interest in mathematics and science which had been motivated in Greek times by the study of the physical world was at a nadir. The intellectuals of medieval Europe were devoted seekers of truths but these they sought in revelation and in the study of the Scriptures. Hence medieval thinkers did not adduce additional evidence for the mathematical design of nature. However, late medieval philosophy did support the belief in the regularity and uniformity of nature's behavior, though this was thought to be subject to the will of God.

Late medieval Europe was shaken and altered by a number of revolutionary influences. Among the many which converted the medieval civilization into the modern, the most important for our present concern was the acquisition and study of Greek works. These became known through the Arabic translations and through Greek works which had been kept intact in the Byzantine Empire. In fact, when the Turks conquered this empire in 1453 many Greek scholars fled westward with their books. It was from Greek works that the leaders of the intellectual revitalization of Europe learned nature is mathematically designed and this design is harmonious, aesthetically pleasing, and the inner truth about nature. Nature not only is rational and orderly but acts in accordance with inexorable and immutable laws. European scientists began their study of nature as the children of ancient Greece.

That the revival of Greek ideals induced some to take up the study of nature is indubitable. But the speed and intensity of the revival of mathematics and science were due to many other factors. The forces which overthrew one culture and fostered the development of a new one are numerous and complicated. The rise of science has been studied by many scholars and much history has been devoted to pinpointing the causes. We shall not attempt here to do more than name them.

The rise of a class of free artisans, and a consequent interest in materials, skills, and technology, generated scientific problems. Geographical explorations, motivated by the search for raw materials and gold, introduced knowledge of strange lands and customs which challenged medieval European culture. The Protestant revolution rejected some Catholic doctrines, thereby fostering controversy and even scepticism concerning both religions. The Puritan emphasis on work and utility of knowledge to mankind, the introduction of gunpowder, which raised new military problems such as the motion of projectiles, and the problems raised by the navigations over thousands of miles of sea out of sight of land all motivated the study of nature. The invention of printing permitted the spread of knowledge which the Church had been able to control. Though authorities differ on the degree to which one or more of these forces may have influenced the investigation of nature, it suffices for our purposes to note their multitude and the universally accepted fact that the pursuit of science did become the dominant feature of modern European civilization.

The Europeans generally did not respond immediately to the new forces and influences. During the period often labelled humanistic the study and absorption of Greek works were far more characteristic than active pursuit of the Greek objectives. But by about A.D. 1500 minds infused with Greek goals—the application of reason to the study of nature and the search for the underlying mathematical design—began to act. However, they faced a serious problem. The Greek goals were in conflict with the prevailing culture. Whereas the Greeks believed in the mathematical design of nature, nature conforming invariably and unalterably to some ideal plan, late medieval thinkers ascribed all plan and action to the Christian God. He was the designer and creator, and all the actions of nature followed the plan laid down by this agency. The universe was the handiwork of God and subject to His will. The mathematicians and scientists of the Renaissance and several succeeding centuries were orthodox Christians and so accepted this doctrine. But Catholic teachings by no means included the Greek doctrine of the *mathematical* design of nature. How then was the attempt to understand God's universe to be reconciled with the search for the mathematical laws of nature? The answer was to add a new doctrine, that the Christian God had designed the universe mathematically. Thus the Catholic doctrine postulating the supreme importance of seeking to understand God's will and His creations took the form of a search for God's mathematical design of nature. Indeed the work of 16th-, 17th-, and most 18th-century mathematicians was, as we shall soon see more clearly, a religious quest. The search for the mathematical laws of nature was an act of devotion which would reveal the glory and grandeur of His handiwork. Mathematical knowledge, the truth about God's design of the universe, was as sacrosanct as any line of Scripture. Man could not hope to perceive the divine plan as clearly as God Himself understood it, but man could with humility and modesty seek at least to approach the mind of God and so understand God's world.

One can go further and assert that these mathematicians were sure of the existence of mathematical laws underlying natural phenomena and persisted in the search for them because they were convinced a priori that God had incorporated them into the construction of the universe. Each discovery of a law of nature was hailed as evidence of God's brilliance rather than the investigator's. The beliefs and attitudes of the mathematicians and scientists exemplify the larger cultural phenomenon that swept Renaissance Europe. The recently rediscovered Greek works confronted a deeply devout Christian world, and the intellectual leaders born in one and attracted to the other fused the doctrines of both.

Perhaps the most impressive evidence that the Greek doctrine of the mathematical design of nature coupled with the Renaissance belief in God's authorship of that design had taken hold in Europe is furnished by the work of Nicolaus Copernicus and Johannes Kepler. Up to the 16th century, the only sound and useful astronomical theory was the geocentric system of Hipparchus and Ptolemy. This was the theory accepted by professional astronomers and applied to calendar-reckoning and navigation. Work on a new astronomical theory was begun by Copernicus (1473–1543). At the University of Bologna, which he entered in 1497, he studied astronomy. In 1512 he assumed his duties as canon of the Cathedral of Frauenberg in East Prussia. This work left Copernicus with plenty of time to make astronomical observations and to think about the relevant theory. After years of reflection and observation, Copernicus evolved a new theory of planetary motions which he incorporated in a classic work, *On the Revolutions of the Heavenly Spheres.* He had written his first version in 1507 but feared to publish it because it would antagonize the Church. The book appeared in 1543, the year in which he died.

When Copernicus began to think about astronomy, the Ptolemaic theory had become somewhat more complicated. More epicycles had been added to those introduced by Ptolemy in order to make the theory fit the increased amount of observational data gathered largely by the Arabs. In Copernicus's time the theory required a total of seventy-seven circles to describe the motion of the sun, moon, and the five planets known then. To many astronomers the theory, as Copernicus says in his Preface, was scandalously complex.

Copernicus had studied the Greek works and had become convinced that the universe was mathematically and harmoniously designed. Harmony demanded a more pleasing theory than the complicated extension of Ptolemaic theory. He had read that some Greek authors, notably Aristarchus (3rd century B.C.), had suggested the possibility that the sun was stationary and that the earth revolved about the sun and rotated on its axis at the same time. He decided to explore this possibility.

The upshot of his reasoning was that he used the Ptolemaic scheme of deferent and epicycle (Chapter I) to describe the motions of the heavenly bodies, with, however, the all-important difference that the *sun* was at the center of each deferent. The earth itself became a planet moving on an epicycle while rotating on its axis. Thereby he achieved considerable simplification. He was able to reduce the total number of circles (deferents and epicycles) to thirty-four instead of the seventy-seven required in the geocentric theory.

The more remarkable simplification was achieved by Johannes Kepler (1571–1630), one of the most intriguing figures in the history of science. In a life beset by many personal misfortunes and hardships occasioned by religious and political events, Kepler had the good fortune in 1600 to become an assistant to the famous astronomer Tycho Brahe. Brahe was then engaged in making extensive new observations, the first major undertaking since Greek times. These observations and others which Kepler made himself were invaluable to him. When Brahe died in 1601 Kepler succeeded him as Imperial Mathematician to Emperor Rudolph II of Austria.

Kepler's scientific reasoning is fascinating. Like Copernicus he was a mystic, and like Copernicus he believed that the world was designed by God in accordance with some simple and beautiful mathematical plan. He said in his *Mystery of the Cosmos* (1596), the mathematical harmonies in the mind of the Creator furnish the cause "why the number, the size, and the motion of the orbs are as they are and not otherwise." This belief dominated all his thinking. But Kepler also had qualities which we now associate with scientists. He could be coldly rational. Though his fertile imagination triggered the conception of new theoretical systems, he knew that theories must fit observations and, in his later years, saw even more clearly that empirical data may indeed suggest fundamental principles of science. He therefore sacrificed even his most beloved mathematical hypotheses when he saw that they did not fit observational data, and it was precisely this incredible persistence in refusing to tolerate discrepancies any other scientist of his day would have disregarded that led him to espouse radical scientific ideas. He also had the humility, patience, and energy that enable great men to perform extraordinary labor.

Kepler's search for the mathematical laws of nature, which his beliefs assured him existed, led him to spend years in following up false trails. In the Preface to his *Mystery of the Cosmos,* we find him saying: "I undertake to prove that God, in creating the universe and regulating the order of the cosmos, had in view the five regular bodies of geometry as known since the days of Pythagoras and Plato, and that he has fixed according to those dimensions the number of heavens, their proportions, and the relations of their movements." However, the deductions from his attempt to build a theory based on the five regular polyhedra were not in accord with observations, and he abandoned this approach only after he had made extraordinary efforts to apply it in modified form.

But he was eminently successful in later efforts to find harmonious mathematical relations. His most famous and important results are known today as Kepler's three laws of planetary motion. The first two were published in a book of 1609 bearing a long title and often referred to by the first part, *The New Astronomy,* or by the last part, *Commentaries on the Motion of the Planet Mars.* The first of these laws is especially remarkable, for Kepler broke with the tradition of two thousand years that circles or spheres must be used to describe heavenly motions. Instead of resorting to deferent and several epicycles, which both Ptolemy and Copernicus had used to describe the motion of any one planet, Kepler found that a single ellipse would do. His first law states that each planet moves on an ellipse and that the sun is at one (common) focus of each of these elliptical paths (Fig. 2.1). The other focus of each ellipse is merely a mathematical point at which nothing exists. This law is of immense value in comprehending readily the paths of the planets. Of course Kepler, like Copernicus, added that the earth also rotates on its axis as it travels along its elliptical path.

But astronomy had to go much further if it was to be useful. It must tell us how to predict the positions of the planets. If one finds by observation that a planet is at a particular position, *P,* say, in Figure 2.1, he might like to know when it will be at some other position such as a solstice or an equinox, for example. What is needed is the velocity with which the planets move along their respective paths.

Here, too, Kepler made a radical step. Copernicus and the Greeks

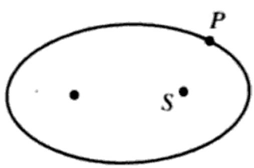


Figure 2.1. Each planet moves in an ellipse about the sun.

MATHEMATICS

Most intelligent people today still believe that mathematics is a body of unshakable truths about the physical world and that mathematical reasoning is exact and infallible. *Mathematics: The Loss of Certainty* refutes that myth. Morris Kline points out that today there is not one universally accepted concept of mathematics—in fact there are many conflicting ones. Yet the effectiveness of mathematics in describing and exploring physical and social phenomena continues to expand. Why?

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The late **Morris Kline** was Professor Emeritus of Mathematics at New York University's Courant Institute of Mathematical Sciences, associate editor of *Mathematics Magazine* and an editor of *Archive for History of Exact Sciences*. His many books include *Mathematics in Western Culture*, *Mathematical Thought from Ancient to Modern Times*, and *Why Johnny Can't Add*.

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