MATHEMATICS

and COMMON SENSE

A Case of Creative Tension

Philip J. Davis

Mathematics and Common Sense

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Preface

The impetus for putting together this book came from a number of questions about mathematics asked of me by a woman I shall call Christina. Christina had the usual mathematical education of students who were not studying science and technology. She married a mathematician and, over the years, developed an increasing desire to know something more about mathematics. She pointed out to me that what she learned from the media left her with an image of the subject as an arcane, esoteric discipline created by unworldly, mentally disturbed geniuses totally lacking in social skills. The media—newspapers, TV, plays, science fiction, etc.—all deal with what might be called the sensational aspect of the subject: famous problems solved after centuries of work, a new prime number, of almost eight million digits, discovered. She felt, and I agree with her, that these aspects of mathematics, while undoubtedly interesting, did not in any way describe the subject as it was pursued by the average professional mathematician, nor how what they did affected our lives.

Christina asked me a number of questions about mathematics. I answered them, and she was pleased to the point of sending me another batch. Her questions and my responses to them make up the first part of this book. The second part is an elaboration of them.

From the age of about ten, I have studied mathematics, learning its history, applying it, creating new mathematics, teaching it, and writing about it. This has left me with a personal and somewhat idiosyncratic view of the subject (or so my colleagues tell me). I would like to share a bit of this view with my readers.

Mathematics is a subject that is one of the finest, most profound intellectual creations of humans, a subject full of splendid architectures of thought. It is a subject that is also full of surprises and paradoxes. Mathematics is said to be nothing more than organized common sense, but the actuality is more complex. As I see it, mathematics and its applications live between common sense and the irrelevance of common sense, between what is possible and what is impossible, between what is intuitive and what is counterintuitive, between the obvious and the esoteric. The tension that exists between these pairs of opposites, between the elements of mathematics that are stable and those that are in flux, is a source of creative strength.

I go on to other paradoxes. Mathematics is a subject based on logic and yet formal logic is not a requirement for a higher degree. It is thought to be precise and objective with no whiff of subjectivity entering to compromise its purity, and yet it is full of ambiguities. It is the "language that nature speaks" but neither scientists nor philosophers have as yet provided cogent reasons for this.

Mathematics is also an attitude and a language that we employ by fiat and in increasing amounts to give order to our social, economic and political lives. It is a language, a method, an attitude that has diffused into medicine, cognitive science, war, entertainment, art, law, sports; which has created schools of philosophy, which has given support to views of cosmology, mysticism, and theology. All this adds up to a spectacular performance for a subject that in elementary grades has amounted to a rigid set of rules that say "do this and do that."

This book is addressed to all who are curious about the nature of mathematics and its role in society. It is neither a textbook nor a specialists' book. It consists of a number of loosely linked essays that may be read independently and for which I have tried to provide a leitmotif by throwing light on the relationship between mathematics and common sense. In these essays I hope to foster a critical attitude towards both the existence of common sense in mathematics and the ambiguous role that it can play.

These essays show how common sense bolsters or creates conflicts within the ideas and constructions of pure mathematics. Among other things, the reader will find discussions of the nature of logic and the uses of inconsistency, discussions of numbers and what to do with them, of conceptions of space, of mathematical intuition and creativity, of what constitutes mathematical proof or evidence. I will also discuss how common sense operates when mathematics together with its operations engages a wide variety of problems posed by the world of objects of people and events.

Writer Anne Fadiman tells how consternation arose when she tried to amalgamate her substantial collection of books with her husband's equally substantial collection. Each had a "common sense" system of filing; but the systems were mutually incompatible. Later, both Anne and her husband were in minor shock when they heard that after their friend's apartment had been redecorated, the decorator, exercising his own common sense, restored all the books to the shelves according to size and color.

There are more than sixty major subjects or branches of mathematics many of which have significant connections to the other branches. There is therefore no way that the mathematical corpus can be put into a linear order that is totally sensible and consistent. Though the present book alludes to only a very few of these subjects, its chapters may display a certain helter-skelter quality in their arrangement and a slight redundancy in their content. I ask for my readers' indulgence.

I hope that the readers of this book will come out with an appreciation of the flights of human imagination that both join and transcend common sense and that have created the mathematical world we live in.

The Further Reading sections in this book contain references to material that is both popular and professional.

Further Reading

Anne Fadiman. *Ex Libris: Confessions of a Common Reader*. Farrar, Straus, Giroux, 1998.

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"Mathematics and Common Sense: What Is Their Relationship?" *SIAM News*, Vol. 28, No. 3, November 1995.

"Mickey Flies the Stealth." *SIAM News*, Vol. 31, No. 3, April 1998.

"Astrology: Early Applied Mathematics?" *SIAM News*, Vol. 33, No. 5, June 2000.

"Playing Ball Probabilistically." *SIAM News*, Vol. 35, No. 3, April 2002.

"Mumbo Math." *SIAM News*, Vol. 37, No. 9, November 2004.

"When Mathematics Says No." In *No Way: The Nature of the Impossible.* Philip J. Davis and David Park, eds., pp. 172 ff. W. H. Freeman, 1987.

"Paraconsistent Thoughts about Consistency." *Mathematical Intelligencer*, Vol. 24, No. 2, Spring 2002.

"Letter to Christine." *Newsletter, European Mathematical Society*, March 2004. Also in *Matilde: Nyhedsbrevfor Dansk Matematisk Forening*, Nummer 20, Juni 2004.

"Applied Mathematics as Social Contract." *Zentralblatt für Didaktik der Mathematik*, 88/I. Also in *The Mathematics Magazine*, Vol. 61, No. 3, June 1988, pp. 139–147.

"When Is a Problem Solved?" In *Current Issues in the Philosophy of Mathematics From the Viewpoint of Mathematicians and Teachers of Mathematics,* Bonnie Gold and Roger Simons, eds. Mathematical Association of America, 2006.

To

Christa and Peter

Christine and Fritz

and

Kay

in friendship

We are enlarged by what estranges.

—Richard Wilbur, "A Wall in the Woods: Cummington"

This is the vision ... to endure ambiguity in the movement of truth and to make light shine through it.

—Jonah Gerondi (c. 1200–1263), *The Gates of Repentance*

Letters to Christina: Answers to Frequently Asked Questions

The Swiss mathematician Leonhard Euler wrote a highly admired three-volume popularization of science entitled *Letters to a Princess of Germany* (published: 1768–1772). Some time ago, my friend Christina (name changed) sent me a letter with a number of questions about mathematics, a subject about which she had heard much but knew little. I believe that her questions are among those frequently asked by the general public.

Christina's questions and my answers provide an overview of the mathematical content of this book and set the stage for the essays that follow.

Q1. What is mathematics?

A1. Mathematics is the science, the art, and a language of quantity, space, and pattern. Its materials are organized into logically deductive and very often computational structures. Its ideas are abstracted, generalized, and applied to topics outside mathematics.

The mix of mathematics and outside topics is called *applied mathematics.* Mathematics has often been called the "handmaiden of the sciences," because of its use in and interactions with the other sciences. For example, for reasons that are by no means clear, mathematics is an indispensable aid to the physical sciences. The expositions of theoretical physics are completely mathematical in character.

Applied mathematics includes *descriptions, predictions,* and *prescriptions.* Description replaces a real-world phenomenon with a mathematical surrogate: A lampshade casts a parabolic shadow on the wall. Prediction makes a statement about future events: A total eclipse of the sun will occur on July 11, 2010; its duration will be 5 minutes and 20 seconds. Prescription (or formatting) organizes our lives and actions along certain lines: Traffic lights control the flow of automobiles in a periodic fashion; tax laws affect us all.

Mathematics has intimate relations with philosophy, the arts, language, and semiotics (the theory of signs and their use). Since the development of mathematics is often inspired and guided by aesthetic considerations, mathematics can be described as "amphibious": It is both a science and one of the humanities.

Q2. Why is mathematics difficult, and why do I spontaneously react negatively when I hear the word?

A2. There are many reasons why the average person finds mathematics difficult. Some of them are poor, uninformed teaching; over-concentration on the deductive aspect of the subject; boring presentation; presentation that fails to connect mathematics with the day-to-day concerns of average people.

Mathematical thinking and manipulations are cerebral activities that, simply, not everyone enjoys. And then, let's face it, the material can be difficult, common sense seems to be irrelevant. While mathematical skills and understanding can be learned and developed, I believe there is such a thing as innate talent for mathematics. Just as not everyone has the talent to create art, write a great book, be a ballet dancer, or break athletic records, not everyone can scale the heights of mathematical understanding. The very fact that professional mathematicians make long lists of unsolved problems attests to the fact that ultimately, all mathematicians reach their own limits of mathematical accomplishment.

Q3. Why should I learn mathematics? History widened my horizons and deepened my "roots." When I learned German, it opened up cultural treasures to me. Karl Marx explained (if not changed) the society I live in. What does mathematics have to offer?

A3. I'll start by providing the time-honored reason. Mathematics trains us to think logically and deductively. I would rephrase this by saying that mathematics opens up the *possibility* of rational thought in contrast to unreasoned or irrational, or mystical, thought.

Mathematics is one of the greatest human intellectual accomplishments. We learn mathematics partly to enable us to function in a complex world with some intelligence, and partly to train our minds to be receptive to intellectual ideas and concepts. Even as natural languages such as German or Chinese embody cultural treasures, so does mathematics: One has to learn the "language" of mathematics to appreciate its treasures. To learn the history of mathematics is to appreciate its growth within the history of general ideas.

Looking about us, if we are perceptive, we can see not only the natural world of rocks, trees, and animals, but also the world of human artifacts and human ideas. The ideas of Karl Marx (and of other thinkers) explain the world along certain lines, each necessarily limited by the thinker's own constraints. Mathematics explains the world in remarkable but limited ways. It is capable of formatting our lives in useful ways, for example, when we take a number for our "next" at the deli, and of allowing us to look into the future and make prudential judgments; for example, the meteorological predictions based on mathematics theories. Every educated person should achieve some appreciation of the historical role that mathematics plays in civilization in order to give the subject both intelligent support and—when it seems necessary—intelligent resistance to its products.

Q4. How has mathematics changed in the last 100 years? What have been the dominant trends?

A4. The number of new mathematical theories produced since 1900 is enormous. During that time, mathematics has become more abstract and deeper than it was in the 19th century. By this I do not mean that simple things such as arithmetic have to be viewed by the general public in a more abstract or deeper way than before, but that the conceptualization of old mathematics and the creation and applications of new mathematics by mathematicians have had that character.

Mathematical logic is now firmly on the scene, as is set theory. A new and more abstract algebra has grown mightily and consolidated itself. Infinitary mathematics (i.e., the calculus and its elaborations) has grown by leaps and bounds, but recently has had increasingly to share the stage with finitary mathematics.

Geometry, which began in antiquity as visual and numerical (lengths, areas, volumes), moved into an abstract axiomatic/deductive mode and now includes such topics as algebraic, combinatorial, and probabilistic geometry.

A major trend since the late 1930s is to view mathematics as the study of deductive structures. Mathematics has shared a structuralist point of view with numerous non-mathematical disciplines ranging from linguistics to anthropology to literary criticism.

There is no doubt that the electronic digital computer that emerged in the middle 1940s, which is in many ways a mathematical instrument, has changed our day-to-day life noticeably. The computer has changed mathematical education. It has revitalized or widened the scope and increased the power of a number of traditional branches of mathematics. It has also created new branches of the subject and, all in all, has had a revolutionary effect throughout science and technology.

Q5. What can you tell me about the "chipification" of mathematics?

A5. A large part of the mathematics that affects our day-to-day lives is now performed automatically. Computer chips are built into our wristwatches, automobiles, cash registers, ATMs, coffee machines, medical equipment, civil-engineering equipment, word processors, electronic games, telephone and media equipment, military equipment, IDs... The list is endless, and the idea of inserting chips to do clever things has become commonplace.

The mathematics in these chips is hidden from view. The average person, though greatly affected by it, needs to pay no conscious attention to its mathematics. That is the job of the corps of experts who design, implement, monitor, repair, and improve such systems. The "chipification of the world" is going forward at a dizzying rate. One of the disturbing side effects of this trend may very well be that in the near future our sons and daughters will not learn to add or read in the conventional way. Will this put us back to several hundred years ago, when literacy and numeracy were relatively rare achievements? Not at all. Modes of communication, interpretation, and social arrangements will change; it does not mean that they will disappear.

Q6. Where are the centers that develop new mathematics? Who is "hip"? A fan of good rock music knows where to go to hear it. Where would a math fan go these days?

A6. There are mathematical centers in all the developed countries of the world. They are located in universities; governmental agencies; research hospitals; scientific, economic or social "think tanks"; and industries. All these centers tend to be organized into groups and tend to concentrate their research on very specific problems or subject matter. There are also many mathematical researchers who work independently and productively outside of groups.

Depending on what sort of "mathematical music" you would like to hear, you would visit one of these groups or individuals. Although the groups exhibit a great deal of *esprit de corps* and self-esteem and can be influential beyond their own walls, there is, in my judgment, no single group that is predominates; this diversity is a very good thing. Closer to home, there are now websites that cater to every level of mathematical interest and engagement.

Q7. How is mathematics research organized? Who is doing it, who is paying for it, and why? Lonely, harmless "riders" or highly efficient, highly organized, secret, threatening groups? Should one be scared?

A7. The forward movement of mathematics is driven by two principal forces: forces on the inside of the subject and forces on the outside.

Forces on the inside perceive certain questions or aspects that are incomplete, unanswered, and call for answers. New mathematical ideas arising from free mental play can come into prominence. "Lonely riders" very often make contributions.

Forces on the outside call for the application of available mathematics to the outside world of people or things. These applications may result in the development of genuinely new mathematics.

The work is paid for by institutions (including universities) or organizations in anticipation of profit, or of scientific, social, national, or cultural gain. The economically independent mathematical researcher exists but is quite rare. The space agency NASA is, for example, one source of considerable mathematical support.

In the years since 1939, and particularly in the United States, a great deal of mathematical research has been paid for by the military. Individual mathematicians have profited from this support independent of their personal views of the United States military or its foreign policy. Since war appears to be endemic to the human situation, this support will undoubtedly continue.

Other support comes from the medical and healthcare sector, and this support is likely to increase in importance. The productivity of the United States gets a considerable boost from the entertainment industry, and today's movies use computer graphics software with a considerable mathematical underlay.

Some of the work of the various groups is restricted or confidential. This may be for reasons of national security or for reasons of industrial confidentiality in a free and competitive market.

In an ideal democratic society, all work must ultimately come into the open so that it may be judged both for its internal operation and its effects on society. Free and complete availability of information is one of the hallmarks of an ideal science. While this has not always been the case in practice, judging from the past several centuries, the record is pretty good. Doing mathematics, developing new mathematics, is simply one type of human activity among thousands of activities. As long as constant scrutiny, judgment, and dissent flourish, fear can be reduced.

Q8. Other sciences have had breakthroughs in the 1980s and 1990s. What breakthroughs does mathematics claim?

A8. We might begin by discussing just what is meant by the phrase "a breakthrough in mathematics." If we equate breakthroughs with prizes, and this equivalence has a certain merit, then the work of, e.g., the Fields Medal winners should be cited. This would cover pure mathematics, where this prize, given since 1936, is often said to be the mathematical equivalent of the Nobel Prize. Similar prizes of substantial value exist in various pure mathematical specialties, applied mathematics, statistics, computer science, etc.

The work of Fields Medal winners over the past decades, people such as K. F. Roth, R. Thom, M. Atiyah, P. J. Cohen, A. Grothendieck,

S. Smale, A. Baker, K. Hironaka, S. P. Novikoff, and J. Thompson, cannot, with some very few exceptions, be easily explained in lay terms. In view of their arcane nature, the public rarely hears about such things. Should a particular accomplishment reach the front page of the newspapers, its meaning is often mutilated by packaging it in the silver paper of sensation. Sensation is always easier for the public to grasp than substance.

Judging from the selection of Fields Laureates, the criteria for the honor seem to be (1) the solution of old and difficult mathematical problems, (2) the unification of several mathematical fields through the discovery of cross connections and of new conceptualizations, and/or (3) new internal developments.

A few years ago the world of research mathematicians was electrified by the public announcement that the Clay Mathematics Institute, a private organization, was offering $1 million each for the solutions of seven famous mathematical problems.

We live in an age of sensation. As a result, only the sensational aspects of mathematics get much space in the newspapers: the solution to Fermat's "Last Theorem," "the greatest prime number now known is...," the solutions of other so-called "big" unsolved problems, international students' contests, etc.

Mathematical applications that are useful, e.g., the programming and chipification of medical diagnostic equipment, are rarely considered to be breakthroughs and make the front pages. Historic research or philosophic discussions of the influence of mathematics on society have yet to be honored in prestigious manners.

From the point of view of societal impact, the major mathematical breakthrough since the end of World War II is the digital computer in all its ramifications. This breakthrough involved a combination of mathematics and electronic technology and is not the brainchild of a single person; hundreds, if not hundred of thousands, of people contributed and still contribute to it.

Q9. Medical doctors fight cancer, AIDS, and SARS. What is now the greatest challenge to modern mathematics?

A9. One might distinguish between "internal" problems and "external" problems. The former are problems that are suggested by the operation of the mathematical disciplines themselves. The latter are problems that come to it from outside applications, e.g., what airplane shape has the minimum drag when the airplane surface is subject to certain geometrical conditions? What are the aerodynamic loads on the plane structure during maneuvers?

In 1900, the mathematician David Hilbert proposed a number of very significant unsolved problems internal to mathematics. Hilbert's reputation and influence was so great that these problems have been worked on steadily, and most of them have been solved. Setting up, as it did, a hierarchy of values as to what was important (every mathematician creates his own list of unsolved problems!), this list has had a considerable influence on the subsequent progress of mathematics. The solvers, in turn, have gained reputations for themselves in the mathematical community.

Within any specific field of mathematics, the practitioners will gladly tell you what they think the major unsolved problem (or challenge) is. Thus, if a topologist is queried, the answer probably is to prove the Poincaré Conjecture in the case $n = 3$. If an analyst is queried, the answer might be "Prove the Riemann Hypothesis."

As for external problems, a fluid dynamicist might say "Devise satisfactory numerical methods for processes [such as occur in turbulence or in meteorology] that develop over long periods of time." A programming theorist might say "Devise a satisfactory theory and economic practice for parallel computation."

If your question is answered in terms of specific problems, it is clear that there are many of them, and there is no agreement on how to prioritize them. Researchers can be drawn to specific problems by the desire for fame, or money, or simply because their past work suggests fruitful approaches to unsolved problems.

Your question can also be answered at a higher level of generality. A pilot assessment of the mathematical sciences prepared for the United States House (of Representatives) Committee on Science, Space, and Technology, identifies five interrelated long-term goals for the mathematical sciences. These are

- to provide fundamental conceptual and computational tools for science and technology;
- to improve mathematics education;

- to discover and develop new mathematics;
- to facilitate technology transfer and modeling;
- to promote effective use of computers.

[*Notices of the American Mathematical Society*, February 1992]

I would like to go up one more rung on the ladder of generality and answer that the greatest challenge to modern mathematics is to keep demonstrating to society that it merits society's continued support. The long history of mathematics exhibits a variety of mathematical intents. Some of these have been to discover the key to the universe, to discover God's will (thought to be formulated through mathematics), to act as a "handmaiden" to science, to act as a "handmaiden" to commerce and trade, to provide for the defense of the realm, to provide social formats of convenience and comfort, to develop a super brain—an intelligence amplifier of macro proportions.

Mathematics can and has flourished as a harmless amusement for a few happy aficionados both at the amateur and professional levels. But to have a long and significant run, mathematics must demonstrate an intent that engages the public. If the intent is simply to work out more and more private themes and variations of increasing complexity and of increasing unintelligibility to the general public, then its support will be withdrawn.

The public demands something in return for its support, but the place and the form of an acceptable return cannot be specified in advance. Perhaps a mathematical model of brain operation will be devised and will lead to insight and ultimately to the alleviation of mental disease. It is not too fatuous to think that many of the common problems that beset humankind such as AIDS, cancer, hunger, hostility, and envy, might ultimately be aided by mathematical methods and computation. But while efforts in these directions are praiseworthy, they may come to naught.

By way of summary, the greatest challenges to mathematicians are to keep the subject relevant and to make sure that its applications promote human values.

Q10. What can you say about the militarization, centralization, regionalization, and politicization of mathematics?

A10. This is a very wide-ranging question: Militarization alone would require several books. Since about 1940, one of the major financial supporters of mathematical research in the United States has been the military or the "military industrial complex"; a similar statement can be made of all the advanced nations. There are essential mathematical underlays to new, sophisticated weaponry, both offensive and defensive, and to military information processing systems. Related economic, demographic, strategic studies and predictions often involve complex mathematics.

Since the end of the Cold War, with the development of a variety of insurgencies and terrorist strategies, military options have been and are being reassessed, which will undoubtedly lead to new developments in mathematics. We can also foresee a time when the development and application of mathematics will be increasingly supported by fields such as medicine, biology, environment, transportation, finance, etc.

Centralization and regionalization: In the pre-computer days it was said that a mathematician didn't require much in the way of equipment: a few reams of paper, a blackboard, and some penetrating ideas. Today, although much, and perhaps most, research is still done that way, we find increasingly that mathematicians, particularly applied mathematicians, require the aid of supercomputers. This is still far less costly than the billions of dollars of laboratory equipment required by high-energy physicists or astrophysicists.

Centralization of mathematics occurs as the result of a number of factors, including the willingness and ability of a society to support mathematical activity and the desire to have mathematicians work in groups or centers. The notion of a critical intercommunicating mass of creative individuals is at work here. New systems of rapid intercommunication and the transport of graphical and printed material may affect the clumping of future centers for research and development.

Politicization: While mathematical content is abstract, mathematics is created by people and is often applied by people to people. It is to be expected, then, that the creation and application of mathematics should be subject to support, pressure, monitoring, and suppression by governmental, political, or even religious institutions. The interaction between mathematics and human institutions has a long and documented history.

Q11. Give me ten points that worry a concerned mathematician.

A11. A concerned mathematician will worry about the abuse, misuse, or misinterpretation of mathematics or its applications. Insofar as we are living in a thoroughly mathematized civilization, the number of concerns is necessarily vast. Many such concerns focus on "life and death" issues. If, for example, a mathematical criterion were developed via encephalography for determining when a person is brain dead, then this would engender a great deal of concern.

Q12. I read a statement attributed to the famous physicist Max Born that the destructive potential of mathematics is an imminent trend. If that is so, why should I, an average person, learn more about mathematics?

A12. You should learn more about it for precisely that reason.

All creative acts have destructive potential. To live is to be at risk, and no amount of insurance can reduce the risk to zero. Moreover, to live at the very edge of risk is thought by some people to make them feel "truly alive." Electrical outlets in the home are not totally risk-free; the destructive or revolutionary potential of graphical arts, or of literature, is well documented. Even as mathematics solves many problems, it creates new problems, both internal and external.

The more the average person knows about mathematics, the better off that person is to make judgments. Some of those judgments will be about how to temper risk with prudence. In a world in which scientific, technological and social changes occur rapidly, a democratic society cannot long endure in the presence of ignorance.

Q13. What is deep mathematics and what is not?

A13. A quick answer is the one given by logician Hao Wang: A deep theorem is a formula that is short but can be established only by long proofs.

I, however, am going to answer differently than Hao Wang by changing your question just a bit and discussing the possibility that Mathematics with a capital M differs from mathematics with a lowercase m. First, a cautionary quotation:

> Insecure intellectuals make a false and basically harmful distinction between "high" and vernacular culture, and then face enormous trouble in trying to determine a status for significant items in between, like Gershwin's *Porgy and Bess* or the best of popular science writings.
>
> —*Crossing Over: Where Art and Science Meet,* Stephen J. Gould and Rosamond Wolff Purcell

Some months ago, I worked out a certain piece of mathematics that gave me much pleasure and that I believed was new and interesting. I thought about building it into a paper and then began to think to what periodical I might appropriately send it. Then I stopped short and said to myself: "You know, there is Mathematics with a big M and mathematics with a small m. What I've done here is of the small m variety. If I send it to periodical *XYZ*, it would be rejected out of hand."

What do I mean by Mathematics with a big M and mathematics with a small m? It would be impossible for me to present a list of criteria, and my criteria would not necessarily be my colleagues' criteria, but as the saying goes: "I know them when I see them." You have most likely heard of Art with a capital A and art with a small a. Possibly you've heard of Opera with a big O (grand Opera: Wagner, Verdi) and opera with a small o (opéra comique: Offenbach, light opera, Broadway musicals). Then there is poetry, verse, light verse, and doggerel. Recent articles have dealt with the movements back and forth within the categories of big and small. The work of Norman Rockwell—a popular American magazine cover artist, once thought to be the Rolls Royce of kitsch—has now been reconsidered and elevated in the minds of art critics.

I will end by adapting a paragraph of the philosopher William James. I have changed a few of James' words so the paragraph relates not to the split of philosophers between the tough- and tender-minded, but to mathematicians.

> It suffices for our immediate purpose that the M and m kinds of material, both exist. Each of you probably knows some well-marked example of each type, and you know what the authors of each type think of the authors on the other side of the line. They have low opinions of each other. Their antagonism, whenever as individuals their temperaments have been intense, has formed in all ages a part of the mathematical atmosphere of the time. It forms part of the mathematical atmosphere today.

Despite all the fuzziness and inconsistency (often in my own mind), the dichotomy remains alive in the mathematical world. It can be the source of professional snobbism: “X is not a ‘real’ mathematician,” “Y doesn’t prove theorems,” “Z only computes.” The split infects the way professional talks are given. It can play a role in job offers, promotions, and in obtaining contracts and grants. Though attitudes change, the dichotomy is not likely to go away soon.

Letters to Christina: Second Round

Christina must have been stimulated by my answers to her questions, because, a few months later, she sent me a second set of questions. I suspect that in the meantime she must have done a bit of independent reading, because to answer her questions now required more thought and elaboration on my part.

Q14. Can mathematicians look at a formula or an expression or an open problem and tell almost immediately whether this is something deep or something easy?

A14. I've often wondered whether a musician or the conductor of an orchestra can look at a new musical score and get a feeling for what the composer has created from the printed page. Along a totally different line, my wife said to me recently, "After all these years of cooking, I think I can now look at a recipe and have an idea how the dish will turn out."

The answer to your question is yes and no and everything in between. Thesis advisors to Ph.D. candidates in mathematics often suggest problems to their students. The suggested problem should be neither too hard nor too easy. In general, the advisor bases his or her recommendations on long experience, knowledge of the mathematics, and a judgment of the capabilities of the student.

When I first went to my thesis advisor for a problem, he suggested one that I was able to solve in a few days. That was the end of that suggestion. Then he suggested a problem that was both difficult and uninteresting for me. And that was the end of that advisor. I looked around—successfully—for a new thesis advisor.

A research mathematician may attempt to solve an open problem that has been around for a while. Unless it happens that no one has else has worked on it, a reasonable assumption is that the problem is fairly difficult.

A research mathematician may very often push forward and make up a problem or a theory of his own. At this stage, he probably has only a vague idea of the depth or the difficulty that the problem presents, or where the problem might lead. He will do what he can with the problem or theory that he has created. Other mathematicians might then come along with more knowledge, technical skills, creative abilities, or just plain luck and push the theory beyond its original conception or context.

Q15. What can you say about independent or "dissident" status of mathematical arguments? There are around us hundreds of paid spin-doctors, thousands of experts, hundreds of thousands of journalists and politicians. They argue for and against many things. Can any single mathematician, on occasion, act like the little boy in the Hans Christian Andersen story of *The Emperor's Clothes*, and say "no way"?

A15. There are many disputes that can be settled by the simple mathematical operations of counting and measuring. A woman decided she would like to move a chest from room A to a place in room B. The husband said it would never fit; the wife insisted it would. The dispute was easily resolved by measuring the widths of the chest and of the space. I wish it were the case that all disputes could be resolved so easily by mathematics!

The great 17th-century mathematician and philosopher Leibniz had law training. He dreamed of a method or a calculus of reasoning that he called the *Characteristica Universalis*. Once humanity was in possession of it, if a dispute arose between individuals, it could be settled easily by a mathematical computation. Well, despite all the developments in mathematical logic since Leibniz's day, the existence of such a calculus seems to be a will-o-the-wisp.

Consider a lawsuit, either civil or criminal. Perhaps the prosecution or the defense has brought in a mathematician to testify as an "expert witness." Very likely the mathematician will produce a statement involving

Universality. Mathematical notations, symbolisms, or formalisms are part of what may be called the language of mathematics. A mathematical sentence consists partly of symbolisms and partly of natural language. This mixture of natural and symbolic languages often promotes ease of understanding by interlarding the symbols with natural language. As a young researcher in mathematics, one of my professors gave me the following piece of advice for writing mathematical papers: Embed every mathematical statement within a grammatically correct English sentence. Here is an example taken from E. C. Titchmarsh's very influential *The Theory of Functions.*

> Let $f(z)$ be an analytic function, regular in a region D and on its boundary C which we take to be a simple closed contour. If $|f(z)| \leq M$ on C, then $|f(z)| < M$ at all interior points unless $f(z)$ is a constant (when, of course, $|f(z)| = M$ everywhere).

While this paragraph could have been written entirely in mathematical symbols, it would have been both tedious and much more opaque, even to a professional reader. Mathematical language has developed over a long period of time. Many older symbolisms were actually abbreviations of natural language expressions. As new theories emerge, new symbolisms are born. Symbolisms can also die. The history of mathematics contains a graveyard of dead symbolisms.

There appears to be a process at work that may be stated as "form follows function," but the reverse is also true: "we make our language, and then our language makes us." This is called the Sapir–Whorf Hypothesis in semiotics.

In recent years, many computer languages have been developed by individuals or by groups. Many of these languages are now dead or obsolescent, replaced by new languages and new hardware. Many programming languages that were once widely used and studied now continue in at most marginal form, in "legacy code," i.e. code that is no longer supported by the manufacturers. But programming languages tend to survive in a marginal way much longer than one might guess.

Someone who knows no English may not be able to read the English part of a mathematical sentence, but since the symbolic part is universally accepted and understood, he will be able to grasp much of its meaning.